How the Increase of the Life Expectancy Affects the Old-Age Dependency Ratio: An Investigation with the Gompertz Distribution

Peter Pflaumer

Department of Statistics
Technical University of Dortmund, Germany

peter.pflaumer@tu-dortmund.de (www.demometrie.de)

Abstract:

The old-age dependency ratio is the ratio of the number of elderly people at an age when they are generally economically inactive, compared to the number of people of working age. It is an indicator of how many potential retirees a potential worker has to support. In the following paper the influence of mortality on the old-age-dependency ratio is investigated with the Gompertz model. Since the mortality of modern developed population is largely the mortality of old age, the Gompertz model provides a good approximation of low mortality life tables. Especially the effect on the ratio of changes in the life expectancy is investigated with approximation formulas using the life table for females of the United States in 2006. It will be shown that an increase in the life expectancy raises the old-age dependency ratio considerably.

Keywords: Mortality, Life Table, Stable Population Model

1. Introduction

Benjamin Gompertz proposed in 1825 a life table function, which is one of the oldest and most famous models of demography. It states that the mortality intensity exponentially increases with age in adulthood. It has been much applied in life table analysis and in insurance mathematics using various modifications (e.g. Gompertz-Makeham law). Due to declining children and youth mortality, it has again become essential in order to describe "modern" life tables with low mortality (cf. Pollard, 1998). The model allows to fully describe the present and future life tables in industrialized countries by using only two parameters, in principle, which are both easy to estimate from data.

In the following paper the influence of mortality and fertility on the old-age-dependency ratio is investigated with the Gompertz model. After an introduction to the Gompertz distribution, approximation formulas for the old-age dependency ratio and its elasticity with respect to the life expectancy are derived. The application is done with data of the U.S. life table for females in 2006.

2. The Old-Age Dependency Ratio in a Stationary Population

The old-age-dependency ratio is the ratio of the number of elderly people at an age when they are generally economically inactive (i.e. aged 60 and over), compared to the number of people of working age (i.e. 20-60 years old). It is an indicator of how many potential retirees a potential worker has to support. Its development significantly affects the financial burden of the social pension insurance. An increase of the ratio causes ceteris paribus an increase in the premium, when the pensions are constant, or a decrease in the pensions, when the premiums stay constant . Therefore it is important, to analytically analyze the influence of mortality on the old-age-dependency ratio. This will is done hereafter with the Gompertz model, since it provides a good approximation for low-mortality life tables (c.f. Pollard, 1991). Especially the effect on the ratio of changes in the life expectancy is analyzed. An increase of the life expectancy will undoubtedly raise the dependency ratio, but by how much?

The formal definition of the old-age-dependency ratio is

$$OADR = \frac{\int_{60}^{\infty} l(x)dx}{\int_{20}^{60} l(x)dx} = \frac{\int_{60}^{\infty} \exp\left(-e^{k\cdot(x-m)}\right)dx}{\int_{20}^{60} \exp\left(-e^{k\cdot(x-m)}\right)dx},$$

where $l(x) = \exp\left(e^{-k\cdot m} - e^{k\cdot(x-m)}\right) \approx \exp\left(-e^{k\cdot(x-m)}\right)$ is the survivor function of the the Gompertz distribution with m>>k>0 (for a detailed presentation of the Gompertz distribution cf., e.g., Pollard, 1991 and 1998 or Carriere, 1992 and 1994).

The mean is

$\mu = m - \frac{\gamma}{k}$, where m is the modal value and $\lambda = 0.57722....$ the Euler-Mascheroni constant.

The variance is

$$\sigma^2 \approx \frac{\pi^2}{6} \cdot \frac{1}{k^2}.$$

Thus, the reciprocal value of k can be regarded as a spread parameter. The parameter k is also the growth rate of the exponential force of mortality function.

