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Newton's and Halley's methods for real polynomials

DISSERTATION

Newton's and Halley's methods for real polynomials

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0.1 Introduction

Newton's method is generally introduced in calculus courses as a useful tool for finding the roots of functions when analytical methods fail. This method works well because if the initial guess is close enough to the actual root, iterations will converge quickly to the root. We are usually warned against picking a point where the derivative is zero because the function that is used in Newton's method is undefined at that point. If we expand our study of the dynamics of Newton's method to the complex plane, we then find lots of interesting properties, fractals, chaos, attracting periodic cycles, Mandelbrot sets, and other phenomena are present, depending on what type of functions we study. We will focus on Newton's method and Halley's method for polynomials of degree $d \geq 2$ with real coefficients and only real (and simple) zeros.

In the first chapter, we recall some definitions and theorems for iteration of rational functions and quasiconformal mappings. In the second chapter we illustrated the definition of Newton's method for a polynomial of degree d, and a characterization of rational map arise as Newton's method for a polynomial. Then we define the special polynomial of degree d and we determine the polynomial of any degree which satisfy the properties of special polynomials. Also we have proved that each immediate basin of the superattracting fixed point is simply connected. Finally we conclude that Julia set is connected, Newton's function is hyperbolic and Julia set is locally connected.

In chapter three, we study the dynamics of H_p (Halley's method associated with polynomials of degree d with real coefficients and only real (and simple) zeros). We also establish certain dynamical properties of Halley's method for a polynomials and simple connectivity of the immediate basins of the superattracting fixed points of Halley's method. But in case of H_p, we can not prove that each immediate basin $A^*(x_k)$, $2 \leq k \leq d-1$, contains exactly 4 critical points, that is degree of H_p on $A^*(x_k)$ is exactly equal 5. Therefore, we can not use the theorem of quasiconformal surgery procedure as in the case of Newton's method, then we just exhibit Halley's method for special polynomials $(H_{p_3}, H_{p_4}, H_{p_5})$, and we show that, we can conjugate them to rational maps R_3, R_4, R_5 respectively with properties that all the critical points are located at the superattracting fixed points. Thus we conclude that $H_{p_3}, H_{p_4},\ and\ H_{p_5}$ are hyperbolic maps and Julia sets of them are locally connected.

In chapter four, we describe Halley's method for a polynomials with multiply real roots, and the dynamics of Householder's method for real polynomials. Finally in chapter five, we have included a brief section which describes the general form of Kœnig's root-finding algorithms.

Chapter 1

Iteration of Rational Functions

Let

$$R(z) = \frac{P(z)}{Q(z)} \tag{1.0.1}$$

be a rational function of degree $d \geq 2$, then the nth iterates of R are denoted by R^n, that is, $R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n \ times}$.

Definition 1.0.1. A rational map $R : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is a holomorphic dynamical system on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Any such map can be written as a quotient $R(z) = P(z)/Q(z)$ of two relatively prime polynomials P and Q. The degree of R can be defined topologically or algebraically; it is the number of pre-images of a typical point z, as well as the maximum of degrees of P and Q.

The fundamental problem in the dynamics of rational maps is to understand the behavior of high iterates $R^n(z) = R \circ R^{n-1}(z)$.

Definition 1.0.2. The periodic points z_0, where

$$z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0) = z_0,$$

can be classified depending on the value of $\lambda = (R^n)'(z_0)$ as follows:

(1) superattracting if $\lambda = 0$;

(2) attracting if $0 < |\lambda| < 1$;

(3) repelling if $|\lambda| > 1$;

(4) neutral if $|\lambda| = 1$.

Remark 1.0.1. The study of Newton's method and all rational functions are interesting because the attracting basins are not the only sets of points present in the plane. There are points in the complex plane that are not members of basins of attraction for attracting fixed points of a rational function R. Based on algebra,

there are solutions to $R^n(p) = p$, yielding periodic points. These points can not be in any fixed point basin, because that would imply that they converge to the fixed point, contradicting the periodicity.
We can bring much of what we have discussed so far, by looking at Figure (1.1). This is the dynamic plane of Newton's method for a polynomial $p(z) = z^3 - 1$. Each

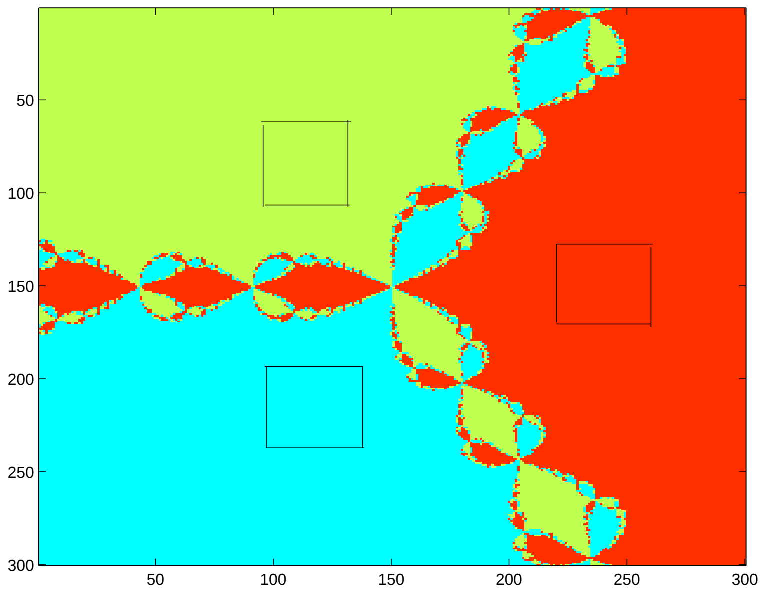


Figure 1.1: Newton's function for the polynomial $p(z) = z^3 - 1$.

pixel is used as an initial value for iterating N_p. Once iteration yields a value within the black square around a root, the initial value pixel is colored according to that root. We can see that the boundary of each basin touches each of the other basins. The boundary is also where we find the Julia set, which has even more interesting behavior.

Definition 1.0.3. The basin of attraction for a fixed point z_0 is:

$$A(z_0) = \{z \in \mathbb{C} : R^n(z) \longrightarrow z_0 \; as \; n \longrightarrow \infty\}.$$

Definition 1.0.4. Let z_0 be a (super)attracting fixed point, then the connected component of $A(z_0)$ containing z_0 is called the immediate basin of attraction of z_0, and denoted by $A^*(z_0)$.

1.1 Fatou and Julia Sets

A complex analytic map always decomposes the plane into two disjoint subsets, one is the stable set and is called the Fatou set, in which the dynamics are relatively tame, and the other one is the Julia set, in which the map is chaotic.

Definition 1.1.1. A family of complex analytic functions $\{F_n\}$, that is defined on a domain D is called a normal family if every infinite sequence of maps from $\{F_n\}$ contains a subsequence which converges uniformly on every compact subset of D.

Definition 1.1.2. The Fatou set F of R is defined to be the set of points $z_0 \in \hat{\mathbb{C}}$ such that $\{R^n\}$ is a normal family in some neighborhood of z_0. The Julia set is the complement of the Fatou set.

Theorem 1.1.1. *([Ste93]page 28) The Julia set (J_R) of a rational function R is nonempty.*

Theorem 1.1.2. *([CG92] page 55) The Julia set (J_R) contains all repelling fixed points and all neutral fixed points that do not correspond to Siegel disks. The Fatou set F contains all attracting fixed points and all neutral fixed points corresponding to Siegel disks.*

Definition 1.1.3. If R is a map of a set X into itself, a subset E of X is:

1. forward invariant if $R(E) \subseteq E$;
2. backward invariant if $R^{-1}(E) \subset E$;
3. completely invariant if $R(E) = E = R^{-1}(E)$.

Theorem 1.1.3. *([Ste93] page 28) The Julia set and the Fatou set are completely invariant.*

Definition 1.1.4. Let R be a rational map of degree at least two with Fatou set F then the forward invariant component F_0 of F is:

(a) an attracting component if it contains an attracting fixed point ζ of R;

(b) a super-attracting component if it contains a super-attracting fixed point ζ of R;

(c) a parabolic component if there is a neutral fixed point ζ of R on the boundary of F_0, and if $R^n \longrightarrow \zeta$ on F_0;

(d) a Siegel disc if $R : F_0 \longrightarrow F_0$ is analytically conjugate to a Euclidean rotation of the unit disc onto itself;

(e) a Herman ring if $R : F_0 \longrightarrow F_0$ is analytically conjugate to a Euclidean rotation of some annulus onto itself.

1.2 Connectivity of the Julia set

There are several papers concerning the connectivity of the Julia set for Newton's method. Przytycki [Prz89] proved that every root of p has a simply connected immediate basin. Meier [Mei98] proved the connectivity in the case of degree 3.
In [Shi90] Shishikura has studied the relationship between the connectivity of Julia sets for rational maps and their fixed points which are repelling or parabolic with multiplier 1, and this study was more complicated than our case. As a consequence, he has proved that if a rational map has only one such fixed point, then its Julia set is connected. This is the case for Newton's method for polynomials, which has one repelling fixed point at infinity. We can get these two corollaries from his paper.

Corollary 1.2.1. *([Shi90] Shishikura) If a rational map has only one fixed point which is repelling or parabolic with multiplier 1, then its Julia set is connected. In other words, every component of the complement of the Julia set is simply connected. In particular, the Julia set of the Newton's method for a non-constant polynomial is connected.*

Corollary 1.2.2. *([Shi90] Shishikura) If the Julia set of a rational map R is disconnected, then there exist two fixed points of R such that each of them is either repelling or parabolic with multiplier 1, and they belong to different components of the Julia set.*

Theorem 1.2.3. *([Bea91] page 81) Let R be a rational map. Then $J(R)$ is connected if and only if each component of $F(R)$ is simply connected.*

Definition 1.2.1. A set $X \subset \mathbb{C}$ is said to be locally connected at a point $x \in X$ if there exist arbitrary small connected neighborhoods of x in X, that is for each neighborhood U of x there exists a neighborhood $V \subset U$ of x such that $V \cap X$ is connected. The set X is said to be locally connected if it is locally connected at each point.

1.3 Critical Points

Critical points and their forward orbits play a key role in complex dynamical systems.
Recall that a critical point of R is a point on the sphere where R is not locally one-to-one. These consist of solutions of $R'(z) = 0$.
The following two theorems and the corollary show the relation between the dynamics of a rational map R and their critical points.

Theorem 1.3.1. *([Bea91] page 43) For any nonconstant rational map R,*

$$\sum [\nu_R(z_0) - 1] = 2\ deg(R) - 2, \tag{1.3.1}$$

where $\nu_R(z_0)$ is called the valency, or order, of R at z_0, and it is the number of solutions of $R(z) = R(z_0)$ at z_0.

Corollary 1.3.2. *([Bea91] page 43) A rational map of positive degree d has at most $2d-2$ critical points. A polynomial of positive degree d has at most $d-1$ finite critical points.*

Theorem 1.3.3. *([Dev94] page 78) Let R be a rational map of degree at least two. Then the immediate basin of each (super)attracting cycle of R contains a critical point of R.*

Definition 1.3.1. The post-critical set is the closure of the set of orbits of the critical points.

Definition 1.3.2. A rational map R is called hyperbolic if its Julia set and its post-critical set are disjoint $P(R) \cap J(R) = \phi$.

Definition 1.3.3. A rational map R is expanding map on its Julia set if there are positive numbers c and λ, with $\lambda > 1$ and

$$|(R^n)'(z)| \geq c\lambda^n, \qquad n \geq 1,$$

on J. We then call λ a dilatation constant of R on J.

1.4 Quasiconformal Surgery Procedure

Quasiconformal surgery can be used to convert attracting fixed points to superattracting fixed points. Therefore quasiconformal maps play an important role in the field of dynamical systems. The main step in this thesis was done by using a theorem on quasiconformal surgery. The idea of quasiconformal maps begin with a generalization of the Cauchy-Riemann equations. Given any function f with continuous partial derivatives in a domain D, then there are two differential operators:

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial z} = \frac{1}{2i}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

Observe that if f is analytic, then:

$$\frac{\partial f}{\partial \overline{z}} = 0, \quad \frac{\partial f}{\partial z} = f'(z).$$

The generalization of these equations is the Beltrami equation:

$$\frac{\partial f}{\partial \overline{z}} = \mu\frac{\partial f}{\partial z}, \tag{1.4.1}$$

where μ is some suitable complex-valued function on D. The basic idea is that if $\mu = 0$ throughout D, then any sufficiently smooth solution of f of (1.4.1) is analytic in D, while in the general solution, μ is taken as a measure of the deviation of a solution f from conformality. Now let a domain D and a Beltrami coefficient μ on D, we say that a homeomorphism map f on D is quasiconformal with complex dilatation μ on D if f is the solution in the distributional sense of (1.4.1) in D. Note that if $|\mu| < 1$, then f preserves orientation.

Theorem 1.4.1. *([CG92] page 106) Let U be a simply connected component of the Fatou set F that contains an attracting fixed point, on which R is m to 1. Then there are a rational function R_0, a quasiconformal homeomorphism ψ of $\hat{\mathbb{C}}$, and a compact subset E of U, such that ψ is analytic on $U \setminus E$, ψ is analytic on all components of F not iterated to U, $\psi \circ R \circ \psi^{-1} = R_0$ on $\psi(U \setminus E)$, and R_0 on $\psi(U)$ is conjugate to ζ^m on the unit disc Δ, and $degR_0 = degR$.*

Definition 1.4.1. The maps $R : U \longrightarrow U$ and $\tilde{R} : V \longrightarrow V$ are called conformally conjugate if there exists a conformal homeomorphism $\phi : U \longrightarrow V$ such that $\phi \circ R = \tilde{R} \circ \phi$.

Proposition 1.4.2. *If polynomials $p, \tilde{p}$ have Newton's functions $N_p, N_{\tilde{p}}$, which are conformal conjugate by $\varphi(z) = az + b$, i.e. $N_{\tilde{p}} = \varphi^{-1} \circ N_p \circ \varphi(z)$. Then $\tilde{p}(z) = \beta p(\varphi(z))$, where β is an constant.*

Proof. Since $\varphi(z) = az + b$, then $\varphi^{-1}(z) = \frac{z-b}{a}$, and $N_{\tilde{p}}(z) = \varphi^{-1} \circ N_p \circ \varphi(z)$, then $N_{\tilde{p}}(z) = z - \frac{p(\varphi(z))}{ap'(\varphi(z))}$, implies $(\log \tilde{p}(z))' = (\log p(\varphi(z)))'$, thus $\tilde{p}(z) = \beta p(\varphi(z))$. $\square$

Chapter 2

Newton's method and complex dynamical system

Let
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d \tag{2.0.1}$$
be a polynomial with real coefficients and only real (and simple) zeros x_k, $1 \leq k \leq d$. Let
$$N(z) = z - \frac{p(z)}{p'(z)} \tag{2.0.2}$$
be Newton's function associated with p. Thus the fixed points of N are the zeros of p together with ∞. Differentiating we find that:
$$N'(z) = \frac{p(z)p''(z)}{(p'(z))^2}, \tag{2.0.3}$$
and $N'(x_k) = 0$, this means that the zeros of p are super-attracting fixed points of N. If $|z|$ is large, $N(z) \sim z(1 - \frac{1}{d})$, where d is the degree of p, so ∞ is a repelling fixed point of N.

If $N'(z) = 0$, then $p(z) = 0$ at $x_1, x_2, \ldots, x_d \in \mathbb{R}$ or $p''(z) = 0$ at $\zeta_2, \zeta_3, \ldots, \zeta_{d-1} \in \mathbb{R}$. Thus $N(z)$ has $2d - 2$ critical points.

