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A well-posed hysteresis model for flows in porous media and applications to fingering effects

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Abstract: We investigate flow problems in unsaturated porous media with hysteresis effects in the capillary pressure relation. The model expands the Richards equation, gravity is included and the space dimension is arbitrary. The hysteresis model has been suggested by experimentalists, static hysteresis is incorporated with a play-type model and additional dynamic effects are included. We propose a Galerkin scheme for these equations, show the convergence of the corresponding approximate solutions and the existence of weak solutions to the original problem. We include numerical results that show the effect of gravity driven fingering in porous media.

1 Introduction

The standard model for the description of saturation distributions in porous media is the Richards equation. We consider a bounded domain $\Omega \subset \mathbb{R}^n$, occupied by the porous material, and a time interval $[0,T)$. Denoting the fluid pressure by $p : \Omega \times [0,T) \to \mathbb{R}$ and the saturation by $s : \Omega \times [0,T) \to \mathbb{R}$ (volume fraction of pore space that is filled with fluid), the combination of mass conservation and Darcy's law for the velocities yields the Richards equation

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_n]) \,. \tag{1.1}$$

In this equation, a normalization of porosity, density, and gravity are performed, the acceleration of gravity is 1 and points in direction $-e_n$. The permeability $k = k(s)$ is a given function $k : \Omega \times \mathbb{R} \to [0,\infty)$.

The interesting modelling problem regards the relation between pressure p and saturation s. The simplest possibility (and the standard choice for the Richards equation) is a functional dependence, $p = p_c(s)$, where the capillary pressure function p_c is monotonically increasing in s. We continue here the analysis of a more complex model which includes hysteresis. We investigate

$$p \in p_c(s) + \gamma \,\mathrm{sign}(\partial_t s) + \tau \partial_t s, \tag{1.2}$$

where $\mathrm{sign}(\xi) := [-1,1]$ for $\xi = 0$ and $\mathrm{sign}(\xi) \in \{\pm 1\}$ for $\xi \neq 0$. This hysteresis relation between s and p was suggested in [8].

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For $\tau = 0$, relation (1.2) encodes hysteresis of play-type between 's and p. In this case, relation (1.2) demands that the pressure p is always in the s-dependent interval $[p_c(s) - \gamma, p_c(s) + \gamma]$. Furthermore, for p strictly between $p_c(s) - \gamma$ and $p_c(s) + \gamma$, the time derivative $\partial_t s$ necessarily vanishes. Given a time evolution of p and an initial condition for the saturation, (1.2) can be solved for s, we refer to [28] for an analysis in appropriate function spaces.

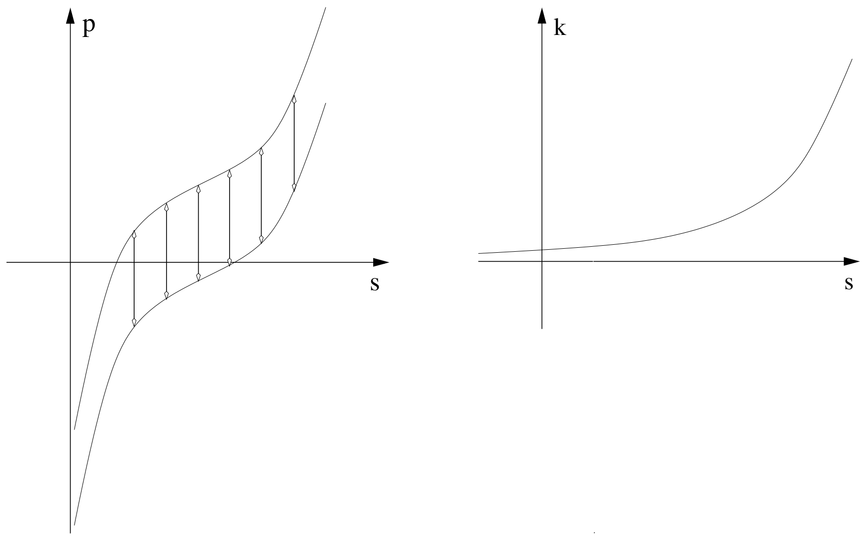


Figure 1: *Illustration of the parameter functions. Left: the play-type hysteresis relation. The curved lines correspond to $p_c(s) + \gamma$ and $p_c(s) - \gamma$. For $\tau = 0$, the pair (s, p) can move along the vertical lines (variable pressure at fixed saturation value). Right: the positive permeability parameter $k(s)$. In the analytical results, the coefficient functions $p_c(s)$ and $k(s)$ are extended to the real line $s \in \mathbb{R}$.*

Our main result is an existence theorem for system (1.1)–(1.2) of partial differential equations. In the proof of this result we provide a Galerkin scheme, show the well-posedness of the scheme and a priori estimates, compactness results for the solution sequence and the solution property for weak limits. One of the main analytical difficulties is to derive compactness of the sequence of saturations in $L^2(\Omega \times (0,T))$, which is needed in order to take limits in the nonlinear terms. The equation allows to control time derivatives of the saturation and space derivatives of the pressure — but since there is no functional dependence, we cannot apply a Lions-Aubin lemma to conclude the compactness. We derive compactness via a careful analysis of the hysteresis relation with the compactness theorem of Riesz-Frechet-Kolmogorov.

We include numerical results for the above model. In our calculations we use initial and boundary conditions that correspond to experiments regarding gravity fingering. While the system without hysteresis cannot predict gravity fingers, our numerical results for the hysteresis system (1.1)–(1.2) show the appearance of gravity fingers.

1.1 Hysteresis models for the capillary pressure

Hysteresis is an important effect in porous media and it must be modelled correctly for quantitative calculations and predictions. Additionally, even qualitative features of flow in porous media depend on the capillary hysteresis. One example is the development of gravity fingers.

It is well known that the capillary pressure is different for imbibition and drainage processes, which means that a single function $s \mapsto p_c(s)$ is inappropriate for the description of processes where $\partial_t s$ changes sign. Repeating imbibition and drainage procedures, one can explore a complex family of scanning curves between the two principle lines [11, 7]. Instead of prescribing a great number of scanning curves, it is desirable to develop a simple (thermodynamically consistent) model which includes hysteresis [15]. The most simple model is play-type hysteresis with vertical scanning curves. It is contained in (1.2) as the special case $\tau = 0$. We use here the simplifying assumption that the imbibition curve is given by $p_c^{im}(s) = p_c(s) + \gamma$ and the drainage curve by $p_c^{dr}(s) = p_c(s) - \gamma$. Note that p_c is the pressure of a non-wetting phase, therefore p_c is increasing.

The play type hysteresis has the disadvantage that all interior scanning curves are vertical and that, in particular, the pair (s, p) can move back and forth along the same line. These facts do not correspond to experiments. Nevertheless, if play type operators with different values of γ are averaged, then a Prandtl-Ishlinskii operator replaces the play-type operator and scanning curves are no longer vertical. The Prandtl-Ishlinskii model was derived by rigorous homogenization from the play-type model in [22]. In this sense, imposing locally play-type hysteresis does not contradict experiments. The local use of the Prandtl-Ishlinskii operator is discussed also in [8].

The play type hysteresis is rate independent and models static hysteresis effects. With different co-authors, Hassanizadeh included dynamic hysteresis effects with the parameter $\tau > 0$ (in underlying models with and without static hysteresis). The full model appears as equation (11) in [8], if the above assumption is made on p_c^{im} and p_c^{dr}, and the wetting phase is not modelled, $p^w \equiv 0$. The model was also used in [9], see their relations (4) and (5). Both contributions regard explicit one-dimensional solutions and thermodynamic considerations.

We emphasize that the play type hysteresis relation can (to a certain extend) be justified by an analysis of a pore-scale model. The relevant mechanism is the bottleneck effect: the interface between wetting and non-wetting fluid in a single pore (with given contact angle) can move between narrow and wide regions of the pore. Such a change of the position of all the interfaces induces a macroscopic change of the pressure, but no macroscopic change of the saturation. In simple geometries, the argument can be made precise to provide a justification of the play-type hysteresis, see [20] and [21]. Regarding recently developed further extensions of the hysteresis model we mention [16] and the references therein.

1.2 Existence results for extended Richards equations

Even in the case without hysteresis, i.e. with an algebraic relation $p = p_c(s)$ instead of (1.2), the Richards equation is an interesting mathematical object due to the possible degeneracies $k(s) = 0$ for some s and $p_c(s_k) \to \pm\infty$ for s_k tending to critical saturation values. Existence results are obtained e.g. in [1] and [2], uniqueness is treated e.g. in [18] and [10], the physical outflow boundary conditions are treated e.g. in [3] and [23].

The relation (1.2), without the coupling to a partial differential equation, poses also interesting questions regarding a functional analytic description and we refer to [28] for the corresponding discussion. In both cases, $\tau = 0$ (rate-independent) and $\tau > 0$ (rate-dependent), the hysteresis relation may be considered as a functional relation $s(t) = \mathcal{B}(t, p|_{[0,t]})$, where $\mathcal{B}$ maps the history of p to a value $s(t)$, given initial values s_0. We emphasize that, even in the equilibrium situation $\partial_t s = 0$, we cannot determine $p(t)$ from $s|_{[0,t]}$. In this sense, the hysteresis relation cannot be inverted.

An existence result for the Richards equation with hysteresis was provided in [22] in the case that the partial differential equation is linear, i.e. in the case that $k(.)$ is not depending on s and that $p_c(.)$ is an affine function. In this situation, it was possible to treat the case $\tau = 0$. We note here that a positive parameter τ has a regularizing effect in the system and helps considerably in the analysis of the equations.

Existence results for the nonlinear system with hysteresis were obtained in [4] and [5] under very general assumptions: coefficients can be degenerate, the hysteresis relation can be more general than play-type, and outflow boundary conditions are treated. Nevertheless, our problem is not covered by these contributions. In [4], the hysteresis relation $s(t) = \mathcal{B}(t, p|_{[0,t]})$ is inverted (compare relation (3.7) of [4]), which is not possible in our model. In [5], another regularization of the equations was introduced, namely a time-convolution in the determination of the permeability coefficient k, which is not a common modelling approach.

1.3 Gravity fingering

Fingering is an effect that can occur when a more saturated layer is above a less saturated region of a porous medium. Under the influence of gravity, the fluid will move downward. For appropriately designed initial and boundary conditions, this downward flow happens in the form of fingers, i.e. the fluid moves mainly along long and thin subdomains of high saturation. The effect is known at least since the 1960'ies, recent studies are presented e.g. in [25, 6]. The phenomenon is similar to viscous fingering in Hele-Shaw cells, but it is more challenging in the sense that the standard models cannot explain the effect, see [14, 26]. Consequently, other models have been proposed, among them different "non-equilibrium Richards equations", studied e.g. in [13, 17]. These models are related to a positive parameter τ.

The parameter $\tau > 0$ introduces a form of *dynamic hysteresis* (we remark that the terminology is somewhat misleading, since the word hysteresis is usually reserved for rate independent processes). We include here both effects, *dynamic hysteresis* and *static hysteresis*. The static effects are modelled with the play-type relation and are certainly relevant in unsaturated porous media. One may even argue that the static hysteresis is the more important for the development of fingers: let us consider a point far away from the finger tip. The flow situation there is almost stationary. The fact that the finger does not widen at later stages must be a consequence of static hysteresis. Another argument that shows the importance of stationary hysteresis for fingering effects is the following instability result: planar wetting fronts under purely static hysteresis (i.e. $\tau = 0$) are unstable, see [24].

Let us summarize the advantages of (1.1)–(1.2): the model includes static and dynamic hysteresis, the static contribution is of play-type, which is the most simple hysteresis model, it is consistent with thermodynamics and with experiments, and it has a clear modelling foundation (bottle neck effect). The numerical results for this model show the development of fingers in qualitative agreement with experiments. In agreement with the one-dimensional analysis, the fingers develop a saturation maximum at the tip (non-monotone saturation profile) only for positive values of τ.

2 Main result and preliminaries

2.1 Coefficient functions and main result

The unknowns in the porous media model (1.1)–(1.2) are pressure and saturation. A typical choice of function spaces is

$$p \in L^2(0,T;H^1(\Omega,\mathbb{R})), \qquad s \in L^\infty(0,T;L^2(\Omega,\mathbb{R})). \tag{2.1}$$

We will actually achieve a solution with improved regularity, see Proposition 4.2. The solution depends on the initial values $s_0 \in L^2(\Omega)$ for the saturation s, and on the boundary data $g \in L^2(0,T;H^1(\Omega))$ for the pressure p; we will actually demand more regularity on these data. Furthermore, the solution depends on the coefficient functions $p_c(x,s)$, $k(x,s)$, and $\gamma(x)$. We assume $\tau > 0$ and for constants $0 < \rho < \rho_1$, $\kappa_1 > 0$, $0 < \kappa < \kappa_0$,

$$p_c, \partial_s p_c \in C(\Omega \times \mathbb{R}, \mathbb{R}), \quad \|p_c\|_{\mathrm{Lip}} \le \rho_1,\ \partial_s p_c \ge \rho, \tag{2.2}$$

$$k, \partial_s k \in C(\Omega \times \mathbb{R}, \mathbb{R}), \quad \kappa_0 \ge k(x,s) \ge \kappa,\ |\partial_s k(x,s)| \le \kappa_1, \tag{2.3}$$

$$\gamma \in C^{0,1}(\Omega, [0,\infty)). \tag{2.4}$$

We emphasize that the Lipschitz continuity of p_c is assumed in x and s.

Remarks on the coefficient functions. The physically relevant range of saturation values is the interval $[0,1]$. Since comparison principles are available for the above scalar parabolic problem, upon conditions on the coefficient function p_c, the pressure boundary values g and the saturation initial values s_0, the saturation will only take values in $[0,1]$. In this case, it is sufficient to consider p_c and k as functions on $\Omega \times [0,1]$.

The assumptions on the coeficient functions are used in the following steps. The bounds $\kappa_0 \ge k(x,s) \ge \kappa > 0$ in Lemma 3.1 and in the energy estimate of Proposition 3.2, the Lipschitz-continuity of $k(x,\cdot)$ in Lemma 3.1, the Lipschitz-continuity of $\gamma(\cdot)$ in Lemma 3.3, the Lipschitz-continuity of p_c in Lemma 3.1 and Proposition 3.2. The differentiability of k and p_c in s is only required for the higher order estimates in Section 4.

