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Dispersive homogenized models and coefficient formulas for waves in general periodic media

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Abstract

We analyze a homogenization limit for the linear wave equation of second order. The spatial operator is assumed to be of divergence form with an oscillatory coefficient matrix a^ε that is periodic with characteristic length scale ε; no spatial symmetry properties are imposed. Classical homogenization theory allows to describe solutions u^ε well by a non-dispersive wave equation on fixed time intervals $(0, T)$. Instead, when larger time intervals are considered, dispersive effects are observed. In this contribution we present a well-posed weakly dispersive equation with homogeneous coefficients such that its solutions w^ε describe u^ε well on time intervals $(0, T\varepsilon^{-2})$. More precisely, we provide a norm and uniform error estimates of the form $\|u^\varepsilon(t) - w^\varepsilon(t)\| \le C\varepsilon$ for $t \in (0, T\varepsilon^{-2})$. They are accompanied by computable formulas for all coefficients in the effective models. We additionally provide an ε-independent equation of third order that describes dispersion along rays and we present numerical examples.

Keywords: wave equation, large time homogenization, dispersive model, Bloch analysis

MSC: 35B27, 35L05

1 Introduction

Waves in heterogeneous media exhibit dispersion. This fact is well-known in physics and it can be observed also for waves that are described (microscopically) by the classical, non-dispersive wave equation. Our aim in this contribution is to cast the effect in mathematical terms, to present a well-posed, dispersive effective wave equation, and to provide computable formulas for the (homogeneous) coefficients in the effective equation.

Our analysis concerns solutions $u^\varepsilon : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$, $n \in \{1, 2, 3\}$, of the linear wave equation in periodic media,

$$\partial_t^2 u^\varepsilon(x, t) = \nabla \cdot (a^\varepsilon(x) \nabla u^\varepsilon(x, t)) . \tag{1.1}$$

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The medium is characterized by a positive, symmetric coefficient matrix field $a^\varepsilon : \mathbb{R}^n \to \mathbb{R}^{n\times n}$. We are interested in periodic media with a small periodicity length-scale $\varepsilon > 0$, and assume that $a^\varepsilon(x) = a_Y(x/\varepsilon)$ where $a_Y : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is periodic with the periodicity of the unit cell $Y = (-\pi,\pi)^n$. Except for positivity, matrix symmetry, and periodicity, no assumptions on $a_Y(.)$ are made (in contrast to our earlier paper [12], where certain spatial symmetries are exploited). Our interest is to describe the solutions u^ε for large times, $t \sim \varepsilon^{-2}$. For classical homogenization results (derivation of effective equations on fixed time intervals) we refer to [6, 17] and mention here that, due to energy conservation, even classical homogenization results for the wave equation are much more involved than corresponding results e.g. for the heat equation (the "intermediate case", the wave equation with damping is considered in [19]). To simplify the exposition, we work here with smooth coefficients a^ε, noting that the regularity of the coefficient is crucial in observability results, see [7].

In order to have a well-defined object u^ε, we must complement the wave equation with an initial condition. For notational convenience, we restrict our analysis to a vanishing initial velocity, i.e. to initial data

$$u^\varepsilon(x,0) = f(x), \quad \partial_t u^\varepsilon(x,0) = 0\,. \tag{1.2}$$

In our mathematical results, we will assume smoothness of f. More precisely, we assume that $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has the Fourier representation

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F_0(k)\, e^{+\mathrm{i}k\cdot x}\, dk\,, \tag{1.3}$$

where $F_0 : \mathbb{R}^n \to \mathbb{C}$ has compact support $K \subset \mathbb{R}^n$.

We note that our assumptions imply the smoothness $f \in C^\infty(\mathbb{R}^n)$. Less regular initial data can be treated with the help of our results, exploiting the linearity of the equations: Decomposing initial data with bounded energy into two parts, our results can be applied to the smooth part, while the other part generates an error that is, for all times, small in energy norm.

Known results on dispersive models

The contribution [16] started a series of articles [13, 14, 15, 16] which is concerned with the derivation of dispersive models for the wave equation. The authors perform asymptotic (two-scale) expansions of u^ε in ε and obtain with their formal calculations a fourth order equation of the form

$$\partial_t^2 U^\varepsilon = AD^2 U^\varepsilon - \varepsilon^2 C D^4 U^\varepsilon\,, \tag{1.4}$$

where A and C are homogeneous coefficients and D denotes spatial derivatives. They call this equation "bad Boussinesq equation", a well-chosen name, considering the fact that the equation is ill-posed (in the homogenization process, a positive matrix A and a non-positive tensor C appear). We note that in the earlier article [21] this equation also appears (with a sign typo) as a result of a Bloch analysis, but it is not further analyzed in [21]. In [13, 14, 15, 16] various approaches for a further (analytical and numerical) exploitation of equation (1.4) are investigated: regularizations, non-local approximations, and multiple time scales.

The first rigorous result that establishes a dispersive model for the wave equation (1.1) appeared in [18]. In that work, which is concerned with the one-dimensional case, the well-posed dispersive equation (1.10) below is formulated and an error estimate similar to (2.1) is derived. The method of proof is very different from our approach here (which is as in [12]): Adaption operators are constructed and used to adapt smooth solutions of the homogeneous dispersive system to the periodic medium. After the adaption, direct energy procedures can be applied.

Another mathematical derivation of dispersive limits is performed in [4, 5]. The wave equation is scaled as in our setting (time scales of order ε^{-2} are investigated), but the initial data are assumed to be oscillatory at scale ε and are described by Bloch wave packets. In this setting, the effective diffraction can be described by a Schrödinger equation for the envelope function. Another scaling of the system is analyzed in [2], where large potentials instead of large time spans are considered.

Bloch analysis

The central tool in a Bloch analysis is the Bloch expansion of an arbitrary function (in our case the solution u^ε). While in a Fourier expansion one uses the dual variable $k \in \mathbb{R}^n$, the Bloch expansion uses two dual variables, k and m. Since $m \in \mathbb{N}_0$ is an additional parameter, the other parameter varies only in a restricted domain, the Brillouin zone, $k \in Z := (-1/2, 1/2)^n$. The basis functions $e^{ik\cdot x}$ of the Fourier analysis are replaced by solutions $\psi_m(.,k)$ of the Bloch eigenvalue problem

$$-(\nabla_y + ik)\cdot(a_Y(y)(\nabla_y + ik)\psi_m(y,k)) = \mu_m(k)\psi_m(y,k)\,. \tag{1.5}$$

Here $\psi_m(.,k) : Y \to \mathbb{C}$ is a periodic function, $\psi_m(.,k) \in H^1_{\mathrm{per}}(Y)$, $0 \le \mu_0(k) \le \mu_1(k) \le \ldots$ are the ordered, real eigenvalues.