2.1 Basic properties of Newton's function

Throughout this section, p will denote a polynomial from $\mathbb{R}$ to $\mathbb{R}$. We will start with the following assumption:

(i) If $p(x) = 0$, then $p'(x) \neq 0$. If $p'(x) = 0$, then $p''(x) \neq 0$. As we have said before
$$N(x) = x - \frac{p(x)}{p'(x)} \tag{2.1.1}$$

denotes the Newton's function associated with p, the fundamental property of N is that it transforms the problem of finding roots of p into a problem of finding attracting fixed points of N.

(ii) $p(x) = 0$ if and only if $N(x) = x$. Moreover, if $p(\alpha) = 0$, then $N'(\alpha) = 0$ so $N^k(x) \longrightarrow \alpha$ for all x near α.

Definition 2.1.1. If $c_1 < c_2$ are consecutive roots of $p'(x)$, then the interval (c_1, c_2) is called a band for N.

Definition 2.1.2. If $p'(x)$ has the largest (respectively, smallest) root c (respectively, b), then the interval (c, ∞) (respectively, $(-\infty, b)$) is called an extreme band for N.

(iii) If (c_1, c_2) is a band for N that contains a root of $p(x)$, then

$$\lim_{x \longrightarrow c_1^+} N(x) = +\infty, \qquad \lim_{x \longrightarrow c_2^-} N(x) = -\infty.$$

Proof. (iii) Since $\frac{p(x)}{p'(x)} < 0$ in (c_1, x_k), thus $N(x) > x$ in (c_1, x_k) and $\lim_{x \longrightarrow c_1^+} N(x) = +\infty$. Also $\frac{p(x)}{p'(x)} > 0$ in (x_k, c_2), thus $N(x) < x$ in (x_k, c_2) and $\lim_{x \longrightarrow c_2^-} N(x) = -\infty$. $\square$

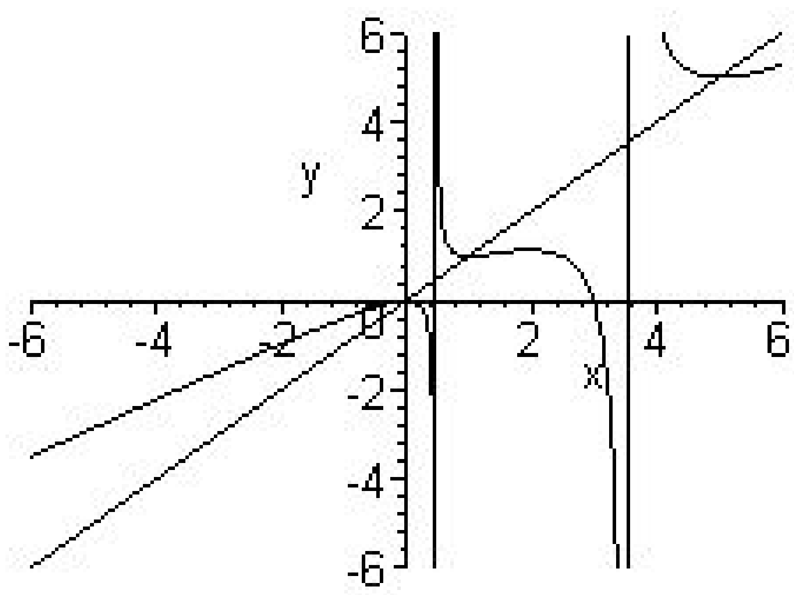


Figure 2.1: Newton's function for the polynomial $p(x) = (x-1)x(x-5)$.

2.2 Special polynomial

In this section we want to determine all polynomials with distinct real zeros, $-1, x_2, x_3, \ldots, x_{d-1}, 1$, where $x_2, x_3, \ldots, x_{d-1} \in (-1, 1)$, such that $p''(x_k) = 0$, where $2 \leq k \leq d-1$. Let us start with $d = 3$, then we have

$$p(z) = (z^2 - 1)(z - a), \tag{2.2.1}$$

then

$$p'(z) = 3z^2 - 2az - 1,$$

and

$$p''(z) = 6z - 2a.$$

Since we need $p''(z) = 0$ at $z = a$, it follows that $a = 0$, then the special polynomial of degree 3 under the above conditions is

$$p(z) = (z^2 - 1)z = z^3 - z. \tag{2.2.2}$$

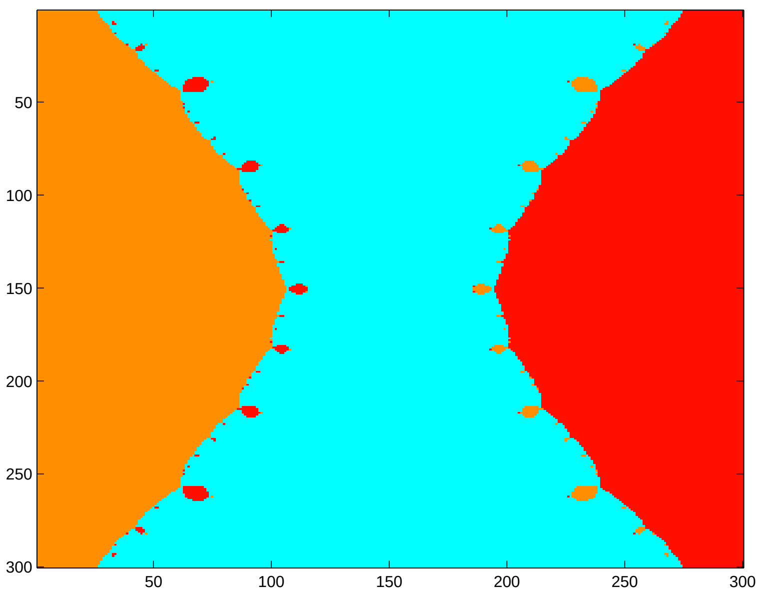


Figure 2.2: Iteration of Newton's function for the polynomial $p(z) = z^3 - z$.

Now take $d = 4$, so $p(z)$ will be in this form

$$p(z) = (z^2 - 1)(z - a)(z - b), \tag{2.2.3}$$

then

$$p'(z) = 4z^3 - 3(a + b)z^2 + 2(ab - 1)z + (a + b),$$

and

$$p''(z) = 12z^2 - 6(a + b)z + 2(ab - 1),$$

again we need $p''(a) = 0$ and $p''(b) = 0$ it follows that

$$0 = 12a^2 - 6(a+b)a + 2(ab-1),$$

and

$$0 = 12b^2 - 6(a+b)b + 2(ab-1),$$

by solving the last two equations, we find out that $a = \pm b$ and since the roots are distinct, we exclude $a = b$, then we have

$$0 = 12a^2 - 6(a+b)a + 2(ab-1),$$

take $b = -a$, we get $a = \pm\frac{1}{\sqrt{5}}$ and $b = -a = \mp\frac{1}{\sqrt{5}}$.
Thus the special polynomial of degree 4 under the above conditions is

$$p(z) = (z^2 - 1)(z^2 - \frac{1}{5}). \tag{2.2.4}$$

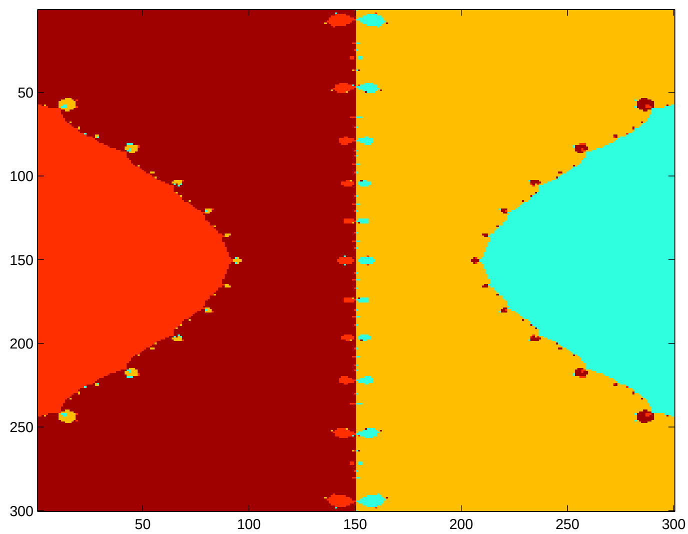


Figure 2.3: Iteration of Newton's function for the polynomial $p(z) = (z^2 - 1)(z^2 - \frac{1}{5})$.

Now suppose we have a polynomial $p(z)$ of degree d with distinct real zeros $-1, x_2, x_3, \ldots, x_{d-1}, 1$, where $x_2, x_3, \ldots, x_{d-1} \in (-1, 1)$, such that $p''(x_k) = 0$, where $2 \leq k \leq d-1$. Then it is easy to see that the function

$$\frac{p''(z)}{p(z)}(z^2 - 1), \tag{2.2.5}$$

does not have zeros or poles. Thus

$$\frac{p''(z)}{p(z)}(z^2 - 1) = constant, \tag{2.2.6}$$

and since

$$\lim_{z \longrightarrow \infty} \frac{p''(z)}{p(z)}(z^2 - 1) = d(d-1),$$

then

$$\frac{p''(z)}{p(z)}(z^2 - 1) = d(d-1).$$

Follows, equivalently

$$p''(z)(z^2 - 1) - d(d-1)p(z) = 0, \tag{2.2.7}$$

or

$$p(z) = \frac{p''(z)(z^2 - 1)}{d(d-1)}. \tag{2.2.8}$$

So by equaling coefficients we have two cases:

1. If d is even, then $a_d = 1$, $a_{d-(2b-1)} = 0$, where $b = 1, 2, ..., \frac{d}{2}$, and the other coefficients are given by

$$a_i = \frac{(i+2)(i+1)a_{i+2}}{i(i-1) - d(d-1)}. \tag{2.2.9}$$

2. If d is odd, then $a_d = 1$, $a_{d-(2b-1)} = 0$, $b = 1, 2, ..., \frac{d+1}{2}$, and the other coefficients are given by

$$a_i = \frac{(i+2)(i+1)a_{i+2}}{i(i-1) - d(d-1)}. \tag{2.2.10}$$

Suppose that d=5, then the special polynomial solution of the equation (2.2.7) exist uniquely and is given by

$$p_5(z) = a_5 z^5 + a_3 z^3 + a_1 z, \tag{2.2.11}$$

where $a_5 = a_d = 1$, by using the equation (2.2.10), we find that $a_3 = \frac{-10}{7}$ and $a_1 = \frac{3}{7}$, thus

$$p_5(z) = z^5 - \frac{10}{7}z^3 + \frac{3}{7}z. \tag{2.2.12}$$

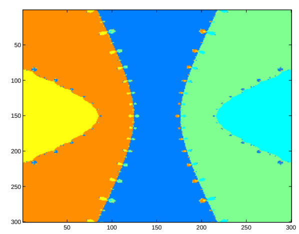


Figure 2.4: Iteration of Newton's function for the polynomial $p_5(z) = z^5 - \frac{10}{7}z^3 + \frac{3}{7}z$.

Now consider $d = 6$, then the special polynomial solution takes the form

$$p_6(z) = z^6 - \frac{5}{3}z^4 + \frac{5}{7}z^2 - \frac{1}{21}. \qquad (2.2.13)$$

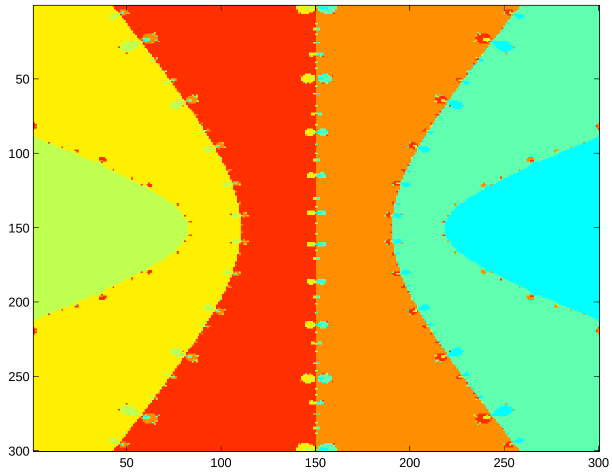


Figure 2.5: Iteration of Newton's function for the polynomial $p_6(z) = z^6 - \frac{5}{3}z^4 + \frac{5}{7}z^2 - \frac{1}{21}$.

Thus now we can find the special polynomial solution of the equation (2.2.7) for any degree, for example take a look at figure (2.6) and figure (2.7).

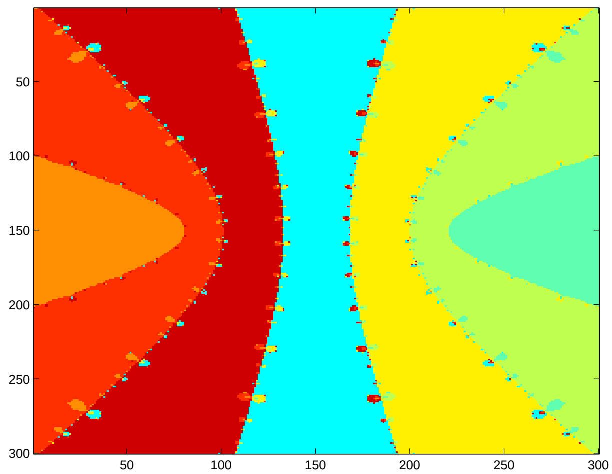


Figure 2.6: Iteration of Newton's function for the polynomial $p_7(z) = z^7 - \frac{21}{11}z^5 + \frac{35}{33}z^3 - \frac{5}{33}z$, where $p_7(z)$ is a polynomial solution of equation (2.2.7).

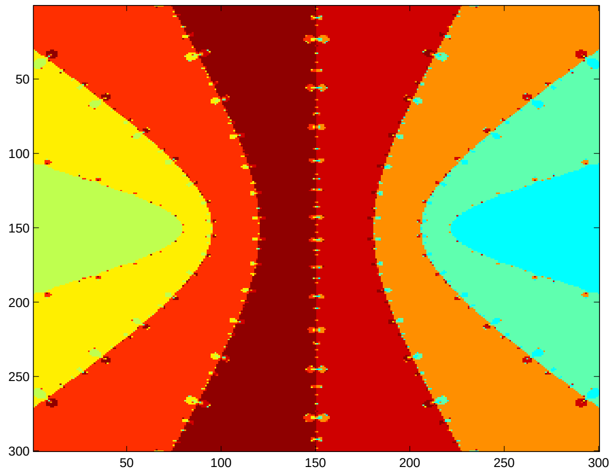


Figure 2.7: Iteration of Newton's function for the polynomial $p_8(z) = z^8 - \frac{28}{13}z^6 + \frac{210}{143}z^4 - \frac{140}{429}z^2 + \frac{5}{429}$, where $p_8(z)$ is a polynomial solution of equation (2.2.7).

2.3 Immediate Basins of Newton's Function

In this section, we want to show that each immediate basin of fixed points of Newton's function for a real polynomial is simply connected, so that we can apply theorem (1.4.1) and we can use the quasiconformal surgery.

Theorem 2.3.1. *Let*

$$p(z) = a_d z^d + a_{d-1}z^{d-1} + \ldots + a_1 z + a_0$$

be a polynomial with real coefficient and only real (and simple) zeros x_k, $1 \leq k \leq d$, and let (c_1, c_2) be a band containing x_k and ζ_k (zero of $p''(x)$ which is a free critical point), so the interval $[x_k, \zeta_k] \subset (c_1, c_2)$. Then the interval between the fixed point x_k and free critical point ζ_k is mapped by N into itself, i.e. $N[x_k, \zeta_k] \subset [x_k, \zeta_k]$.