We treat here only the system with $\tau > 0$. If one regularizes the multivalued sign-function to a single valued function Φ, then one may insert p from (1.2) into (1.1) to obtain an equation for s. Such an equation (usually with the linear operator $\Delta\partial_t s$ on the right hand side) is called pseudo-parabolic. We do not use here the theory of pseudo-parabolic systems since we are interested in the multi-valued sign function (i.e. we allow $\delta = 0$ in the function Φ_δ^τ below).

The positivity assumption $\partial_s p_c > \rho > 0$ can actually be replaced by the much weaker assumption that there exists a primitive

$$P_c, \partial_s P_c \in C(\Omega \times \mathbb{R}, \mathbb{R}), \quad P_c(x,s) \ge 0,\ \partial_s P_c(x,s) = p_c(x,s). \tag{2.5}$$

This means that we solve the pseudo-parabolic equation ($\tau > 0$) even if the corresponding parabolic equation has (partially) backward character.

Main Theorem. The aim of this contribution is to suggest and to analyze a spatial discretization of the above system and to show the convergence of the corresponding approximate solutions to a solution of the hysteresis problem.

Our analysis is – unfortunately – restricted to the case $\tau > 0$. A positive coefficient τ has a regularizing effect in the equations and provides better a priori estimates on solutions. A positive time delay coefficient τ makes the problem rate dependent. Even though a rate independent system is more desirable in some applications, there are experiments that support the positivity of τ.

Theorem 2.1 (Approximation procedure and existence result). *Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain, $T > 0$, and $\tau > 0$. Let the coefficients p_c, k, γ satisfy* (2.2)–(2.4) *and let initial and boundary data be given by $s_0 \in L^2(\Omega)$ and $g \in L^2(0,T;H^1(\Omega))$. Then there exists a weak solution (s,p) of the hysteresis system (1.1)–(1.2) according to the solution concept of Definition 2.2.*

The Galerkin discretization of the equations from Definition 2.4 provides approximate solutions, depending on the grid size parameter $h > 0$ and, possibly, on a regularization parameter $\delta \geq 0$. In the limit $h \to 0$ and $\delta \to 0$, the approximate solutions converge to a weak solution of the limit system.

Section 3 is devoted to the analysis of the discrete scheme. We show the solvability of the discrete system in Lemma 3.1 and provide a priori estimates of energy type in Proposition 3.2. The estimates control time derivatives of s and space derivatives of p. The compactness of the family of saturations relies on the hysteresis relation and is shown in Lemma 3.3. Finally, the weak limits of the approximate solutions are weak solutions to the hysteresis problem by Lemma 3.4.

Section 4 is devoted to uniform and to higher order estimates. Such estimates are not needed in the existence proof presented in Section 3. Nevertheless, we find it an interesting observation that, despite the nonlinearities in the equations, some higher order estimates are available (on arbitrary time intervals $(0,T)$ and not only locally). These estimates guarantee, in particular, the strong convergence of the pressure approximations $\tilde{p}^h$.

Section 5 shows results of an implementation of the scheme. We describe the performed time discretization and linearization and observe numerical convergence rates in an adaptive calculation. Our main interest is the application of the model to gravity fingers in unsaturated porous media. The numerical results show the fingering effect for small perturbations of the initial data. This confirms that the hysteresis model (1.1)–(1.2) can be used for the description of the fingering effect.

2.2 Solution concepts

We introduce a space discretization in order to define a sequence of approximate solutions (s^h, p^h). Energy estimates provide integral bounds for $\partial_t s^h$ and ∇p^h. We select weakly convergent subsequences in the corresponding function spaces and find, in the limit $h \to 0$, weak solutions to the original problem. It will be one of the key observations that s^h actually has a compactness property. It is worth noting here that the hysteresis relation (which has p as an input and s as an output) does not allow to conclude any compactness of the sequence p^h.

In the definition of the solution concept, the main point is to avoid nonlinear expressions in either p^h or $\partial_t s^h$. This prohibits, in particular, to use $\mathrm{sign}(\partial_t s)$ or $\mathrm{sign}^{-1}(p - p_c(s))$ in any variant. Our approach is to demand the linear relations explicitely in the distributional sense, and to encode non-linear relations with the help of a variational inequality.

Definition 2.2 (Weak solution). *Let (s,p) be a pair of functions with*

$$s \in L^\infty(0,T;L^2(\Omega)),\ \partial_t s \in L^2(0,T;L^2(\Omega)), \tag{2.6}$$
$$p \in L^2(0,T;H^1(\Omega)), \tag{2.7}$$

satisfying, in the sense of traces, the initial condition $s = s_0$ on $\Omega \times \{0\}$ and the boundary condition $p = g$ on $\partial\Omega \times (0,T)$. We call (s,p) a weak solution *of the Richards equation with hysteresis when the following three conditions are satisfied.*

1. *The relation $\partial_t s = \nabla \cdot (k(s)[\nabla p + e_n])$ holds in the sense of distributions.*
2. *The relation $p(x,t) - p_c(x,s(x,t)) - \tau\partial_t s(x,t) \in [-\gamma(x),\gamma(x)]$ holds for almost every (x,t).*
3. *There holds*

$$\begin{aligned} 0 \geq &\int_0^T \int_\Omega (p_c(x,s(x,t)) - g(x,t))\, \partial_t s(x,t)\ dx\, dt \\ &+ \int_0^T \int_\Omega \left\{\tau\, |\partial_t s(x,t)|^2 + \gamma(x)\, |\partial_t s(x,t)|\right\}\ dx\, dt \\ &+ \int_0^T \int_\Omega k(x,s(x,t))\, (\nabla p(x,t) + e_n)\, \nabla(p(x,t) - g(x,t))\, dx\, dt. \end{aligned} \tag{2.8}$$

Corollary 2.3. *Weak solutions are also strong solutions in the following sense: the functions $s, \partial_t s, p$, and ∇p are in $L^2(\Omega \times (0,T))$, equation* (1.1) *holds in the sense of distributions,* (1.2) *holds pointwise almost everywhere, the initial condition for s and the boundary condition for p are satisfied in the sense of traces. Theorem 2.1 therefore provides the existence of strong solutions to the hysteresis system.*

Proof. We only have to show that (1.2) holds almost everywhere. For weak solutions, the distribution $\partial_t s = \nabla\cdot(k(s)[\nabla p + e_n])$ is actually an $L^2(\Omega\times(0,T))$ function, hence we can perform an integration by parts in the third integral of (2.8). Then the inequality (2.8) simplifies to

$$\begin{aligned} 0 \geq &\int_0^T \int_\Omega [p_c(x,s(x,t)) - p(x,t)]\partial_t s(x,t)\, dx\, dt \\ &+ \int_0^T \int_\Omega \left\{\tau\, |\partial_t s(x,t)|^2 + \gamma(x)\, |\partial_t s(x,t)|\right\}\ dx\, dt. \end{aligned}$$

We write this as

$$\int_0^T \int_\Omega \gamma\, |\partial_t s| \leq \int_0^T \int_\Omega [p - p_c(.,s) - \tau\partial_t s]\partial_t s.$$

By the second property of weak solutions, the integrand on the right hand side satisfies $[p - p_c(.,s) - \tau\partial_t s]\partial_t s \le \gamma|\partial_t s|$ almost everywhere, hence it is (pointwise) less or equal than the integrand on the left hand side. The inequality of the integrals therefore implies that the integrands coincide almost everywhere, i.e. $[p - p_c(.,s) - \tau\partial_t s]\partial_t s = \gamma\,|\partial_t s|$ almost everywhere. In points (x,t) with $\partial_t s(x,t) \neq 0$ we conclude $p(x,t) - p_c(x, s(x,t)) - \tau\partial_t s(x,t) \in \gamma(x)\,\mathrm{sign}(\partial_t s(x,t))$. Instead, in points (x,t) with $\partial_t s(x,t) = 0$, the relation $p(x,t) - p_c(x, s(x,t)) - \tau\partial_t s(x,t) \in \gamma(x)\,\mathrm{sign}(0)$ holds by the second property of weak solutions. □

2.3 Regularization and discretization

We recall that three positive (and possibly small) physical parameters appear in the equations: the number $\kappa > 0$ is a lower bound for the permeability, $\rho > 0$ is a lower bound for the slope of p_c, and $\tau > 0$ is the time delay parameter. We introduce two further positive parameters, which allow to introduce a well-posed approximate problem: a regularization parameter for the sign-function, $\delta > 0$, and a space-discretization parameter $h > 0$.

In the case $\tau = 0$, the play-type relation (1.2) can be written with the function $\Phi_0^0(\sigma) := \gamma\,\mathrm{sign}(\sigma)$ as $p \in p_c(s) + \Phi_0^0(\partial_t s)$. For the general case $\tau \ge 0$ we introduce $\Phi_0^\tau := \Phi_0^0 + \tau\,\mathrm{id}$, i.e.

$$\Phi_0^\tau(\sigma) := \begin{cases} [-\gamma(x), \gamma(x)] & \text{for} \quad \sigma = 0 \\ \gamma(x) + \tau\sigma & \text{for} \quad \sigma > 0 \\ -\gamma(x) + \tau\sigma & \text{for} \quad \sigma < 0 \end{cases} \tag{2.9}$$

This function is multivalued. The inverse function is denoted by $\Psi_0^\tau := (\Phi_0^\tau)^{-1}$, it is multivalued for $\tau = 0$. For positive τ, the function Ψ_0^τ is single-valued with maximal slope τ^{-1}. In this sense, a positive parameter τ introduces a regularization.

Regularization. In numerical schemes, it can be advantageous to introduce another regularization. With a parameter $\delta > 0$, we introduce the regularized functions

$$\Phi_\delta^\tau(\sigma) := \begin{cases} \left(\frac{\gamma(x)}{\delta} + \tau\right)\sigma & \text{for} \quad \sigma \in [-\delta, \delta] \\ \gamma(x) + \tau\sigma & \text{for} \quad \sigma > \delta \\ -\gamma(x) + \tau\sigma & \text{for} \quad \sigma < -\delta \end{cases} \tag{2.10}$$

such that $\Phi_\delta^\tau(\pm\delta) = \gamma(x) \pm \tau\delta = \Phi_0^\tau(\pm\delta)$ and such that the relation $\Phi_\delta^\tau = \Phi_\delta^0 + \tau\,\mathrm{id}$ remains valid. The inverse of Φ_δ^τ is denoted as Ψ_δ^τ. For $\tau, \delta > 0$, both functions Φ_δ^τ and Ψ_δ^τ are single-valued, globally Lipschitz continuous, and strictly monotonically increasing. In the above notation, we suppressed everywhere the dependence on the parameter $\gamma \ge 0$.

Spatial discretization. We now discretize the spatial domain Ω in order to construct a Galerkin scheme. We assume that we are given a triangulation $\mathcal{T}_h$ of the polygonal domain Ω, where the triangles $A \in \mathcal{T}_h$ satisfy uniform bounds on the angles and where the diameter of the largest triangle is bounded by $h > 0$. To every triangle

$A \in \mathcal{T}_h$ we associate a corner $x \in \Omega_h$, where Ω_h is a subset of N corners. This map defines a projection $X^h : \Omega \to \Omega_h = \{x_1, \ldots, x_N\}$, $X^h : A_k \ni x \mapsto x_k$. The map X^h additionally defines an invertible map which identifies $\mathbb{R}^N$ with piecewise constant functions,

$$J : \mathbb{R}^N \equiv \{f : \Omega_h \to \mathbb{R}\} \to \{f : \Omega \to \mathbb{R} \textit{ piecewise constant}\} =: \mathcal{P}_0(\Omega, \mathcal{T}_h), \tag{2.11}$$

by $(Jf)(x) = f(X^h(x))$. We will furthermore use the L^2-orthogonal projection $P := P_h : L^2(\Omega) \to L^2(\Omega)$ to the space of piecewise constant functions $\mathcal{P}_0(\Omega, \mathcal{T}_h)$. In addition, we discretize the parameter γ as $\gamma^h := J(\gamma|_{\Omega_h})$. To this piecewise constant parameter function γ^h we denote the corresponding regularization of $\gamma^h \mathrm{sign} + \tau\, \mathrm{id}$ by $\Phi^\tau_{\delta,h}(\sigma) = \Phi^\tau_{\delta,h}(\sigma; x)$ and its inverse by $\Psi^\tau_{\delta,h}(z) = \Phi^\tau_{\delta,h}(z; x)$. We are now in a position to define our Galerkin scheme.

Definition 2.4 (Galerkin scheme)**.** *We consider the following system of ordinary differential equations for $p^h : \Omega \times [0,T] \to \mathbb{R}$ and $s^h : \Omega \times [0,T] \to \mathbb{R}$ piecewise constant.*

$$\partial_t s^h(x_k, t) = -\Psi^\tau_{\delta,h}(p_c(x_k, s^h) - p^h(x_k, t)) \qquad \forall x_k \in \Omega_h \tag{2.12}$$

$$s^h(.,0) = P_h s_0\,, \tag{2.13}$$

where we suppressed the explicit dependence of $\Psi^\tau_{\delta,h}$ on x_k. The pressure p^h is reconstructed from s^h as follows. We solve for $\tilde{p}^h(.,t) : \Omega \to \mathbb{R}$ the elliptic problem

$$\nabla \cdot \big(k(x, s^h)(\nabla \tilde{p}^h + e_n)\big) = -\Psi^\tau_{\delta,h}\big(p_c(x, s^h) - P_h \tilde{p}^h\big) \ \textit{in } \Omega \tag{2.14}$$

$$\tilde{p}^h(\cdot, t) = g(\cdot, t) \ \textit{on } \partial\Omega, \tag{2.15}$$

for all $t \in [0,T]$, and set $p^h = P_h \tilde{p}^h$.

For later reference we note that the ordinary differential equation (2.12) can be written also with the (for $\delta = 0$ multivalued) functions $\Phi^0_{\delta,h}$ and $\Phi^\tau_{\delta,h}$ as

$$\Phi^\tau_{\delta,h}(\partial_t s^h(x_k, t)) = \Phi^0_{\delta,h}(\partial_t s^h(x_k, t)) + \tau \partial_t s^h(x_k, t) \ni p^h(x_k, t) - p_c(x_k, s^h). \tag{2.16}$$

3 Well-posedness, uniform estimates, compactness, and limit procedure

3.1 Well-posedness of the Galerkin scheme

We have to show that the elliptic equation (2.14) with boundary condition (2.15) defines a Lipschitz-continuous map $s^h \mapsto p^h = P_h \tilde{p}^h$. Once this is shown, the Galerkin scheme is reduced to the ordinary differential equation (2.12). The subsequent lemma performs this analysis of (2.14), where we write $u + g$ instead of $\tilde{p}$ and denote test-functions by v. Our aim is to solve $A_h u = 0$.