The life expectancy at age x can be approximated by

$e(x) = -\frac{\frac{\gamma + k \cdot (x-m) - \exp(k \cdot (x-m))}{k}}{\exp\left(e^{-k \cdot m} - e^{k \cdot (x-m)}\right)}$ (cf., e.g., Carriere, 1992 and 1994),

Substituting the values of the life expectancy and the survivor function at age x in the transformed the formula of the old-age dependency ratio

$$OADR = \frac{e_{60} \cdot l_{60}}{e_{20} \cdot l_{20} - e_{60} \cdot l_{60}} = \frac{1}{\frac{e_{20} \cdot l_{20}}{e_{60} \cdot l_{60}} - 1}$$

yields a good approximation of the ratio, if the modal age m is not too low (m>70)

$$OADR_1 = \frac{1}{\frac{\gamma - k \cdot (m-20) - \exp(-k \cdot (m-20))}{\gamma - k \cdot (m-60) - \exp(-k \cdot (m-60))} - 1}.$$

The function is nearly linear and independent of k, since

$$\lim_{m \to \infty} \frac{dOADR_1}{dm} = \frac{1}{40}.$$

If the modal value m is still getting higher and higher, then the ratio finally tends to

$$OADR_2 = \frac{1}{\frac{\gamma - k \cdot (m-20)}{\gamma - k \cdot (m-60)} - 1} = \frac{m - \frac{\gamma}{k} - 60}{40} = \frac{e_0 - 60}{40}.$$

In order to examine the quality of the approximation formulas a Gompertz distribution has been fitted to the life table for females of the United States in 2006. The results are given in Table 1. In Figure 1 the actual values of the survivor function l(x) and the death density function $d(x) = -\frac{dl(x)}{dx}$ are compared with the estimated values. The Gompertz model provides a good approximation of a life table with low mortality.

Table 1: Estimates of the Gompertz parameters in the life table for females of the United States in 2006

			Lower	**Upper**
Parameter	**Estimate**	**Std Error**	**95% C.I.**	**95% C.I.**
k	0.0938	7.837E-04	0.0922	0.0953
m	87.047	0.0751	86.898	87.196

Data Source: United States Life Tables 2006, NVSR, Volume 58, Number 21, 2010

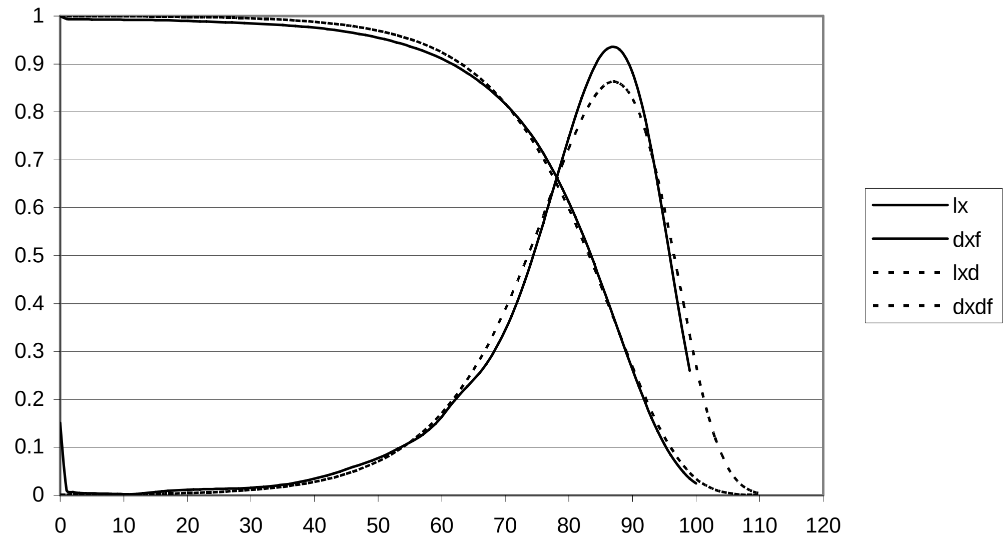

Figure 1: Comparison between actual (solid lines) and estimated (dotted lines) survival and death density (multiplied by a factor of 25) functions

Figure 2 and Table 2 show the quality of the approximation formulas of the old-age dependency ratios. The exact ratio has been calculated by numerical integration. There is virtually no difference between the exact and the approximative values if the modal ages are high. Even the very simple approximation formula leads to satisfactory results at very old ages.

