Frequency estimation by DFT interpolation: A comparison of methods

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Abstract

This article comments on a frequency estimator which was proposed by [6] and shows empirically that it exhibits a much larger mean squared error than a well known frequency estimator by [8]. It is demonstrated that by using a heuristical adjustment [2] the performance can be greatly improved. Furthermore, references to two modern techniques are given, which both nearly attain the Cramér-Rao bound for this estimation problem.

1 Introduction

The problem of achieving a precise estimation of the fundamental frequency of a noisy signal has been researched for a considerable amount of time. It has many applications in different areas of engineering, not least in audio processing and musicology. Quite a number of algorithms has been published [5, 4] until today. These methods often differ strongly regarding their general idea, area of application, preciseness and time complexity. This work deals with an empirical comparison of two such algorithms for frequency estimation of simple sinudoids with noise. The basic tool for this task is the discrete Fourier transform (DFT) and its efficient implementation the fast Fourier transform (FFT).

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2 Frequency estimation

Let the signal be a simple sinsusoid

$$X_t = \mu + A\sin(\omega t + \phi) + \epsilon_t, \quad t = 0, 1, \ldots, T-1$$

with additive noise ϵ_t, sampled at T points in time. In general the noise is assumed to be Gaussian white noise with $E[\epsilon_t] = 0$ and $Var[\epsilon_t] = \sigma^2$ (although some cited papers generalize upon this [7, 8, 9]). While all other parameters are unknown, a frequency estimator $\hat{\omega}$ is sought, which is as accurate as possible and can be computed efficiently in time. Minimizing the squared error seems to be a reasonable first approach:

$$SSE = \sum_{t=0}^{T-1} [X_t - \mu - A\sin(\omega t + \phi)]^2 \quad .$$

This equates to a maximum-likelihood approach. The Cramér-Rao bound for unbiased estimators $\hat{\omega}$ is known and amounts to:

$$Var(\hat{\omega}) \geq \frac{6\sigma^2}{T(T^2-1)A^2} \quad ,$$

which contains apart from the cubic term $T(T^2-1)$ the signal-to-noise ratio (SNR) $A^2/2\sigma^2$ (see [4] for a derivation).

2.1 Interpolation of Fourier coefficients

There are quite a number of algorithms for frequency estimation, which more or less directly interpolate the complex-valued DFT coefficients

$$Y_j = \sum_{t=0}^{T-1} X_t \exp(-i2\pi jt/T)$$

or the real-valued coefficients of the periodogram

$$I(\lambda) = \frac{2}{T} \left| \sum_{t=0}^{T-1} X_t \exp(-it\lambda) \right|^2 \quad ,$$

where $I(\lambda)$ is in practice only available at the discrete Fourier frequencies $2\pi jt/T$:

$$P_k = \frac{2|Y_k|^2}{T} = I(2\pi jt/T)$$

The reason for this course of action is that the maximum of the periodogram

$\hat{\omega}_{\max} = \text{argmax } I(\lambda)$ is an asymptotically unbiased estimator for ω, so that also asymptotically holds [8]:

$$T^{3/2}(\hat{\omega}_{\max} - \omega) \sim N\left(0, \frac{48\pi f(\omega)}{A^2}\right) \ ,$$

where $f(\omega)$ is the spectral density of the noise ϵ_t. However, the global maximum of $I(\lambda)$ is hard to compute numerically (in an efficient way) because of the many local optima of the oscillating $I(\lambda)$. One can neither use the maximiser of the discrete periodogram P_k as an approximation - as the resolution of P_k is not good enough, even for signals without noise, to get sufficiently precise estimations for applications [6] - nor does choosing the maximiser of P_k as starting point for an iterative optimisation of $I(\lambda)$ guarantee convergence to the global maximum of $I(\lambda)$. But as the maximum of the periodogram and its two neighbours already contain about 85% of the power of the spectrum, an interpolation of these three points seems appropriate.

2.2 Algorithm by Ligges

Ligges [6] presents a heuristically motivated procedure for frequency estimation in the area of musicology. As this algorithm should also be used for signals with overtones or multiple tones and implements further practical aspects, it is more comprehensive than the following estimators given below, but in the here considered case of simple tones it can be reduced to the following: A parabola is fitted to the maximum of the periodogram and its neighbouring frequencies and the peak position of the parabola is used as an estimator for the tone frequency.

$$\hat{\omega}_{Ligges1} = \frac{\lambda^{**} + \lambda^{*}}{2} + \frac{(\lambda^{**} - \lambda^{*})(I(\lambda^{**}) - I(\lambda^{*}))}{2I(\lambda^{*}) - I(\lambda^{**}) - I(\lambda^{***})} = \lambda^{*} + \frac{2\pi}{T}\frac{P_{k+1} - P_{k-1}}{4P_k - 2P_{k+1} - 2P_{k-1}}$$

Here, λ^{*} denotes the maximising frequency of the periodogram, λ^{**} the neighbouring frequency with the larger and λ^{***} the neighbouring frequency with the smaller value in the periodogram.