Proof. We know that $N(x_k) = x_k$, $N'(x_k) = 0$, $N'(\zeta_k) = 0$, and we have to consider two cases which are $x_k < \zeta_k$ or $x_k > \zeta_k$. So let us start with the first case where $x_k < \zeta_k$. In the previous section we have proved that

$$\lim_{x \longrightarrow c_1^+} N(x) = +\infty, \qquad \lim_{x \longrightarrow c_2^-} N(x) = -\infty.$$

Then from $N(x) > x_k$ in (c_1, x_k), it follows that

$$N(x) > x_k \quad in \quad (x_k, \zeta_k), \tag{2.3.1}$$

and since $N(x) < \zeta_k$ in (ζ_k, c_2), then

$$N(x) < \zeta_k \quad in \quad (x_k, \zeta_k), \tag{2.3.2}$$

from (2.3.1) and (2.3.2), it follows

$$N[x_k, \zeta_k] \subset [x_k, \zeta_k]. \tag{2.3.3}$$

The second case is proved in the same manner. It also follows that $\zeta_k \in A^*(x_k)$, where $A^*(x_k)$ is the immediate basin of attraction of x_k. $\square$

Theorem 2.3.2. *Let $p(z) = a_d z^d + a_{d-1} z^{d-1} + \ldots + a_1 z + a_0$ be a polynomial with distinct and real roots x_k, $1 \leq k \leq d$, and let N be Newton's function associated with $p(z)$. Then the immediate basin of each x_k, $1 \leq k \leq d$, is simply connected.*

Proof. We know that x_k, $1 \leq k \leq d$, are superattracting fixed points, and ζ_k, $2 \leq k \leq d-1$, are free critical points of $N_p(z)$. Let $V = \Delta(x_k, \epsilon) = \{z : |z - x_k| < \epsilon\} \subset A^*(x_k)$, such that $N(\zeta_k) \notin V$. Let V_1 be the component of $N^{-1}(V)$ which contains x_k. Since $\zeta_k \notin V_1$, it follows $N : V_1 \xrightarrow{2:1} V$, and by the Riemann Hurwitz formula, it follows that V_1 has the connectivity number $n_1 = 1$. Let V_μ be the component of $N^{-\mu}(V)$ which contains x_k, let μ be the unique integer such that $N(\zeta_k) \in V_\mu$ and $\zeta_k \notin V_\mu$, then V_μ is simply connected too. Since $V_{\mu+1}$ contains x_k and ζ_k, $N : V_{\mu+1} \xrightarrow{3:1} V_\mu$ has degree 3 with two critical points, by the previous formula, it follows that V_{i+1} has the connectivity number $n_{\mu+1} = 1$. Since there is no more critical points in $A^*(x_k)$, thus $A^*(x_k)$ is simply connected. $\square$

2.4 The Conjugation of Newton's function

In this section, we want to conjugate Newton's function for $p(z)$ to Newton's function for a spacial polynomial which is a unique solution for the equation (2.2.7).

Theorem 2.4.1. *Let $N(z)$ be the Newton's function associated with $p(z)$, where $p(z)$ is a polynomial with distinct real (and simple) roots x_k, $1 \leq k \leq d$. Then N is quasiconformally conjugate to some function R with real superattracting fixed points $\tilde{x}_k$, $1 \leq k \leq d$, and $R'(\tilde{x}_k) = R''(\tilde{x}_k) = 0$.*

Remark 2.4.1. The proof will be based on theorem (1.4.1). We have to choose ψ which is in theorem (1.4.1) to be symmetric with the real line. So when we apply ψ on U all simple and real super-attracting fixed points will remain real.

Proof. (**Theorem**(2.4.1)). Now we have to apply theorem (1.4.1) on each simply connected component U of the Fatou set of our Newton's function N. Since we have $d-2$ basins, each one contains two critical points, one is a super-attracting fixed point x_k, $1 \leq k \leq d$, and the other one is a free critical point ζ_k, $2 \leq k \leq d-1$. So by applying above theorem $d-2$ times, and since ψ is symmetric with the real line then each free critical point ζ_k, $2 \leq k \leq d-1$ will be located at the super-attracting fixed point $\tilde{x}_k$, $2 \leq k \leq d-1$. Thus we have a new rational function R having d distinct real super-attracting fixed points, $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_d$, with $R'(\tilde{x}_k) = R''(\tilde{x}_k) = 0$. $\square$

Note that the Julia set of R is the ψ-image of the Julia set of N, and the dynamics of R outside $\psi(U)$ are the same as those of N outside U.

Proposition 2.4.2. *The rational function R which we get from theorem (2.4.1) is the Newton's function of some polynomial $\tilde{p} = \prod_{k=1}^{d}(z - \tilde{x}_k)$.*

Proof. Since $z = \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_d$ and ∞ are fixed points of R with $R'(\tilde{x}_j) = 0$, $j = 1, 2, \ldots, d$ and $R''(\tilde{x}_j) = 0$, $j = 2, 3, \ldots, d-1$, R must be of the form

$$R(z) = z - \left(\sum_{j=1}^{d} \frac{c_j}{z - \tilde{x}_j}\right)^{-1}. \tag{2.4.1}$$

Then after some elementary calculation we obtain

$$R'(\tilde{x}_1) = 1 - \frac{c_1(\tilde{x}_1 - \tilde{x}_2)^2(\tilde{x}_1 - \tilde{x}_3)^2 \ldots (\tilde{x}_1 - \tilde{x}_d)^2}{[c_1(\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_1 - \tilde{x}_3) \ldots (\tilde{x}_1 - \tilde{x}_d)]^2}.$$

But we have $R'(\tilde{x}_1) = 0$, hence

$$\frac{c_1(\tilde{x}_1 - \tilde{x}_2)^2(\tilde{x}_1 - \tilde{x}_3)^2 \ldots (\tilde{x}_1 - \tilde{x}_d)^2}{[c_1(\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_1 - \tilde{x}_3) \ldots (\tilde{x}_1 - \tilde{x}_d)]^2} = 1,$$

which implies that $c_1 = 1$. In the same way we can prove that $c_j = 1$, for all $j = 1, 2, \ldots, d$, thus

$$R(z) = z - \left(\sum_{j=1}^{d} \frac{1}{z - \tilde{x}_j}\right)^{-1}. \tag{2.4.2}$$

Now its easy to see that

$$\left(\sum_{j=1}^{d} \frac{1}{z - \tilde{x}_j}\right)^{-1} = \left(\frac{\tilde{p}'}{\tilde{p}}\right)^{-1},$$

where

$$\tilde{p}(z) = (z - \tilde{x}_1)(z - \tilde{x}_2) \ldots (z - \tilde{x}_d).$$

Then

$$R(z) = z - \frac{\tilde{p}(z)}{\tilde{p}'(z)},$$

which is Newton's function of some polynomial $\tilde{p} = \prod_{k=1}^{d}(z - \tilde{x}_k)$. □

Proposition 2.4.3. *Let R be as in theorem (2.4.1), associated with $\tilde{p}(z) = \prod_{k=1}^{d}(z - \tilde{x}_k)$, then $\tilde{p}$ is (up to normalization) the special polynomial of degree d.*

Proof. We know that R is Newton's function associated with a polynomial $\tilde{p}(z)$ of degree d with d distinct real zeros $\tilde{x}_k$, $1 \le k \le d$, and $\tilde{p}''$ has $d-2$ distinct real zeros $\tilde{x}_k$, $2 \le k \le d-1$. Then it follows that

$$\frac{\tilde{p}''(z)}{\tilde{p}(z)}(z-\tilde{x}_1)(z-\tilde{x}_d)$$

does not have any zero or pole, thus

$$\frac{\tilde{p}''(z)}{\tilde{p}(z)}(z-\tilde{x}_1)(z-\tilde{x}_d) = constant = d(d-1), \tag{2.4.3}$$

and

$$\tilde{p}''(z)(z-\tilde{x}_1)(z-\tilde{x}_d) - d(d-1)\tilde{p}(z) = 0. \tag{2.4.4}$$

Now we can move the interval $[\tilde{x}_1, \tilde{x}_d]$ to the interval $[-1, 1]$ by using the following transformation

$$\phi(\tilde{x}) = \frac{2\tilde{x}}{\tilde{x}_d - \tilde{x}_1} + \frac{\tilde{x}_1 + \tilde{x}_d}{\tilde{x}_1 - \tilde{x}_d}. \tag{2.4.5}$$

So by the above transformation we can figure out a new polynomial $p(z) \not\equiv 0$ which satisfy the following equation

$$p''(z)(z^2-1) - d(d-1)p(z) = 0. \tag{2.4.6}$$

This equation equals (2.2.7), and the polynomial solution is unique. If d is even, then

$$p(z) = z^d + \sum_{b=1}^{\frac{d}{2}} a_{d-2b} z^{d-2b}, \tag{2.4.7}$$

and if d is odd, then

$$p(z) = z^d + \sum_{b=1}^{\frac{d-1}{2}} a_{d-2b} z^{d-2b}, \tag{2.4.8}$$

where

$$a_{d-2b} = a_i = \frac{(i+2)(i+1)a_{i+2}}{i(i-1) - d(d-1)}. \tag{2.4.9}$$

Thus p is the normalized special polynomial of degree d. □

Conclusion 1. Now we have conjugated Newton's function for a real polynomial by a quasiconformal mapping to Newton's function for a special polynomial. Since we have proved that each component of Fatou set of N_p is simply connected, we have also that all critical points of N_p are located at the superattracting fixed points of N_p. Thus the following theorem applies.

Theorem 2.4.4. *([Bea91] page 225) Suppose that the forward orbit of each critical point of a rational function R accumulates at a (super)attracting cycle of R. Then R is expanding on J.*

Now we can say that N_p is expanding on J. On the other hand we can use the following proposition.

Proposition 2.4.5. *([Ste93] page 118) If $J \cap C^+ = \phi$, then f is expanding on J, i.e., there exist numbers $\delta > 0$ and $k > 1$ such that*

$$|(f^n)'(z)| \geq \delta k^n$$

for $n = 1, 2, \ldots$ and $z \in J$.

Applying the following.

Theorem 2.4.6. *([Mil99] page 191) If the Julia set of a hyperbolic map is connected then it is locally connected.*

Implies that Julia set of N_p is locally connected.

2.5 Appendix. Good starting points

It is a fundamental problem to find all roots of a complex polynomial p. Newton's method is one of the most widely known numerical algorithms for finding the roots of complex polynomials. For most starting points of most polynomials, it does not take to long to find good approximations of a root. But there are problems, since there are starting points which never converge to a root under iteration of Newton's method (for example all the points on the boundaries of the basins of all the roots). Therefore, we suggest good, in other word convenient, starting points to find all real roots of a polynomials of degree d with d distinct real roots. These good starting points are ζ_k, $2 \leq k \leq d-1$, where ζ_k are zeros of $p''(z)$, and that is because we have proved that $N^n(\zeta_k) \to x_k$. So we can describe the algorithm as follows:

1. If d is even, then we find the zeros of $p^{(d-2)}(z)$, where $p^{(d-2)}(z)$ is the derivative of $p(z)$, $d-2$ times. So they are two good starting points for iteration of $N_p(z)$ to find the two roots in the middle which are $x_{\frac{d}{2}}, x_{\frac{d}{2}+1}$. Then we find the zeros of $p^{(d-4)}(z)$, so we get another two starting points for iteration of $N_p(z)$ to find another two roots which are $x_{\frac{d}{2}-1}, x_{\frac{d}{2}+2}$. We keep going by the same steps until we find the zeros of $p''(z)$, then we have two starting points for iteration of N_p to find the two roots x_2, x_{d-1}. Now for the last two roots which are x_1, x_d, we can start the iteration of $N_p(z)$ with any points $x < x_1$ to get the root x_1, and with any point $x > x_d$ to get the root x_d.

2. If d is odd, then we find the zero of $p^{(d-1)}(z)$, so it will be a good starting point for iteration of $N_p(z)$ to find the middle root which is $x_{\frac{d+1}{2}}$. Then we find the zeros of $p^{(d-3)}(z)$, so we get another two starting points for iteration of $N_p(z)$ to find the two roots which are $x_{\frac{d-1}{2}}, x_{\frac{d+3}{2}}$. Now we follow the same steps until we find the zeros of $p''(z)$, then we have two starting points for iteration of N_p to find the two roots x_2, x_{d-1}. Finally for the two roots x_1, x_d, we can find them as in case one.

2.6 Newton's method with multiply fixed points

In this section, we are enthusiastic in studying the dynamics of Newton's function of polynomial with multiply real zeros.

Remark 2.6.1. Let $p(z) = \prod_{i=1}^{d}(z - x_i)^{n_i}$, then $N_p(z)$ has the following properties:

(i) The points x_i are (super)attracting fixed points with multipliers $\frac{n_i - 1}{n_i}$, and when $n_i = 1$ the local degree of N_p at x_i is at least 2.

(ii) The point ∞ is a repelling fixed point with multiplier $\frac{d}{d-1}$.

2.6.1 Relaxed Newton's Method for a Double Root

In this subsection, we have a polynomial p of degree d, and there is one double root. Then we have to prove that the immediate basin of a double root is simply connected and to do that, we first prove the following proposition.

Proposition 2.6.1. *Let a_j be the only double real root of p, and let the other roots be real and simply roots. Then the interval $[\mu_j, \zeta_j]$ is mapped into itself, $N_p[\mu_j, \zeta_j] \subset [\mu_j, \zeta_j]$, where μ_j, ζ_j are two free critical points belonging to the immediate basin of a_j.*

Proof. Assume that $\mu_j < \zeta_j$, we know that $N(a_j) = a_j$, $N'(\mu_j) = 0$, $N'(\zeta_j) = 0$, and in the previous section, we have proved that

$$\lim_{x \longrightarrow c_1^+} N(x) = +\infty, \qquad \lim_{x \longrightarrow c_2^-} N(x) = -\infty.$$

Then from $N(x) > \mu_j$ in (c_1, μ_j), it follows that

$$N(x) > \mu_j \ \ in \ \ (\mu_j, \zeta_j), \tag{2.6.1}$$

and since $N(x) < \zeta_j$ in (ζ_j, c_2), it follows that

$$N(x) < \zeta_j \ \ in \ \ (\mu_j, \zeta_j). \tag{2.6.2}$$

From (2.6.1) and (2.6.2) we obtain

$$N[\mu_k, \zeta_j] \subset [\mu_j, \zeta_j]. \tag{2.6.3}$$

It also follows that $\mu_j, \zeta_j \in A^*(a_j)$, where $A^*(a_j)$ is the immediate basin of attraction of a_j. □

Theorem 2.6.2. *Let a_j be as in proposition (2.6.1), then the immediate basin of a_j is simply connected.*

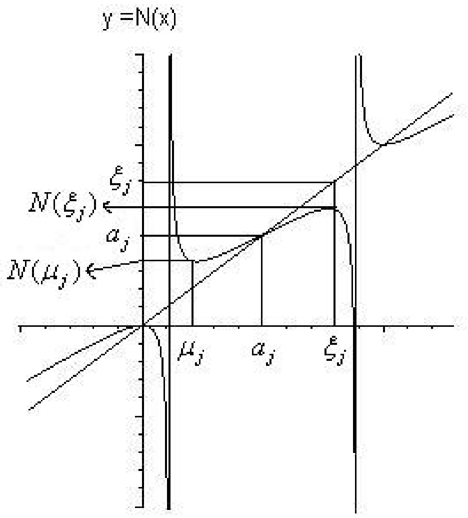


Figure 2.8:

Proof. The idea of this proof is the same as it is in the proof of theorem (2.3.2). But we have to note that, in this case the fixed point a_j of N is not critical point, and we also have to keep in mind that a_j has two preimages. □

Hence we can conjugate N_p to N_{p_1} of degree $d-1$ and p_1 has $d-1$ real (simple) roots, it follows that N_{p_1} can be conjugated to $N_{\hat{p}}$, where $\hat{p}$ is a special polynomial. Thus we obtain the same conclusion (1).

