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A rational problem from elementary number theory

Abstract

We solve a bit more then the half of the XV.th. proposition of Leonardo Pisano who was also named as Fibonacci. We do it using only elementary tools of number theory. Our problem is explicitly what square number and the sum of a *p* prime of the form 4*k*+1 give a square number again.

The problem arises according to a really old book. After solving the problem we thought that it is possible to work on it with talented school children or with first grade university students as well. The best place is in number theory more precisely after the lecture on Pythagorean triple.

Leonardo of Pisa (Leonardo Pisano) known as Fibonacci also dedicated his Book of Square Numbers (1225, “Liber quadratorum”) to Emperor Frederick II. His PROPOSITION XV. was “To find a square number which, being increased or diminished by 5, gives a square number.” [1], [2] He solved the problem in a very elegant way. Our simply task is only to find two numbers when the difference of their squares is five or more generally their difference is a *p* prime of the form of 4*k* +1.

The first question is what do you mean on numbers. There are no problem if the numbers are real. On the other hand it is easy to see that there is only one solution if the numbers are the natural numbers $3^2 - 2^2 = 5$.

The problem is a bit harder if I mean the numbers as rational numbers ($\mathbb{Q}$). In this situation we get

$$\frac{z^2}{y^2} - \frac{x^2}{u^2} = 5 \tag{1}$$

where *x*, *y*, *z* and *u* are positive natural numbers ($y \neq 0$ and $u \neq 0$) the solution is not trivial.

The problem is quite the same as the Pythagorean-triples which was solved in a lot of different ways [3]. We will get three infinite classes of solutions, and the proof is completely the same for all prime *p* where $p = 4k + 1$ for some *k* positive integer so we discuss this situation.

Theorem: The $\frac{z^2}{y^2}-\frac{x^2}{u^2}=p$ equation where p prime is $p=4k+1$ for k is a positive integer is solvable in the natural numbers if $y=u$ and

$$x=|pr_1^2-s_1^2|,\quad y=2r_1s_1,\quad z=pr_1^2+s_1^2$$

where r_1 and s_1 are natural numbers and $5r_1$ and s_1 are coprime and exatly one of them is even or

$$x=\left|\frac{pr_2^2-s_2^2}{2}\right|,\quad y=r_2s_2,\quad z=\frac{pr_2^2+s_2^2}{2}$$

where r_2 and s_2 are natural numbers and $5r_2$ and s_2 are coprime and both of them is odd.

Proof: Let us denote (a,b) the greatest common divisor of a and b. We can suppose that $(y,z)=1$ and $(x,u)=1$. If we multiply the equation (1) by u^2y^2 then we get

$$z^2u^2-x^2y^2=py^2u^2. \tag{2}$$

Here $u^2\,|\,py^2u^2 \Rightarrow u^2\,|\,x^2y^2$ but $(x,u)=1 \Rightarrow u^2\,|\,y^2$.

Completely same way $y^2\,|\,py^2u^2 \Rightarrow y^2\,|\,z^2u^2$ but $(y,z)=1 \Rightarrow y^2\,|\,u^2$. Thus we get $y=u$ and we can write

$$\frac{z^2}{y^2}-\frac{x^2}{y^2}=p \tag{3}$$

where $(y,z)=1$ and $(x,y)=1$. Multiplying by y^2 we can get the

$$x^2+py^2=z^2. \tag{4}$$

Let us denote $d=(x,z)$ then $d\,|\,py^2$. If $d=p$ then $p^2\,|\,z^2-x^2=py^2$ and we get that $p\,|\,y$ what is contradiction. If $d\,|\,y$ then $d\,|\,(x,y)=1$ so we get then x, y and z are pairwise coprime.

Exactly one of x and y is even. If both of them are odd then $z^2=x^2+py^2\equiv 2 \mod (4)$ what is impossible.

There are two cases

a) x is odd and y is even;

b) x is even and y is odd.

The case a)

First let us suppose that *y* is even and *x* is odd. Then we can write $y = 2y_1$ and

$$py_1^2 = \frac{z+x}{2} \cdot \frac{z-x}{2} \tag{5}$$

Let $d = \left(\frac{z+x}{2}, \frac{z-x}{2}\right)$ then $d \mid (x,z) \Rightarrow d = 1$. We can conclude that $py^2 = pr_1^2 \cdot s_1^2$ where $(pr_1, s_1) = 1$. Here we have two possibilities again.

$$\left.\begin{aligned} z &= pr_1^2 + s_1^2 \\ x &= pr_1^2 - s_1^2 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} z &= pr_1^2 + s_1^2 \\ x &= s_1^2 - pr_1^2 \end{aligned}\right\} \quad \text{and} \quad y = 2r_1 s_1 \quad \text{in both cases} \tag{6}$$

We now that *x* and *z* are odd numbers which implies that r_1 and s_1 are different parities. We can write the solutions in a common form

$$\left.\begin{aligned} x &= | pr_1^2 - s_1^2 | \\ y &= 2r_1 s_1 \\ z &= pr_1^2 + s_1^2 \end{aligned}\right\} \quad \text{where} \quad (pr_1, s_1) = 1 \quad \text{and} \quad 2 \mid r_1 s_1 . \tag{7}$$

For example, if $p = 5$, $r_1 = 1$ and $s_1 = 2$ then the $\{x, y, z\}$ triple is $\{1, 4, 9\}$.

If $r_1 = 2$ and $s_1 = 1$ then $\{x, y, z\} = \{19, 4, 21\}$, and $19^2 + 5 \cdot 4^2 = 21^2$.

If $r_1 = 2$ and $s_1 = 3$ then $\{x, y, z\} = \{11, 12, 29\}$, and $11^2 + 5 \cdot 12^2 = 29^2$.

The case b)

Let us suppose that *x* is even and *y* and *z* are odd numbers.

$$py_1^2 = (z + x) \cdot (z - x) \tag{8}$$

Let $d = (z + x, z - x)$ where *d* is odd number. We get that $d \mid 2z \Rightarrow d \mid z$.
Moreover $d \mid 2x \Rightarrow d \mid x$ which implies $d \mid (x, z) \Rightarrow d = 1$.

Similar argument then in case a) shows that $py^2 = pr_2^2 \cdot s_2^2$ where $(pr_2, s_2) = 1$ and both of them are odd.

$$\left.\begin{aligned} x &= \frac{| pr_2^2 - s_2^2 |}{2} \\ y &= 2r_2 s_2 \\ z &= \frac{pr_1^2 + s_1^2}{2} \end{aligned}\right\} \quad \text{where} \quad (pr_2, s_2) = 1 \quad \text{and} \quad 2 \nmid r_2 s_2 . \tag{9}$$

For example, if $p=5$, $r_2=1$ and $s_2=2$ then the $\{x,y,z\}$ triple is $\{2,1,3\}$.

If $r_2=3$ and $s_2=1$ then the $\{x,y,z\}=\{22,3,23\}$, when $22^2+5\cdot 3^2=23^2$.

If $r_2=1$ and $s_2=3$ then the $\{x,y,z\}=\{2,3,7\}$, when $2^2+5\cdot 3^2=7^2$.

References

[1] *The American Mathematical Monthly*, Vol. 26, No. 1. (Jan., 1919), pp. 1-8.

[2] Sigler, L. E. *The book of squares* an annotated translation into modern English (original Fibonacci, Leonardo *Liber quadratorum*). Boston: Academic Press, 1987

[3] Niven, I. – Zuckerman, H. S. *An Introduction to the Theory of Numbers.* New York: John Wiley & Sons, Inc. 1966.