Table 2: Old-age dependency ratios (k=0.0938)

m	e_0	OADR	$OADR_1$	$OADR_2$
70	63.8	0.211	0.223	0.096
75	68.8	0.301	0.306	0.221
80	73.8	0.401	0.403	0.346
85	78.8	0.508	0.509	0.471
90	83.8	0.621	0.622	0.596
95	88.8	0.738	0.738	0.721
100	93.8	0.858	0.858	0.846

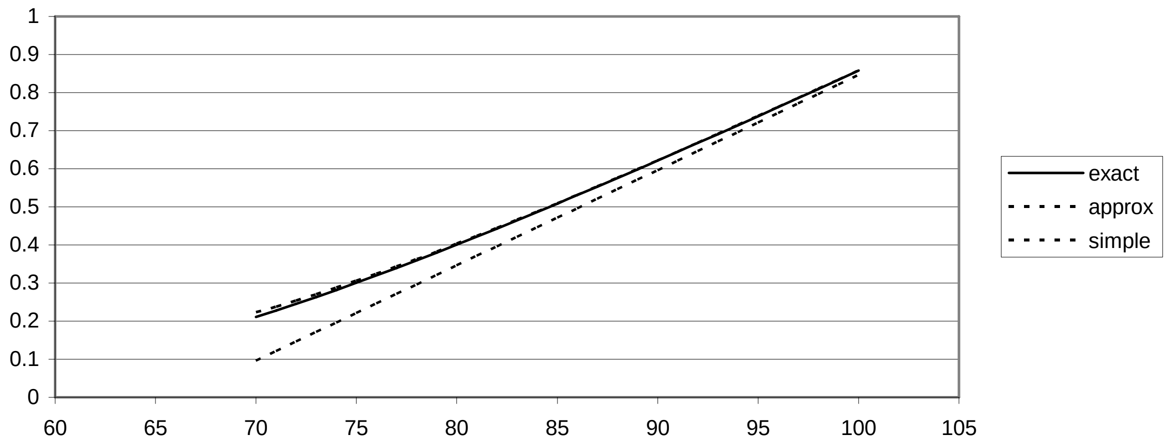

Figure 2: Old-age dependency ratios as a function of m (k=0.938; dotted lines represent the approximation formulas)

In order to analyze the influence of a change in the life expectancy on the old-age-dependency ratio, elasticities are computed. Elasticity is the ratio of the percent change in one variable to the percent change in another variable.
Mathematically, elasticity is defined as

$$\varepsilon\left(OADR, e_0\right) = \frac{dOADR_1}{de_0} \cdot \frac{e_0}{OADR_1}.$$

Using above approximation formula leads to a rather complicated formula

$$\varepsilon\left(OADR_1, e_0\right) = \frac{k \cdot e^{\gamma+e_0 \cdot k}\left(40 \cdot k \cdot e^{\gamma+e_0 \cdot k} - e^{20 \cdot k}\left(e^{40 \cdot k}\left(e_0 \cdot k - 20k + 1\right) - e_0 \cdot k + 60k - 1\right)\right)}{\left(40 \cdot k \cdot e^{\gamma+e_0 \cdot k} + e^{20 \cdot k}\left(1 - e^{40 \cdot k}\right)\right) \cdot \left(k \cdot e^{\gamma+e_0 \cdot k}\left(e_0 - 60\right) + e^{60 \cdot k}\right)} \cdot e_0,$$

where

$$e_0 = m - \frac{\gamma}{k}.$$

With the simple approximation formula one gets a very simple elasticity formula

$$\varepsilon\left(OADR_2, e_0\right) = \frac{d\left(\frac{e_0 - 60}{40}\right)}{de_0} \cdot \frac{e_0}{\frac{e_0 - 60}{40}} = \frac{e_0}{e_0 - 60}.$$

Table 3 and Figure 3 show the elasticities depending on the expectancy of life. In order to investigate the influence of k on the elasticity, calculations have been repeated with k=0.15 and the same modal age m=87. The exact elasticities have been obtained by numerical integration, again From the results it can be concluded that an increase in the life expectancy of 1 percent leads to an increase in the old-age-dependency ratio of about 3 percent in a stationary population of today with low mortality. The higher the growth rate k of the force of mortality function is, the higher will be the elasticity.