Lemma 3.1 (Local existence result for the discrete system)**.** *For $\tau > 0$, $h > 0$ and $s \in L^\infty(\Omega)$ piecewise constant we define two nonlinear operators $A_0, A_h : H^1_0(\Omega) \to H^{-1}(\Omega)$ by*

$$\langle A_h u, \varphi \rangle := \big\langle k\big(x, s\big)(\nabla u + \nabla g + e_n), \nabla \varphi \big\rangle_{L^2(\Omega)} - \big\langle \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big), \varphi \big\rangle_{L^2(\Omega)}$$

$$\langle A_0 u, \varphi \rangle := \big\langle k\big(x, s\big)(\nabla u + \nabla g + e_n), \nabla \varphi \big\rangle_{L^2(\Omega)} - \big\langle \Psi^\tau_{\delta,h}\big(p_c(x,s) - (u+g)\big), \varphi \big\rangle_{L^2(\Omega)}$$

for all $\varphi \in H^1_0(\Omega)$, where P_h ist the projection to piecewise constant functions, the operator A_0 is the operator A_h without the projection. We suppress the s-dependence of the operators $A_0 = A^s_0$ and $A_h = A^s_h$ if it is not relevant. There exists $h_0 = h_0(\tau, g) > 0$ such that the following holds:

1. *The operator A_0 is monotone, coercive and continuous on finite dimensional subspaces. There exists a continuous solution operator $A_0^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$.*

2. *For every $f \in L^2(\Omega)$ and every $h \in [0, h_0]$ there exists a unique $u \in H^1_0(\Omega)$ satisfying $A_h u = f$. The following inequality holds:*

$$\|u\|_{H^1(\Omega)} \leq C(s, g)\left(1 + \|f\|_{L^2(\Omega)}\right), \tag{3.1}$$

   where the constant C depends on $\|s\|_{L^2(\Omega)}$ and $\|g\|_{L^2(\Omega)}$ and not on h.

3. *For every $h \in [0, h_0]$ the map $L^\infty(\Omega) \ni s \mapsto \left(A^s_h\right)^{-1}(0) \in H^1_0(\Omega)$ is locally Lipschitz continuous.*

Item 3 of Lemma 3.1 shows that the the elliptic equation (2.14)–(2.15) defines a Lipschitz continuous map $s^h(t) \mapsto p^h(t) = P_h(u + g)$. In particular, (2.12)–(2.13) of Definition 2.4 describes a system of ordinary differential equtions. We conclude that for $\delta \geq 0$ and sufficiently small $h > 0$ there exists a unique local solution (p^h, s^h) to the Galerkin scheme of Definition 2.4.

Proof. Item 1. For the monotonicity of A_0 we calculate

$$\begin{aligned}
&\langle A_0 u - A_0 v, u - v\rangle = \big\langle k\big(x, s\big)(\nabla u - \nabla v), \nabla u - \nabla v\big\rangle_{L^2(\Omega)} \\
&- \big\langle \Psi^\tau_{\delta,h}\big(p_c(x, s) - (u + g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x, s) - (v + g)\big), u - v\big\rangle_{L^2(\Omega)} \\
&\geq \kappa\langle \nabla(u - v), \nabla(u - v)\rangle_{L^2(\Omega)}.
\end{aligned}$$

We exploited here the lower bound $k \geq \kappa$ and the monotonicity of $\Psi^\tau_{\delta,h}(\cdot)$. The right hand side is non-negative and we conclude the (strict) monotonicity of A_0. A similar calculation, using additionally the Poincaré inequality, yields the coerciveness of A_0.

$$\langle A_0 u, u\rangle \geq C_1 \kappa \|u\|^2_{H^1(\Omega)} - C_2(\tau, s, g)\|u\|_{H^1(\Omega)},$$

where $C_1 > 0$ depends on the Poincaré constant and $C_2(\tau, s, g) \in \mathbb{R}$ is independent of h. We conclude

$$\frac{\langle A_0 u, u\rangle}{\|u\|_{H^1(\Omega)}} \to \infty \text{ for } \|u\|_{H^1(\Omega)} \to \infty,$$

and thus the coercivity of A_0. The continuity of A_0 on finite dimensional subspaces follows from the continuity of Ψ^τ_δ. The theory of monotone operators yields the existence of a unique solution u to $A_0 u = f$.

The continuity of the solution operator A_0^{-1} is shown with a similar calculation: We consider $A_0 u = f_1$ and $A_0 v = f_2$ and calculate as for the monotonicity $\|u - v\|_{H^1(\Omega)} \leq C\|f_1 - f_2\|_{H^{-1}(\Omega)}$.

Item 2. We now prove inequality (3.1) for solutions of $A_h u = f$. We write the relation $\langle A_h u, u\rangle = \langle f, u\rangle_{L^2(\Omega)}$ as

$$\int_\Omega k(x,s)\,(\nabla u + \nabla g + e_n)\,\nabla u\;dx - \int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)u\;dx = \int_\Omega f\,u\;dx.$$

The first term on the left hand side is estimated in the standard way as

$$\int_\Omega k(x,s)\,(\nabla u + \nabla g + e_n)\,\nabla u\;dx \geq C_1\kappa\|u\|^2_{H^1(\Omega)} - C_2(\kappa_0, g)\|\nabla u\|_{L^2(\Omega)},$$

where $C_1 > 0$ depends on the Poincaré constant and $C_2(\kappa_0, g) \in \mathbb{R}$ depends on the uniform bound $k \leq \kappa_0$. For the second term we calculate

$$\begin{aligned}
&-\int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)u\;dx\\
&= -\int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)\,\big(P_h u + (u - P_h u)\big)\;dx\\
&= \int_\Omega \big[\Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h g\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)\big]\,P_h u\;dx\\
&\quad - \int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h g\big)P_h u\;dx - \int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)\,(u - P_h u)\;dx\\
&\geq -\int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h g\big)P_h u\;dx - \int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)\,(u - P_h u)\;dx\\
&\geq -C(\tau,g,s)\|P_h u\|_{L^2(\Omega)} - \frac{1}{\tau}\left(C(s,g) + \|P_h u\|_{L^2(\Omega)}\right)\|u - P_h u\|_{L^2(\Omega)}\\
&\geq -C(\tau,g,s)\|u\|_{L^2(\Omega)} - \frac{1}{\tau}\left(C(s,g) + \|u\|_{H^1(\Omega)}\right)Ch\|u\|_{H^1(\Omega)},
\end{aligned}$$

where we exploited the monotonicity of $\Psi^\tau_{\delta,h}$, the Lipschitz bound for $p_c(x,.)$, and for the projection P_h the facts $\|P_h u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ and $\|u - P_h u\|_{L^2(\Omega)} \leq Ch\|u\|_{H^1(\Omega)}$. For h sufficiently small (depending on $\tau > 0$), we can then absorb the negative quadratic term $-\frac{Ch}{\tau}\|u\|^2_{H^1(\Omega)}$ into the positive term $C_1\kappa\|u\|^2_{H^1(\Omega)}$. Similarly, linear terms in $\|u\|_{H^1(\Omega)}$ can be absorbed. Applying the Hölder inequality to $\int_\Omega f\,u\;dx$ provides the claimed estimate (3.1).

In our next step we prove the existence of a solution u to $A_h u = f$ for $h > 0$. We use that for $h = 0$ a unique solution exists and apply the Schauder fixed point Theorem to the operator T, which we define as

$$\begin{aligned}
T(u) :&= A_0^{-1}\left(f - A_h(u) + A_0(u)\right)\\
&= A_0^{-1}\left(f + \Psi^\tau_{\delta,h}\big(p_c(x,s) - (u+g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s) - P_h(u+g)\big)\right)
\end{aligned}$$

We note that a fixed point of T solves $A_h u = f$, which is the desired relation. In order to apply the Schauder fixed point Theorem we show: (i) The operator T is compact. (ii) There exists $R > 0$ such that $T : H^1(\Omega) \supset B_R(0) \to B_R(0)$, where $B_R(0)$ denotes the ball with radius R with respect to the $H^1(\Omega)$-norm.

Concerning (i): The embedding $id : H^1(\Omega) \to L^2(\Omega)$ is compact and the operator T is continuous as an operator from $L^2(\Omega)$ to $H^1(\Omega)$. Thus the composition $T \circ \mathrm{id} : H^1(\Omega) \to H^1(\Omega)$ is compact.

Concerning (ii): Exploiting estimate (3.1) for A_0 we calculate for u with $\|u\|_{H^1} \le R$

$$\begin{aligned}
&\|T(u)\|_{H^1(\Omega)} \\
&\le C(s)\left(1 + \left\|f + \Psi^\tau_{\delta,h}(p_c(x,s) - (u+g)) - \Psi^\tau_{\delta,h}(p_c(x,s) - P_h(u+g))\right\|_{L^2(\Omega)}\right) \\
&\le C(s)\left(1 + \|f\|_{L^2(\Omega)} + \frac{1}{\tau}\|P_h(u+g) - (u+g)\|_{L^2(\Omega)}\right) \\
&\le C(s)\left(C(f) + \frac{1}{\tau} Ch\left(\|u\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\right)\right) \le C(s)\left(C(f,g,\tau) + \frac{Ch}{\tau}R\right) \le R
\end{aligned}$$

for $R > 0$ sufficiently large and $h > 0$ sufficiently small, depending on τ and s.

Item 3. We now show the local Lipschitz continuity of the map $s \mapsto (A^s_h)^{-1}(0)$ and, in particular, that the map $s \mapsto (A^s_h)^{-1}(0)$ is well defined. We consider two discrete saturation variables $s_1, s_2 \in L^\infty(\Omega)$. Since we are interested in a local Lipschitz property, we assume the boundedness $\|s_1\|_{L^\infty}, \|s_2\|_{L^\infty} \le R$ for some $R > 0$. We consider the corresponding operators $A^{s_1}_h, A^{s_2}_h$ and two solutions u_1 and u_2 of $A^{s_1}_h u_1 = 0$ and $A^{s_2}_h u_2 = 0$. Then

$$\begin{aligned}
0 &= \langle A^{s_1}_h u_1 - A^{s_2}_h u_2, u_1 - u_2\rangle \\
&= \left\langle k\big(x,s_1\big)(\nabla u_1 + \nabla g + e_n) - k\big(x,s_2\big)(\nabla u_2 + \nabla g + e_n), \nabla u_1 - \nabla u_2\right\rangle_{L^2(\Omega)} \\
&\quad - \left\langle \Psi^\tau_{\delta,h}\big(p_c(x,s_1) - P_h(u_1+g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s_2) - P_h(u_2+g)\big), u_1 - u_2\right\rangle_{L^2(\Omega)} \\
&=: Z_1 + Z_2
\end{aligned}$$

We treat the term Z_1 as follows

$$\begin{aligned}
Z_1 &= \left\langle \big(k\big(x,s_1\big) - k\big(x,s_2\big)\big)\left(\nabla g + e_n\right), \nabla u_1 - \nabla u_2\right\rangle_{L^2(\Omega)} \\
&\quad + \langle k(x,s_1)\left(\nabla u_1 - \nabla u_2\right), \nabla u_1 - \nabla u_2\rangle_{L^2(\Omega)} \\
&\quad + \langle (k(x,s_1) - k(x,s_2))\, \nabla u_2, \nabla u_1 - \nabla u_2\rangle_{L^2(\Omega)} \\
&\ge -C(g)\|s_1 - s_2\|_{L^\infty(\Omega)}\|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)} \\
&\quad + \kappa\|\nabla u_1 - \nabla u_2\|^2_{L^2(\Omega)} - \kappa_1\|s_1 - s_2\|_{L^\infty(\Omega)}\|\nabla u_2\|_{L^2(\Omega)}\|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)} \\
&\ge C\kappa\|u_1 - u_2\|^2_{H^1(\Omega)} - C(R,g)\|s_1 - s_2\|_{L^\infty(\Omega)}\|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)},
\end{aligned}$$

where we have used the Poincaré inequality, the Lipschitz constant κ_1 of $k(x,\cdot)$ and the a priori estimate (3.1) for u_2. The lower order term Z_2 can be estimated as follows

$$\begin{aligned}
Z_2 = &- \left\langle \Psi^\tau_{\delta,h}\big(p_c(x,s_1) - P_h(u_1+g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s_2) - P_h(u_2+g)\big),\right. \\
&\qquad\qquad\qquad\qquad \left.(\mathrm{id} - P_h)(u_1 - u_2)\right\rangle_{L^2(\Omega)} \\
&- \left\langle \Psi^\tau_{\delta,h}\big(p_c(x,s_1) - P_h(u_1+g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s_2) - P_h(u_2+g)\big), P_h(u_1 - u_2)\right\rangle_{L^2(\Omega)} \\
\ge &-\frac{1}{\tau}\|p_c(x,s_1) - P_h u_1 - p_c(x,s_2) + P_h u_2\|_{L^2(\Omega)}\|(\mathrm{id} - P_h)(u_1 - u_2)\|_{L^2(\Omega)} \\
&- \left\langle \Psi^\tau_{\delta,h}\big(p_c(x,s_1) - P_h(u_1+g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s_1) - P_h(u_2+g)\big), P_h(u_1 - u_2)\right\rangle_{L^2(\Omega)} \\
&- \left\langle \Psi^\tau_{\delta,h}\big(p_c(x,s_1) - P_h(u_2+g)\big) - \Psi^\tau_{\delta,h}\big(p_c(x,s_2) - P_h(u_2+g)\big), P_h(u_1 - u_2)\right\rangle_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{1}{\tau}\left(C\rho_1\|s_1-s_2\|_{L^\infty(\Omega)}+\|P_hu_1-P_hu_2\|_{L^2(\Omega)}\right)Ch\|u_1-u_2\|_{H^1(\Omega)}\\
&\quad -\left\langle\Psi^\tau_{\delta,h}\big(p_c(x,s_1)-P_h(u_2+g)\big)-\Psi^\tau_{\delta,h}\big(p_c(x,s_2)-P_h(u_2+g)\big),P_h(u_1-u_2)\right\rangle_{L^2(\Omega)}\\
&\geq -Ch\frac{1}{\tau}\left(\|s_1-s_2\|_{L^\infty(\Omega)}+\|u_1-u_2\|_{H^1(\Omega)}\right)\|u_1-u_2\|_{H^1(\Omega)}\\
&\quad -\frac{1}{\tau}C\|s_1-s_2\|_{L^\infty(\Omega)}\|u_1-u_2\|_{H^1(\Omega)},
\end{aligned}
$$

where we used the monotonicity of $\Psi^\tau_{\delta,h}$ and the Lipschitz continuity of $\Psi^\tau_{\delta,h}$ and $p_c(x,\cdot)$.