Remark 2.6.2. By starting with an initial approximation z_0 sufficiently close to a root of p, if the root is simple, the sequence of iterates $z_{k+1} = N(z_k)$, will converge quadratically to the root. But if the root is a multiple one, the convergence is only linear [Pei89].

Remark 2.6.3. If $p(z)$ is known to have a multiple of order exactly m, then apply Newton's method to $\sqrt[m]{p(z)}$ to obtain

$$N_m(z) = z - \frac{p(z)^{\frac{1}{m}}}{\frac{1}{m}p(z)^{\frac{1}{m}-1}p'(z)} = z - \frac{mp(z)}{p'(z)}. \tag{2.6.4}$$

This is called Newton's method for a root of order m or the relaxed Newton's method. This relaxed Newton's method will converge quadratically to a root of order m.

(William J. Gilbert) [Wal48] using the above remarks in order to prove the following theorem.

Theorem 2.6.3. *The relaxed Newton's method, N_2 for a double root, applied to any cubic equation with a double root is conjugate, by a linear fractional transformation on the Riemann sphere, to the iteration of the quadratic $p(z) = z^2 - \frac{3}{4}$.*

Example 1. Let

$$p(z) = (z+2)^2(z+1)^3z^4(z-1)^2(z-2)^2(z-3)^3.$$

Since we have in this case $n-2$ simply connected immediate basins, where n is

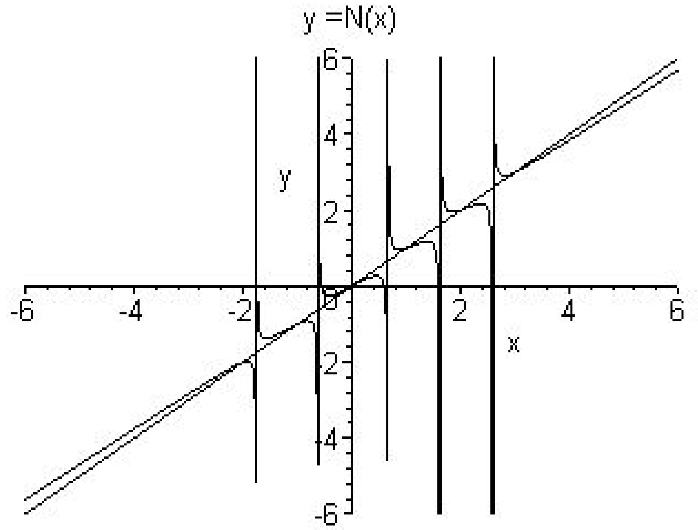


Figure 2.9: Newton's function for the polynomial $p(x) = (x+2)^2(x+1)^3x^4(x-1)^2(x-2)^2(x-3)^3$.

the number of the roots, each one contains two free critical points and an attracting fixed point. So by applying theorem (1.4.1), $n-2$ times and follow the same steps in section (2.4). We will come with a new rational function R, and that R is a Newton's function (as we have proved in proposition (2.4.2)), for a polynomial $p_1(z)$ with only real (and simply) zeros a_j, $j = 1, 2, \ldots, n$. Then we will have the same situation as we mentioned before. Thus we can conjugate Newton's function for p_1 to the Newton's function for a unique polynomial solution of the equation

$$p''(z)(z^2-1) - n(n-1)p(z) = 0. \tag{2.6.5}$$

But the problem is we can not study the general form of multiplicities because its in general not known.

Note that the degree of Newton's function for a polynomial of degree d, with multiple roots will go down and equal to the number of the roots without multiplicity.

Chapter 3

Halley's method for real Polynomials

3.1 Introduction

Halley's method is an elegant method for finding roots. There are two methods called Halley's method, one is called the irrational method and the other is called rational method. The rational method is simpler and has some advantages over the irrational method since the latter involves taking the square root. And since this thesis concerned with Halley's rational method, then we will talk about Halley's method as a Halley's rational method.

Whereas Newton's method is second order, we will show that Halley's method is a third-order algorithm. Such an algorithm converges cubically insofar as the number of significant digits eventually triples with each iteration. And not only does the first derivative of a third-order iteration vanish at a fixed point, but so does the second derivative.

We can derive Halley's method by using a second-degree Taylor approximation

$$R(x) \approx R(x_n) + R'(x_n)(x - x_n) + \frac{R''(x_n)}{2}(x - x_n)^2,$$

where x_n is again an approximate root of $R(x) = 0$. As for Newton's method, the goal is to determine a point x_{n+1} where the graph of R intersects the x-axis, that is, to solve the equation

$$0 = R(x_n) + R'(x_n)(x_{n+1} - x_n) + \frac{R''(x_n)}{2}(x_{n+1} - x_n)^2, \qquad (3.1.1)$$

for x_{n+1}. Following Frame [S.44] and others, we factor $x_{n+1} - x_n$ from the last two terms of (3.1.1) to obtain

$$0 = R(x_n) + (x_{n+1} - x_n)\left(R'(x_n) + \frac{R''(x_n)}{2}(x_{n+1} - x_n)\right),$$

from which it follows that

$$x_{n+1} - x_n = -\frac{R(x_n)}{R'(x_n) + \frac{R''(x_n)}{2}(x_{n+1} - x_n)}. \tag{3.1.2}$$

Approximating the difference $x_{n+1} - x_n$ remaining on the right-hand side of (3.1.2) by Newton's correction $x_{n+1} - x_n = -\frac{R(x_n)}{R'(x_n)}$, we obtain

$$x_{n+1} = x_n - \frac{2R(x_n)R'(x_n)}{2R'(x_n)^2 - R(x_n)R''(x_n)}, \tag{3.1.3}$$

which is widely known as Halley's method.
Note that Bateman [H38] was the first to point out that Halley's method for a function $f(z)$ is obtained by applying Newton's method to $\frac{f(z)}{\sqrt{f'(z)}}$ in the sense that

$$N_{\frac{f(z)}{\sqrt{f'(z)}}}(z) = z - \frac{\frac{f(z)}{\sqrt{f'(z)}}}{(\frac{f(z)}{\sqrt{f'(z)}})'} = H_f(z) \quad for\ all\ z.$$

3.2 Halley's method for real polynomials

In this section, our objective is to study the iteration of Halley's function associated with a polynomial p of degree d with real coefficients and only real (and simple) zeros x_k, $1 \leq k \leq d$. This method is equivalent to iterating the rational map

$$H_p(z) = z - \frac{2p'(z)p(z)}{2(p'(z))^2 - p(z)p''(z)}, \tag{3.2.1}$$

where

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{d-1}z^{d-1} + a_d z^d.$$

So if $p(z)$ has degree d and has distinct roots, then by a simple calculation $H_p(z)$ is a rational map of degree $2d - 1$. As for the case of Newton's method, the roots of $p(z)$ are fixed points of $H_p(z)$, although other fixed points exist as well. Since we are assuming that the roots of $p(z)$ are distinct, the critical points of $p(z)$ are also fixed points under Halley's method.

3.3 Derivative of Halley's method

The derivative of Halley's method is

$$H_p'(z) = -\frac{(p(z))^2 S[p](z)}{2\left(p'(z) - \frac{p(z)p''(z)}{2p'(z)}\right)^2}, \tag{3.3.1}$$

where $S[p](z)$ is the Schwarzian derivative of $p(z)$, that is

$$S[p](z) = \frac{p'''(z)}{p'(z)} - \frac{3}{2}\left(\frac{p''(z)}{p'(z)}\right)^2. \tag{3.3.2}$$

From expression (3.3.1), we can see that the roots are super-attracting fixed points, but of one degree higher order than for Newton's method.
As we know that the degree of Halley's function is $2d-1$, where d is the degree of the polynomial p, there are $4d-4$ critical points, $2d$ of them coincide with the roots x_k, and $2d-4$ are free critical points placed at points where the Schwarzian derivative of $p(z)$ vanishes.

Remark 3.3.1. The second derivative of H_p vanishes at x_k, where as the second derivative of N_p does not, the graph of H_p is flatter than that of N_p near the fixed point. This accounts for the difference in speed at which the two algorithms converge (see [ST95], [E.49] for details). In general, the higher the order, the flatter the graph, the faster convergence.

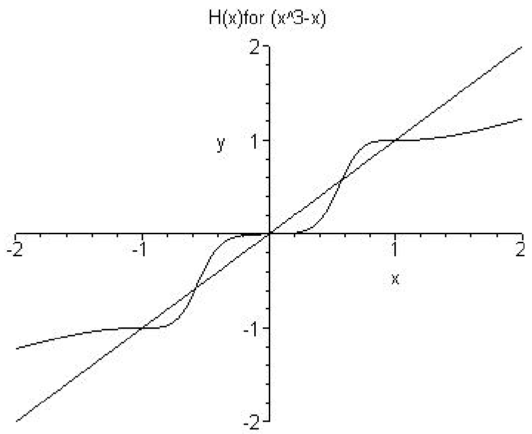


Figure 3.1: Halley's function for the polynomial $p(x) = x^3 - x$.

3.4 Properties of Halley's function

Remark 3.4.1. Kœnig gave in [BH03] the general iteration

$$z_{n+1} = z_n + (\alpha+1)\left(\frac{(\frac{1}{p})^{(\alpha)}}{(\frac{1}{p})^{(\alpha+1)}}\right),$$

where α is an integer and $(\frac{1}{p})^{(\alpha)}$ is the derivative of order α of the inverse of the polynomial p. This iteration has convergence of order $(\alpha+2)$. For example $\alpha=0$ has quadratic convergence (order 2) and the formula gives back Newton's iteration, while $\alpha=1$ has cubical convergence (order 3) and gives again Halley's method. Just like Newton's method, a good starting point is required to insure convergence.

Proposition 3.4.1. *Let $p : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial of degree d with real coefficients and real (and simple) zeros. Then:*

1. *Halley's method associated with p is a rational map, it has a repelling fixed point at ∞ with multiplier $\frac{d+1}{d-1}$.*
2. *The local degree of H_p at the roots of p is exactly 3.*
3. *The rational map H_p has $d-1$ repelling fixed points in $\mathbb{R}$ and their multipliers are equal to 3.*

Proof. **(1)** When $|z|$ tends to ∞, we have

$$p(z) \sim \lambda z^d,$$

$(\frac{1}{p(z)})' \sim \frac{-d}{\lambda z^{d+1}}$ and $(\frac{1}{p(z)})'' \sim \frac{d(d+1)}{\lambda z^{d+2}}$, we can write H_p in this form

$$H_p(z) = z + 2\frac{\left(\frac{1}{p(z)}\right)'}{\left(\frac{1}{p(z)}\right)''},$$

it follows

$$H_p(z) \sim z - \frac{2z}{d+1},$$

and

$$H_p'(z) \sim \frac{d-1}{d+1},$$

as we know that the multiplier λ at ∞ is equal to

$$\lim_{z\longrightarrow\infty} \frac{1}{H_p'(z)} = \frac{d+1}{d-1},$$

it follows that ∞ is a fixed point of H_p with multiplier $\frac{d+1}{d-1}$. This concludes the proof of (1).

(2) The zeros of p are double critical points. Assume $U_i = \{z : |z - x_i| < \epsilon\}$ is contained in $A^*(x_i)$, such that U_i does not contain $H_p(z_0)$, where z_0 is a free critical point. Then from (3.3.1) and theorem (3.4.3) which show that $S[p](z) < 0$, it follows that U_i contains exactly two critical points located at x_i. Let U_{i+1} be a component of $H_p^{-1}(U_i)$ contains x_i. Then by using the Riemann Hurwitz Formula on U_i, U_{i+1} we will find the degree of H_p equals 3 on U_i, then The local degree of H_p at the roots of p is exactly 3.

(3) Let x_i, $i = 1, 2, \cdots, d$, be the zeros of p which are real and simple. The fixed points of H_p are ∞, the points x_i and the zeros of the rational map $g = (\frac{1}{p})'$, where H_p written in the following form

$$H_p(z) = z + 2\frac{(\frac{1}{p})'}{(\frac{1}{p})''}.$$

For any rational map, the number of zeros is equal to the number of poles. The poles of g are the points x_i, and g has a zero of order $d + 1$ at ∞, thus g has $2d$ poles, it follows that g has $2d - (d + 1) = d - 1$ zeros, consequently, H_p has $d - 1$ repelling fixed points. For any rational map, the number of fixed points counted with multiplicities equal to the degree plus 1. As we know in our case that the fixed points of H_p are real and simple, then there are d super-attracting fixed points, one repelling fixed point at ∞ and $d-1$ repelling fixed points in $\mathbb{R}$. Therefore the degree of H_p is $2d - 1$.
Now we have

$$H_p(z) = z + \frac{2g}{g'},$$

where $g = (\frac{1}{p})'$. Thus

$$H_p'(z) = 3 - \frac{2gg''}{g'^2},$$

and at the repelling fixed points $p' = 0$ gives $g = \frac{-p'}{p^2} = 0$. Thus $H_p' = 3$ at each repelling fixed point.

□

Theorem 3.4.2. *Let*

$$H_p(z) = z - \frac{2p'(z)p(z)}{2(p'(z))^2 - p(z)p''(z)},$$

where p is a polynomial with real (and simply) distinct zeros. Then H_p has no real pole.

Proof. We will show that

$$(p')^2 - pp'' > 0 \qquad on \ \ \mathbb{R},$$

which is known as Polya's result.
Write
$$(p')^2 - pp'' = p^2\frac{(p')^2}{p^2} - p^2\frac{pp''}{p^2} = p^2\left(\left(\frac{p'}{p}\right)^2 - \frac{p''}{p}\right).$$
We know that
$$\left(\frac{p'}{p}\right)^2 = \left(\sum_{j=1}^{d}\frac{1}{z-x_j}\right)^2,$$
where x_j are roots of p, $1 \le j \le d$, hence
$$\frac{p''}{p} = \left(\sum_{j=1}^{d}\frac{1}{z-x_j}\right)^2 - \sum_{j=1}^{d}\frac{1}{(z-x_j)^2}.$$
From
$$\sum_{j=1}^{d}\frac{1}{(z-x_j)^2} > 0, \qquad z \in \mathbb{R},$$
it follows that
$$\left(\frac{p'}{p}\right)^2 > \frac{p''}{p},$$
hence
$$2(p')^2 - pp'' > 0.$$
Thus H_p does not have any real pole. □

Theorem 3.4.3. *Let $H_p(z)$ be a Halley's function for a polynomial $p(z)$, then $H_p'(z) \ge 0$ on $\mathbb{R}$.*

Proof. We know that
$$H_p'(z) = -\frac{(p(z))^2 S[p](z)}{2\left(p'(z) - \frac{p(z)p''(z)}{2p'(z)}\right)^2},$$
where $S[p](z)$ is the Schwarzian derivative of $p(z)$, that is
$$S[p](z) = \frac{p'''(z)}{p'(z)} - \frac{3}{2}\left(\frac{p''(z)}{p'(z)}\right)^2 = \frac{2p'p''' - 3(p'')^2}{2(p')^2}.$$
To show that $H_p'(z) \ge 0$, we have to prove that $S[p](z) < 0$. By the same proof as before, we can see that $(p'')^2 - p'p''' > 0$, then $2p'p''' - 3(p'')^2 < 0$. Thus $S[p](z) < 0$, implies $H_p'(z) \ge 0$. □

Conclusion 2. From theorems (3.4.2) and (3.4.3), we conclude that H_p is an increasing rational homeomorphism map on $\mathbb{R}$.