Bloch wave homogenization theory establishes that the effective behavior of u^ε in the limit $\varepsilon \to 0$ is characterized solely by the behavior of the smallest eigenvalue $\mu_0(k)$ in a neighborhood of $k = 0 \in Z$. For such results in classical homogenization settings, we refer to [3, 8, 9, 10, 11]. In Fig. 1 we plot the Bloch wave functions $\psi_0(y,k)$ for $k = (1/2, 0)$ and $k = (1/2, 1/2)$. For an illustration of the eigenvalue structure see Fig. 2 (a).

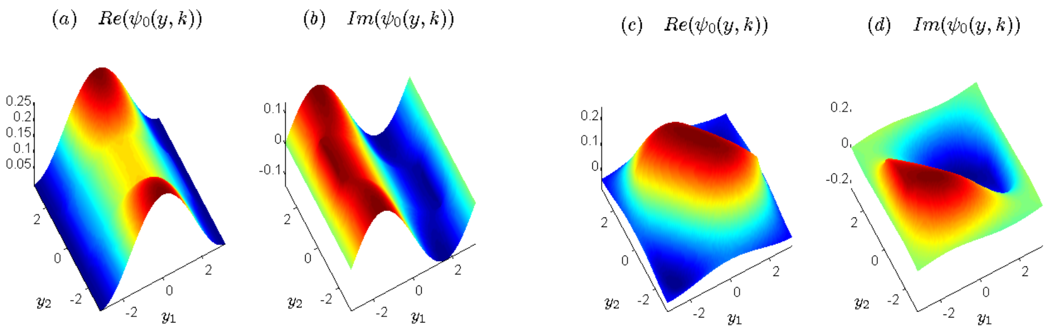

Figure 1: The Bloch wave $\psi_0(y,k)$ at $k = (1/2, 0)$ in (a) and (b) and at $k = (1/2, 1/2)$ in (c) and (d) corresponding to μ_0 for a_Y from (4.8).

The Bloch wave homogenization method was used for an analysis of higher order effects of the heterogeneity of the medium in the influential article [21]. That article does not formulate a well-posed dispersive effective equation (hence, in particular, it does not provide an error estimate), but it gives a lot of insight into the dispersive limit: Even the effective *long time* behavior of u^ε is characterized by $\mu_0(k)$ and its behavior near $k = 0$. Expanding $\mu_0(k)$ in a Taylor series around $k = 0$, we may write

$$\mu_0(k) = \sum A_{lm} k_l k_m + \sum C_{lmnq} k_l k_m k_n k_q + O(|k|^6)\,, \tag{1.6}$$

where odd derivatives vanish due to the symmetry $\mu_0(k) = \mu_0(-k)$, sums are over repeated indices. The matrix A and the tensor C provide the coefficients in the formal equation (1.4), where CD^4 is the spatial fourth order operator

$$CD^4 = \sum C_{ijkl} \partial_i \partial_j \partial_k \partial_l\,. \tag{1.7}$$

While A is positive definite and symmetric, C turns out to be negative semi-definite, a fact that is shown in [10]. As a consequence, the differential operator AD^2 is negative and the operator $-\varepsilon^2 CD^4$ is non-negative. For this reason, equation (1.4) is ill-posed.

Let us be more precise about the arguments in the Bloch analysis: We start with the Bloch expansion of the solution u^ε. Using the coefficients $\hat{f}^\varepsilon_m(k)$ of a Bloch expansion of the initial values f and the Bloch eigenfunctions $\tilde{\psi}_m$ that have $L^2(Y)$-norm 1 we may write

$$u^\varepsilon(x,t) = \sum_{m=0}^{\infty} \int_{Z/\varepsilon} \hat{f}^\varepsilon_m(k) \tilde{\psi}_m(x/\varepsilon, \varepsilon k) e^{\mathrm{i}k\cdot x}\, \mathrm{Re}\left(e^{\mathrm{i}t\sqrt{\mu_m(\varepsilon k)}/\varepsilon}\right) dk\,. \tag{1.8}$$

For a justification, see Lemma 2.1 of [12]. In the next steps, this formula is simplified for small $\varepsilon > 0$: One realizes, to leading order in ε, that only $m = 0$ has to be considered, that $\hat{f}^\varepsilon_m$ can be replaced by the Fourier transform F_0 of the initial values, and that $\tilde{\psi}_0$ can be replaced by the constant $(2\pi)^{-n/2}$. Expanding finally $\mu_0(\varepsilon k)$ in ε, one finds the following expression, which can be used to define an approximate solution v^ε.

$$\begin{aligned} v^\varepsilon(x,t) := (2\pi)^{-n/2} \frac{1}{2} \sum_{\pm} \int_K F_0(k) e^{\mathrm{i}k\cdot x} \exp\left(\pm \mathrm{i}t\sqrt{\sum A_{lm} k_l k_m}\right) \\ \times \exp\left(\pm \frac{\mathrm{i}\varepsilon^2}{2} t \frac{\sum C_{lmnq} k_l k_m k_n k_q}{\sqrt{\sum A_{lm} k_l k_m}}\right) dk\,. \end{aligned} \tag{1.9}$$

The equation (1.4) with tensors A and C is constructed in such a way that (formally) the function v^ε is a solution up to errors of order ε^4.

Rigorous approximation results

A rigorous mathematical analysis can be performed when the equation (1.4) is transformed into a well-posed equation, using the replacement $AD^2 U^\varepsilon \approx \partial_t^2 U^\varepsilon$ to re-write the operator CD^4. The first utilization of this trick for a rigorous result seems to be in the treatment [18] in the one-dimensional case. More recently, we were able to exploit the same trick in arbitrary dimension in [12] (under quite strong spatial symmetry assumptions on the coefficient field $a_Y(.)$).

In that contribution, we use a rigorous Bloch wave analysis to show that v^ε of (1.9) approximates u^ε in appropriate norms. In a second step, we show with energy methods that v^ε is close to the solution w^ε of the well-posed, weakly dispersive equation

$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon \,. \tag{1.10}$$

The positive semi-definite and symmetric tensors E and F are constructed in such a way that

$$-CD^4 = ED^2 AD^2 - FD^4 \,. \tag{1.11}$$

Together, the two estimates provide an estimate for $u^\varepsilon - w^\varepsilon$. This shows that the weakly dispersive equation (1.10) is a valid replacement for the original equation (1.1) on large time intervals.

New results

In the article at hand, we obtain the long-time homogenization result for very general coefficient fields $a_Y(.)$. In particular, we show that the well-posed equation (1.10) provides the effective description of solutions for large times, characterizing dispersion in arbitrary dimension and without spatial symmetry assumptions. Furthermore, we provide explicit formulas for the effective coefficients.

Decomposition lemma and approximation result. In order to show the approximation result, we can rely on the Bloch wave analysis of [12]. The only new ingredient is a considerably developed decomposition lemma: Lemma 2.5 below yields that, without any structural assumptions on C, the differential operator CD^4 can be written as in (1.11) for appropriate semi-definite and symmetric tensors E and F (using the given positive, symmetric matrix A). In particular, the lemma allows to decompose the operator CD^4 also when the coefficients $a_Y(.)$ have no spatial symmetries.