For sufficiently small $h>0$ the quadratic negative term $-\frac{Ch}{\tau}\|u_1-u_2\|^2_{H^1(\Omega)}$ can be absorbed into the positive term $C\kappa\|u_1-u_2\|^2_{H^1(\Omega)}$ of Z_1. We thus conclude from $Z_1+Z_2=0$ the estimate

$$
\|u_1-u_2\|^2_{H^1(\Omega)}\leq C(R,g,\tau)\|s_1-s_2\|_{L^\infty(\Omega)}\|u_1-u_2\|_{H^1(\Omega)},
$$

and hence the local Lipschitz continuity of the map $s\mapsto(A^s_h)^{-1}(0)$. □

3.2 Energy estimates

Our next aim is to derive for s^h estimates that do not depend on the regularization parameters h and δ. One consequence of the subsequent proposition is that we can extend the local solutions of the Galerkin scheme to the whole interval $[0,T]$.

Proposition 3.2 (Energy estimates and global solutions)**.** *Every solution $\tilde p^h$, s^h to the scheme of Definition 2.4 satisfies the uniform bounds*

$$
\|\tilde p^h\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_ts^h\|^2_{L^2(0,T;L^2(\Omega))}\leq C, \tag{3.2}
$$

where the constant C depends on initial and boundary data, on κ, τ, κ_0, and ρ_1, but it is independent of $h>0$ and $\delta\geq 0$.

Proof. We start by writing the Galerkin evolution equation in a form that is similar to the original formulation. We apply the operator J (identification with piecewise constant functions) to (2.12) and use (2.14) to write

$$
\begin{aligned}
&\partial_ts^h-\nabla\cdot\big(k(x,s^h)(\nabla\tilde p^h+e_n)\big)\\
&\qquad=-J\Psi^\tau_{\delta,h}(p_c(x,s^h)-p^h)+\Psi^\tau_{\delta,h}\big(p_c(x,s^h)-p^h\big).
\end{aligned}\tag{3.3}
$$

Multiplication with $\tilde p^h-g$ and integration over Ω yields

$$
\begin{aligned}
&\int_\Omega\partial_ts^h\left(\tilde p^h-g\right)\,dx+\int_\Omega k(x,s^h)(\nabla\tilde p^h+e_n)\left(\nabla\tilde p^h-\nabla g\right)\,dx\\
&\quad=-\int_\Omega\left[J\Psi^\tau_{\delta,h}(p_c(x,s^h)-p^h)-\Psi^\tau_{\delta,h}\big(p_c(x,s^h)-p^h\big)\right]\left(\tilde p^h-g\right)\,dx.
\end{aligned}\tag{3.4}
$$

We will conclude the a priori energy estimate (3.2) from equation (3.4). We will identify several positive terms on the left hand side, the right hand side will turn out to be small.

In order to treat the time derivative in the first integral, we use the hysteresis differential equation in the form (2.16) and apply the operator J,

$$J\Phi^0_{\delta,h}(\partial_t s^h(x,t)) + \tau\partial_t s^h(x,t) \ni p^h(x,t) - Jp_c(x,s^h). \tag{3.5}$$

Using $P_h\tilde{p}^h = p^h$ and the orthogonality $\langle \partial_t s^h, \tilde{p}^h\rangle_{L^2} = \langle \partial_t s^h, p^h\rangle_{L^2}$, we find, exploiting the monotonicity of $\Phi^0_{\delta,h}$,

$$\begin{aligned}\int_\Omega \partial_t s^h \tilde{p}^h\,dx &\in \int_\Omega \partial_t s^h\left[J\Phi^0_{\delta,h}(\partial_t s^h) + \tau\partial_t s^h + Jp_c(x,s^h)\right]\,dx\\ &\geq \tau\|\partial_t s^h\|^2_{L^2(\Omega)} + \int_\Omega \partial_t s^h Jp_c(x,s^h)\,dx.\end{aligned}$$

In order to analyze the last integral we use a primitive $P_c(x,\cdot)$ of the function $p_c(x,\cdot)$, i.e. $\partial_s P_c(x,s) = p_c(x,s)$. Due to $\rho > 0$ (or by using (2.5)), we can assume the non-negativity of P_c. With this primitive, we calculate

$$\int_\Omega \partial_t s^h Jp_c(x,s^h)\,dx = \int_\Omega J\left(\frac{d}{dt}P_c(x,s^h)\right)\,dx = \partial_t\int_\Omega JP_c(x,s^h)\,dx.$$

For the second term on the left hand side of (3.4) we calculate

$$\int_\Omega k(x,s^h)(\nabla\tilde{p}^h + e_n)\nabla\tilde{p}^h\,dx \geq \kappa\|\nabla\tilde{p}^h\|^2_{L^2(\Omega)} - C\|\nabla\tilde{p}^h\|_{L^2(\Omega)} \geq \frac{\kappa}{2}\|\nabla\tilde{p}^h\|^2_{L^2(\Omega)} - \frac{2C^2}{\kappa},$$

where C depends on κ_0, the uniform bound for k. The terms on the left hand side of (3.4) containing the boundary data g are treated as follows.

$$\begin{aligned}&\left|\int_\Omega \partial_t s^h\,g\,dx\right| + \left|\int_\Omega k(x,s^h)(\nabla\tilde{p}^h + e_n)\nabla g\,dx\right|\\ &\leq \frac{\tau}{2}\|\partial_t s^h\|^2_{L^2(\Omega)} + \frac{2}{\tau}\|g\|^2_{L^2(\Omega)} + \frac{\kappa}{4}\|\nabla\tilde{p}^h\|^2_{L^2(\Omega)} + \frac{C}{\kappa}\|\nabla g\|^2_{L^2(\Omega)} + C\|\nabla g\|_{L^2(\Omega)}\\ &= \frac{\tau}{2}\|\partial_t s^h\|^2_{L^2(\Omega)} + \frac{\kappa}{4}\|\nabla\tilde{p}^h\|^2_{L^2(\Omega)} + C(g)\frac{1}{\kappa},\end{aligned}$$

where the constant C depends on the uniform bound κ_0 of k. Consequently, the $s^h, \tilde{p}^h$-dependent terms can be absorbed.

We finally treat the error term on the right hand side of (3.4). We consider an arbitrary point x in a triangle with associated vertex $x_k = X^h(x)$.

$$\begin{aligned}&\left|\left[J\Psi^\tau_{\delta,h}(p_c(\cdot,s^h) - p^h) - \Psi^\tau_{\delta,h}\big(p_c(\cdot,s^h) - p^h\big)\right](x)\right|\\ &= \left|\Psi^\tau_{\delta,h}(p_c(x_k,s^h) - p^h) - \Psi^\tau_{\delta,h}\big(p_c(x,s^h) - p^h\big)\right|\\ &\leq \frac{1}{\tau}\left|(p_c(x_k,s^h) - p_c(x,s^h)\right| \leq \frac{1}{\tau}\rho_1 h,\end{aligned}$$

where we exploit the Lipschitz continuity of $\Psi^\tau_{\delta,h}$ and the Lipschitz continuity of p_c in x. With this pointwise estimate, equation (3.4) yields

$$\begin{aligned}&\frac{\kappa}{4}\|\nabla\tilde{p}^h\|^2_{L^2(\Omega)} + \frac{\tau}{2}\|\partial_t s^h\|^2_{L^2(\Omega)} + \partial_t\int_\Omega JP_c(x,s^h)\,dx\\ &\leq \frac{2C^2 + C(g)}{\kappa} + \frac{\rho_1 h}{\tau}\left(\|\tilde{p}^h\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right).\end{aligned}$$

Using the Poincaré inequality, the last product can also be absorbed. An integration over $(0,T)$ provides estimate (3.2). □

3.3 Bounds and compactness from the hysteresis relation

Lemma 3.3 (Regularity and compactness from the hysteresis relation)**.** *Let s^h and $\tilde{p}^h$ satisfy the ordinary differential equation of the hysteresis relation* (2.12)*, i.e.*

$$\begin{aligned} \partial_t s^h(x_k, t) &= -\Psi^\tau_{\delta,h}(p_c(x_k, s^h) - p^h(x_k, t)) \qquad \forall x_k \in \Omega_h \\ s^h(x_k, 0) &= P_h s_0(x_k) \end{aligned}$$

for $p^h = P_h \tilde{p}^h$. Then, for every $q \in [1, \infty]$ and $s_0 \in L^q(\Omega)$, we find the estimate

$$\|\partial_t s^h\|_{L^2(0,T;L^q(\Omega))} + \|s^h\|_{L^2(0,T;L^q(\Omega))} \le C\|\tilde{p}^h\|_{L^2(0,T;L^q(\Omega))}, \tag{3.6}$$

where the constant C does not depend on h and δ.

Let additionally the following estimate hold with C independent of h and δ,

$$\|\tilde{p}^h\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{3.7}$$

Then the family s^h is pre-compact in the space $L^2(\Omega \times (0, T))$.

Proof. We recall the following characterization of the L^2-orthogonal projection $P = P_h$ onto piecewise constant functions with the help of averages. For an arbitrary L^2-function $\tilde{u}$, the projection $u = P\tilde{u}$ satisfies

$$u(x) = \frac{1}{|A_k|} \int_{A_k} \tilde{u}(x)\, dx \qquad \forall x \in A_k, \tag{3.8}$$

for each simplex $A_k \in \mathcal{T}_h$. This can easily be verified from the fact that u minimizes the L^2-distance to $\tilde{u}$. A consequence of this characterization of P and Jensens's inequality is the estimate

$$\|P\tilde{u}\|^q_{L^q(\Omega)} = \sum_k \int_{A_k} |P\tilde{u}|^q = \sum_k |A_k| \fint_{A_k} \left| \fint_{A_k} \tilde{u} \right|^q \le \sum_k |A_k| \fint_{A_k} |\tilde{u}|^q = \|\tilde{u}\|^q_{L^q(\Omega)}.$$

In particular, the pressure functions satisfy $\|p^h(t)\|_{L^q(\Omega)} \le \|\tilde{p}^h(t)\|_{L^q(\Omega)}$.

Estimates. We now study solutions s^h of the hysteresis relation. We start with an arbitrary point $x_k \in \Omega_h$ and note that

$$\partial_t |s^h(x_k, t)| \le \left|\Psi^\tau_{\delta,h}(p_c(x_k, s^h) - p^h(x_k, t))\right| \le C|s^h(x_k, t)| + C|p^h(x_k, t)|.$$

An application of Gronwalls inequality yields pointwise, with $C = C(\tau, T)$ independent of h and δ, the bound $|s^h(x_k, t)| \le C|p^h(x_k, t)|$. Using a second time the evolution equation, we also find an estimate for the time derivative,

$$|\partial_t s^h(x_k, t)| + |s^h(x_k, t)| \le C|p^h(x_k, t)|.$$

Taking the q-th power and adding the inequalities over the triangles yields

$$\|\partial_t s^h(t)\|^q_{L^q(\Omega)} + \|s^h(t)\|^q_{L^q(\Omega)} \le C \sum_k |A_{h,k}| \left|p^h(x_k, t))\right|^q = C\|p^h\|^q_{L^q(\Omega)} \le C\|\tilde{p}^h\|^q_{L^q(\Omega)}.$$

Taking an appropriate power and integrating over time, this provides the desired estimate (3.6).

Compactness. Estimate (3.6) allows to conclude from the bound (3.7) for the pressures the boundedness of a time derivative of the saturation,

$$\|\partial_t s^h\|_{L^2(0,T;L^2(\Omega))} \le C\|\tilde{p}^h\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.9}$$

We remark that, for the Galerkin scheme used here, this time regularity of s was already a direct consequence of the energy estimate. It remains to transfer the spatial regularity of the pressures from (3.7) to the saturation in order to conclude the compactness rom the compactness theorem of Riesz-Frechet-Kolmogorov.

Step 1. Spatial shifts of the pressure inputs. In the following we will consider functions $f \in L^2(\Omega)$ also as functions $f \in L^2(\mathbb{R}^n)$ by the trivial extension. For such functions in the (spatial) domain $\mathbb{R}^n$ and an arbitrary vector $y \in \mathbb{R}^n$ we introduce the shift operator $L_y : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $f \mapsto L_y f$ through $L_y f(x) = f(x-y)$.

A bounded family $f^h \in H^1(\Omega)$, indexed with h, satisfies $\sup_h \|L_y f^h - f^h\|_{L^2(\mathbb{R}^n)} \to 0$ for $y \to 0$. This can be concluded either in an abstract way via the characterization of compact subsets of $L^2(\mathbb{R}^n)$ through the Riesz-Frechet-Kolmogorov theorem. The more direct proof is to write $f(x-y) - f(x) = \int_0^1 \nabla f(x-ty)\cdot y\,dt$, and to integrate over x. Concerning the boundary layer where $f(x) \neq 0$ is compared with the value 0 of the extended function, we note that $\|f^h\|_{L^q(\mathbb{R}^n)}$ is bounded for some $q > 2$, hence the boundary layer gives only a small contribution for small $|y|$. In this way, we also achieve an estimate $\|L_y f^h - f^h\|^2_{L^2(\mathbb{R}^n)} \le C|y|^\alpha \|f^h\|^2_{H^1(\Omega)}$ for some fixed $\alpha > 0$. This estimate remains valid after an integration over $(0,T)$ and we find

$$\sup_h \|L_y \tilde{p}^h - \tilde{p}^h\|_{L^2(0,T;L^2(\mathbb{R}^n))} \to 0 \qquad \text{for } |y| \to 0. \tag{3.10}$$

In the following, we write only $\|.\|_{L^2L^2}$ in order to indicate the above norm.