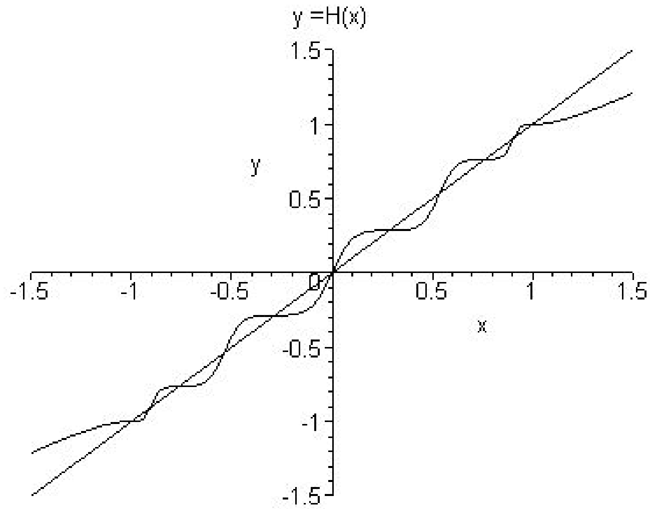


Figure 3.2: Halley's function for the polynomial $p(x) = x^6 - \frac{5}{3}x^4 + \frac{5}{7}x^2 - \frac{1}{21}$.

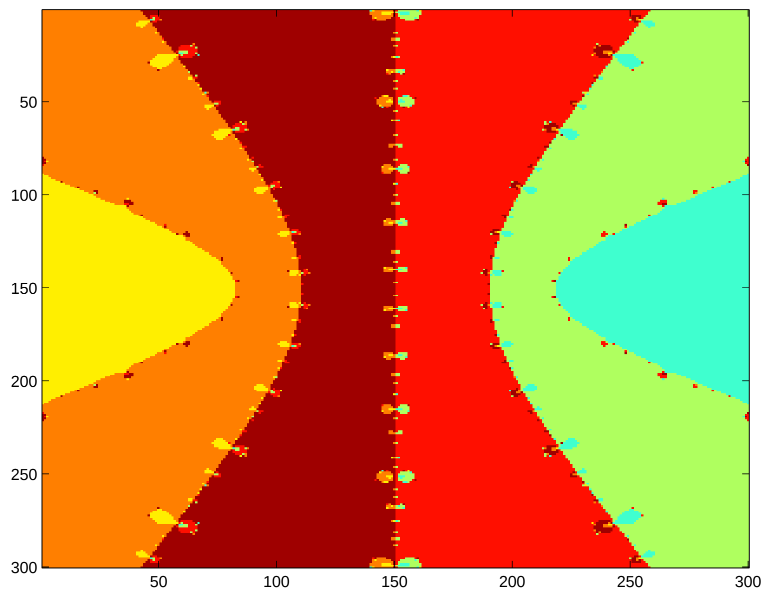


Figure 3.3: Iteration of Halley's function for the polynomial $p(z) = z^6 - \frac{5}{3}z^4 + \frac{5}{7}z^2 - \frac{1}{21}$.

3.5 Immediate basins

We can not follow the same steps in the part of Newton's method to prove that each component of the Fatou set of Halley's method is simply connected, because in Halley's method the free critical points are non-real numbers, so we can not work on the real line. But results in [Prz89] can be used to prove the immediate basins of a superattracting fixed points of Halley's method are simply connected.

Lemma 3.5.1. *([Prz89]) Let A be the immediate basin of attraction to a fixed point for a rational map $R : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$. Assume that A is not simply connected. Then there exist in $\hat{\mathbb{C}}$ two disjoint domains U_0 and U_1 intersecting A, such that $V = R(U_0) = R(U_1) \supset \overline{U_0} \cup \overline{U_1}$, $R(\partial U_i) = \partial V \subset A$ for $i = 0, 1$, $V \cup A = \hat{\mathbb{C}}$ and V is homeomorphic to a disc.*

Theorem 3.5.2. *The immediate basins of attraction to the roots of any polynomial with real coefficients and only real (and simply) zeros x_k, $1 \leq k \leq d$, for Halley's method, are simply connected.*

Proof. In [Prz89] Feliks Przytycki has proved that the immediate basins of attraction for N_p is simply connected. We can apply the same proof, so we can assume that A is a multiply connected immediate basin of attraction for $R = H_p$ to a root $x \in \mathbb{R}$ of a polynomial p. Choose $z \in V \cap A$, V given by Lemma (3.5.1), and branches R^{-1}, so that $w_i = R^{-1}(z) \in U_i \cap A$. Join z with w_i by a curve $\gamma_i^0 \subset V \cap A$. Take care additionally to have $\gamma_i^0 \cap cl(\bigcup\limits_{n>0} R^n(critR)) = \phi$. Define by induction $\gamma_i^n = R^{-1}(\gamma_i^{n-1})$, where R^{-1} is the extension of the preliminary branch along the curve $\bigcup\limits_{j=0}^{n-1} \gamma_i^j$. Define $\gamma_i = \bigcup\limits_{n=0}^{\infty} \gamma_i^n$. The curve γ_i converges to a fixed point $\zeta_i \in U_i$ of R. The reason is that $R_{v_i}^{-1} \circ \ldots \circ R_{v_i}^{-1}$, n times, $n = 0, 1, \ldots,$ is a normal family of functions on a neighborhood of γ_i with the set of limit functions on boundary of A which is nowhere dense. So all limit functions are constant, hence $\lim\limits_{n\to\infty} diam(\gamma_i^n) = 0$. Therefore all limit points of the sequence of curves γ_i^n are fixed points for R. On the other hand they must be isolated from each other. So we actually have only one limit point. The conclusion is that the boundary of A contains two different fixed points ζ_0, ζ_1 belonging to two different components of the boundary of A. But the only fixed points for H_p are the roots of p (real), the roots of p' (real), and ∞. Since we have proved that H_p is continuous on $\mathbb{R}$. Thus $A \cap \mathbb{R}$ is an interval. We arrived at a contradiction. □

Theorem 3.5.3. *([Pom86]) Let $\zeta \neq \infty$ be a repelling fixed point of some rational function f. For $\mu = 1, 2, \ldots, m$, let G_μ be different simply connected invariant components of $F(f)$, let h_μ map $\mathbb{D}$ conformally onto G_μ, and let $\sigma_{\mu v}$ be distinct fixed points of $\varphi_\mu = h_\mu^{-1} \circ f \circ h_\mu$ with*

$$h_\mu(\sigma_{\mu v}) = \zeta \quad for \quad v = 1, \ldots, q_\mu.$$

Then

$$\sum_{\mu=1}^{m}\sum_{v=1}^{q_\mu}\frac{1}{\log\varphi'_\mu(\sigma_{\mu v})}\leq\frac{2}{\log|f'(\zeta)|}.$$

Let $A^*(x_k)$, where $k=1,2,\ldots,d$ be different simply connected invariant components of $F(H_p)$, H_p has $d-1$ repelling fixed points on $\mathbb{R}$ with multiplier equal to 3, then $G_k=A^*(x_k)$, let h_k map $\mathbb{D}$ conformally onto $A^*(x_k)$, and let σ_{kv} be distinct fixed points of $\varphi_k=h_k^{-1}\circ H_p\circ h_k$ with

$$h_k(\sigma_{kv})=\zeta\quad for\quad v=1,\ldots,m,$$

where $\zeta\neq\infty$ is repelling fixed point of H_p. So from the above theorem we can prove the following theorem.

Theorem 3.5.4. *Let $p(z)$ be a polynomial of degree $d\geq 2$ with real coefficients and only real (and simply) zeros, $H_p(z)$ be the Halley's function associated with $p(z)$, and $A^*(x_1),A^*(x_2),\ldots,A^*(x_d)$ are the immediate basins of superattracting fixed points of $H_p(z)$. Then there are at least two free critical points in each second immediate basin.*

Proof. Denote by $x_1,x_2,\ldots,x_{d-1},x_d$ the d superattracting fixed points of $H_p(z)$. We know that each immediate basin $A^*(x_i)$, $1\leq i\leq d$, is simply connected and the local degree of H_p at each x_i is exactly 3. Now take $A^*(x_i),A^*(x_{i+1})$ and assume that both of them do not contain any free critical points; then the degree of $H_p(z)$ in both basins is equal to 3. It follows that there exists a conformal representation $h^{-1}:A^*(x_k)\longrightarrow\mathbb{D}$, $k=i,i+1$, conjugating $H_p:A^*(x_k)\longrightarrow A^*(x_k)$ to the mapping $\varphi_k:\mathbb{D}\to\mathbb{D}$, where the degree of φ_k is exactly 3. Then there are at least two external rays which land at ζ, where $\zeta\in(x_i,x_{i+1})$ is a repelling fixed point of H_p. By the inequality in theorem (3.5.3), we get

$$\sum_{k=i}^{i+1}\sum_{v=1}^{m}\frac{1}{\log\varphi'_k(\sigma_{kv})}<\frac{2}{\log|H'(\zeta)|}=\frac{2}{\log 3}.$$

Since $\varphi_k=h_k^{-1}\circ H_p\circ h_k$ with

$$h_k(\sigma_{kv})=\zeta\quad for\quad v=1,\ldots,m,$$

and $H_p'(\zeta)=3$, it follows that $\varphi'_k(\sigma_{kv})=3$, and

$$\frac{2m}{\log 3}<\frac{2}{\log 3}.$$

Then we get a contradiction because $m\geq 2$. Therefore if the degree of $H_p(z)$ in $A^*(x_i)$ equals 3, then the degree of H_p in $A^*(x_{i+1})$ must be ≥ 5. Since it is odd, then each second immediate basin of superattracting fixed point contains at least two free critical points. $\square$

Proposition 3.5.5. *If polynomials $p, \tilde{p}$ have Halley's functions $H_p, H_{\tilde{p}}$ which are conformal conjugate, i.e. $H_{\tilde{p}} = \varphi^{-1} \circ H_p \circ \varphi(z)$, then $\tilde{p}(z) = c\, p(\varphi(z))$, where c is a constant.*

Proof. Consider $\varphi(z) = az + b$, then $\varphi^{-1}(z) = \frac{z-b}{a}$, and we can write Halley's function as follow

$$H_p = z + 2\frac{g}{g'},$$

where $g = (\frac{1}{p})'$ and $g' = (\frac{1}{p})''$, by simple calculation we have

$$H_{\tilde{p}}(z) = z + 2\frac{g(\varphi(z))}{ag'(\varphi(z))},$$

so it follows that

$$\frac{\tilde{g}(z)}{\tilde{g}'(z)} = \frac{g(\varphi(z))}{ag'(\varphi(z))},$$

we have $a = \varphi'(z)$ then

$$\log \tilde{g}(z) = \log g(\varphi(z)) \;\; \textit{implies} \;\; \tilde{g}(z) = c\, g(\varphi(z)).$$

Thus

$$\tilde{p}(z) = c\, p(\varphi(z)).$$

□

3.6 The Conjugation of Halley's method to a rational function

Since all free critical points of $H_p(z)$ are complex numbers, we can not prove directly that each $A^*(x_k)$ contains exactly four critical points, that is, that the degree of $H_p(z)$ on $A^*(x_k)$ is exactly equal to 5. We also do not know that H_p is hyperbolic. But we can look for a rational map $R(z)$ which satisfies the following conditions: $degR(z) = degH_p(z)$, and $R(z)$ has d real superattracting fixed points, with property that all the critical points of $R(z)$ are located at the superattracting fixed points of $R(z)$. Now we consider the following three subsections as examples to find rational maps R_3, R_4, R_5 with the above conditions and property, so that we can conjugate Halley's function H_{p_d}, where $d = 3, 4, 5$ is the degree of p, to the rational map R_d. For simplification, we will consider $p_d(z)$ to be the special polynomials of degree $d = 3, 4, 5$.

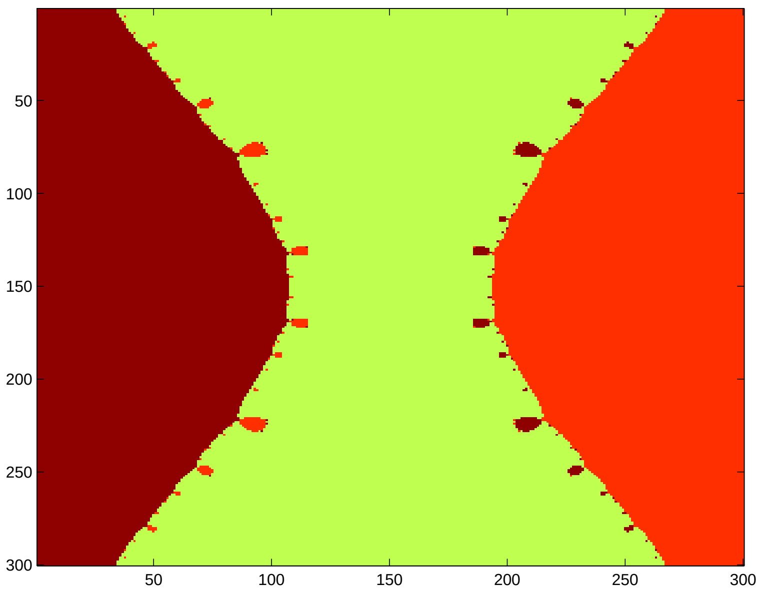


Figure 3.4: Iteration of Halley's function for the polynomial $p_3(z) = z^3 - z$.

3.6.1 Example

In this subsection, we explore the dynamics of Halley's method applied to cubic polynomials (p_3) with real coefficients and only real (and simply) zeros x_k, $k = 1, 2, 3$. Consider p_3 a special polynomial of degree three

$$p_3(z) = (z - 1)(z + 1)z.$$

Then the Schwarzian derivative $S[p_3](z)$ vanishes at $\rho_\pm = \pm\frac{i}{\sqrt{6}}$, which are the two free critical points. Note that in this example, the free critical points of Halley's method are symmetric about the free critical points of Newton's method, here we also have that $Re(\rho_-) = Re(\rho_+) = 0$. Since its easy to see that $H_{p_3}(i\mathbb{R}) \subseteq i\mathbb{R}$ and $0 \in i\mathbb{R}$, therefore $i\mathbb{R} \subset A^*(0)$ (the immediate basin of attraction for H_{p_3} to the root $z = 0$ of p_3), then $\rho_\pm \in A^*(0)$, which means that $\rho_\pm$ converge to zero under iteration of H_{p_3}. So we then have two critical points at a root $z = 1$, two critical points at a root $z = -1$, and the other four critical points belong to $A^*(0)$, two of them are at the root $z = 0$. So we are now looking for a rational map $R_3(z)$ which satisfy these conditions:
$degR_3(z) = 5$, $R_3(\pm 1) = \pm 1$, $R_3(0) = 0$, $R_3'(0) = 0$, $R_3'(\pm 1) = 0$, $R_3''(0) = 0$, $R_3''(\pm 1) = 0$, $R_3^{(3)}(0) = 0$, and $R_3^{(4)}(0) = 0$.
By using Maple with the above conditions, we found a rational map

$$R_3(z) = \frac{8z^5}{15z^4 - 10z^2 + 3},$$

with $degR_3 = degH_{p_3}$ and it has the same fixed points of H_{p_3} which are $0, 1, -1, \pm 0.654$ (approximately), and ∞, where the critical points of $R_3(z)$ are

$$0, 0, 0, 0, 1, 1, -1, -1.$$

By theorem (1.2.3), $J(H_p)$ is connected. Since $A^*(0)$ contains a pair of free critical

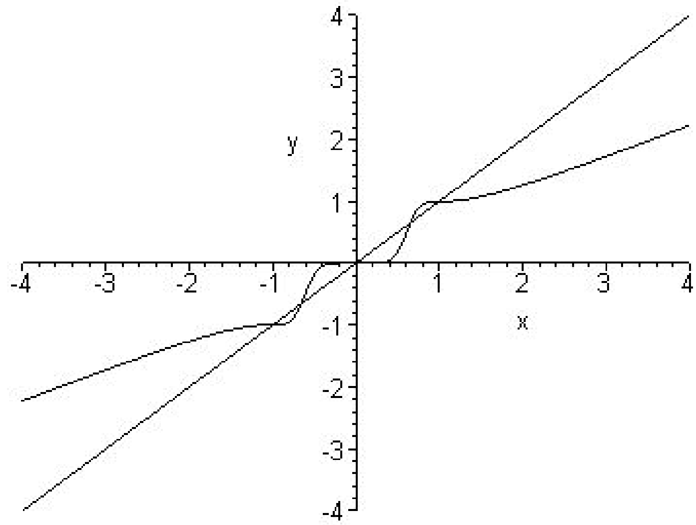


Figure 3.5: Rational function $R_3(x)$.

points, it follows $P(H_{p_3}) \cap J(H_{p_3}) = \phi$. Thus H_{p_3} is hyperbolic and $J(H_{p_3})$ is locally connected. H_{p_3} is quasiconformally conjugate to R_3 with 3 superattracting fixed points and all critical points located at these fixed points.