Once the decomposition lemma is established, we can apply results of [12]. We obtain that (1.10) is well-posed and that solutions w^ε approximate the solutions u^ε. More precisely, from the analysis of [12], we obtain an error estimate of the form $\|u^\varepsilon(t) - w^\varepsilon(t)\| \leq C_0\varepsilon$ uniformly in $t \in (0, T\varepsilon^{-2})$ in general periodic media. This approximation result is stated and proved in Section 2.

An ε-independent third order dispersive equation. Our aim in Section 3 is to provide a simplified model in which no ε-dependence occurs. The approximation result of Theorem 2.2 below allows to analyze, instead of the solution u^ε of the original problem, the solution w^ε of the weakly dispersive equation (1.10). In Section 3 we analyze w^ε in two dimensions in polar coordinates. On every ray through the origin, for an appropriate scaling of the solution, we can perform the limit $\varepsilon \to 0$. The result is an equation that determines the shape of pulses, namely a linear third order equation (a linearized KdV-equation). All coefficients in this equation are computable from the coefficient field a_Y.

Algorithms to compute effective quantities. In Section 4 we present an algorithm that provides the homogeneous coefficients in all effective equations, i.e.

A and C (in a more direct form than as derivatives of the Bloch eigenvalue), the coefficients E and F, and the coefficients of the linearized KdV-equation.

The numerical results of Section 4 compare solutions to the original problem with solutions to the effective problems. We find a remarkable qualitative and quantitative agreement also for moderate ε, and the correct experimental convergence rates for $\varepsilon \to 0$.

2 A weakly dispersive effective equation

In Section 2.1 we formulate our main result. The main part of its proof can be obtained by applying results of [12]; this is the subject of Subsection 2.2. The new ingredient is the decomposition lemma, which is shown in Subsection 2.3.

2.1 Main approximation result

We emphasize that the set $Y \subset \mathbb{R}^n$, the reciprocal cell $Z := (-1/2, 1/2)^n \subset \mathbb{R}^n$, and the support $K \subset \mathbb{R}^n$ are fixed data of the problem. Given is also the coefficient field a_Y that determines $a^\varepsilon(x) = a_Y(x/\varepsilon)$.

Assumption 2.1. *The coefficient field $a_Y : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is Y-periodic for the cube $Y := (-\pi, \pi)^n \subset \mathbb{R}^n$ and has the regularity $a_Y \in C^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$. The matrix $a_Y(y)$ is symmetric for every point $y \in \mathbb{R}^n$, i.e. $(a_Y(y))_{ij} = (a_Y(y))_{ji}$ for all $i, j \in \{1, ..., n\}$. The field is positive definite: for some $\gamma > 0$ there holds $\sum_{i,j=1}^n (a_Y(y))_{ij}\xi_i\xi_j \geq \gamma|\xi|^2$ for every $y \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$.*

Our main approximation result is stated in the following theorem. The result is very similar to the main theorem in [12]. The difference is that we do not assume any spatial symmetry of $a_Y(.)$ (such as a reflection symmetry in each coordinate direction or symmetry with respect to exchanging coordinate axes). We note that dimensions $n > 3$ can be treated by assuming higher regularity properties.

Theorem 2.2 (Approximation). *Let $\varepsilon = \varepsilon_l \to 0$ be a sequence of positive numbers and $n \in \{1, 2, 3\}$ be the dimension. Let the medium $a_Y : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy Assumption 2.1 and let the initial data $f : \mathbb{R}^n \to \mathbb{R}$ be as in* (1.3)*. We use the coefficient matrices A and C defined in* (1.6)*. Let E and F be positive semi-definite such that* (1.11) *holds (the existence is established in Lemma 2.5 below). Then the following holds:*

1. **Well-posedness** *Equation* (1.10) *with initial condition* (1.2) *has a unique solution w^ε for all positive times (see Theorem 2.4 below for function spaces).*

2. **Error estimate** *Let w^ε be the solution of* (1.10)*, and let u^ε be the solution of* (1.1) *for the same initial condition* (1.2)*. Then, with a constant $C_0 = C_0(a_Y, T_0, f)$, there holds the error estimate*

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|u^\varepsilon(., t) - w^\varepsilon(., t)\|_{L^2(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \leq C_0\varepsilon\,. \tag{2.1}$$

Here we use, for two Banach spaces with norms $\|.\|_X$ and $\|.\|_Y$, the weaker norm $\|u\|_{X+Y} := \inf\{\|u_1\|_X + \|u_2\|_Y : u = u_1 + u_2\}$. Such a norm appears in our main theorem 2.2, since different contributions to the error $u^\varepsilon - w^\varepsilon$ are measured in different norms.

2.2 Proof of Theorem 2.2

The following corollary is the central result of the Bloch analysis. It is derived with mathematical rigor in [12]; it provides a comparison between the solution u^ε of the heterogeneous wave equation with the explicitely defined function v^ε.

Theorem 2.3 (Corollary 2.5 of [12]). *Let Assumption 2.1 be satisfied. Let u^ε be the solution of* (1.1) *and let v^ε be defined by* (1.9)*. Then*

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|u^\varepsilon(.,t) - v^\varepsilon(.,t)\|_{L^2(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \le C_0\varepsilon. \tag{2.2}$$

The following theorem is based on energy methods. It provides the comparison between the solution w^ε of the weakly dispersive (homogeneous) equation and the explicit function v^ε.

Theorem 2.4 (Theorem 3.3 of [12]). *Let A, C, E, F be tensors with the properties: $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, $\sum_{ij} A_{ij}\xi_i\xi_j \ge \gamma|\xi|^2$ for some $\gamma > 0$, $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$ are positive semi-definite and symmetric, $C \in \mathbb{R}^{n\times n\times n\times n}$ allows the decomposition* (1.11)*. Then the following holds.*

1. Well-posedness. *For the initial datum $f \in H^2(\mathbb{R}^n)$, the equation*

$$\begin{aligned} &\partial_t^2 w^\varepsilon - AD^2 w^\varepsilon - \varepsilon^2\partial_t^2 ED^2 w^\varepsilon + \varepsilon^2 FD^4 w^\varepsilon = 0\,, \\ &w^\varepsilon(.,0) = f, \qquad \partial_t w^\varepsilon(.,0) = 0 \end{aligned} \tag{2.3}$$

has a unique solution $w^\varepsilon \in L^\infty(0, T_0\varepsilon^{-2}; H^2(\mathbb{R}^n)) \cap W^{1,\infty}(0, T_0\varepsilon^{-2}; H^1(\mathbb{R}^n))$.

2. Approximation. *Let v^ε be defined by* (1.9) *where F_0 and f are related by* (1.3)*. Let w^ε be a solution of* (2.3)*. Then there holds*

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|\partial_t(v^\varepsilon - w^\varepsilon)(.,t)\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,T_0\varepsilon^{-2}]} \|\nabla(v^\varepsilon - w^\varepsilon)(.,t)\|_{L^2(\mathbb{R}^n)} \le C_0\varepsilon^2 \tag{2.4}$$

with a constant C_0 that is independent of ε.