Table 3: Elasticites of the old-age dependency ratio with respect to the life expectancy

e_0	exact		approximation		simple approx.
	k=0.0938	k=0.15	k=0.0938	k=0.15	
70	4.24	5.50	4.03	5.43	7.00
75	3.74	4.49	3.67	4.47	5.00
80	3.35	3.80	3.32	3.80	4.00
85	3.04	3.32	3.03	3.32	3.40
90	2.79	2.96	2.79	2.96	3.00
95	2.59	2.70	2.59	2.70	2.71
100	2.43	2.49	2.43	2.49	2.50

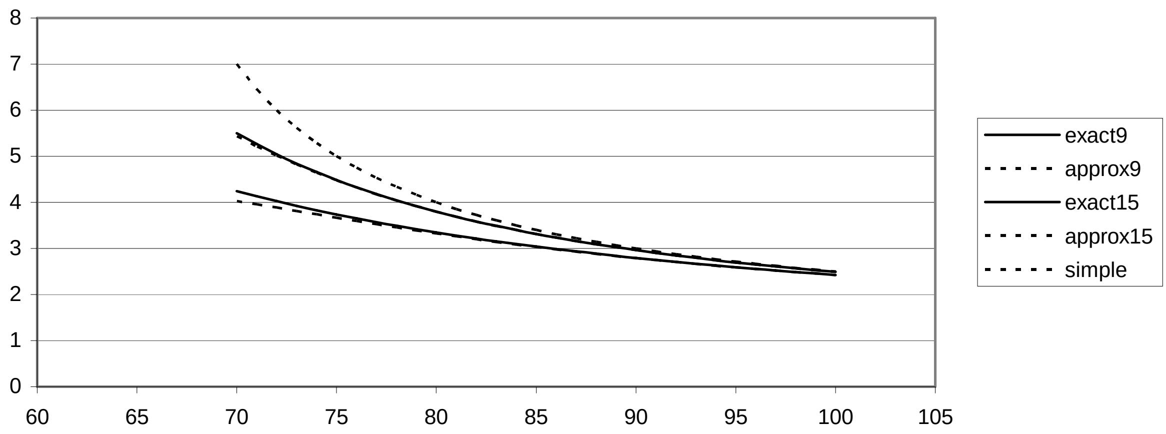

Figure 3: Elasticites of the old-age dependency ratio with respect to the life expectancy (k=0.0938 or k=0.15; dotted lines represent approximation formulas)

3. The Old-Age-Dependency Ratio in a Stable Population

A population with an invariable age structure and a fixed rate of increase is called a stable population.

If one would like to know the old-age-dependency ratio in a stable population with a growth rate of r, one has to compute

$$OADR(r) = \frac{\int_{60}^{\infty} e^{-r\cdot x} l(x)dx}{\int_{20}^{60} e^{-r\cdot x} l(x)dx}.$$

This expression can be approximated based on Keyfitz (1977) by

$$OADR(r) \approx OADR(0)\cdot e^{-T\cdot r},$$

where T is the difference between the mean age of the two generations in the age classes 20 to 60 and 60 to ω.

Substituting the approximation formulas for the old-age-dependency ratio yields

$$OADR_1(r) \approx \frac{1}{\frac{\gamma - k\cdot(m-20) - \exp(-k\cdot(m-20))}{\gamma - k\cdot(m-60) - \exp(-k\cdot(m-60))} - 1} \cdot e^{-T\cdot r} \text{ and } OADR_2(r) \approx \frac{e_0 - 60}{40} \cdot e^{-T\cdot r}$$

Using only the simple approximation of the old-age dependency ratio results in the following elasticity formula.

$$\varepsilon(\mathrm{OADR}_2, e_0) \approx \frac{e_0}{e_0 - 60} \cdot e^{-T\cdot r}.$$

Elasticities are now dependent on the growth rate r and the life expectancy e_0.
In a stable population with a positive (negative) growth rate the elasticities are lower (higher) than in a stationary population by the factor $e^{-T\cdot r}$

Table 4 shows the old-age dependency ratios for different growth rates. The exact values and the mean generation time T have been obtained by numerical integration. The results confirm the quality of the approximation formulas.