3.6.2 Example

We want to study the dynamics of Halley's method applied to polynomials of degree four with real coefficients and real (and simple) zeros x_k, $k = 1, 2, 3, 4$. Let p_4 be the special polynomial of degree four

$$p_4(z) = (z^2 - 1)(z^2 - 1/5),$$

the zeros of p_4 *are* $\pm 1, \pm\frac{1}{\sqrt{5}}$ and the zeros of p_4' are 0, and approximately ± 0.7745966, they are fixed points of H_{p_4}. Since

$$H_{p_4}(z) = z - \frac{2p_4'(z)p_4(z)}{2(p_4'(z))^2 - p_4(z)p_4''(z)},$$

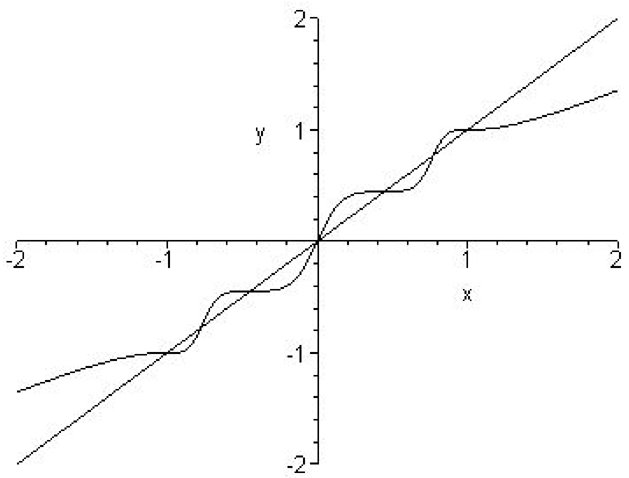


Figure 3.6: Halley's function for the special polynomial of degree four ($H_{p_4}(x)$).

and

$$H'_{p_4}(z) = -\frac{(p_4(z))^2 S[p](z)}{2(p'_4(z) - \frac{p_4(z)p''_4(z)}{2p'_4(z)})^2},$$

there are two critical points at each zero of $p_4(z)$. In total we have 12 critical points, thus there are 4 free critical points, where the Schwarzian derivative $S[p_4](z)$ vanishes; these zeros are approximately equal to $\pm 0.4406405322 \pm 0.2723308257i$. Now we have to look for a rational map $R_4(z)$ which satisfies the following conditions: $deg R_4(z) = 7$, $R_4(\pm 1) = \pm 1$, $R_4(0) = 0$, $R_4(\pm\frac{1}{\sqrt{5}}) = \pm\frac{1}{\sqrt{5}}$, and

$$R'_4(\pm\frac{1}{\sqrt{5}}) = R'_4(\pm 1) = R''_4(\pm 1) = R''_4(\pm\frac{1}{\sqrt{5}}) = R_4^{(3)}(\pm\frac{1}{\sqrt{5}}) = R_4^{(4)}(\pm\frac{1}{\sqrt{5}}) = 0.$$

Again by using Maple we found the following rational map

$$R_4(z) = \frac{225z^7 - 195z^5 + 11z^3 + 23z}{350z^6 - 430z^4 + 138z^2 + 6}.$$

So we now have a rational map $R_4(z)$ with $deg R_4(z) = deg H_{p_4}(z)$ and $R_4(z)$ has superattracting fixed points $\pm 1, \pm\frac{1}{\sqrt{5}}$ and repelling fixed points 0 and ± 0.8246211251 (approximately) with multiplier approximately equal to $\lambda = 3.36$. The interesting thing about properties of $R_4(z)$ is that all of its critical points are

$$1, 1, -1, -1, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}},$$

which means that all the critical points are located at the superattracting fixed points of $R_4(z)$. Now since each immediate basin $A^*(x_k)$ is simply connected, thus julia set of H_{p_4} is connected. If each $A^*(x_k)$ contains a pair of free critical points, it follows $P(H_{p_4}) \cap J(H_{p_4}) = \phi$. Thus H_{p_4} is hyperbolic and $J(H_{p_4})$ is locally connected, and H_{p_4} is quasiconformally conjugate to R_4.

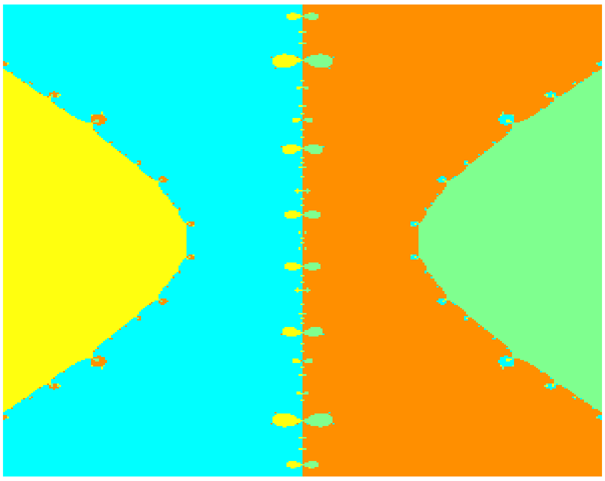


Figure 3.7: Iteration of Halley's function for the polynomial $p_4(z) = (z^2 - 1)(z^2 - 1/5)$.

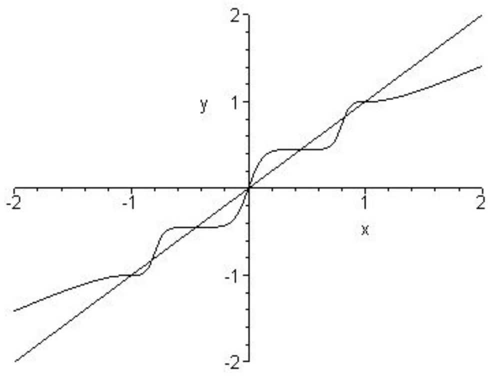


Figure 3.8: Rational function $R_4(x)$.

3.6.3 Example

In this example, we describe dynamics of Halley's method for polynomials of degree five of real coefficients and real (and simple) zeros x_k, $k = 1, 2, 3, 4, 5$. As an example, let p_5 be the special polynomial of degree 5, that is,

$$p_5 = (z^2 - 1)z(z^2 - \frac{3}{7}),$$

then the zeros of p_5 are $\pm 1, 0, \pm\sqrt{\frac{3}{7}}$, and the zeros of p_5' are approximately

$$\pm 0.8611363116, \pm 3399810436.$$

Since $degH_{p_5} = 9$ there are 16 critical critical points, 6 of them are free, zeros of the Schwarzian derivative $S[p_5](z)$. They are approximately equal to

$$\pm 0.2442000522i, \pm 0.648112265 \pm 0.1819061817i$$

. Now by using Maple we find the rational map

$$R_5(z) = \frac{\frac{1708}{27}z^9 - \frac{836}{9}z^7 + \frac{716}{21}z^5 + \frac{20}{21}z^3}{\frac{805}{9}z^8 - \frac{4160}{27}z^6 + \frac{1714}{21}z^4 - \frac{88}{7}z^2 + 1},$$

which satisfy the following conditions: $degR_5(z) = degH_{p_5}(z)$, $R_5(\pm 1) = \pm 1$, $R_5\left(\pm\sqrt{\frac{3}{7}}\right) = \pm\sqrt{\frac{3}{7}}$, $R_5(0) = 0$, $R_5'(\pm 1) = R_5''(\pm 1) = 0$ and $R_5^{(n)}(x_k) = 0$, where $n = 1, 2, 3, 4$, $x_k = 0, \pm\sqrt{\frac{3}{7}}$. So we have a rational map $R_5(z)$ with property that all of its critical points are located at the superattracting fixed points. Now since each immediate basin $A^*(x_k)$ is simply connected, thus julia set of H_{p_5} is connected. If each $A^*(x_k)$ contains a pair of free critical points, it follows $P(H_{p_5}) \cap J(H_{p_5}) = \phi$. Thus H_{p_5} is hyperbolic and $J(H_{p_5})$ is locally connected, and H_{p_5} is quasiconformally conjugate to R_5.

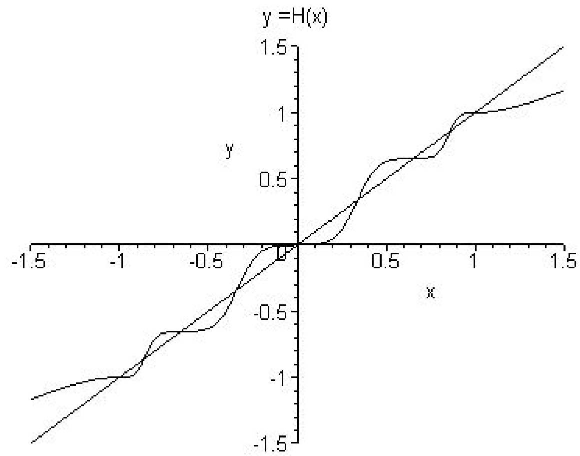


Figure 3.9: Halley's function for the special polynomial of degree five ($H_{p_5}(x)$).

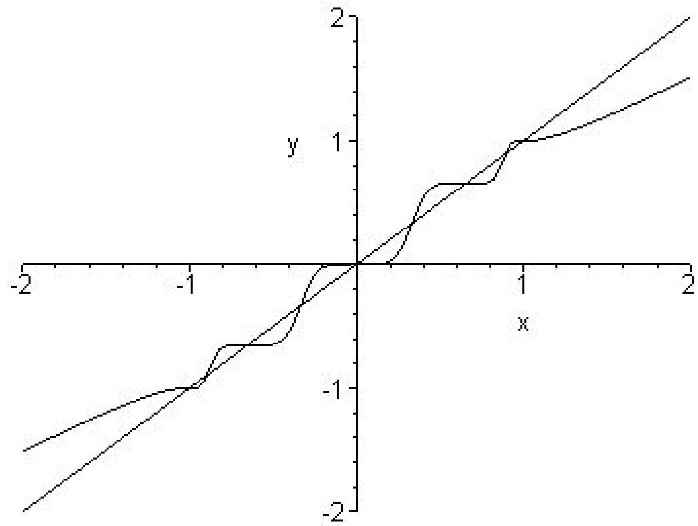


Figure 3.10: Rational function $R_5(x)$.

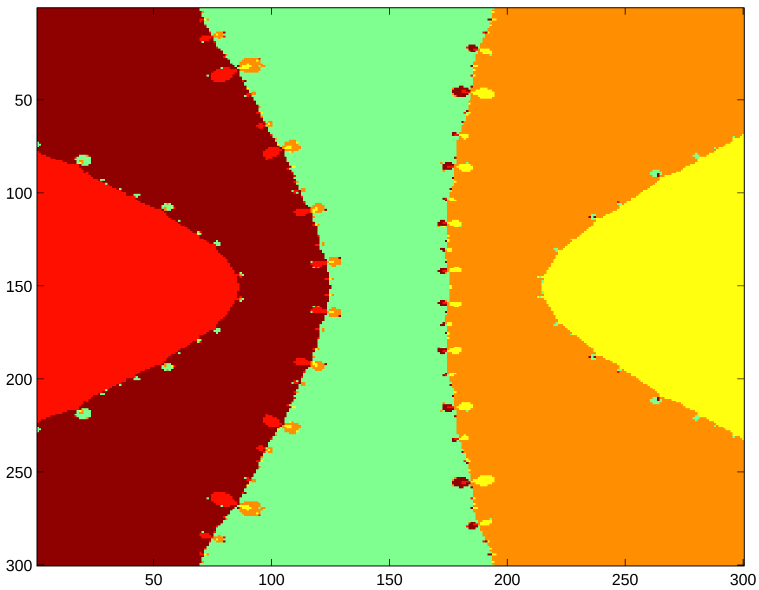


Figure 3.11: Iteration of Halley's function for the polynomial $p_5(z) = (z^2 - 1)z(z^2 - \frac{3}{7})$.

Chapter 4

Rational Homeomorphisms

In this section, we want to describe the dynamics of the iterates of the family $A = \{$Rational maps which are homeomorphism on $\mathbb{R}$ and all fixed points $\in \mathbb{R} \cup \infty\}$. Since we have shown in the previous chapter that H_p (Halley's function for polynomial with real coefficients and only real (and simple) zeros) is a homeomorphism rational map, then its a subset of A. Note that H_p has a superattracting fixed points at the zeros of p, that is the multiplier ($\lambda = 0$), and the repelling fixed points at the zeros of $S[p]$ (Schwarzian derivative), so by elementary calculation we found that at the repelling fixed points of H_p the multiplier ($\lambda = 3$), then H_p would be a good example to describe dynamics of a homeomorphism rational maps $R \subset A$, where all the fixed points of R are superattracting, or repelling, that is, $\lambda = 0$, $\lambda > 1$.

4.1 Describing Halley's functions with attracting fixed points

We know that the class of Halley's functions with real attracting fixed points belong to the family A. Now we want to study Halley's method for a polynomial p of degree d with real zeros and some of them with multiplicity ≥ 2. We then obtain Halley's function with some of its fixed points attracting fixed points, and some superattracting. Let

$$H_p(z) = z + 2\frac{(\frac{1}{p})'}{(\frac{1}{p})''}.$$

If p of degree d has only real zeros, it follows that each root of p with multiplicity ≥ 2 is an attracting fixed point of H_p.

Definition 4.1.1. The extraneous fixed points of H_p are the fixed points which are zeros of $(\frac{1}{p})'$.

Lemma 4.1.1. *Let $p : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial of degree $d \geq 2$ with real zeros. If ζ is a zero of $(\frac{1}{p})'$ with multiplicity m, then it is a repelling fixed point of H_p with multiplier $(1 + \frac{2}{m})$.*

Proof. We have

$$H_p(z) = z + \frac{2g(z)}{g'(z)} \quad \textit{where} \;\; g = (\frac{1}{p})'.$$

If ζ is a zero of g of order m, then there exists a $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$g(z) = \lambda(z - \zeta)^m + O(|z - \zeta|^{m+1}),$$

and

$$g'(z) = \lambda m(z - \zeta)^{m-1} + O(|z - \zeta|^m).$$

Thus

$$\begin{aligned} H_p(z) &= z + \frac{2\lambda(z-\zeta)^m}{\lambda m(z-\zeta)^{m-1}} + O(|z-\zeta|^2) \\ &= \zeta + z - \zeta + \frac{2(z-\zeta)}{m} + O(|z-\zeta|^2) \\ &= \zeta + (z-\zeta)(1 + \frac{2}{m}) + O(|z-\zeta|^2), \end{aligned}$$

therefore, ζ is a repelling fixed point of $H_p(z)$ and its multiplier is $(1 + \frac{2}{m})$. □

Remark 4.1.1. In the case of Newton's Method for polynomial there are no extraneous fixed points, but as for Halley's method case for a polynomial, the extraneous fixed points are the critical points of p which are not zeros of p.