Theorem 2.2 is a consequence of the above results and a new decomposition lemma that is shown in the next subsection.

The estimate (2.2) provides that $\|u^\varepsilon - v^\varepsilon\|$ is of order ε. The norms coincide with the ones in the claim (2.1). It therefore remains to estimate the difference $\|v^\varepsilon - w^\varepsilon\|$.

We define A and C through (1.6) and note that A is positive definite and symmetric. The decomposition result of Lemma 2.5 below allows to construct E and F such that (1.11) is satisfied. This means that Theorem 2.4 can be applied. It provides the well-posedness claim and the estimate (2.4), which shows that norms of derivatives of $v^\varepsilon - w^\varepsilon$ are of order ε^2. The norms can be transformed with an interpolation lemma (see Lemma 3.4 in [12]); the result is an estimate for $\sup_{t\in[0,T_0\varepsilon^{-2}]} \|v^\varepsilon(.,t) - w^\varepsilon(.,t)\|_{L^2+L^\infty}$ of order ε. We therefore obtain (2.1).

2.3 Decomposition lemma

Our aim is to construct coefficient tensors $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$ such that the differential operator CD^4 can be re-written as in (1.11). We impose that E and F are positive semi-definite and symmetric, i.e.

$$\sum_{i,j,k,l=1}^{n} F_{ijkl}\xi_{ij}\xi_{kl} \geq 0 \quad \text{for every } \xi \in \mathbb{R}^{n\times n}\,, \qquad F_{ijkl} = F_{klij}\,, \tag{2.5}$$

and similarly $\sum_{i,j=1}^{n} E_{ij}\eta_i\eta_j \geq 0$ for every $\eta \in \mathbb{R}^n$ and $E_{ij} = E_{ji}$. The symmetry relations must hold for all indices $i,j,k,l \in \{1,...,n\}$. In this section, $n \in \{1,2,3,4,\dots\}$ can also be larger than 3.

Lemma 2.5 (Decomposability)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix and let $C \in \mathbb{R}^{n\times n\times n\times n}$ be arbitrary. Then there exist symmetric and positive semi-definite tensors $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$ such that the differential operator CD^4 can be written as in* (1.11).

The proof of the above lemma consists of two steps. In the first and essential step we show that the decomposability result holds for diagonal matrices $A \in \mathbb{R}^{n\times n}$ and with rotated derivative operators. In the second step, general matrices A are treated by diagonalization.

Lemma 2.6 (Decomposability for diagonal matrices A)**.** *Let A be a positive definite diagonal matrix, $A = \mathrm{diag}(a_1, a_2, ..., a_n)$. Let $S \in SO(n)$ be an orthogonal matrix and let $C \in \mathbb{R}^{n\times n\times n\times n}$ be arbitrary. Then there exist symmetric and positive semi-definite tensors $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$ such that*

$$-C\tilde{D}^4 = E\tilde{D}^2A\tilde{D}^2 - F\tilde{D}^4\,. \tag{2.6}$$

Here, the operator $\tilde{D} := SD$ denotes a rotated derivative.

Proof of Lemma 2.6. Step 1: Reduction to matrices C with only one non-trivial entry. We note that the relation (2.6) is additive in the following sense: Let $C^{(1)}$ and $C^{(2)}$ be two matrices, and let (2.6) be satisfied for $C^{(m)}$ with tensors $E^{(m)}$ and $F^{(m)}$, $m \in \{1,2\}$. Then (2.6) holds for $C^{(1)} + C^{(2)}$ with the two tensors $E^{(1)} + E^{(2)}$ and $F^{(1)} + F^{(2)}$. We exploit here that the sum of symmetric, semi-definite tensors is again symmetric and semi-definite.

This observation implies that it is sufficient to consider a tensor C that has only one non-trivial entry. We denote a tensor with only one entry 1 in the canonical way as $e_\alpha \otimes e_\beta \otimes e_\gamma \otimes e_\delta$. An arbitrary tensor C can be written as a sum, $C = \sum_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}\, e_\alpha \otimes e_\beta \otimes e_\gamma \otimes e_\delta$. After constructing tensors $E^{(\alpha,\beta,\gamma,\delta)}$ and $F^{(\alpha,\beta,\gamma,\delta)}$ according to the one-entry tensor $C_{\alpha\beta\gamma\delta}\, e_\alpha \otimes e_\beta \otimes e_\gamma \otimes e_\delta$, we find E and F according to C by a summation, $E = \sum_{\alpha\beta\gamma\delta} E^{(\alpha,\beta,\gamma,\delta)}$ and $F = \sum_{\alpha\beta\gamma\delta} F^{(\alpha,\beta,\gamma,\delta)}$.

In the following construction, we restrict ourselves to a fixed choice of indices, $(\alpha,\beta,\gamma,\delta) \in \{1,...,n\}^4$. For a number $c \in \mathbb{R}$, we can consider a tensor C of the form $C = c\, e_\alpha \otimes e_\beta \otimes e_\gamma \otimes e_\delta$.

Our aim is to re-write the differential operator $-C\tilde{D}^4 = c\,\tilde{D}_\alpha\tilde{D}_\beta\tilde{D}_\gamma\tilde{D}_\delta$. We use here $\tilde{D}_i := \sum_{j=1}^{n} S_{ij}\partial_{x_j}$ for the i-th component of the rotated gradient. In the following, $\{a\}_+ := \max\{0, a\}$ denotes the positive part of a number $a \in \mathbb{R}$.

Step 2: Construction of E and F for $C = c\, e_\alpha \otimes e_\beta \otimes e_\gamma \otimes e_\delta$, where at least two indices coincide.

Case 1. The indices $\alpha, \beta, \gamma, \delta$ contain two different pairs, i.e. $(\alpha, \beta, \gamma, \delta) = (i, i, j, j)$ or $(\alpha, \beta, \gamma, \delta) = (i, j, i, j)$ or $(\alpha, \beta, \gamma, \delta) = (i, j, j, i)$ for $i, j \in \{1, ..., n\}$. We restrict ourselves to $(\alpha, \beta, \gamma, \delta) = (i, i, j, j)$, the permutations define the same operator $C\tilde{D}^4$. We define the tensors $E = E^{(\alpha,\beta,\gamma,\delta)}$ and $F = F^{(\alpha,\beta,\gamma,\delta)}$ through

$$E_{ii} := \frac{\{-c\}_+}{a_j}\,, \qquad F_{ijij} := \{c\}_+\,, \qquad F_{imim} := \frac{\{-c\}_+}{a_j} a_m\,,$$

for all $m \in \{1, ..., n\}$ with $m \neq j$. All other entries of E and F are set to zero.