Table 4: Old-age dependency ratios as a function of the stable growth rate (k=0.0938, m=87, T=73.9-39.8=34.1)

r	exact	approximation	simple approx.
-0.02	1.086	1.095	1.031
-0.015	0.919	0.924	0.869
-0.01	0.777	0.779	0.733
-0.005	0.656	0.657	0.618
0	0.553	0.554	0.521
0.005	0.466	0.467	0.439
0.01	0.392	0.394	0.371
0.015	0.330	0.332	0.312
0.02	0.277	0.280	0.263
0.025	0.232	0.236	0.222
0.03	0.195	0.199	0.187

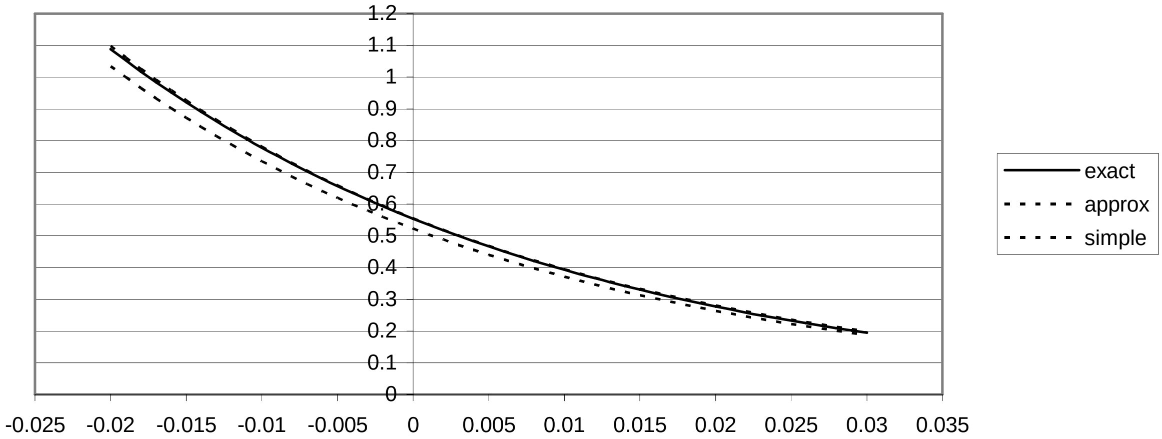

Figure 4: Old-age dependency ratios as a function of the stable growth rate (k=0.0938, m=87, T=73.9-39.8=34.1; dotted lines represent approximation formulas)

In order to demonstrate extreme or limiting cases, old-age-dependency ratios for the de-Moivre life table $\left(l(x)=1-\frac{x}{\omega} \quad 0 \le x \le \omega\right)$ and a complete rectangular life table $\left(l(x)=1 \quad 0 \le x \le \omega\right)$ have been calculated. The derived formulas are

$$OADR_M(r) = \frac{\int_{60}^{100} e^{-r\cdot x}\left(1-\frac{x}{100}\right)dx}{\int_{20}^{60} e^{-r\cdot x}\left(1-\frac{x}{100}\right)dx} = \frac{e^{-40r}\left(e^{40r}\left(40\cdot r-1\right)+1\right)}{e^{40r}\left(40\cdot r-1\right)-40\cdot r+1} \quad r \ne 0$$

and

$$OADR_R(r) = \frac{\int_{60}^{100} e^{-r\cdot x}dx}{\int_{20}^{60} e^{-r\cdot x}dx} = e^{-40\cdot r},$$

where in both cases it is assumed that the upper limit is $\omega = 100$. The results are shown in Table 5 and Figure 5.