Proposition 4.1.2. *Let p be a real polynomial of degree $d \geq 2$. Denote by x_i, $1 \leq i \leq d$, its zeros and by n_i their multiplicities. Then the fixed points of H_p are either superattracting, attracting or repelling. The superattracting and attracting fixed points are exactly the zeros x_i and their multipliers are $1 - \frac{2}{n_i+1}$, when $n_i = 1$, the local degree of H_p at x_i equals at least 3.*

Remark 4.1.2. From lemma (4.1.1), we know that the extraneous fixed points of H_p are exactly the zeros of $(\frac{1}{p})'$. If ζ_j is a zero of $(\frac{1}{p})'$ with multiplicity m_j, then it is a repelling fixed point of H_p with multiplier $(1 + \frac{2}{m_j})$.

Example 2. Consider

$$p(z) = (z^2 - 1)z^3,$$

and

$$H_p(z) = z - \frac{2(\frac{1}{p})'}{(\frac{1}{p})''}.$$

Then the superattracting fixed points of H_p are ± 1 with multiplicity $n = 1$ and multiplier $\lambda = 1 - \frac{2}{n+1} = 1 - \frac{2}{2} = 0$, the attracting fixed point is zero with multiplicity

$n = 3$ and multiplier $\lambda = 1 - \frac{2}{3+1} = \frac{1}{2}$, and the repelling fixed points of $H_p(z)$ are the zeros of $(\frac{1}{p})'$ which are approximately equal to ± 0.774596692 with multiplier $\lambda = 3$. Moreover, p has $N = 3$ distinct real roots, then there are $4N - 4 = 8$ critical points of $H_p(z)$ which are $1, 1, -1, -1$, and four non-real free critical points which are approximately equal to $\pm 0.5460815697 \pm 0.2195565548i$. Now since each immediate basin is simply connected, then Julia set is connected. If $A^*(0)$ contains the four free critical points, then H_p is hyperbolic and $J(H_p)$ is locally connected. We can also conjugate $H_p(z)$, where $p(z) = z^5 - z^3$, to the rational function:

$$R_3(z) = \frac{8z^5}{15z^4 - 10z_2 + 3},$$

which we have discussed in the previous chapter.

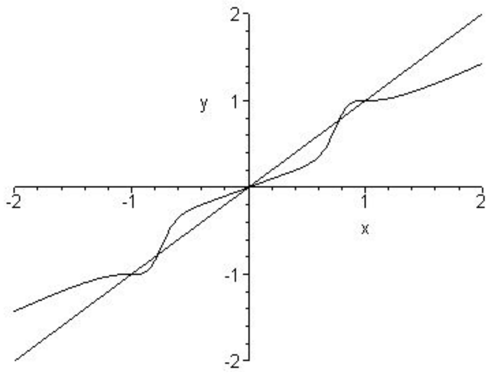


Figure 4.1: Halley's function for the polynomial $p(x) = x^5 - x^3$.

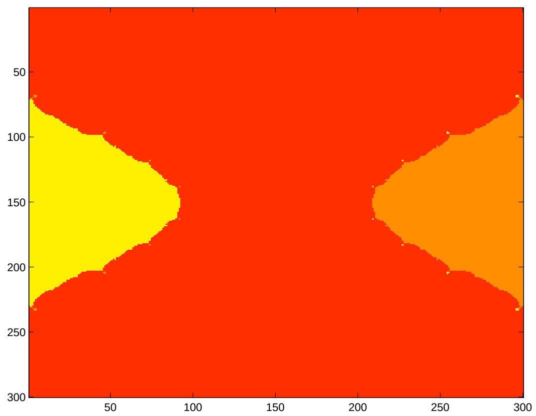


Figure 4.2: Iteration of Halley's function for the polynomial $p(z) = z^5 - z^3$.

4.2 Appendix. Householder's method

Let
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d,$$
be a polynomial with real coefficients and only real (and simple) zeros x_k, $1 \leq k \leq d$. The method is equivalent to iterating the rational map

$$h_p(z) = z - \frac{2p(z)(p'(z))^2 + (p(z))^2 p''(z)}{2(p'(z))^3}, \tag{4.2.1}$$

is called Householder's method associated with p. So the fixed points of h_p are given by

$$\frac{2(p'(z))^2 p(z) + (p(z))^2 p''(z)}{2(p'(z))^3} = 0, \quad \text{or} \quad p(z)[2(p'(z))^2 + p(z)p''(z)] = 0.$$

Thus the fixed points of h_p are the zeros of p at x_k, $1 \leq k \leq d$, the zeros of $[2(p'(z))^2 + p(z)p''(z)]$ at $x_k^{'}$, $1 \leq k \leq d$, and at $x_k^{''}$, where $2 \leq k \leq d-1$, and ∞. The degree of h_p is $3d-2$, where d is the degree of $p(z)$.

Definition 4.2.1. If $c_1 < c_2$ are consecutive roots of $p'(x)$, then the interval (c_1, c_2) is called a band for h_p.

Definition 4.2.2. If $p'(x)$ has the largest (respectively, smallest) root c (respectively, b), then the interval (c, ∞) (respectively, $(-\infty, b)$) is called an extreme band for h_p.

So in each band (c_k, c_{k+1}) which contains x_k there are two repelling fixed points $x_k^{'}, x_k^{''}$ and one superattracting fixed point x_k, where $2 \leq k \leq d-1$, and there are two fixed points in each extreme band.

4.2.1 Derivative of Householder's function

Since
$$h_p'(z) = -\frac{(p(z))^2[p'(z)p^{'''}(z) - 3(p''(z))^2]}{2(p'(z))^4}, \tag{4.2.2}$$
we see that the zeros of p are superattracting fixed points of h_p, and there are double critical points at each superattracting fixed point. The number of critical points is $6d-6$, and $4d-6$ of them are free critical points which are non-real, so we are in the same situation of Halley's method. But in Householder's method there are $d-1$ poles as in Newton's method. And there are $2d-2$ real repelling fixed points which are the zeros of $(2(p'(z))^2 + p(z)p''(z))$.

Proposition 4.2.1. *Let $h_p(z)$ be Householder's function associated with $p(z)$, where $p(z)$ is a polynomial with real coefficients and only real (and simple) zeros, then $h_p'(z) \geq 0$ on each band (c_k, c_{k+1}), where $2 \leq k \leq d-1$.*

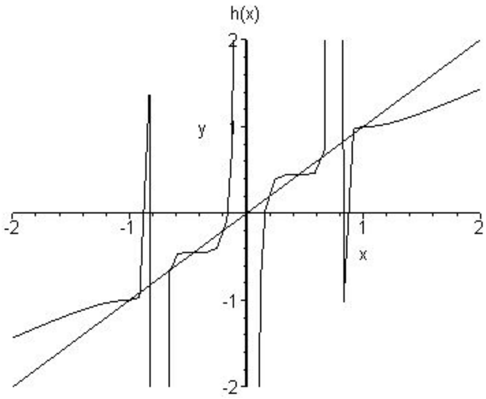


Figure 4.3: Householder's function for the polynomial $p_4(x) = (x^2 - 1)(x^2 - \frac{1}{5})$.

Proof. See (4.2.1). By Polya's result we have

$$p'^2 - pp'' > 0 \quad on \quad \mathbb{R},$$

hence also

$$p''^2 - p'p''' > 0,$$

and this implies $h_p'(x) \geq 0$ on $\mathbb{R}$ except at zeros of p'. $\square$

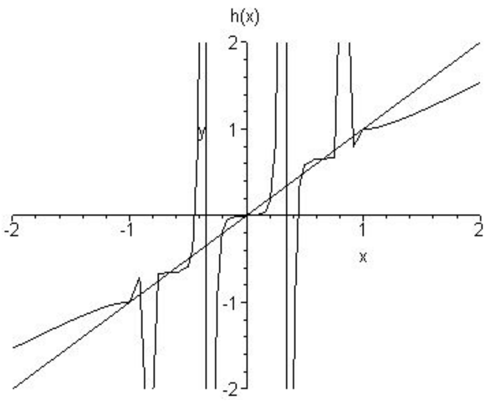


Figure 4.4: Householder's function for the polynomial $p_5(x) = x^5 - \frac{10}{7}x^3 + \frac{3}{7}x$.

Theorem 4.2.2. *Immediate basins of attraction of the roots of polynomial with real coefficients and only real (and simple) zeros x_k, $1 \leq k \leq d$, for the Householder's method are simply connected.*

Proof. We know that in each band (c_k, c_{k+1}) there are two real repelling fixed points which are zeros of $2p'^2 - pp''$. Let $x_k^{'}, x_k^{''}$ be two consecutive real repelling fixed points of $h_p(z)$, then there exists a superattracting fixed point $x_k \in [x_k^{'}, x_k^{''}] \subset (c_k, c_{k+1})$. Now we want to show that $[x_k^{'}, x_k^{''}]$ is mapped into itself under iteration of h_p. We have shown that $h_p' \geq 0$ on each band, hence $h_p(z)$ is increasing and continuous on each band. So if $x \geq x_k^{'}$ then $h_p(x) \geq x_k^{'}$, and if $x \leq x_k^{''}$ then $h_p(x) \leq x_k^{''}$. It follows that

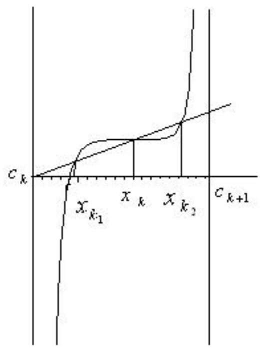
$$h_p[x_k^{'}, x_k^{''}] = [x_k^{'}, x_k^{''}].$$

Thus

$$A^*(x_k) \cap [x_k^{'}, x_k^{''}] = [x_k^{'}, x_k^{''}].$$

Now since $h_p(x)$ is continuous on $[x_k^{'}, x_k^{''}]$, we can follow the same path as in the proof of theorem (3.5.2), which shows that each immediate basins of Halley's method is simply connected. □

Since there are free critical points which are non-real. Then we are in the same situation of Halley's method.



Chapter 5

Kœnig's root-finding algorithms

In this chapter, we first recall the definition of a family of Kœnig's root-finding algorithms known as Kœnig's algorithms ($K_{p,n}$) [BH03] for polynomials. In the whole chapter p has degree $d \geq 2$ with real coefficients and real (and simple) zeros x_k , $1 \leq k \leq d$. Now we want to discuss Kœnig's algorithms in details where $n = 4$, $(K_{p,4}(z))$.

Definition 5.0.3. Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d,$$

be a polynomial with real coefficients and real (and simple) zeros x_k, $1 \leq k \leq d$, and $n \geq 2$ is an integer. Kœnig's method of p of order n is defined by the formula

$$K_{p,n}(z) = z + (n-1)\frac{(\frac{1}{p})^{[n-2]}}{(\frac{1}{p})^{[n-1]}}, \tag{5.0.1}$$

where $(\frac{1}{p})^{[n]}$ is the nth derivative of $\frac{1}{p}$.

For $n = 2$ the map $K_{p,n}$ is Newton's method of p, for $n = 3$ the map $K_{p,n}$ is Halley's method of p, and Householder's method

$$h_p(z) = K_{p,2}(z) - \frac{p}{2p'} K'_{p,2}$$

which we have discussed all of them in the previous chapters.

5.1 Kœnig's root-finding algorithms of order four

Let p be a polynomial with real coefficients and real (and simple) zeros x_k, $1 \leq k \leq d$, then

$$K_{p,4} = z - 3\frac{p^2 p'' - 2pp'^2}{6pp'p'' - 6p'^3 - p^2 p'''}, \tag{5.1.1}$$

defined as Kœnig's function of order four associated with p. The fixed points of $K_{p,4}$ are given by the zeros of $p^2p'' - 2pp'^2$. Since we have proved $pp'' - 2p'^2 < 0$ on $\mathbb{R}$, the fixed points of $K_{p,4}$ are the zeros of p together with ∞, and from proposition (5.4.1) the rational map $K_{p,4}$ has degree $3d - 2$.

Proposition 5.1.1. *Let $p : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial of degree d, then Kœnig's method $K_{p,4}$ is a rational map, it has a repelling fixed point at ∞ with multiplier $(d+2)/(d-1)$.*

Proof. When $|z|$ tends to ∞, we have

$$p(z) \sim \lambda z^d,$$

we know

$$K_{p,4} = z + 3\frac{(\frac{1}{p})''}{(\frac{1}{p})'''},$$

where

$$\left(\frac{1}{p}\right)'' \sim \frac{d(d+1)}{\lambda z^{d+2}},$$

and

$$\left(\frac{1}{p}\right)''' \sim \frac{-d(d+1)(d+2)}{\lambda z^{d+3}},$$

then

$$K_{p,4}(z) \sim z - 3\frac{z}{d+2},$$

$$K'_{p,4}(z) \sim 1 - \frac{3}{d+2} \sim \frac{d-1}{d+2},$$

as we know that the multiplier λ at ∞ is equal to

$$\lim_{n\to\infty} \frac{1}{K'_{p,4}(z)} = \frac{d+2}{d-1}.$$

□

5.2 Derivative of Kœnig's method of order four

The derivative of Kœnig's method $K_{p,4}$ is

$$K'_{p,4} = \frac{p^3(4pp'''^2 - 24p'p''p''' + 6p'^2p^{(4)} + 18p''^3 - 4pp''p^{(4)})}{(6pp'p'' - 6p'^3 - p^2p''')^2}, \tag{5.2.1}$$

from (5.2.1), we can see that the roots of $p(z)$ are superattracting fixed points of $K_{p,4}$, but of one degree higher order than for Halley's method. There are three critical points at each fixed point of $K_{p,4}$. The rational map $K_{p,4}$ has $2(3d-2)-2 = 6d-6$ critical points, and $3d - 6$ of them are free critical points. Also from proposition (5.4.1), the local degree of $k_{p,4}$ at the roots of p is exactly equal to four.

Remark 5.2.1. Let x be a simple zero of p, then $K_{p,4}(x) = x$ and from (5.4.1) $K'_{p,4}(x) = K''_{p,4}(x) = K'''_{p,4}(x) = 0$, while $K^{(4)}_{p,4}(x) \neq 0$. Thus $K_{p,4}$ is of order four for simple roots. Since $p(x) = 0$, it follows that $N_p(x) = H_p(x) = K_{p,4}(x) = x$, and this fixed point is superattracting fixed point for the three methods because $N'_p(x) = H'_p(x) = K'_{p,4}(x) = 0$. And since the third derivative of $K_{p,4}$ vanishes, whereas the third derivative of H_p does not, the graph of $K_{p,4}$ is flatter than that of H_p near the fixed point. Thus $K_{p,4}$ is faster convergence to the fixed point than H_p. From figures (5.1,5.2), Kœnig's function ($K_{p,4}$) looks like Newton's function but ($K_{p,4}$), where $p(z) = z^3 - z$, has non-real critical points whereas Newton's function does not.

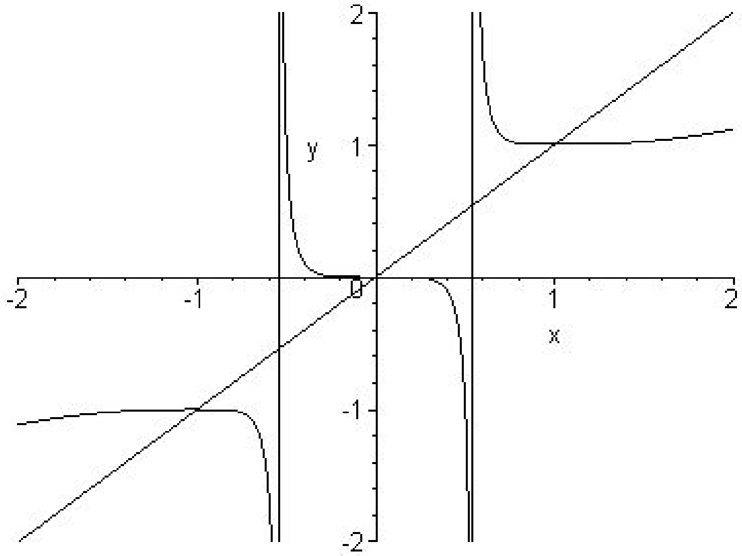


Figure 5.1: Kœnig's function for the polynomial $p(x) = x^3 - x$.