Properties of E and F. By definition E and F are symmetric and positive semi-definite. A direct calculation yields the decomposition property:

$$\begin{aligned}
&E\tilde{D}^2 A\tilde{D}^2 - F\tilde{D}^4 \\
&\quad = \left(\frac{\{-c\}_+}{a_j}\tilde{D}_i^2\right)\left(\sum_m a_m \tilde{D}_m^2\right) - \{c\}_+\tilde{D}_i^2\tilde{D}_j^2 - \sum_{m\neq j}\frac{\{-c\}_+}{a_j} a_m \tilde{D}_i^2\tilde{D}_m^2 \\
&\quad = \{-c\}_+\tilde{D}_i^2\tilde{D}_j^2 + \sum_{m\neq j}\frac{\{-c\}_+}{a_j} a_m \tilde{D}_i^2\tilde{D}_m^2 - \{c\}_+\tilde{D}_i^2\tilde{D}_j^2 - \sum_{m\neq j}\frac{\{-c\}_+}{a_j} a_m \tilde{D}_i^2\tilde{D}_m^2 \\
&\quad = \left(\{-c\}_+ - \{c\}_+\right)\tilde{D}_i^2\tilde{D}_j^2 = -c\tilde{D}_i^2\tilde{D}_j^2 = -C\tilde{D}^4\,.
\end{aligned}$$

Case 2. The indices $\alpha, \beta, \gamma, \delta$ contain three identical entries, i.e. $(\alpha, \beta, \gamma, \delta) = (i, i, i, j)$ or $(\alpha, \beta, \gamma, \delta) = (i, i, j, i)$ or $(\alpha, \beta, \gamma, \delta) = (i, j, i, i)$ or $(\alpha, \beta, \gamma, \delta) = (j, i, i, i)$ for $i, j \in \{1, ..., n\}$ with $i \neq j$. We restrict ourselves to $(\alpha, \beta, \gamma, \delta) = (i, i, i, j)$ since the other cases define the same differential operator. We define the tensors $E = E^{(\alpha,\beta,\gamma,\delta)}$ and $F = F^{(\alpha,\beta,\gamma,\delta)}$ through

$$\begin{aligned}
&E_{ij} := E_{ji} := -\frac{c}{2a_i} =: \tilde{c}\,, \quad E_{ii} := E_{jj} := |\tilde{c}| \\
&F_{imim} := F_{jmjm} := |\tilde{c}|\, a_m \qquad \text{for all } m \in \{1, ..., n\} \\
&F_{imjm} := F_{jmim} := \tilde{c} a_m \qquad \text{for all } m \in \{1, ..., n\} \text{ with } m \neq i.
\end{aligned}$$

All other entries of E and F are set to zero.

Properties of E and F. By definition, E and F are symmetric. Concerning the positive semi-definiteness of E and F we calculate, for arbitrary $\xi \in \mathbb{R}^n$ and $\zeta \in R^{n\times n}$,

$$\begin{aligned}
&\sum_{l,m} E_{lm}\xi_l\xi_m = |\tilde{c}|\xi_i^2 + |\tilde{c}|\xi_j^2 + 2\tilde{c}\xi_i\xi_j \geq |\tilde{c}|\xi_i^2 + |\tilde{c}|\xi_j^2 - \left(|\tilde{c}|\xi_i^2 + |\tilde{c}|\xi_j^2\right) = 0, \\
&\sum_{l,m,p,q} F_{lmpq}\zeta_{lm}\zeta_{pq} = \sum_m |\tilde{c}| a_m \zeta_{im}^2 + \sum_m |\tilde{c}| a_m \zeta_{jm}^2 + \sum_{m\neq i} 2\tilde{c} a_m \zeta_{im}\zeta_{jm} \\
&\qquad \geq \sum_m |\tilde{c}| a_m \zeta_{im}^2 + \sum_m |\tilde{c}| a_m \zeta_{jm}^2 - \sum_{m\neq i}(|\tilde{c}| a_m \zeta_{im}^2 + |\tilde{c}| a_m \zeta_{jm}^2) \geq 0\,.
\end{aligned}$$

Concerning the decomposition property we calculate

$$
\begin{aligned}
&E\tilde D^2 A\tilde D^2 - F\tilde D^4\\
&\quad= \left(|\tilde c|\tilde D_i^2 + |\tilde c|\tilde D_j^2 - \frac{c}{a_i}\tilde D_i\tilde D_j\right)\left(\sum_m a_m\tilde D_m^2\right)\\
&\qquad - \sum_m |\tilde c|a_m(\tilde D_i^2\tilde D_m^2 + \tilde D_j^2\tilde D_m^2) + \sum_{m\neq i} a_m\frac{c}{a_i}\tilde D_i\tilde D_j\tilde D_m^2\\
&\quad= \sum_m |\tilde c|a_m(\tilde D_i^2\tilde D_m^2 + \tilde D_j^2\tilde D_m^2) - c\tilde D_i^3\tilde D_j - \sum_{m\neq i} a_m\frac{c}{a_i}\tilde D_i\tilde D_j\tilde D_m^2\\
&\qquad - \sum_m |\tilde c|a_m(\tilde D_i^2\tilde D_m^2 + \tilde D_j^2\tilde D_m^2) + \sum_{m\neq i} a_m\frac{c}{a_i}\tilde D_i\tilde D_j\tilde D_m^2 = -c\tilde D_i^3\tilde D_j = -C\tilde D^4\ .
\end{aligned}
$$

Case 3. The indices $\alpha,\beta,\gamma,\delta$ contain two identical entries, the other entries are different. We restrict ourselves to the case $(\alpha,\beta,\gamma,\delta) = (i,i,j,k)$ with three different indices $i,j,k\in\{1,...,n\}$, permutations of the indices define the same operator. We define the tensors $E = E^{(\alpha,\beta,\gamma,\delta)}$ and $F = F^{(\alpha,\beta,\gamma,\delta)}$ through

$$
\begin{aligned}
E_{jk} := E_{kj} := -\frac{c}{2a_i} =: \tilde c,&\quad E_{kk} := E_{jj} := |\tilde c|\\
F_{kmkm} := F_{jmjm} := |\tilde c|a_m,&\quad \text{for all } m\in\{1,...,n\}\\
F_{kmjm} := F_{jmkm} := \tilde c a_m&\quad \text{for all } m\in\{1,...,n\} \text{ with } m\neq i.
\end{aligned}
$$

All other entries of E and F are set to zero.