Step 2. Spatial shifts of projected pressures. We next consider the projected pressures $p^h = P\tilde{p}^h$ with the aim to show

$$\sup_h \|L_y p^h - p^h\|_{L^2L^2} \to 0 \qquad \text{for } |y| \to 0. \tag{3.11}$$

We first study the case that h is large compared to $|y|$. We claim that

$$\sup_{h \ge |y|^{1/2}} \|L_y p^h - p^h\|_{L^2L^2} \to 0 \qquad \text{for } |y| \to 0. \tag{3.12}$$

In this situation we consider a piecewise constant function and compare this function with its (slightly) shifted version. We use the triangles A_k and the selected points $x_k \in \bar{A}_k$ corresponding to A_k, with k in a finite index set. Since the function $(L_y p^h - p^h)|_{A_k}$ is non-vanishing only on a set of measure $C|y|h^{n-1}$, we can calculate, using that all triangles have comparable size,

$$\begin{aligned}
\|L_y p^h - p^h\|^2_{L^2L^2} &= \int_0^T \left\{ \sum_k \int_{A_k} |p^h(x-y) - p^h(x)|^2 \right\} dt \\
&\le C \int_0^T \left\{ |y| h^{n-1} \sum_k |p^h(x_k)|^2 \right\} dt \le C\frac{|y|}{h} \int_0^T \left\{ \sum_k h^n |p^h(x_k)|^2 \right\} dt \\
&\le C\frac{|y|}{h} \|p^h\|^2_{L^2L^2}\,.
\end{aligned}$$

Due to the uniform bound for $p^h \in L^2L^2$ from (3.7), this implies (3.12).

We now treat the case $h \le |y|^{1/2}$. In this situation we insert the function $PL_y\tilde{p}^h$ and use the triangle inequality to calculate

$$\begin{aligned}\|L_y p^h - p^h\|_{L^2L^2} &\le \|L_y P\tilde{p}^h - PL_y\tilde{p}^h\|_{L^2L^2} + \|P(L_y\tilde{p}^h - \tilde{p}^h)\|_{L^2L^2}\\ &\le \|L_y P\tilde{p}^h - PL_y\tilde{p}^h\|_{L^2L^2} + \|L_y\tilde{p}^h - \tilde{p}^h\|_{L^2L^2}.\end{aligned}$$

Given (3.10), it suffices to treat the first term on the right hand side. We use the fact that the operator $f \mapsto L_{-y}PL_y f$ is a projection of the function f onto a piecewise constant function, but corresponding to a shifted grid. We can use h-dependent projection estimates to calculate

$$\begin{aligned}&\|L_y P\tilde{p}^h - PL_y\tilde{p}^h\|_{L^2L^2} = \|P\tilde{p}^h - L_{-y}PL_y\tilde{p}^h\|_{L^2L^2}\\ &\quad\le \|P\tilde{p}^h - \tilde{p}^h\|_{L^2L^2} + \|\tilde{p}^h - L_{-y}PL_y\tilde{p}^h\|_{L^2L^2} \le C(h + (h+|y|)^\alpha),\end{aligned}$$

where the constant C depends on the uniform bound (3.7), and $C(h+|y|)^\alpha$ covers the contributions of boundary layers. Combined with (3.12), we find the property (3.11).

Step 3. Spatial shifts of the saturation outputs. The remainder of the compactness proof follows the ideas of the proof of the s^h-estimates. In what follows, we have to deal with the explicit dependence of Ψ_δ^τ on the parameter $\gamma(x)$. We thus write $\Psi_\delta^{\tau,\gamma(x)} := \Psi_\delta^\tau$ and note that $\Psi_\delta^{\tau,\gamma^h(.)} = \Psi_{\delta,h}^\tau$.

For every point $x \in \mathbb{R}^n$ we have the differential equation

$$\begin{aligned}&\partial_t(L_y s^h - s^h)(t)\\ &\quad= -L_y\left(J\Psi_\delta^{\tau,\gamma^h(.)}(p_c(.,s^h) - p^h)(t)\right) + J\Psi_\delta^{\tau,\gamma^h(.)}(p_c(.,s^h) - p^h)(t),\end{aligned} \tag{3.13}$$

where the extension operator J indicates that we evaluate p_c and p^h only in points $x_k \in \Omega_h$ and then extend by a piecewise constant function.

Let us firstly consider the special case that $\gamma(x)$ is a constant function, $\gamma(x) = \gamma^h(x) \equiv \gamma$. In this case, the shift operator commutes with the x-independent function $\Psi_\delta^{\tau,\gamma}$ and we can calculate with the Lipschitz continuity of $\Psi_\delta^{\tau,\gamma}$ and of $x \mapsto p_c(x,s)$

$$\begin{aligned}&\left|\partial_t(L_y s^h - s^h)(t)\right| \le \left|\Psi_\delta^{\tau,\gamma}(L_yJ(p_c(.,s^h) - p^h))(t) - \Psi_\delta^{\tau,\gamma}(J(p_c(.,s^h) - p^h)(t))\right|\\ &\quad\le C\left(|L_yJp_c(.,s^h) - Jp_c(.,s^h)| + |L_yp^h - p^h|\right)(t)\\ &\quad\le C\left(|X^h(x) - X^h(x-y)| + |L_ys^h - s^h| + |L_yp^h - p^h|\right)(t).\end{aligned} \tag{3.14}$$

Setting $X^h(z) = 0$ for $z \notin \Omega$, the function $f_h(x,y) := |X^h(x) - X^h(x-y)|$ satisfies $f_h(X,Y) \le C\max\{h,|y|\}$. This provides immediately

$$\sup_{h\le|y|^{1/2}} \|f_h(.,y)\|_{L^2(\mathbb{R}^n)} \le C(|y|^{1/2} + |y|).$$

On the other hand, for $|y|^{1/2} \le h$, the set $\Sigma_h = \{x \in \Omega : f_h(x,y) \ne 0\}$ has a small volume, namely $|\Sigma_h| \le C|y|/h \le C|y|^{1/2}$. We therefore find

$$\sup_h \|f_h(.,y)\|^2_{L^2(\mathbb{R}^n)} \le C|y|^{1/2}. \tag{3.15}$$

for $|y| \le 1$. Using Gronwall's inequality, exploiting (3.11) and (3.15), we find

$$\sup_h \|L_y s^h - s^h\|_{L^2L^2} \to 0 \qquad \text{for } |y| \to 0. \tag{3.16}$$

Let us now consider the general case, that $\gamma(x)$ is a non-negative Lipschitz continuous function. By relation (3.13) we have to study, for the function $u : \Omega_h \to \mathbb{R}$, $u(x_k) = (p_c(x_k, s^h) - p^h)(x_k))(t)$, the expression

$$\begin{aligned} &L_y J\Psi_\delta^{\tau,\gamma^h}(u) - J\Psi_\delta^{\tau,\gamma^h}(u) = \Psi_\delta^{\tau,L_y\gamma^h}(L_y Ju) - \Psi_\delta^{\tau,\gamma^h}(Ju) \\ &= \Big(\Psi_\delta^{\tau,L_y\gamma^h}(L_y Ju) - \Psi_\delta^{\tau,L_y\gamma^h}(Ju)\Big) + \Big(\Psi_\delta^{\tau,L_y\gamma^h}(Ju) - \Psi_\delta^{\tau,\gamma^h}(Ju)\Big) \end{aligned}$$

The first term is treated as in (3.14), exploiting the Lipschitz continuity of $\Psi_\delta^{\tau,L_y\gamma^h}$. The second difference can be treated with the Lipschitz continuity of γ, which implies $|L_y\gamma^h(x) - \gamma^h(x)| \le C(h + |y|)$. This carries over to the corresponding Ψ-functions, $\|\Psi_\delta^{\tau,L_y\gamma^h} - \Psi_\delta^{\tau,\gamma^h}\|_\infty \le C(h + |y|)$, with C depending on τ, but not on h or y. This, together with the above arguments for $|y| < h^2$, implies that (3.16) is satisfied also in in the case of a general Lipschitz continuous $\gamma(\cdot)$. Considering (3.9), we found the compactness property of s^h. □

3.4 Limit procedure and existence result

We consider now limit functions to the solution sequence $(s^h, \tilde{p}^h)$ for $h \to 0$. If a regularization of the sign-function was performed, we assume that the regularization parameter satisfies $\delta \to 0$ for $h \to 0$. Due to the uniform estimates of Proposition 3.2 we find a subsequence $h \to 0$ and limit functions s, p such that

$$\tilde{p}^h \rightharpoonup p \quad \text{in} \quad L^2(0,T;H^1(\Omega)), \tag{3.17}$$

$$s^h \rightharpoonup s,\ \partial_t s^h \rightharpoonup \partial_t s \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{3.18}$$

Furthermore, by the compactness result of Lemma 3.3, we find the strong convergence

$$s^h \to s \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{3.19}$$

The following lemma concludes the proof of the main theorem.

Lemma 3.4. *The limit pair $(s,p) \in L^2(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega))$ is a weak solution to the hysteresis system according to Definition 2.2.*

Proof. The limit pair is contained in the desired function spaces. The weak convergence allows to take limits in the initial and boundary conditions, hence they are satisfied by the limit pair. We have to check the three conditions of Definition 2.2.

Item 1. We prove that $\partial_t s = \nabla \cdot (k(s)[\nabla p + e_n])$ in the sense of distributions. Indeed, let $\phi \in C_c^\infty((0,T) \times \Omega)$. We start from the Galerkin ordinary differential equation (2.12) and the elliptic equation (2.14) and write

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t s^h \phi \, dx\, dt + \int_0^T \int_\Omega k(x,s^h) \left[\nabla \tilde{p}^h + e_n\right] \nabla\phi \, dx\, dt \\ &= -\int_0^T \int_\Omega \left[J\Psi_{\delta,h}^\tau(p_c(.,s^h) - p^h) - \Psi_{\delta,h}^\tau\big(p_c(x,s^h) - p^h\big)\right] \phi \, dx\, dt. \end{aligned} \tag{3.20}$$

As in the proof of Proposition 3.2 the error term on the right hand side of (3.20) can be estimated as

$$\left|J\Psi^\tau_{\delta,h}(p_c(.,s^h)-p^h)(x)-\Psi^\tau_{\delta,h}\big(p_c(x,s^h)-p^h\big)\right|\le C(\tau)h,$$

since p_c is Lipschitz continuous in x. Therefore, the right hand side of (3.20) tends to zero.

On the left hand side we can also pass to the limit. Due to the strong convergence (3.19) we have $s^h\to s$ pointwise and thus, exploiting the continuity and the boundedness of k, the strong convergence $k(x,s^h)\to k(x,s)$ in $L^2(0,T;L^2(\Omega))$. We find $\partial_t s=\nabla\cdot(k(s)[\nabla p+e_n])$ in the sense of distributions.

Item 2. Relation (2.16) implies

$$p^h(x_k,t)-p_c(x_k,s^h(x_k,t))-\tau\partial_t s^h(x_k,t)\in[-\gamma(x_k),\gamma(x_k)]$$

for all $x_k\in\Omega_h$ and all $t\in[0,T]$. We apply the operator J,

$$p^h(x,t)-Jp_c(.,s^h)(x,t)-\tau\partial_t s^h(x,t)\in[-\gamma^h(x),\gamma^h(x)]$$

and insert $\tilde p^h$ and $-p_c(x,s^h)$ to obtain

$$\begin{aligned}&\tilde p^h(x,t)-p_c(x,s^h(x,t))-\tau\partial_t s^h(x,t)\\&\quad+\big(p_c(x,s^h(x,t))-Jp_c(x,s^h)(x,t)\big)+\left(p^h-\tilde p^h\right)(x,t)\\&\quad+r^h(x,t)\in[-\gamma(x),\gamma(x)]\end{aligned}\tag{3.21}$$

for almost all $x\in\Omega$ and $t\in(0,T)$. The error term $r^h(x,t)$ concerns the replacement of γ by γ^h. It satisfies $|r^h(x,t)|\le|\gamma^h(x)-\gamma(x)|\le Ch$ due to the Lipschitz continuity of γ.

Since $[-\gamma(x),\gamma(x)]\subset\mathbb{R}$ is a convex set, in order to conclude the inclusion of item 2, it suffices to show that the left hand side of (3.21) converges weakly to $p(x,t)-p_c(x,s(x,t))-\tau\partial_t s(x,t)$.

In the first line of (3.21) we can pass to the limit due to the global Lipschitz continuity of $p_c(x,\cdot)$ and the pointwise convergence $s^h\to s$. The error terms in the second line converge to zero, since

$$\left|p_c(x,s^h)-Jp_c(.,s^h)\right|\le Ch,$$

where C is the Lipschitz constant for $p_c(\cdot,s^h)$. For the second term, we note that $p^h-\tilde p^h=P\tilde p^h-\tilde p^h$ and use the inequality $\|P\tilde p^h-\tilde p^h\|_{L^2(\Omega)}\le Ch\|\tilde p^h\|_{H^1(\Omega)}$ and the uniform bound for $\tilde p^h$. This shows that (s,p) satisfies the second condition of the weak formulation.

