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History of teaching the concept of a function in Russia

L.Euler formulated the new definition of a function in his foreword to the "Differential calculus" (1755): "When quantities depend on others in such a way that at the change of the last they are also changed the first are called functions of the second ones" (see Yushkevich, 1970, p. 617).

In B.L.van der Warden's classical textbook in 1930 the, we already see the quite modern definition of a mapping: "If to each element from some set M by any rule a unique (generally speaking, new) object $\mathcal{P}(x)$ is put in a correspondence then this correspondence $\mathcal{P}$ is called a function. If all objects $\mathcal{P}(x)$ belong to some set N, the correspondence $x \quad \mathcal{P}(x)$ is called also a mapping from M into N » (van der Waerden, 1930).

F.Klein (1977, p. 292) complained that "…the school mostly ignores all development of a science which took place after Euler" and offered: "… we wish that the general concept of a function … has entered as the enzyme into all the teaching of mathematics at school; but it should be introduced not in the form of abstract definition but rather on concrete examples … in order to make this concept a living property of a pupil". F.Klein noted that "it would be desirable that among numerous teachers there was at least a small number of independently working people who would be familiar also with the newest concepts of the theory of sets" (ibid.).

However, modern definitions of a function did not soon find the path to the educational practice and literature not only in secondary but also in the higher school in our country. Up to the beginning of 21-th century only the first part of F.Klein's appeal has been in essence executed: Euler's definition of a function has taken a strong place in school and university mathematical curricula. The second part – taking into account the development of mathematics after L.Euler and use of set-theoretic concepts - actually is not executed till now at the school level and even not completely at undergraduate level.

For example, in Soviet schools up to the middle of 60-th the textbook of algebra written by A.P.Kiselev prior to the October Revolution with the following definition of a function was used:

"That variable whose numerical values change depending on numerical values another one is called a dependent variable or a function of that other variable" (Kiselev, 1964, p. 25).

In the textbook the emphasis was made on the analytical expression of a functional dependence (in the form of the formula).

The same situation was observed in undergraduate textbooks. So, in A.K.Sushkevich's textbook (1941) it is supposed by default that a function is an expression *f(x)*, where *x* is variable quantity (p. 86).

At the same time in A.Sushkevich's textbook there is (in a little bit archaic language) the quite modern definition of a group homomorphism with the requirement that to each element of the first group one has put in a correspondence a unique element of the second group, i.e. with the requirement to a homomorphism to be a mapping in the modern sense (p. 353).

In the textbook of V.V.Stepanov (1953) on differential equations the definition of a function is absent at all.

On the other hand, in the second edition of the textbook of L.S.Pontryagin (1965) the special appendix was added containing the modern definition of a mapping (p. 292-293).

Note that the greatest mathematicians of the rank of N.N.Luzin, A.N.Kolmogorov, P.S.Aleksandrov, L.S.Pontryagin, apparently, were the first to realize the necessity of the introduction of the modern definition of a mapping into the scientific and educational literature.

Such definition of a mapping is used in books of N.N.Luzin (1948), P.S.Aleksandrov and A.N.Kolmogorov (1948), A.N.Kolmogorov and S.V.-Fomin (1954). Note that all these descriptions characterizing a function as a rule of correspondence were not strict definitions and left the concept of a function (mapping) undefined.

At the same time some great scientists still did not introduce the general concept of a function and its definition, limiting themselves to special cases. So did A.I.Maltsev (1956) and I.M.Gelfand (1971) in their textbooks on linear algebra. Apparently, it was implicitly supposed, that mastering special cases of the concept of a mapping is enough for mastering the appropriate themes of mathematics, and it is not necessary "to multiply entities". The mathematics educator G.V.Dorofeev (1978, p. 21) expressed similar educational ideas when he in a discussion article even protected a thesis about uselessness of the definition of a function: "Pupils have, basically, the correct substantial view of a function as a mathematical object, but experience significant difficulties when they encounter the definition of this object… This situation, namely the possession of a concept without knowledge of its exact definition is not strange at all… it is typical in the majority of kinds of human activity…".

A.N.Kolmogorov (1978, p.29) in his reaction, however, indicated: "… G.V.Dorofeev … at school in general … allocates to any version of the set-theoretical definition of function a modest place (basically only for optional lessons). I think, however, that for school textbooks … rules (composing

the definition of the concepts of a function. - I.S)… should be given to pupils early enough and should be coordinated with some certain final definition".

A.N.Kolmogorov supervised the reform of school mathematics teaching at the end of 60-eth.

Meanwhile, in the textbook for upper secondary school edited by A.N.Kolmogorov a rather concrete definition of a function is used, and authors consider only numerical functions:

«A correspondence with a domain D where to each number x from the set D a unique number y is corresponds by some law, is called a numerical function» (Kolmogorov et al., 1990, p. 20).

Thus, "Kolmogorov" reform did not aim at giving to teaching of mathematics abstract and formal character of what it was severely accused by opponents. The purposes were to eliminate archaic language and character of teaching, to correct the scientific level of mathematical education. The great attention was given to the didactical maintenance of the reform.

The general definition of a mapping has been introduced into the first textbooks corresponding to reformed curricula and published under the edition of A.A.Markushevich (1975) - the prominent mathematician and educator, the ally of A.N.Kolmogorov in the reforming of school mathematics. In 1960-70-eth didacticians have developed also methods of the teaching the concept of a function at school (Kolyagin et al., 1977), and concluded that the concept of a function is expedient for studying at school quite strictly, consistently introducing concepts of the set, of the ordered pair, the direct product of sets and of correspondences.

Appropriate steps have been undertaken for the preparation of school teachers: in particular, into curricula of pedagogical institutes new subjects have been introduced: "Scientific foundations of school mathematics", "Modern foundations of school mathematics". Note that the introduction of these subjects has played a revolutionary role in reforming also undergraduate mathematical courses, having set new standards of strictness and modern mathematical language.

Nevertheless, in 80-s, after the publication of the notorious article of L.S.-Pontryagin in the magazine "Communist", directed against Kolmogorov reforms, the general concept of a mapping, as well as other general set-theoretic concepts, has been expelled from school curricula, and Euler's definition of a function occupied a strong position in school mathematics.

In our opinion, it was a mistake, because the absence of the strict definition often complicates mastering the concept of a mapping by students in their further study in universities. We believe that, not demanding from pupils

the faultless possession of strict definitions, it is necessary to attain nevertheless that they would be aware of the modern definition of function.

Thus, the final opinion about the introduction of modern strict definition of a mapping is not reached concerning not only school, but even undergraduate textbooks.

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