Properties of E and F. By definition, E and F are symmetric. Concerning the positive semi-definiteness of E and F, we calculate for arbitrary $\xi\in\mathbb{R}^n$ and $\zeta\in R^{n\times n}$

$$
\begin{aligned}
&\sum_{l,m} E_{lm}\xi_l\xi_m = |\tilde c|\xi_k^2 + |\tilde c|\xi_j^2 + 2\tilde c\xi_k\xi_j \geq 0\,,\\
&\sum_{l,m,p,q} F_{lmpq}\zeta_{lm}\zeta_{pq} = \sum_m |\tilde c|a_m\zeta_{km}^2 + \sum_m |\tilde c|a_m\zeta_{jm}^2 + \sum_{m\neq i} 2\tilde c a_m\zeta_{km}\zeta_{jm} \geq 0\,.
\end{aligned}
$$

Regarding the decomposition property we calculate

$$
\begin{aligned}
&E\tilde D^2 A\tilde D^2 - F\tilde D^4\\
&\quad= \left(|\tilde c|\tilde D_k^2 + |\tilde c|\tilde D_j^2 - \frac{c}{a_i}\tilde D_k\tilde D_j\right)\left(\sum_m a_m\tilde D_m^2\right)\\
&\qquad - \sum_m |\tilde c|a_m(\tilde D_k^2\tilde D_m^2 + \tilde D_j^2\tilde D_m^2) + \sum_{m\neq i} a_m\frac{c}{a_i}\tilde D_k\tilde D_j\tilde D_m^2\\
&\quad= \sum_m |\tilde c|a_m(\tilde D_k^2\tilde D_m^2 + \tilde D_j^2\tilde D_m^2) - c\tilde D_k\tilde D_j\tilde D_i^2 - \sum_{m\neq i} a_m\frac{c}{a_i}\tilde D_k\tilde D_j\tilde D_m^2\\
&\qquad - \sum_m |\tilde c|a_m(\tilde D_k^2\tilde D_m^2 + \tilde D_j^2\tilde D_m^2) + \sum_{m\neq i} a_m\frac{c}{a_i}\tilde D_k\tilde D_j\tilde D_m^2 = -c\tilde D_k\tilde D_j\tilde D_i^2 = -C\tilde D^4\ .
\end{aligned}
$$

Step 3: Construction of E and F for $C = c\,e_\alpha\otimes e_\beta\otimes e_\gamma\otimes e_\delta$, where no two indices coincide. We treat now the case $(\alpha,\beta,\gamma,\delta) = (i,j,k,l)$ with pairwise different indices $i, j, k, l \in \{1, ..., n\}$; our aim is to rewrite the operator $C\tilde{D}^4 = c\tilde{D}_i\tilde{D}_j\tilde{D}_k\tilde{D}_l$. We obtain corresponding matrices in two steps: (a) We define a positive semi-definite, symmetric tensor $\hat{F} \in \mathbb{R}^{n\times n\times n\times n}$ in such a way that $\bar{C} := C - \hat{F}$ has only non-trivial entries at positions with repeated indices. (b) We apply Step 2 of this proof to the remainder $\bar{C}$, which provides $\bar{E}$ and $\bar{F}$ with $-\bar{C}\tilde{D}^4 = \bar{E}\tilde{D}^2A\tilde{D}^2 - \bar{F}\tilde{D}^4$. The desired tensors $E = E^{(\alpha,\beta,\gamma,\delta)}$ and $F = F^{(\alpha,\beta,\gamma,\delta)}$ are then obtained as $E := \bar{E}$ and $F := \bar{F} + \hat{F}$.

We set

$$\hat{F}_{ijkl} := \hat{F}_{klij} := \frac{1}{2}c\,, \qquad \hat{F}_{ijij} := \hat{F}_{klkl} := \frac{1}{2}|c|\,. \tag{2.7}$$

All other entries of $\hat{F}$ are set to zero. The symmetry of $\hat{F}$ is obvious; regarding positivity we calculate

$$\sum_{m,p,q,r} \hat{F}_{mpqr}\zeta_{mp}\zeta_{qr} = \frac{1}{2}|c|\zeta_{ij}^2 + \frac{1}{2}|c|\zeta_{kl}^2 + c\zeta_{ij}\zeta_{kl} \geq 0\,.$$

It remains to check that Step 2 of this proof can be applied to the remainder $\bar{C} := C - \hat{F}$. We evaluate

$$\begin{aligned} CD^4 - \hat{F}D^4 &= c\,\tilde{D}_i\tilde{D}_j\tilde{D}_k\tilde{D}_l - \left[c\,\tilde{D}_i\tilde{D}_j\tilde{D}_k\tilde{D}_l + \frac{1}{2}|c|\tilde{D}_i^2\tilde{D}_j^2 + \frac{1}{2}|c|\tilde{D}_k^2\tilde{D}_l^2\right] \\ &= -\frac{1}{2}|c|\left(\tilde{D}_i^2\tilde{D}_j^2 + \tilde{D}_k^2\tilde{D}_l^2\right). \end{aligned}$$

This operator has nontrivial entries only for repeated indices, it therefore posesses a decomposition by Step 2. This concludes the proof of the lemma. □

With Lemma 2.6 at hand we are in the position to prove the general decomposition result of Lemma 2.5.

Proof of Lemma 2.5. The symmetry of A implies that A is diagonalizable: there exists an orthogonal matrix $S \in SO(n)$ such that

$$A = S^T\tilde{A}S \quad \text{with} \quad \tilde{A} = \mathrm{diag}(a_1, a_2, ..., a_n).$$

Since A is positive definite, the eigenvalues a_i are positive. Our aim is to apply Lemma 2.6 with the diagonal matrix $\tilde{A}$ and a tensor $\tilde{C}$, which is defined from C with the transformation S. Lemma 2.6 provides tensors $\tilde{E}$ and $\tilde{F}$ that can be transformed back into the desired tensors E and F.

Step 1: Construction of $\tilde{C}$. Here, we denote the space of matrices by $M := \mathbb{R}^{n\times n}$. The tensor C defines a linear map $C : M \to M$ through $C(e_i\otimes e_j))_{kl} = C_{ijkl}$. We define a transformed tensor $\tilde{C} : M \to M$ through

$$\tilde{C} : M \ni B \mapsto S\cdot C(S^TBS)\cdot S^T \in M\,. \tag{2.8}$$

As we show next, with $\tilde{D} := SD$, the corresponding differential operators coincide, $CD^4 = \tilde{C}\tilde{D}^4$. We use the convention that sums are over repeated indices.

$$\begin{aligned} \tilde{C}\tilde{D}^4 &= \sum(\tilde{C}(e_i\otimes e_j))_{kl}\tilde{D}_i\tilde{D}_j\tilde{D}_k\tilde{D}_l \\ &= \sum S_{k\alpha}\,C(S^T\cdot e_i\otimes e_j\cdot S)_{\alpha\beta}S_{l\beta}\tilde{D}_i\tilde{D}_j\tilde{D}_k\tilde{D}_l \\ &= \sum S_{k\alpha}S_{i\gamma}S_{j\delta}\,C(e_\gamma\otimes e_\delta)_{\alpha\beta}S_{l\beta}\;S_{i\xi}D_\xi\,S_{j\zeta}D_\zeta\,S_{k\eta}D_\eta\,S_{l\theta}D_\theta \\ &= \sum C(e_\gamma\otimes e_\delta)_{\alpha\beta}D_\alpha D_\beta D_\gamma D_\delta = CD^4\,. \end{aligned} \tag{2.9}$$

Step 2: Application of Lemma 2.6. Since $\tilde{A}$ is diagonal, we can apply Lemma 2.6 with the data $\tilde{A}$, S, and $\tilde{C}$. We find symmetric, positive semi-definite tensors $\tilde{E} \in \mathbb{R}^{n\times n}$ and $\tilde{F} \in \mathbb{R}^{n\times n\times n\times n}$ such that, with $\tilde{D} := SD$,

$$-\tilde{C}\tilde{D}^4 = \tilde{E}\tilde{D}^2\tilde{A}\tilde{D}^2 - \tilde{F}\tilde{D}^4. \tag{2.10}$$

We can now define the desired tensors E and F through $E := S^T\tilde{E}S$ and

$$F : M \ni B \mapsto S^T \cdot \tilde{F}(SBS^T) \cdot S \in M\,. \tag{2.11}$$

Since $\tilde{F}$ is obtained from F by the same formula as $\tilde{C}$ is obtained from C, cf. (2.8), the calculation of (2.9) yields $\tilde{F}\tilde{D}^4 = FD^4$.