Item 3. We have to prove the variational inequality (2.8) for (s,p). To this end, we multiply the elliptic equation (2.14) by $\tilde p^h-g$ and integrate over $(0,T)\times\Omega$ to arrive at

$$\begin{aligned}0=&\int_0^T\int_\Omega k(x,s^h)\left(\nabla\tilde p^h+e_n\right)\left(\nabla\tilde p^h-\nabla g\right)\,dx\,dt\\&-\int_0^T\int_\Omega\Psi^\tau_{\delta,h}\big(p_c(x,s^h)-p^h\big)(\tilde p^h-g)\,dx\,dt.\end{aligned}\tag{3.22}$$

Concerning the first term on the right hand side we use the strong convergence $k(x,s^h) \to k(x,s)$ and the weak convergence of $\nabla \tilde{p}^h$ we obtain, for $h \to 0$,

$$\int_0^T \int_\Omega k(x,s^h) \left(\nabla \tilde{p}^h + e_n\right) \nabla g \; dx\, dt \to \int_0^T \int_\Omega k(x,s) \left(\nabla p + e_n\right) \nabla g \; dx\, dt,$$
$$\int_0^T \int_\Omega k(x,s^h) e_n \nabla \tilde{p}^h \; dx\, dt \to \int_0^T \int_\Omega k(x,s) e_n \nabla p \; dx\, dt.$$

For the term $\int_0^T \int_\Omega k(x,s^h) \left|\nabla \tilde{p}^h\right|^2 \, dx\, dt$ we use lower semi-continuity of norms. We apply Egorov's Theorem to the pointwise convergent subsequence $k(x,s^h)$ and find for every $\varepsilon > 0$ a subset $\Omega_\varepsilon \subseteq \Omega$ such that $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and $k(x,s^h) \to k(x,s)$ uniformly on Ω_ε. By non-negativity of k we find

$$\begin{aligned}
&\liminf_{h\to 0} \int_0^T \int_\Omega k(x,s^h) \left|\nabla \tilde{p}^h\right|^2 \; dx\, dt \\
&\geq \liminf_{h\to 0} \int_0^T \int_{\Omega_\varepsilon} k(x,s^h) \left|\nabla \tilde{p}^h\right|^2 \; dx\, dt \geq \liminf_{h\to 0} \int_0^T \int_{\Omega_\varepsilon} k(x,s) \left|\nabla \tilde{p}^h\right|^2 \; dx\, dt \\
&\qquad - \liminf_{h\to 0} \left| \int_0^T \int_{\Omega_\varepsilon} \left(k(x,s^h) - k(x,s)\right) \left|\nabla \tilde{p}^h\right|^2 \; dx\, dt \right| \\
&\geq \int_0^T \int_{\Omega_\varepsilon} k(x,s) \left|\nabla p\right|^2 \; dx\, dt,
\end{aligned}$$

where we have used the uniform convergence of $k(x,s^h)$ on Ω_ε and the weak lower semicontinuity of $\tilde{p}^h \mapsto \int_0^T \int_{\Omega_\varepsilon} k(x,s) \left|\nabla \tilde{p}^h\right|^2 \, dx\, dt$ in the last line. Since $k(x,s) \left|\nabla p\right|^2 \in L^1(\Omega)$ for almost every $t \in (0,T)$ we can now let ε tend to zero to obtain

$$\liminf_{h\to 0} \int_0^T \int_\Omega k(x,s^h) \left|\nabla \tilde{p}^h\right|^2 \; dx\, dt \geq \int_0^T \int_\Omega k(x,s) \left|\nabla p\right|^2 \; dx\, dt.$$

This concludes the limit procedure in the first term on the right hand side of (3.22).

Next, we consider the term $-\int_0^T \int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s^h) - p^h\big)(\tilde{p}^h - g) \, dx\, dt$. We exploit relation (2.12) and apply the operator J,

$$\partial_t s^h(x,t) = -J \Psi^\tau_{\delta,h}(p_c(.,s^h) - p^h).$$

This allows to write

$$\begin{aligned}
-\int_0^T \int_\Omega \Psi^\tau_{\delta,h}\big(p_c(x,s^h) - p^h\big)(\tilde{p}^h - g) \; dx\, dt &= \int_0^T \int_\Omega \partial_t s^h (\tilde{p}^h - g) \; dx\, dt \\
+ \int_0^T \int_\Omega \left[J\Psi^\tau_{\delta,h}(p_c(.,s^h) - p^h) - \Psi^\tau_{\delta,h}\big(p_c(x,s^h) - p^h\big)\right] &(\tilde{p}^h - g) \; dx\, dt.
\end{aligned} \tag{3.23}$$

As noted already in Item 1, the second term on the right hand side of (3.23) tends to zero. We treat the term $\int_0^T \int_\Omega \partial_t s^h (\tilde{p}^h - g) \, dx\, dt$ as follows. While

$$\int_0^T \int_\Omega \partial_t s^h g \; dx\, dt \to \int_0^T \int_\Omega \partial_t s \, g \; dx\, dt$$

due to the weak convergence $\partial_t s^h \rightharpoonup \partial_t s$, we note that for $\int_0^T \int_\Omega \partial_t s^h \tilde{p}^h \, dx\, dt$ the following holds

$$\begin{aligned} \int_0^T \int_\Omega \partial_t s^h \tilde{p}^h \, dx\, dt &= \int_0^T \int_\Omega \partial_t s^h p^h \, dx\, dt \\ &\overset{(3.5)}{\in} \int_0^T \int_\Omega \partial_t s^h \left(J\Phi^0_{\delta,h}(\partial_t s^h) + \tau \partial_t s^h + J p_c(.,s^h) \right) \, dx\, dt. \end{aligned} \tag{3.24}$$

The limit procedure in the last two terms of (3.24) is straightforward: the weak lower semicontinuity of the L^2-norm implies

$$\liminf_{h\to 0} \int_0^T \int_\Omega \tau \left| \partial_t s^h \right|^2 \, dx\, dt \ge \int_0^T \int_\Omega \tau \left| \partial_t s \right|^2 \, dx\, dt,$$

and the last term can be written as

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t s^h J p_c(.,s^h) \, dx\, dt \\ &\quad = \int_0^T \int_\Omega \partial_t s^h p_c(x,s^h) \, dx\, dt + \int_0^T \int_\Omega \partial_t s^h \left(J p_c(.,s^h) - p_c(x,s^h) \right) \, dx\, dt, \end{aligned}$$

where the first term converges to $\int_0^T \int_\Omega \partial_t s \, p_c(x,s) \, dx\, dt$ due to the strong convergence $p_c(.,s^h) \to p_c(.,s)$ and the weak convergence $\partial_t s^h \rightharpoonup \partial_t s$. The second term tends to zero, since $\left| J p_c(.,s^h) - p_c(x,s^h) \right| \le C\, h$, where C is the Lipschitz constant of p_c.

It remains to pass to the limit in the first term on the right hand side of (3.24), $\int_0^T \int_\Omega \partial_t s^h J\Phi^0_{\delta,h}(\partial_t s^h) \, dx\, dt$. We distinguish between $\delta = 0$ and $\delta > 0$.

The case $\delta = 0$ is straightforward, since $\partial_t s^h J\Phi^0_{0,h}(\partial_t s^h) = \gamma^h |\partial_t s^h|$ and thus

$$\begin{aligned} \liminf_{h\to 0} \int_0^T \int_\Omega \partial_t s^h(x,t) J\Phi^0_{0,h}(\partial_t s^h)(x,t) \, dx\, dt &\ge \liminf_{h\to 0} \int_0^T \int_\Omega \gamma(x) |\partial_t s^h(x,t)| \, dx\, dt \\ &\quad - \liminf_{h\to 0} \left| \int_0^T \int_\Omega (\gamma^h(x) - \gamma(x)) \, |\partial_t s^h(x,t)| \, dx\, dt \right|, \end{aligned}$$

where the last term vanishes, since $|\gamma^h(x) - \gamma(x)| \le Ch$ and $\|\partial_t s^h\|_{L^2(0,T;L^2(\Omega))}$ is bounded. The first term satisfies

$$\liminf_{h\to 0} \int_0^T \int_\Omega \gamma(x) |\partial_t s^h(x,t)| \, dx\, dt \ge \int_0^T \int_\Omega \gamma(x) |\partial_t s(x,t)| \, dx\, dt$$

due to the convexity of $|\cdot|$.

In the case $\delta > 0$ we first observe that $J\Phi^0_{\delta,h}(\partial_t s^h) = \Phi^0_{\delta,h}(\partial_t s^h)$ and that

$$\Phi^0_{\delta,h}(\cdot) = \Phi^0_{0,h}(\cdot) - \Theta_\delta(\cdot),$$

where $|\Theta_\delta| \le \gamma$ and $\Theta_\delta(\sigma) = 0$ for $|\sigma| \ge \delta$. Consequently, $\left| \Theta_\delta(\partial_t s^h) \partial_t s^h \right| \le \gamma\, \delta$ and thus

$$\int_0^T \int_\Omega \partial_t s^h J\Phi^0_{\delta,h}(\partial_t s^h) = \int_0^T \int_\Omega \partial_t s^h \Phi^0_{0,h}(\partial_t s^h) - \int_0^T \int_\Omega \partial_t s^h \Theta_\delta(\partial_t s^h).$$

where the second term on the right hand side tends to zero as $\delta \to 0$ and, since the parameters are coupled, also for $h \to 0$. The first term was treated in the case $\delta = 0$.

Inserting all these convergences in (3.22), using (3.23), we conclude the variational inequality (2.8) for the limit (s,p). This shows that the pair (s,p) is indeed a weak solution of the Richards equation with hysteresis. □

4 Higher order estimates

Our aim here is to show additional uniform estimates for the sequence of approximate solutions (s^h, p^h). Our main results are uniform L^∞-bounds for s^h, p^h, and $\partial_t s^h$, and the boundedness

$$\|\partial_t \tilde{p}^h\|_{L^2(0,T;H^1(\Omega))} \le C(T, s_0, g).$$

for sufficiently regular boundary data. The main assumption will be that initial and boundary data are uniformly bounded. The above estimate implies, in particular, the strong convergence in $L^2(\Omega_T)$ of the pressures p^h.

We start with the observation that uniform estimates for s^h, $\partial_t s^h$, and p^h are available due to L^∞-bounds for the elliptic equation.

Lemma 4.1 (Uniform bounds). *Let $s^h, \tilde{p}^h$ satisfy* (2.12)–(2.15) *on a fixed time interval $(0,T)$. Additionally to the previous assumptions, we assume $s_0 \in L^\infty(\Omega)$ and $g \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega \times (0,T))$. Then there exists $C > 0$ independent of $h > 0$ and $\delta \ge 0$, such that*

$$\|s^h\|_{L^\infty(\Omega\times(0,T))} + \|\partial_t s^h\|_{L^\infty(\Omega\times(0,T))} + \|p^h\|_{L^\infty(\Omega\times(0,T))} \le C. \tag{4.1}$$

Proof. Due to the global Lipschitz continuity of p_c and of $\Psi^\tau_{\delta,h}$ (independent of δ), the ordinary differential equation (2.12) implies the estimate

$$\|s^h\|_{L^\infty(\Omega\times(0,T))} + \|\partial_t s^h\|_{L^\infty(\Omega\times(0,T))} \le C\|p^h\|_{L^\infty(\Omega\times(0,T))}. \tag{4.2}$$

The constant C depends on T and $\|s_0\|_{L^\infty}$, and is obtained with the help of a Gronwall inequality.

We additionally exploit the elliptic relation (2.14). The measurable and bounded coefficient $k(x, s^h)$ is uniformly bounded from below by the number $\kappa > 0$. Uniform estimates for the elliptic system with uniformly bounded boundary data (which can be obtained with a testing procedure) provide, for some $q < \infty$,

$$\|\tilde{p}^h(.,t)\|_{L^\infty(\Omega)} \le C \left\| \Psi^\tau_{\delta,h}\big(p_c(., s^h) - p^h\big)\right\|_{L^q(\Omega)}. \tag{4.3}$$

We use now the fact that the projection P can only reduce the L^∞-norm, estimate (4.3) for all t, and the elementary interpolation estimate $\|f\|_{L^q} \le C\|f\|^\Theta_{L^2}\|f\|^{1-\Theta}_{L^\infty}$ for some $\Theta \in (0,1)$, the energy estimate of Proposition 3.2, and (4.2).

$$\begin{aligned}
\|p^h\|_{L^\infty(\Omega\times(0,T))} &\le \|\tilde{p}^h\|_{L^\infty(\Omega\times(0,T))} \le C \left\| \Psi^\tau_{\delta,h}\big(p_c(x, s^h) - p^h\big)\right\|_{L^q(\Omega\times(0,T))} \\
&\le C \left(\|s^h\|_{L^q(\Omega\times(0,T))} + \|p^h\|_{L^q(\Omega\times(0,T))}\right) \\
&\le C \left(\|s^h\|_{L^2(\Omega\times(0,T))} + \|p^h\|_{L^2(\Omega\times(0,T))}\right)^\Theta \left(\|s^h\|_{L^\infty(\Omega\times(0,T))} + \|p^h\|_{L^\infty(\Omega\times(0,T))}\right)^{1-\Theta} \\
&\le C \left(\|s^h\|_{L^\infty(\Omega\times(0,T))} + \|p^h\|_{L^\infty(\Omega\times(0,T))}\right)^{1-\Theta} \le C\|p^h\|^{1-\Theta}_{L^\infty(\Omega\times(0,T))}.
\end{aligned}$$

This implies the uniform bound for the pressure. From this, (4.2) provides the bounds for saturation variables. □

Proposition 4.2 (Higher order estimates)**.** *Let coefficient functions and boundary data be as in Subsection 2.1 with $s_0 \in L^\infty(\Omega)$ and $g \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega \times (0,T))$ with $\partial_t g \in L^2(0,T;H^1(\Omega))$. Moreover, we assume that $p_c(\cdot, s)$ is piecewise constant (for all grids along the sequence) and that $\gamma(x) \equiv \gamma$ is a constant function. Then every solution $\tilde{p}^h, s^h$ to the scheme of Definition 2.4 satisfies the higher order estimate*

$$\|\partial_t \tilde{p}^h\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t^2 s^h\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{4.4}$$

where C depends on coefficient functions and boundary data, but not on δ or h.

Proof. We recall that the evolution system written in (3.3) as

$$\partial_t s^h - \nabla \cdot \big(k(x, s^h)(\nabla \tilde{p}^h + e_n)\big) = F, \tag{4.5}$$

with

$$F = -J\Psi^\tau_{\delta,h}(p_c(x, s^h) - p^h) + \Psi^\tau_{\delta,h}\big(p_c(x, s^h) - p^h\big). \tag{4.6}$$

Since $p_c(\cdot, s)$ is assumed to be piecewise constant we have $Jp_c(., s^h) = p_c(x, s^h)$ and the right hand side vanishes, $F = 0$.