Proposition 5.2.1. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d with real coefficients and real (and simple) zeros. Then the rational map $K_{p,4}$ has $2d - 2$ repelling fixed points in $\mathbb{C}$ and their multipliers are all equal to four.*

Proof. Let

$$K_{p,4}(z) = z + 3\frac{g(z)}{g'(z)},$$

where

$$g = (\frac{1}{p})'' = \frac{2p'^2 - pp''}{p^3}, \tag{5.2.2}$$

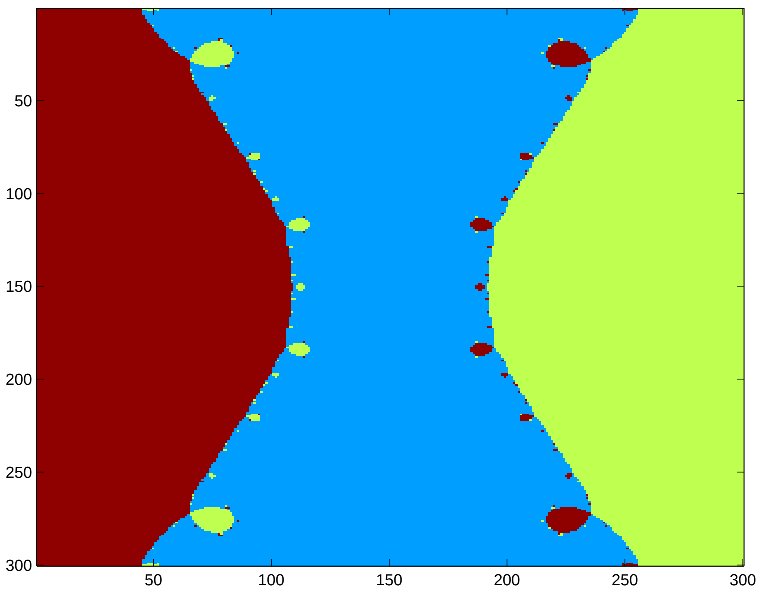


Figure 5.2: Iteration of Kœnig's function for the polynomial $p(z) = z^3 - z$.

$$g' = (\frac{1}{p})''' = \frac{6pp'p'' - 6p'^3 - p^2p'''}{p^4}.$$

Let x_k, $1 \leq x_k \leq d$, be the zeros of p which are real and simple. The fixed points of $K_{p,4}(z)$ are ∞, the points x_k and the zeros of the rational map $g = (\frac{1}{p})''$. From (5.2.2), we can see that g has $3d$ poles. When $z \to \infty$, then $p(z) \sim \lambda z^d$ and it follows that g has a zero of order $d+2$ at ∞. Since the number of zeros for any rational map is equal to the number of poles, then g has $3d-(d+2) = 2d-2$ finite zeros. Since we have proved that $2p'^2 - pp'' > 0$ on $\mathbb{R}$, $2d-2$ zeros of g are non-real repelling fixed points of $K_{p,4}$. Now we have

$$K'_{p,4} = 4 - \frac{3gg''}{g'^2},$$

and at the repelling fixed points of $K_{p,4}$, $g = 0$. Thus $K'_{p,4} = 4$ at each repelling fixed point. □

Definition 5.2.1. If $c_1 < c_2$ are consecutive real poles of $K_{p,4}$, then the interval (c_1, c_2) is called a band for $K_{p,4}$.

Proposition 5.2.2. *If (c_1, c_2) is a band for $K_{p,4}$ that contains a root of $p(x)$, then*

$$\lim_{x\to c_1^+} K_{p,4}(x) = +\infty, \qquad \lim_{x\to c_2^-} K_{p,4}(x) = -\infty.$$

Proof. From

$$K_{p,4} = z - 3\frac{p(pp'' - 2p'^2)}{6p'(pp'' - p'^2) - p^2p'''},$$

and $pp'' - 2p'^2 < 0$ on $\mathbb{R}$, it follows that, if $p > 0$ in (c_1, x_k), then $p' < 0$ and $p''' < 0$, and if $p < 0$ in (c_1, x_k), then $p' > 0$ and $p''' > 0$. Thus

$$\frac{p(pp'' - 2p'^2)}{6p'(pp'' - p'^2) - p^2p'''} < 0 \qquad in \ \ (c_1, x_k),$$

it follows that $K_{p,4}(x) > x$ in (c_1, x_k), thus

$$\lim_{x \to c_1^+} K_{p,4}(x) = +\infty.$$

Similarly, we have

$$\frac{p(pp'' - 2p'^2)}{6p'(pp'' - p'^2) - p^2p'''} > 0 \quad in \ \ (x_k, c_2),$$

so $K_{p,4}(x) < x \quad in \ \ (x_k, c_2)$, thus

$$\lim_{x \to c_2^-} K_{p,4}(x) = -\infty.$$

□

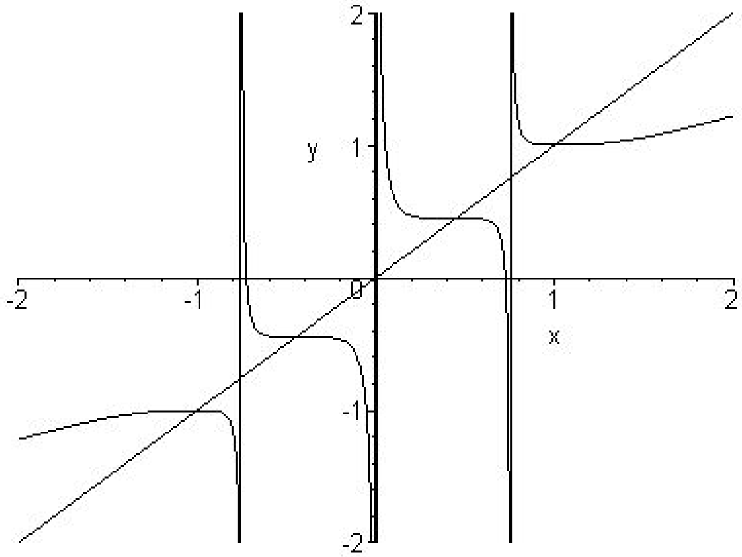


Figure 5.3: Kœnig's function for the polynomial $p(x) = (x^2 - 1)(x^2 - 1/5)$.

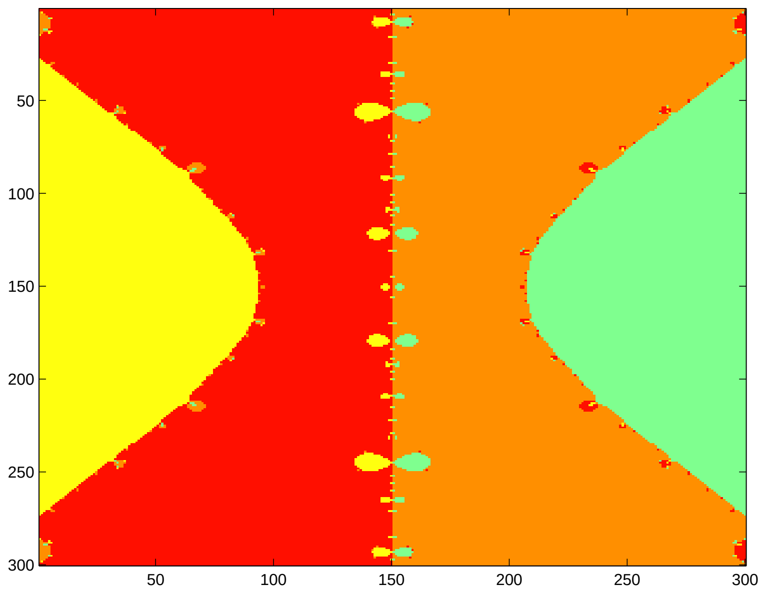


Figure 5.4: Iteration of Kœnig's function for the polynomial $p(z) = (z^2 - 1)(z^2 - 1/5)$.

5.3 Immediate basins of Kœnig's method of order four

In this section, we want to prove that each component of Fatou set of Kœnig's method $K_{p,4}$ is simple connected.

Theorem 5.3.1. *Immediate basins of attraction of Kœnig's function $K_{p,4}$ are simply connected, whenever p is a complex polynomial with real coefficients and only real and simple zeros x_k , $1 \leq k \leq d$.*

Proof. We follow the same steps of proof of theorem (3.5.2) with some changes. In this case we work on the interval (a_1, a_2), where a_1, a_2 are two consecutive poles of $K_{p,4}$ instead of the interval (r_1, r_2), where r_1, r_2 are two repelling fixed points of H_p. Assume that A is a non simply connected immediate basin of attraction for $K_{p,4}$ to a root $x \in \mathbb{R}$ of a polynomial p. We follow the same proof of theorem (3.5.2) until we arrive to the conclusion that boundary A contains two different fixed points belonging to two different components of boundary of A. But the only fixed points for $K_{p,4}$ are the roots of p and ∞. We arrived at a contradiction. $\square$

Since $K_{p,4}$, (where for simplicity $p(z) = z^3 - z$), has non real free critical points, then we are in the same situation of Halley's method.

5.4 General form of Kœnig's method

The following rational map

$$K_{p,n}(z) = z + (n-1)\frac{(\frac{1}{p})^{[n-2]}}{(\frac{1}{p})^{[n-1]}},$$

is the general form of Köing's function. We end this chapter with some general remarks describe, without proof, the dynamics of the general form of Kœnig's function $K_{p,n}$. We will consider p be a special polynomial of degree $d \geq 2$ which is a complex polynomial with real coefficients and real (and simple) zeros x_k, $1 \leq k \leq d$, and $p'(x_k) = p''(x_k) = 0$.

Proposition 5.4.1. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. Then for any $n \geq 2$,*

(a) *The rational map $K_{p,n}$ has degree $(n-1)(d-1)+1$.*

(b) *If p has d distinct roots, then $K_{p,n}$ has $(n-2)(d-1)$ repelling fixed points in $\mathbb{C}$.*

(c) *The local degree of $K_{p,n}$ at the roots of p is exactly n.*

(d) *Kœnig's method $K_{p,n}$ is a rational map, it has a repelling fixed point at ∞ with multiplier $1 + \frac{n-1}{d-1}$.*

Proof. For details proof see [BH03]. □

In general case of the map $K_{p,n}$, $n \geq 2$ and p is special polynomial of degree $d \geq 2$ with real coefficients and real (and simple) zeros, we have two cases.

Case (1) If n is even, then the map $K_{p,n}$ has $nd-2$ real critical points, and $(n-2)(d-2)$ non-real critical points which are distributed as follows; each basin of x_k, $2 \leq k \leq d-1$, contains n real critical points and $n-2$ non-real critical points, symmetric to the real line; the two basins of x_1, x_d each contains $(n-1)$ real critical points. And there are $(d-1)$ real poles of $K_{p,n}$.

Case (2) If n is odd then the map $K_{p,n}$ has $(n-1)d$ real critical points and $(n-1)(d-2)$ non-real critical points, where each basin x_k, $1 \leq k \leq d$, contains $(n-1)$ real critical points and each basin x_k, $2 \leq k \leq d-1$, contains (n-1) non-real critical points. And there are no real poles.

The following figures show how the critical points (c.p) distributed around the fixed points of the map $K_{p,n}$, where p is special polynomial.

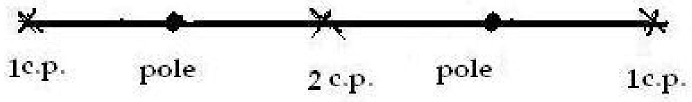


Figure 5.5: $n = 2, d = 3$ (Newton), number of critical points $2d - 2$.

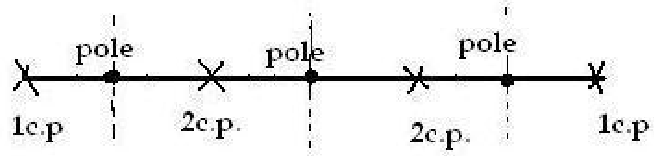


Figure 5.6: $n = 2, d = 4$ (Newton), number of critical points $2d - 2$.

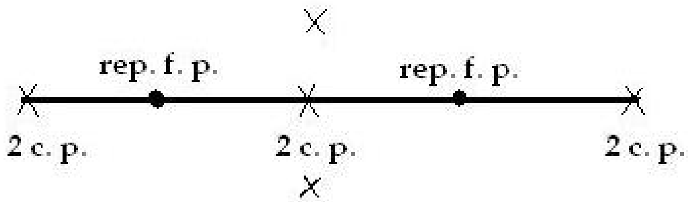


Figure 5.7: $n = 3, d = 3$ (Halley), number of critical points $4d - 4$.

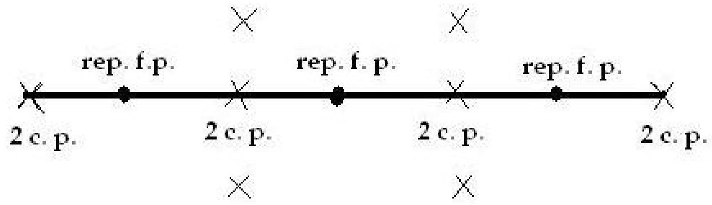


Figure 5.8: $n = 3, d = 4$ (Halley), number of critical points $4d - 4$.

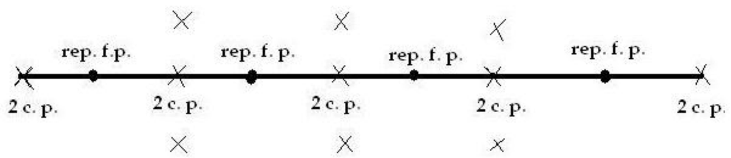


Figure 5.9: $n = 3, d = 5$ (Halley), number of critical points $4d - 4$.

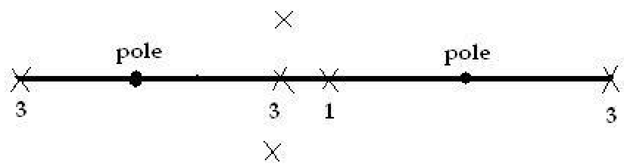


Figure 5.10: $n = 4, d = 3$ ($K_{p,4}$), number of critical points $6d - 6$.

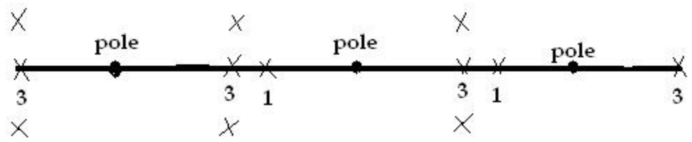


Figure 5.11: $n = 4, d = 4$ ($K_{p,4}$), number of critical points $6d - 6$.

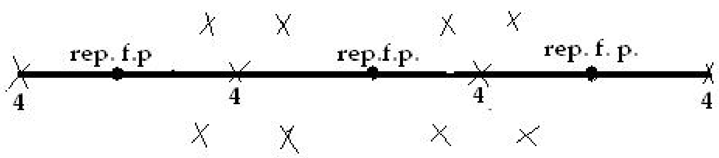


Figure 5.12: $n = 5, d = 4$ ($K_{p,5}$), number of critical points $8d - 8$.

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