Regarding the operator $\tilde{E}\tilde{D}^2$ we calculate

$$\begin{aligned}\tilde{E}\tilde{D}^2 &= \sum (SES^T)_{ij}\tilde{D}_i\tilde{D}_j = \sum S_{i\alpha}E_{\alpha\beta}S_{j\beta}S_{ik}D_kS_{jl}D_l\\ &= \sum E_{kl}D_kD_l = ED^2\,.\end{aligned}$$

The calculation can also be applied to $A = S^T\tilde{A}S$ and provides $\tilde{A}\tilde{D}^2 = AD^2$. We conclude that relation (2.10) coincides with

$$-CD^4 = ED^2AD^2 - FD^4\,.$$

This is the decomposition result for the tensors C and A. We remark that, since $S \in SO(n)$ is an orthogonal matrix, the symmetry and positive semi-definiteness of $\tilde{E}$ and $\tilde{F}$ carry over to E and F. This concludes the proof of the general decomposition lemma. □

3 An ε-independent effective equation

The aim of this section is to carry the analysis of the weakly dispersive effective equation one step further. We will introduce moving frame coordinates which will allow us to follow the main pulse of w^ε along rays through the origin. Performing the limit $\varepsilon \to 0$ in distributional sense will provide an ε-independent linear third order equation (a linearized KdV-equation) that describes the effective shape of the pulse in dependence on the ray direction.

In the following we will restrict ourselves to the analysis of the two-dimensional case with even symmetry,

$$a_Y(y_1, y_2) = a_Y(-y_1, y_2) = a_Y(y_1, -y_2) \text{ for all } y \in \mathbb{R}^2. \tag{3.1}$$

The above symmetry assumption, which is in particular satisfied in the case of a laminated structure, guarantees that the effective coefficients A and C have the form (cf. proof of Lemma 2.6 in [12])

$$A = \operatorname{diag}(a_1, a_2), \tag{3.2}$$

$$C_{iiii} =: \alpha_i, \quad C_{ijij} = C_{ijji} = C_{iijj} =: \beta \text{ for } i, j \in \{1, 2\} \text{ with } i \neq j. \tag{3.3}$$

All other entries of C, that are not mentioned above, vanish.

We expect that the main pulse of w^ε, solution to the weakly dispersive equation, propagates with a direction-dependent speed according to the anisotropic matrix $A = \mathrm{diag}(a_1, a_2)$. We introduce appropriate elliptic coordinates (r, φ) through

$$(x_1, x_2) = (r\sqrt{a_1}\cos\varphi, r\sqrt{a_2}\sin\varphi). \tag{3.4}$$

The above coordinate transform is chosen in such a way that the main pulse of w^ε at time t is located along the ellipse that is given as the level set $\{x \in \mathbb{R}^2 | r = t\}$. In order to perform a fine analysis of the dynamics of the pulse, we rewrite w^ε as a function of (r, φ, t) and use a moving frame in the radial variable r. More precisely, given the solution w^ε, we define W^ε through

$$W^\varepsilon(r, \varphi, t) := \begin{cases} w^\varepsilon\left(r + \frac{t}{\varepsilon^2}, \varphi, \frac{t}{\varepsilon^2}\right) & \text{for } r > -t\varepsilon^{-2} \\ 0 & \text{for } r \le -t\varepsilon^{-2}\,. \end{cases} \tag{3.5}$$

The time scaling t/ε^2 accounts for the fact that the dispersive effects of w^ε are weak, i.e. slow in time. The main result of this section is the following. Provided that W^ε has a distributional limit W, then W is characterized by an ε-independent linearized KdV-eqation.

Proposition 3.1 (Effective behavior in the moving frame)**.** *Let the medium $a_Y : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ be evenly symmetric in the sense of* (3.1)*. Let $w^\varepsilon(r, \varphi, t)$ be the solution to the weakly dispersive wave equation* (2.3)*, expressed in elliptic coordinates. Let $W^\varepsilon(r, \varphi, t)$ be defined by* (3.5)*. Assume that W^ε has a limit in the sense of distributions,*

$$W^\varepsilon \rightharpoonup W \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R} \times (0, T)).$$

Then the distribution $U := \partial_r W$ satisfies the following linearized cylindrical Korteweg-de-Vries-equation (linearized KdV-equation) in distributional sense

$$\partial_t U + \frac{1}{2t} U - \frac{1}{2}\kappa(\varphi)\partial_r^3 U = 0. \tag{3.6}$$

Here, the dispersion coefficient κ is given by

$$\kappa(\varphi) := 6\beta\frac{\cos^2(\varphi)}{a_1}\frac{\sin^2(\varphi)}{a_2} + \alpha_1\frac{\cos^4(\varphi)}{a_1^2} + \alpha_2\frac{\sin^4(\varphi)}{a_2^2}. \tag{3.7}$$

The angle-dependent coefficient $\kappa(\varphi)$ can also be expressed with C,

$$\kappa(\varphi) = \sum_{ijkl} C_{ijkl}\xi_i\xi_j\xi_k\xi_l \quad \text{for} \quad \xi = \left(\frac{1}{\sqrt{a_1}}\cos(\varphi), \frac{1}{\sqrt{a_2}}\sin(\varphi)\right). \tag{3.8}$$

It can be understood as a measure for the amount of dispersion along the ray $R_\varphi := \{x = (r\sqrt{a_1}\cos\varphi, r\sqrt{a_2}\sin\varphi) \,|\, r \in (0,\infty)\}$. Note that, due to the negative semi-definiteness of C, the dispersion coefficient is always nonpositive: $\kappa(\varphi) \le 0$ for every angle φ.

In the proof of the above proposition we will need some elementary formulas, collected in the following remark.