Table 5: Old-age dependency ratios for the rectangular, Gompertz (k=0.0938, m=87, T=73.9-39.8=34.1) and de Moivre life table as a function of the stable growth rate

r	de-Moivre	Gompertz	Rectangular
-0.02	0.674	1.095	2.226
-0.015	0.566	0.924	1.822
-0.01	0.475	0.779	1.492
-0.005	0.398	0.657	1.221
0	0.333	0.554	1.000
0.005	0.279	0.467	0.819
0.01	0.233	0.394	0.670
0.015	0.195	0.332	0.549
0.02	0.162	0.280	0.449
0.025	0.135	0.236	0.368
0.03	0.113	0.199	0.301

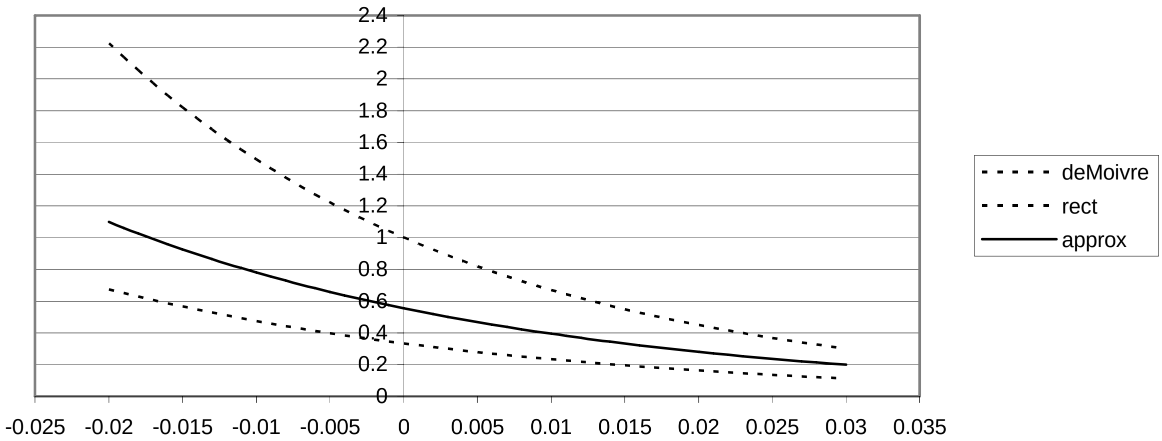

Figure 5: Old-age dependency ratios for the rectangular, Gompertz (k=0.0938, m=87, T=73.9-39.8=34.1) and de Moivre life table as a function of the stable growth rate (dotted lines represent approximation formulas)

4. Conclusion

The Gompertz distribution is a suitable model for low mortality life tables. Only two parameters are necessary in order to describe the complete life table with all of its demographic parameters. In general these two parameters are the modal value m and the growth rate k of the force of mortality function.

A simple formula for the old-age dependency ratio was derived. With this formula it is possible to study analytically the influence of the life expectancy on the old-age dependency ratio. It has been shown that in a stable population with a life expectancy of about 80 years the influence of the mortality on the old-age dependency ratio is considerable. In a stationary population with low mortality an increase of the life expectancy of 1 percent causes an increase of the old-age dependency of approximately 3 percent. In a population with a decreasing growth rate this influence is even higher.

References

Gompertz, B (1825): On the nature of the function expressive of the law of human mortality, Philosophical Transactions of the Royal Society of London, Ser. A, 115, 513-585, reprint: Haberman S; Sibbett, TA (1995), Vol. II, 119-191.

Carriere, JF (1992): Parametric models for life tables, Transactions of Society of Actuaries, 44, 77-99.

Carriere, JF (1994): An investigation of the Gompertz law of mortality, Actuarial Research Clearing House, 2, 161-177.

Keyfitz, N (1977): Applied mathematical demography, 2nd ed., New York.

Pflaumer, P (2007): Life table forecasting with the Gompertz distribution, Proceedings of the Joint Statistical Meetings, Section on Social Statistics [CD-ROM] , Salt Lake City, 3564-3571.