Practically the same method was published by Voglewede in [10]. A further heuristical adaptation to improve the estimation [6, 1] yields:

$$\hat{\omega}_{Ligges2} = \lambda^{*} + \frac{\lambda^{**} - \lambda^{*}}{2}\sqrt{\frac{I(\lambda^{**})}{I(\lambda^{*})}}$$

In his PhD thesis, Ligges [6] conducts an experimental evaluation of these two estimators and reports a mean error of approximately 2.2 Hz for the fundamental frequency when using the second algorithm. He also compares his method to known model-based approaches. Asymptotic distributions are not indicated.

2.3 Algorithm by Quinn

[8] uses an estimator, which in a similar efficient fashion interpolates three Fourier coefficients, although he does not use the periodogram but instead directly works on the complex DFT coefficients Y_j of the data.

1. Let k be the maximising index of $|Y_j^2|$.
2. Let $\alpha_1 = \Re(Y_{k_T-1}/Y_{k_T})$ and $\alpha_2 = \Re(Y_{k_T+1}/Y_{k_T})$,
   and $\delta_1 = \alpha_1/(1-\alpha_1)$ and $\delta_2 = -\alpha_2/(1-\alpha_2)$
3. If both $\delta_1, \delta_2 > 0$, then $\delta = \delta_2$, else $\delta = \delta_1$
4. $\hat{\omega}_{Quinn} = 2\pi(k_T + \delta)/T$

Quinn does not assume the noise to be i.i.d Gaussian, but derives under quite weak and somewhat technical assumptions (ϵ_t strictly stationary and ergodic, $E[\epsilon_t] = 0$, $E[\epsilon_t^2] < \infty$, for further details see [8]) the following asymptotic distribution by using a central limit theorem:

$$\frac{T^{3/2}}{v_T}(\hat{\omega}_{Quinn} - \omega) \sim \text{N(0,1)}$$

where

$$v_T^2 = \frac{16\pi^5\delta^2(1-|\delta|)^2(2\delta^2 - 2|\delta| + 1)f(\omega)}{A^2 \sin^2(\pi\delta)}$$

and $f(\omega)$ is the spectral density of the noise.

2.4 Algorithm by Jacobsen

As the following empirical comparison shows, neither $\hat{\omega}_{Ligges1}$ nor $\hat{\omega}_{Ligges2}$ produce very good results. $\hat{\omega}_{Ligges1}$ can be rectified, if one employs the same general idea as in 2.3 and switches to the complex-valued DFT coefficients:

$$\hat{\omega}_{Jacobsen} = \lambda^* + \frac{2\pi}{T}\Re\left(\frac{Y_{k-1} - Y_{k+1}}{2Y_k - Y_{k+1} - Y_{k-1}}\right)$$

Please note the other slight changes in the formula as well. This method was published by Jacobsen in [3]. There are no further theoretical results concerning the distribution of the estimator. Also, it's not very evident, why exactly this adaptation results in such an improvement[Jacobsen 2009, personal communication].

3 Empirical comparison

1000 frequencies for ω between 2 and 32 generated, while the noise was varied from $\sigma^2 = 0$ to 1. The amplitude A was set to 1 (so SNR was at least 0.5) and the phase ϕ was selected randomly for the resulting 30000 sinusoids. Every signal was sampled 128 times. Figure 1 shows the error distributions for all four estimators. It is clearly visible in fig. 1 that the simple quadratic interpolation results in the worst accuracy. Both algorithms by Ligges exhibit a much larger variance than the methods by Quinn and Jacobsen, also the main mass of the distribution is not even centred around zero. $\hat{\omega}_{Ligges2}$ does improve on the variance to some extend, but does not change the general, problematic shape of the error function. Jacobsen's estimator seems to have about the same quality as the one by Quinn.