We differentiate (4.5) with respect to t,

$$\partial_t^2 s^h - \nabla \cdot \Big(k(x, s^h)\nabla\partial_t \tilde{p}^h + \partial_s k(x, s^h)\partial_t s^h \big(\nabla \tilde{p}^h + e_n\big)\Big) = 0.$$

Multiplication with $\partial_t(\tilde{p}^h - g)$ and integration over $(0,T) \times \Omega$ yields

$$\begin{aligned}
Z_1 + Z_2 + Z_3 :=& \\
&\int_0^T \int_\Omega \partial_t^2 s^h \partial_t \tilde{p}^h \,dx\,dt + \int_0^T \int_\Omega k(x, s^h)|\nabla \partial_t \tilde{p}^h|^2 \,dx\,dt \\
&+ \int_0^T \int_\Omega \partial_s k(x, s^h)\partial_t s^h \big(\nabla \tilde{p}^h + e_n\big)\nabla\partial_t \tilde{p}^h \,dx\,dt \\
= &\int_0^T \int_\Omega \partial_t^2 s^h \partial_t g \,dx\,dt + \int_0^T \int_\Omega \Big(k(x, s^h)\nabla\partial_t \tilde{p}^h + \partial_s k(x, s^h)\partial_t s^h \big(\nabla \tilde{p}^h + e_n\big)\Big)\nabla\partial_t g \,dx\,dt.
\end{aligned} \tag{4.7}$$

Our aim is to show that Z_1 and Z_2 have positivity properties. The positivity of Z_2 is straightforward,

$$Z_2 = \int_0^T \int_\Omega k(x, s^h)|\nabla\partial_t \tilde{p}^h|^2 \,dx\,dt \ge \kappa \|\nabla \partial_t \tilde{p}^h\|^2_{L^2(0,T;L^2(\Omega))}.$$

In order to treat the first integral Z_1, we first write the hysteresis relation for $\delta > 0$ as

$$P_h \tilde{p}^h = p^h = J\Phi^\tau_\delta(\partial_t s^h) + Jp_c(., s^h). \tag{4.8}$$

We differentiate with respect to t,

$$\partial_t p^h = J\Big(D\Phi^\tau_\delta(\partial_t s^h) \cdot \partial_t^2 s^h + \partial_s p_c(., s^h) \cdot \partial_t s^h\Big),$$

and insert this expression into term Z_1,

$$\begin{aligned}
Z_1 =& \int_0^T \int_\Omega \partial_t^2 s^h \partial_t \tilde{p}^h \, dx\, dt = \int_0^T \int_\Omega \partial_t^2 s^h \partial_t p^h \, dx\, dt \\
=& \int_0^T \int_\Omega \partial_t^2 s^h J\Big(D\Phi_\delta^\tau(\partial_t s^h) \cdot \partial_t^2 s^h + \partial_s p_c(x, s^h) \cdot \partial_t s^h \Big) \, dx\, dt \\
=& \int_0^T \int_\Omega |\partial_t^2 s^h|^2 J\left(D\Phi_\delta^\tau(\partial_t s^h)\right) \, dx\, dt + \int_0^T \int_\Omega J\left(\partial_s p_c(x, s^h)\right) \partial_t s^h \partial_t^2 s^h \, dx\, dt \\
\geq& \tau \|\partial_t^2 s^h\|^2_{L^2(0,T;L^2(\Omega))} - C \|\partial_t s^h\|_{L^2(0,T;L^2(\Omega))} \|\partial_t^2 s^h\|_{L^2(0,T;L^2(\Omega))} \\
\geq& \frac{\tau}{2} \|\partial_t^2 s^h\|^2_{L^2(0,T;L^2(\Omega))} - \frac{2C^2}{\tau} \|\partial_t s^h\|^2_{L^2(0,T;L^2(\Omega))} \\
\geq& \frac{\tau}{2} \|\partial_t^2 s^h\|^2_{L^2(0,T;L^2(\Omega))} - C,
\end{aligned}$$

where we use the uniform bound (3.2) for $\partial_t s^h$ in the last line.

The term Z_3 has no positivity properties. It can be estimated with the help of Lemma 4.1 as follows

$$\begin{aligned}
|Z_3| =& \left| \int_0^T \int_\Omega \partial_s k(x, s^h) \partial_t s^h \big(\nabla \tilde{p}^h + e_n\big) \nabla \partial_t \tilde{p}^h \, dx\, dt \right| \\
\leq& C \, \|\partial_t s^h\|_{L^\infty(0,T;L^\infty(\Omega))} \, \|\nabla \tilde{p}^h + e_n\|_{L^2(0,T;L^2(\Omega))} \, \|\nabla \partial_t \tilde{p}^h\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

The first term is uniformly bounded (uniformly in δ and h) by Lemma 4.1. The second term is uniformly bounded by the energy estimate. The last factor appears squared in the term Z_2. Hence the error term Z_3 can be absorbed. The terms containing g on the right hand side of (4.7) can be controlled and absorbed in a straightforward way with expressions similar to those appearing in Z_3. We find the higher regularity estimate (4.4). □

5 Numerical treatment

In this section, we present some details of a finite element discretization of the system (1.1)–(1.2). We restrict our investigations to the following case: (i) the capillary pressure function p_c is the identity, $p_c(s) = s$, hence, in particular, independent of $x \in \Omega$. (ii) the function $\gamma : \Omega \to [0, \infty)$ is constant, we denote its value by $\gamma \in \mathbb{R}$. (iii) the regularization parameter δ is positive such that Φ_δ^τ is invertible, the inverse is denoted by $\Psi_\delta^\tau : \mathbb{R} \to \mathbb{R}$. System (1.1)–(1.2) then reads

$$\partial_t s = \nabla \cdot (k(s)(\nabla p + e_n)), \quad (x, t) \in \Omega \times (t_0, T), \tag{5.1}$$

$$\partial_t s = \Psi_\delta^\tau(p - s), \quad (x, t) \in \Omega \times (t_0, T). \tag{5.2}$$

The functions Φ_δ^τ and Ψ_δ^τ of (2.10) read

$$\Phi_\delta^\tau = \Phi_\delta^\tau(\sigma) = \begin{cases} \gamma + \tau\sigma & \text{for} \quad \sigma > \delta, \\ (\frac{\gamma}{\delta} + \tau)\sigma & \text{for} \quad \sigma \in [-\delta, \delta], \\ -\gamma + \tau\sigma & \text{for} \quad \sigma < -\delta, \end{cases}$$

and

$$\Psi_\delta^\tau = \Psi_\delta^\tau(z) = \begin{cases} \frac{z-\gamma}{\tau} & \text{for} \quad z > \gamma + \tau\delta, \\ (\frac{\gamma}{\delta} + \tau)^{-1} z & \text{for} \quad z \in [-(\gamma+\tau\delta), \gamma+\tau\delta], \\ \frac{z+\gamma}{\tau} & \text{for} \quad z < -(\gamma+\tau\delta). \end{cases}$$

In all numerical experiments, we use the permeability

$$k(s) = \begin{cases} \kappa & \text{for} \quad s < a, \\ \kappa + (s-a)^2 & \text{for} \quad s \geq a \end{cases}$$

with $a \geq 0$ and $\kappa > 0$. The numerical approach is to discretize the Galerkin scheme of Definition 2.4 and to regard the corresponding set of equations (for every time step) as one large system for saturation and pressure.

In space, we apply linear finite elements to discretize the above system. We discuss adaptivity in space and time and present numerical examples which show the validity of the algorithm, which is implemented in the FEM toolbox AMDiS [27].

Our numerical test regards viscous gravity fingering. As it was shown recently in [24], fingers can develop from arbitrarily small disturbances of planar initial saturations, when the following boundary conditions are studied: low initial saturation, on a time interval (t_0, t_s) large influx on the upper boundary (or large pressure), on a second time interval (t_s, T) smaller influx on the upper boundary. The fact that the numerical tests show the development of fingers prove the stability of the numerical scheme and the relevance of the hysteresis model in an important application.

5.1 Discrete system

For the numerical results presented here, we consider domains $\Omega := (-L, L)^n \subset \mathbb{R}^n$ with $L > 0$. With the definition $\Gamma_\pm := \{x \in \overline{\Omega} : x_n = \pm L\} \subset \partial\Omega$ and given functions $p_- : \Gamma_- \times (t_0, T] \to \mathbb{R}$ and $j_+ : \Gamma_+ \times (t_0, T] \to \mathbb{R}$, we assume Dirichlet boundary conditions $p = p_-$ for $x \in \Gamma_-$ and Neumann boundary conditions

$$j := -k(s)(\nabla p + e_n) \cdot \nu = j_+ \quad \text{for} \quad x \in \Gamma_+, \tag{5.3}$$

$\nu = e_n$ denoting the outer normal to Γ_+. In the lateral directions, i.e. for $x_i \in \{-L, L\}$, $i \in \{1, \dots, n-1\}$, we assume a periodic pressure.

In addition, the time interval $[t_0, T]$ is split by discrete time instants $t_0 < t_1 < \dots$, from which one gets the time steps $\Delta t_m := t_{m+1} - t_m$, $m = 0, 1, \dots$.

Linearization of $\Psi = \Psi_\delta^\tau$ yields (we drop δ and τ in the Ψ-notation)

$$\Psi(p^{(m+1)} - s^{(m+1)}) \approx \Psi'(p^{(m)} - s^{(m)})(p^{(m+1)} - s^{(m+1)}) + \Psi_\mathrm{e}(p^{(m)} - s^{(m)}),$$

where

$$\Psi_\mathrm{e}(p^{(m)} - s^{(m)}) := \Psi(p^{(m)} - s^{(m)}) - \Psi'(p^{(m)} - s^{(m)})(p^{(m)} - s^{(m)}).$$

We therefore approximate with

$$\Psi(p^{(m+1)} - s^{(m+1)}) \approx \begin{cases} \frac{p^{(m+1)} - s^{(m+1)}}{\tau} - \frac{\gamma}{\tau} & \text{for} \quad p^{(m)} - s^{(m)} > \gamma + \tau\delta, \\ \frac{p^{(m+1)} - s^{(m+1)}}{\gamma/\delta + \tau} & \text{for} \quad p^{(m)} - s^{(m)} \in [-(\gamma+\tau\delta), \gamma+\tau\delta], \\ \frac{p^{(m+1)} - s^{(m+1)}}{\tau} + \frac{\gamma}{\tau} & \text{for} \quad p^{(m)} - s^{(m)} < -(\gamma+\tau\delta). \end{cases}$$

To discretize in space, we choose a triangulation of Ω and use linear finite elements (globally continuous, piecewise linear) with standard nodal basis $(\eta_i)_i \subset H^1(\Omega)$, periodic in the lateral directions. With this choice, after a linearization of k, one has to solve in every time step the following discrete system. for the discrete saturation $s^{(m)} = \sum_i S_i^{(m)} \eta_i$ and the discrete pressure is $p^{(m)} = \sum_i P_i^{(m)} \eta_i$.

$$\begin{pmatrix} \frac{1}{\Delta t_m}\boldsymbol{M} + \boldsymbol{B}^{\mathrm{i}} & \boldsymbol{A}^k \\ \frac{1}{\Delta t_m}\boldsymbol{M} + \boldsymbol{\Psi}^{\mathrm{i}} & -\boldsymbol{\Psi}^{\mathrm{i}} \end{pmatrix} \begin{pmatrix} S^{(m+1)} \\ P^{(m+1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta t_m}\boldsymbol{M} S^{(m)} - \mathbf{B}^{\mathrm{e}} \\ \boldsymbol{\Psi}^{\mathrm{e}} + \frac{1}{\Delta t_m}\boldsymbol{M} S^{(m)} \end{pmatrix} \tag{5.4}$$

Here we used the mass matrix $\boldsymbol{M}$, the stiffness matrix $\boldsymbol{A}$, and other matrices as below.

$$\begin{aligned}
\boldsymbol{M} &= (M_{ij}) & M_{ij} &= (\eta_i, \eta_j)_\Omega, \\
\boldsymbol{A}^k &= (A_{ij}^k) & A_{ij}^k &= (k(s^{(m)})\nabla\eta_i, \nabla\eta_j)_\Omega, \\
\boldsymbol{\Psi}^{\mathrm{i}} &= (\Psi_{ij}^{\mathrm{i}}) & \Psi_{ij}^{\mathrm{i}} &= (\Psi'(p^{(m)} - s^{(m)})\eta_i, \eta_j)_\Omega, \\
\boldsymbol{\Psi}^{\mathrm{e}} &= (\Psi_i^{\mathrm{e}}) & \Psi_i^{\mathrm{e}} &= (\Psi_{\mathrm{e}}(p^{(m)} - s^{(m)}), \eta_i)_\Omega, \\
\mathbf{B}^{\mathrm{e}} &= (B_i^{\mathrm{e}}) & B_i^{\mathrm{e}} &= ((k(s^{(m)}) - k'(s^{(m)})s^{(m)}), e_n \cdot \nabla\eta_i)_\Omega, \\
\mathbf{B}^{\mathrm{i}} &= (B_{ij}^{\mathrm{i}}) & B_{ij}^{\mathrm{i}} &= (k'(s^{(m)})\eta_j, e_n \cdot \nabla\eta_i)_\Omega\,.
\end{aligned}$$

System (5.4) is solved by a direct solver (UMFPACK, [12]).

Adaptivity

To obtain satisfactory computational results, a mesh with a sufficiently fine resolution near the saturation front is necessary. Uniform refinement would lead to large computational costs, thus we are naturally led to use local mesh refinement. Since the front is moving, it is indispensable to use some adaptive strategy for local mesh refinement and coarsening. At every time step, the finite element mesh from the previous time step is locally refined and/or coarsened. For local mesh adaptation, we use an L^2-like error indicator for the saturation $s^{(m)}$, which is based on the jump of the normal derivatives of $s^{(m)}$ across the edges of the elements (see [19] and Fig. 2 for typical mesh).

Furthermore, a simple strategy of time adaptivity is used, where the time step Δt_m is inversely proportional to the maximum of $\frac{S^{(m)} - S^{(m-1)}}{\Delta t_{m-1}}$. This is especially helpful due to the fact that we consider boundary flux functions being step functions in time. Therefore, at times when the boundary flux function jumps, short time steps are necessary, while at later times larger time steps are possible.

5.2 Numerical results

For the numerical results presented in the following, we have used a time dependent boundary flux

$$j_+ = \begin{cases} j_+^0 & \text{for} \quad t < t_s, \\ j_+^s & \text{for} \quad t \geq t_s \end{cases}$$

with $t_s > t_0$ and $j_+^0, j_+^s \in \mathbb{R}$. We have used the following set of parameters.

$$\gamma = 4, \tau = \delta = 10^{-7}, \kappa = 10^{-3}, a = 0.32, j_+^0 = 0.524, j_+^s = 0.002, p_- = 4, L = 24.$$

The initial time is $t_0 = -2$ and the switching time is $t_s = 0$. The parameter $\tau = 10^{-7}$ is used everywhere with the exception of section 5.2.4, where we are interested in the effect of a large value of τ.