Pflaumer, P (2010): Measuring the Rectangularization of Survival Functions Using the Gompertz Distribution, Proceedings of the American Statistical Association, Section on Social Statistics [CD-ROM], Vancouver, Canada, 2010, 664-674.

Pflaumer, P (2011): Methods for Estimating Selected Life Table Parameters Using the Gompertz Distribution, Proceedings of the American Statistical Association, Section on Social Statistics [CD-ROM], Miami Beach, Florida, 2011, 733-747.

Pollard, J (1998): An old tool – modern applications, Actuarial Studies and Demography, Research Paper No. 001/98, August.

Pollard, JH (1991): Fun with Gompertz, Genus, XLVII-n.1-2, 1-20.

Appendix:

A1: Approximation Formulas of the Old-Age Dependency Ratio and their Accuracy

The considered approximations are bad for small values of m. Better approximations in these cases can be achieved, when the life expectancy at age x is replaced by the easy-to-calculate median survival time at age x or the Pollard formula, an approximation formula for the expectation of life at age x, which works well over a wide range of ages (cf. Pollard, 1991):

$$e_x^P = \frac{1}{k}\left(\ln\left(1+\frac{k}{\mu(x)}\right)-\frac{1}{2}\left(1+\frac{\mu(x)}{k}\right)^{-2}\right),$$

where $\mu(x) = k \cdot e^{k\cdot(x-m)}$ is the force of mortality at age x.

The median survival time at age x is the time at which half of the survivors l(x) will have died. Formally the median survival time can be computed by

$$\frac{l(x+\tilde{e}_x)}{l(x)} = 0.5\,.$$

Substituting for l(x) the survivor function of the Gompertz distribution leads to

$$\frac{\exp\left(\exp(-k\cdot m)-\exp\left(k\cdot(x+\tilde{e}_x-m)\right)\right)}{\exp\left(\exp(-km)-\exp\left(k\cdot(x-m)\right)\right)} = 0.5\,.$$

Solving this equation for the median survival time at age x yields

$$\tilde{e}_x = \frac{\ln\left(e^{k\cdot x}+e^{k\cdot m}\ln 2\right)}{k} - x = \frac{1}{k}\cdot\ln\left(1+\frac{k\cdot\ln 2}{\mu(x)}\right)$$ (c.f., Pollard (1991),

where $\mu(x) = k \cdot e^{k\cdot(x-m)}$ is the force of mortality at age x.

Table 3 shows the accuracy of the approximations with the old-age dependency ratio based on the Pollard median survival time and the Pollard formula

$$OADR_3 = \frac{\tilde{e}_{60}\cdot l_{60}}{\tilde{e}_{20}\cdot l_{20}-\tilde{e}_{60}\cdot l_{60}} = \frac{1}{\frac{\tilde{e}_{20}\cdot l_{20}}{\tilde{e}_{60}\cdot l_{60}}-1}$$

and

$$OADR_4 = \frac{e_{60}^P\cdot l_{60}}{e_{20}^P\cdot l_{20}-e_{60}^P\cdot l_{60}} = \frac{1}{\frac{e_{20}^P\cdot l_{20}}{e_{60}^P\cdot l_{60}}-1}$$

The replacement by the median survival time leads to good approximation results over the whole range of the modal values (see also Figure A1).

Table A1: Approximations of the old-age dependency ratios

	k=0.09					k=0.15				
m	OADR	$OADR_1$	$OADR_2$	$OADR_3$	$OADR_4$	OADR	$OADR_1$	$OADR_2$	$OADR_3$	$OADR_4$
50	0.010	0.827	-0.404	0.008	0.010	0.001	1.572	-0.346	0.000	0.001
55	0.033	0.252	-0.279	0.026	0.031	0.009	0.203	-0.221	0.007	0.009
60	0.074	0.152	-0.154	0.062	0.069	0.042	0.085	-0.096	0.036	0.040
65	0.134	0.165	-0.029	0.117	0.125	0.107	0.117	0.029	0.097	0.099
70	0.211	0.223	0.096	0.191	0.196	0.196	0.198	0.154	0.188	0.185
75	0.301	0.306	0.221	0.282	0.282	0.301	0.302	0.279	0.301	0.291
80	0.401	0.403	0.346	0.388	0.380	0.415	0.416	0.404	0.424	0.409
85	0.508	0.509	0.471	0.503	0.489	0.535	0.535	0.529	0.552	0.534
90	0.621	0.622	0.596	0.626	0.606	0.657	0.657	0.654	0.681	0.660
$RMSE_{70-90}$		0.0035	0.0538	0.0091	0.0120		0.0011	0.0222	0.0142	0.0073