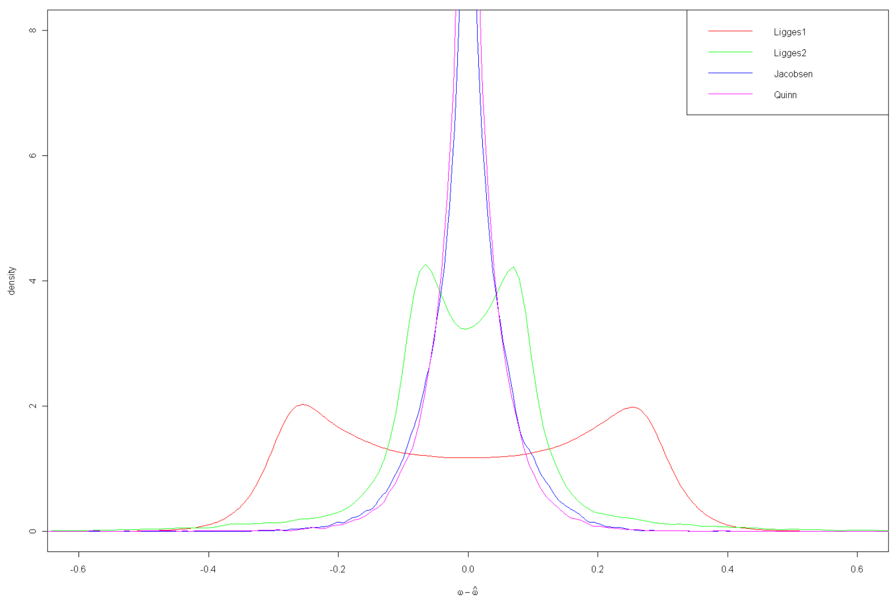

Fig. 1: *Empirical distributions of estimation errors*

To gain some further insights into the estimators, figures 2 and 3 show the development of bias and variance in relation to the true frequency of the signal - first without noise, then with $\sigma^2 = 0.2$ and at last with $\sigma^2 = 0.4$. While both estimators of Ligges are strongly biased and also exhibit a large variance (which of course grows with the noise), the remaining two algorithms of Quinn and Jacobsen seem again roughly equal in quality, bias and variance being slightly different in different areas of the bin.

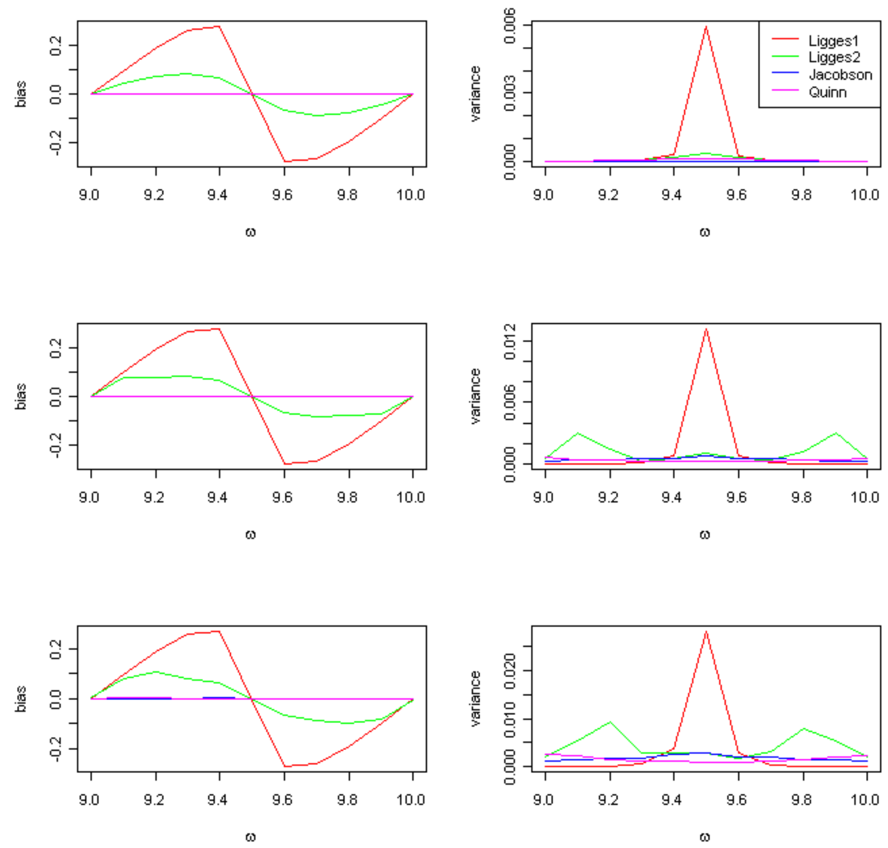

Fig. 2: *Bias and variance of the 4 estimators against the bin number of the true frequency. The three rows correspond to a noise setting of* $\sigma^2 = 0$, $\sigma^2 = 0.2$ *and* $\sigma^2 = 0.4$ *respectively.*