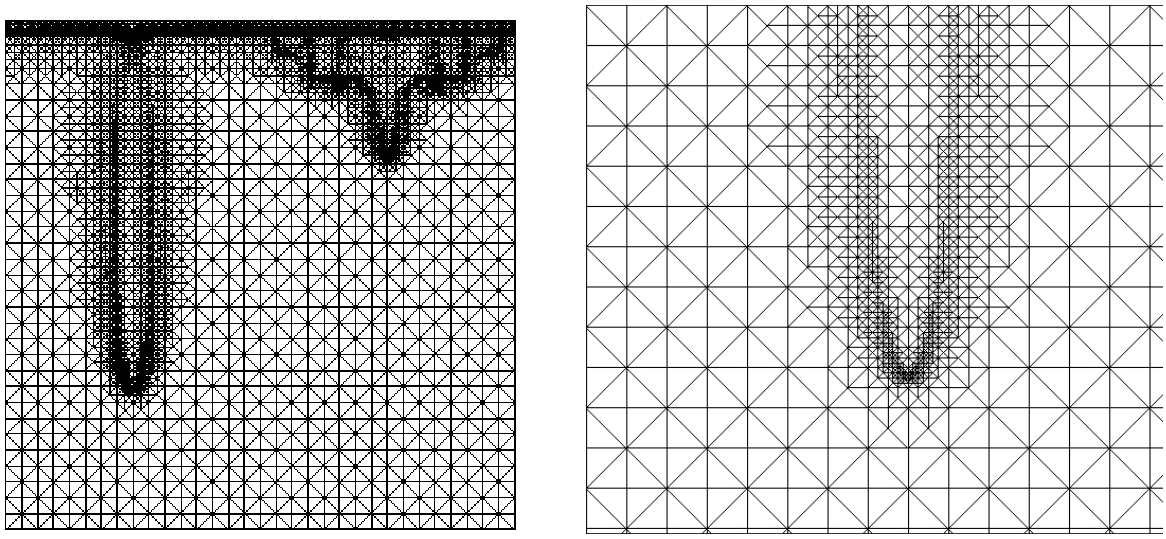


Figure 2: *Typical mesh with zoom in (right).*

5.2.1 1d-examples

For $n = 1$, we have used uniform grids with $N \in \mathbb{N}$ grid points (which corresponds to a spacing $h = 1/(N-1)$). Furthermore, in the boundary flux function, the parameter $j_+^s = 0.01$ has been chosen for this 1d-case. A plot of the discrete saturation for $N - 1 = 2^{16}$ at time $t = 500$ is shown in Fig. 3 with saturation values larger than 0.4 for $12 \lessapprox x \lessapprox 23$. In later 2d-examples the value of the saturation will be displayed by different levels of gray (see e.g. Fig. 4).

We define $d_h := \|s_h - s_{h/2}\|_{L^\infty(\Omega)}$, where s_h denotes the discrete solution with respect to spacing h at time $t = 500$. Tab. 1 shows values of d_h for different numbers of grid points. In Fig. 3 we plot values of $\log d_h$ versus $\log(N-1)$ and show the linear fit to the data. The corresponding slope $\sigma \approx -1.20472$ indicates a slightly superlinear behaviour, and therefore, we experimentally observe $d_h \lessapprox \text{const}\ h^{-\sigma}$ and thus the error

$$\|s_h - s\|_{L^\infty(\Omega)} \lessapprox \text{const}\ h^{-\sigma}$$

where s indicates the (unknown) exact solution at $t = 500$.

$N-1$	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}
d_h	0.0957391	0.0457352	0.0228676	0.0064209	0.0039276

Table 1: $N - 1$ *and* d_h *for different values of* N.

5.2.2 2d-example: Onset of instability

Fig. 4 indicates the instability of the planar front in a two-dimensional calculation. We choose $n = 2$ and the randomly perturbed inital saturation

$$s_0(x) = 0.001 + R(x),$$

where random numbers $R(x) \in [0, 0.002]$ are chosen uniformly in that interval, and independently for every grid point x.

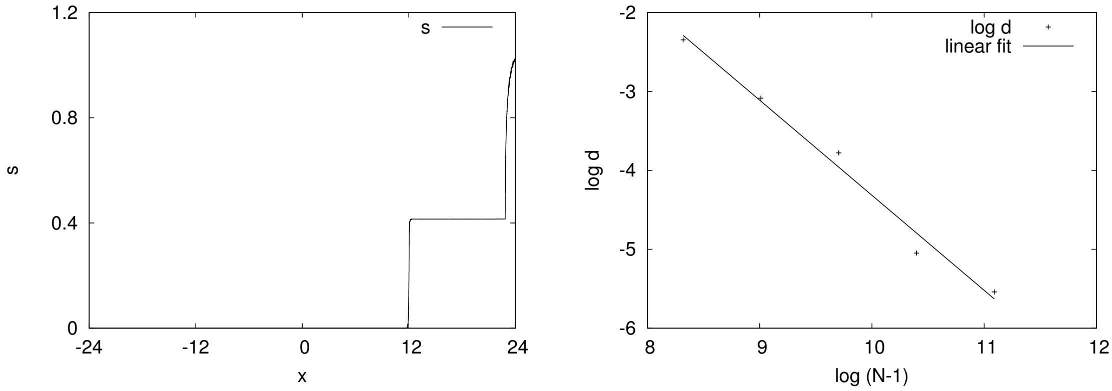


Figure 3: *Left: Discrete saturation $s = s_h$ for $N = 65537$; Right: Plot of $\log d_h$ versus $\log(N-1)$ and linear fit.*

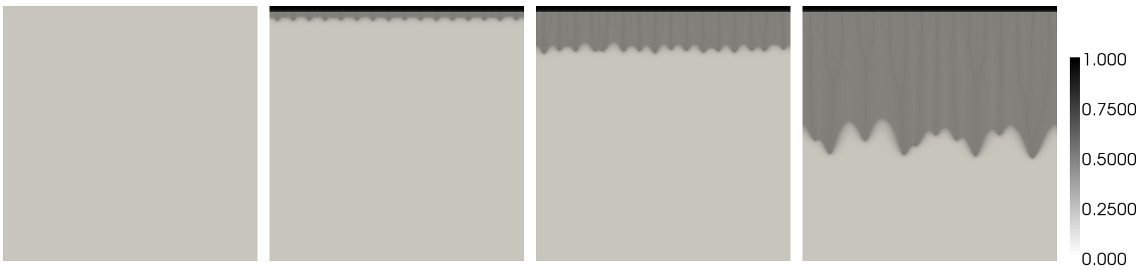


Figure 4: *From left to right: the discrete saturations for $t = t_0 = -2$, $t \approx 509$, $t \approx 2508$, and $t \approx 8487$.*

5.2.3 2d-example: Two fingers

We next study a more pronounced perturbation of the initial condition. Fig. 5 shows numerical results in two dimensions with the inital condition

$$s_0 = \frac{4}{5}(1 - \tanh(3(|x - x_0| - 1/2))) + \frac{1}{10}(1 - \tanh(3(|x - \tilde{x}_0| - 1/2))),$$

where $x_0 = (-12, 23.5)$ and $\tilde{x}_0 = (12, 23.5)$ has been used. The stronger perturbation at x_0 leads to a pronounced single finger. The weaker perturbation at $\tilde{x}_0$ leads to a faster saturation front around that point; eventually, the instability of the front leads to the development of a finger.

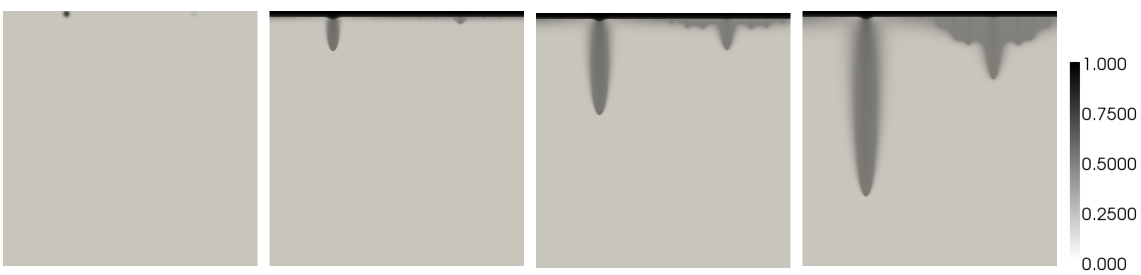


Figure 5: *From left to right: discrete saturation for $t = t_0 = -2$, $t \approx 263$, $t \approx 1032$, and $t \approx 2510$.*

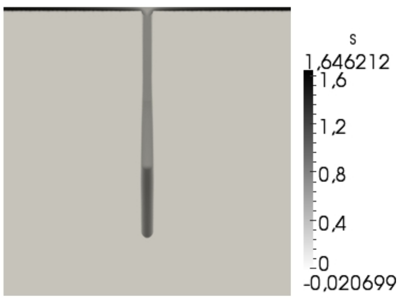
5.2.4 2d-example: Single finger for large τ

In this experiment we are interested in the effect of a large value of τ. We investigate in two dimensions the inital condition

$$s_0 = \frac{4}{5}(1 - \tanh(3(|x - x_0| - 1/2)))$$

with $x_0 = (0, 23.5)$. In contrast to the previous examples, we use the value $\tau = 2$ such that the dynamic hysteresis effect has a visible influence on the solution. The numerical results are shown in Fig. 6 and Fig. 7. The perturbation of the initial values leads to a well defined narrow finger, where the width of the finger is almost constant. In addition, after some transitional time, we observe the "non-monotone saturation profile", the saturation develops a local maximum near the tip of the finger.

Figure 6: *From left to right: discrete saturation for $t = t_0 = -2$, $t \approx 112$, $t \approx 300$, and $t \approx 556$.*



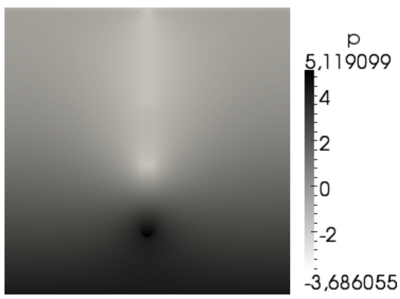


Figure 7: *Discrete saturation (left) and pressure (right) for $t \approx 556$.*

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References

[1] H. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[2] H. Alt, S. Luckhaus, and A. Visintin. On nonstationary flow through porous media. *Ann. Mat. Pura Appl. (4)*, 136:303–316, 1984.

[3] H. W. Alt and E. DiBenedetto. Nonsteady flow of water and oil through inhomogeneous porous media. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):335–392, 1985.

[4] F. Bagagiolo and A. Visintin. Hysteresis in filtration through porous media. *Z. Anal. Anwendungen*, 19(4):977–997, 2000.

[5] F. Bagagiolo and A. Visintin. Porous media filtration with hysteresis. *Adv. Math. Sci. Appl.*, 14(2):379–403, 2004.

[6] T. W. J. Bauters, D. A. DiCarlo, T. S. Steenhuis, and J.-Y. Parlange. Soil water content dependent wetting front characteristic in sands. *Elsevier Journal of Hydrology*, 231-232:244–254, 2000.

[7] J. Bear. *Dynamics of Fluids in Porous Media.* Elsevier, New York, 1972.

[8] A. Y. Beliaev and S. M. Hassanizadeh. A theoretical model of hysteresis and dynamic effects in the capillary relation for two-phase flow in porous media. *Transp. Porous Media*, 43(3):487–510, 2001.

[9] A. Y. Beliaev and R. J. Schotting. Analysis of a new model for unsaturated flow in porous media including hysteresis and dynamic effects. *Comput. Geosci.*, 5(4):345–368 (2002), 2001.

[10] J. Carrillo. On the uniqueness of the solution of the evolution dam problem. *Nonlinear Anal.*, 22(5):573–607, 1994.

[11] R. Collins. *Flow of Fluids through Porous Materials.* Englewood, 1990.

[12] T. A. Davis. Algorithm 832: UMFPACK V4.3—an unsymmetric-pattern multifrontal method. *ACM Trans. Math. Software*, 30(2):196–199, 2004.

[13] A. G. Egorov, R. Z. Dautov, J. L. Nieber, and A. Y. Sheshukov. Stability analysis of gravity-driven infiltrating flow. *Water Resources Research*, 39(9):12–1–12–14, 2003.

[14] M. Eliassi and R. J. Glass. On the continuum-scale modeling of gravity-driven fingers in unsaturated porous media: The inadequacy of the richards equation with standard monotonic constitutive relations and hysteretic equation of state. *Water Resources Research*, 37(8):2019–2035, 2001.

[15] S. M. Hassanizadeh and W. G. Gray. Thermodynamic basis of capillary pressure in porous media. *Water Resour. Res.*, 29(10):3389–3405, 1993.

[16] V. Joekar-Niasar, S. M. Hassanizadeh, and A. Leijnse. Insights into the relationships among capillary pressure, saturation, interfacial area and relative permeability using pore-network modeling. *Transp. Porous Media*, 74(2):201–219, 2008.

[17] J. L. Nieber, R. Z. Dautov, A. G. Egorov, and A. Y. Sheshukov. Dynamic capillary pressure mechanism for instability in gravity-driven flows; review and extension to very dry conditions. *Transp Porous Med*, 58:147–172, 2005.

[18] F. Otto. L^1-contraction and uniqueness for unstationary saturated-unsaturated porous media flow. *Adv. Math. Sci. Appl.*, 7(2):537–553, 1997.

[19] A. Rätz, A. Ribalta, and A. Voigt. Surface evolution of elastically stressed films under deposition by a diffuse interface model. *J. Comput. Phys.*, 214(1):187–208, 2006.

[20] B. Schweizer. Laws for the capillary pressure in a deterministic model for fronts in porous media. *SIAM J. Math. Anal.*, 36(5):1489–1521 (electronic), 2005.

[21] B. Schweizer. A stochastic model for fronts in porous media. *Ann. Mat. Pura Appl. (4)*, 184(3):375–393, 2005.

[22] B. Schweizer. Averaging of flows with capillary hysteresis in stochastic porous media. *European J. Appl. Math.*, 18(3):389–415, 2007.

[23] B. Schweizer. Regularization of outflow problems in unsaturated porous media with dry regions. *J. Differential Equations*, 237(2):278–306, 2007.

[24] B. Schweizer. Instability of gravity wetting fronts for richards equations with hysteresis. Preprint TU Dortmund, 2010.

[25] J. S. Selker, J.-Y. Parlange, and T. S. Steenhuis. Fingered flow in two dimensions. part 2. predicting finger moisture profile. *Wat. Resources Res.*, 28(9):2523–2528, 1992.

[26] C. J. van Duijn, G. J. M. Pieters, and P. A. C. Raats. Steady flows in unsaturated soils are stable. *Transp. Porous Media*, 57(2):215–244, 2004.

[27] S. Vey and A. Voigt. AMDiS — adaptive multidimensional simulations. *Comput. Visual. Sci.*, 10:57–67, 2007.

[28] A. Visintin. *Differential models of hysteresis*, volume 111 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1994.

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