The root mean squared error in the last row of the table was calculated using the following formula:

$$RMSE_{i,70-90} = \sqrt{\frac{1}{5}\sum_{t}\left(OADR_{i,t} - OADR_{1,t}\right)^2}, t = 70, 75, \cdots 90$$

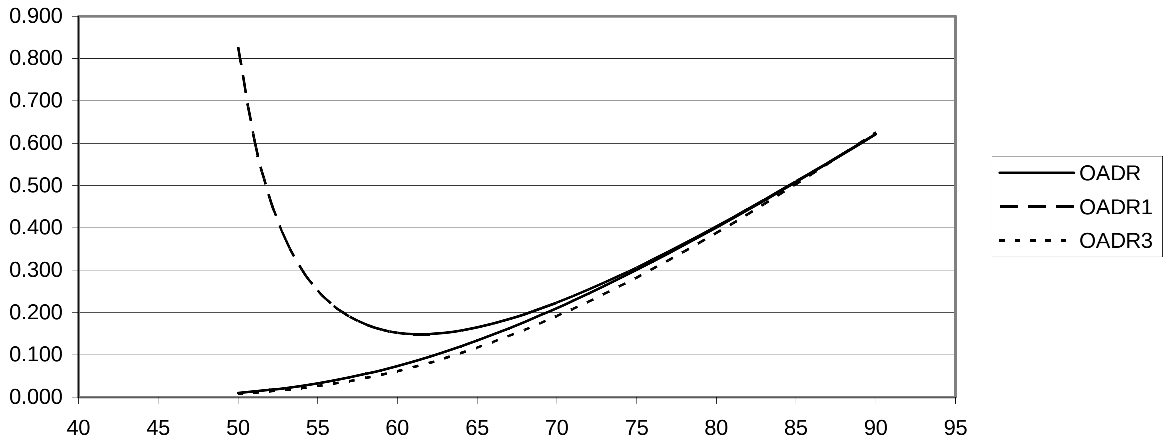

Figure A1: Old-age dependency ratios as a function of m (k=0.938; dotted lines represent the approximation formulas: life expectancy at age x and median survival time)

A2 Influence of Fertility

If one sets $T = h \cdot G$, $h > 0$, where G is the mean generation time, i.e., the time for a population to increase by a factor equal to the net reproduction rate R_0, one obtains

$$OADR(r) \approx \frac{1}{\dfrac{\gamma - k \cdot (m-20) - \exp(-k \cdot (m-20))}{\gamma - k \cdot (m-60) - \exp(-k \cdot (m-60))} - 1} \cdot R_0^{-h} \approx \frac{e_0 - 60}{40} \cdot R_0^{-h},$$

because the net reproduction rate is approximately equal to $R_0 = e^{r \cdot G}$.

The net reproduction rate elasticity of the old-age dependency ratio is

$$\varepsilon(OADR(r), R_0) = \frac{dOADR(r)}{dR_0} \cdot \frac{R_0}{OADR(r)} = -h.$$

An increase in the net reproduction rate of 1 percent reduces the old-age dependency ratio by h percent. If one assumes that T=34 and G=25, then the factor is h=0.735. An increase in the life expectancy raises the old-age dependency ratio over proportionally, whereas a decrease in the net reproduction rate raises it under proportionally. In general, the latter elasticity will be slightly below one.

In a stationary population with a life expectancy of about 80 years the influence of the mortality on the old-age dependency ratio is nearly three times as high as the influence of the fertility, if elasticities are used as measures.