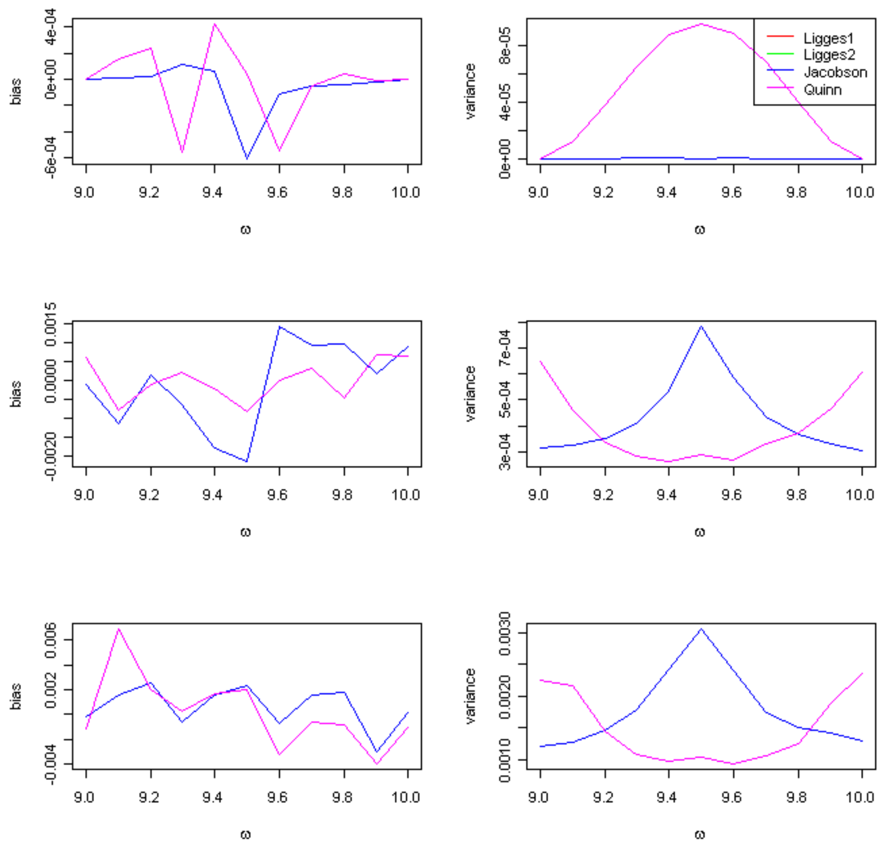

Fig. 3: *Bias and variance of best 2 estimators against the bin number of the true frequency. The three rows correspond to a noise setting of* $\sigma^2 = 0$, $\sigma^2 = 0.2$ *and* $\sigma^2 = 0.4$ *respectively.*

4 Best known methods near the Cramér-Rao bound

Apart from the already discussed methods, there a two other ones, which exhibit an even lower MSE. The first one is an improvement of Quinn's algorithm [9] $\hat{\omega}_{Quinn}$, where δ_1 and δ_2 are combined in a nonlinear fashion to a new δ and not selected depending on their sign:

$$\delta = \frac{\delta_1 + \delta_2}{2} - \kappa(\delta_1^2) + \kappa(\delta_2^2)$$

$$\kappa(x) = \frac{1}{4}\log(3x^2 + 6x + 1) - \frac{\sqrt{6}}{24}\log\left(\frac{x + 1 - \sqrt{2/3}}{x + 1 + \sqrt{2/3}}\right)$$

The second is a further estimator by [7]. MacLeod's estimator is - on a superficial level - similar to Jacobsen's quadratic estimator plus an additional nonlinear correction term:

$$\tau = Y_k, \quad R_i = \Re(Y_i \tau^*) \ ,$$

$$\gamma = \frac{R_{k-1} - R_{k+1}}{2R_k + R_{k+1} + R_{k-1}} \ ,$$

$$\delta = \frac{\sqrt{1+8\gamma^2} - 1}{4\gamma}, \quad \hat{\omega}_{MacLeod} = \lambda^* + \delta \frac{2\pi}{T} \quad ,$$

where k denotes the maximising index of the discrete periodogram and τ^* the complex conjugate. Though one should note the different signs in the denominator of γ and Jacobsen's interpolation quotient and that the Y_k are multiplied by a phase reference to align them better with the phase of the periodogram peak [7].
It can be shown in both cases, that the variance of the estimators is quite close to the Cramér-Rao bound. Jacobsen has published his own empirical evaluation of these methods on his web page [2].

5 Summary

A couple of methods for frequency estimation of noisy sinusoids - all using the interpolation of three Fourier coefficients - were compared. It was demonstrated, that both algorithms by Ligges exhibit a large bias and variance. The first estimator can be corrected and is then equivalent to an estimator by Jacobsen, which has a roughly similar performance as a well known estimator by Quinn. References to two further methods were given, which show an even lower MSE than the two mentioned before and which are provably nearly optimal with regard to the Cramér-Rao bound. In particular, the algorithm by Quinn has the advantage that asymptotic normal distributions are available.

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