Remark 3.2 (Derivatives in elliptic coordinates). *Let (r,φ) denote the elliptic coordinates defined in* (3.4)*. Then the following relationship between derivatives in Cartesian and elliptic coordinates holds:*

$$\partial_{x_1} = \frac{1}{\sqrt{a_1}}\left(\cos\varphi\,\partial_r - \frac{1}{r}\sin\varphi\,\partial_\varphi\right), \tag{3.9}$$

$$\partial_{x_2} = \frac{1}{\sqrt{a_2}}\left(\sin\varphi\,\partial_r + \frac{1}{r}\cos\varphi\,\partial_\varphi\right), \tag{3.10}$$

$$AD^2 = a_1\partial_{x_1}^2 + a_2\partial_{x_2}^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2\,. \tag{3.11}$$

For general derivatives of order $\gamma = \gamma_1+\gamma_2$ with $\gamma_1,\gamma_2\in\mathbb{N}_0$ one has

$$\begin{aligned}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2} &= \frac{1}{(\sqrt{a_1})^{\gamma_1}(\sqrt{a_2})^{\gamma_2}}P^{\gamma_1,\gamma_2}\left(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi,\tfrac{1}{r}\right)\\ &= \frac{1}{(\sqrt{a_1})^{\gamma_1}(\sqrt{a_2})^{\gamma_2}}\left(\cos^{\gamma_1}(\varphi)\sin^{\gamma_2}(\varphi)\partial_r^\gamma + \tilde{P}^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi,\tfrac{1}{r})\right),\end{aligned} \tag{3.12}$$

where P^{γ_1,γ_2} and $\tilde{P}^{\gamma_1,\gamma_2}$ are polynomials and $\tilde{P}^{\gamma_1,\gamma_2}$ can be written as

$$\tilde{P}^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi,\tfrac{1}{r}) = \sum_{k=1}^{\gamma} Q_k^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi)\frac{1}{r^k}\,,$$

where $Q_k^{\gamma_1,\gamma_2}$ is a polynomial in $\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi$ of degree at most 2γ.

Equations (3.9)–(3.11) are elementary identities for elliptic coordinates; (3.12) follows easily by an induction argument. In the decomposition of P^{γ_1,γ_2}, we wrote the term of highest order in ∂_r explicitely, with the result that all other terms (collected in $\tilde{P}^{\gamma_1,\gamma_2}$) contain non-vanishing powers of $\frac{1}{r}$.

A consequence of the formulas (3.9)-(3.12) is the following.

Lemma 3.3. *Let w^ε and W^ε be related by moving frame coordinates as in* (3.5) *and let W be a distributional limit of W^ε as in Proposition 3.1. Let $\gamma_1,\gamma_2\in\mathbb{N}_0$ and $\gamma=\gamma_1+\gamma_2$. Then the following distributional convergence holds*

$$\left(\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}w^\varepsilon\right)\left(r+\frac{t}{\varepsilon^2},\varphi,\frac{t}{\varepsilon^2}\right) \rightharpoonup \frac{\cos^{\gamma_1}(\varphi)\sin^{\gamma_2}(\varphi)}{(\sqrt{a_1})^{\gamma_1}(\sqrt{a_2})^{\gamma_2}}\,\partial_r^\gamma W$$

for $\varepsilon\to 0$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}\times(0,T))$.

Proof. We recall the definition $W^\varepsilon(r,\varphi,t) = w^\varepsilon\left(r+\frac{t}{\varepsilon^2},\varphi,\frac{t}{\varepsilon^2}\right)$. The formula for $\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}$ of Remark 3.2 provides

$$\begin{aligned}&\left(\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}w^\varepsilon\right)\left(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}\right) =\\ &\frac{1}{(\sqrt{a_1})^{\gamma_1}(\sqrt{a_2})^{\gamma_2}}\left[\cos^{\gamma_1}(\varphi)\sin^{\gamma_2}(\varphi)\left(\partial_r^\gamma w^\varepsilon\right)\left(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}\right) + R^\varepsilon(r,\varphi,t)\right],\end{aligned} \tag{3.13}$$

where $R^\varepsilon(r,\varphi,t)$ is of the form

$$\begin{aligned}R^\varepsilon(r,\varphi,t) &:= \sum_{k=1}^{\gamma} \left(Q_k^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi)w^\varepsilon\right)\left(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}\right)\frac{1}{(r+\frac{t}{\varepsilon^2})^k}\\ &= \sum_{k=1}^{\gamma} Q_k^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi)W^\varepsilon(r,\varphi,t)\left(\frac{\varepsilon^2}{\varepsilon^2 r+t}\right)^k.\end{aligned}$$

The last equality holds, since the derivatives $\partial_r,\partial_\varphi$ act on w^ε as they act on W^ε.

Since W^ε converges to W in the sense of distributions, also all derivatives converge in the distributional sense. We conclude $Q_k^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi)W^\varepsilon(r,\varphi,t) \rightharpoonup Q_k^{\gamma_1,\gamma_2}(\cos\varphi,\sin\varphi,\partial_r,\partial_\varphi)W$ in the distributional sense on $\mathbb{R}\times\mathbb{R}\times(0,T)$ for every k. In particular, exploiting that $\frac{\varepsilon^2}{\varepsilon^2 r+t}$ is of order ε^2 for every $t>0$, we obtain $R^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}\times(0,T))$.

The same argument implies for the term containing only r-derivatives

$$\left(\partial_r^\gamma w^\varepsilon\right)\left(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}\right) = \left(\partial_r^\gamma W^\varepsilon\right)(r,\varphi,t) \rightharpoonup \partial_r^\gamma W \quad \text{in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}\times(0,T)).$$

This allows to pass to the distributional limit in (3.13), which concludes the proof of the lemma. □

Proof of Proposition 3.1. We start from the weakly dispersive equation

$$\partial_t^2 w^\varepsilon - AD^2 w^\varepsilon - \varepsilon^2\partial_t^2 ED^2 w^\varepsilon + \varepsilon^2 FD^4 w^\varepsilon = 0.$$

The general idea of the proof is to transform the above equation into the elliptic coordinates of (3.4), to rewrite it in terms of $W^\varepsilon(r,\varphi,t) = w^\varepsilon\left(r+\frac{t}{\varepsilon^2},\varphi,\frac{t}{\varepsilon^2}\right)$, and to pass to the distributional limit.

Step 1: The term $(\partial_t^2 w^\varepsilon - AD^2 w^\varepsilon)$: We start with the evaluation of time-derivatives of w^ε, using the chain rule on $w^\varepsilon(r,\varphi,t) = W^\varepsilon(r-t,\varphi,t\varepsilon^2)$,

$$(\partial_t^2 w^\varepsilon)(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}) = \partial_r^2 W^\varepsilon(r,\varphi,t) - 2\varepsilon^2\partial_t\partial_r W^\varepsilon(r,\varphi,t) + \varepsilon^4\partial_t^2 W^\varepsilon(r,\varphi,t). \tag{3.14}$$

Combining this result with (3.11), we find

$$\begin{aligned}&\left(\partial_t^2 w^\varepsilon - AD^2 w^\varepsilon\right)\left(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}\right) =\\ &\quad = \partial_r^2 W^\varepsilon(r,\varphi,t) - 2\varepsilon^2\partial_t\partial_r W^\varepsilon(r,\varphi,t) + \varepsilon^4\partial_t^2 W^\varepsilon(r,\varphi,t)\\ &\qquad - \left(\partial_r^2 W^\varepsilon(r,\varphi,t) + \frac{1}{r+\frac{t}{\varepsilon^2}}\partial_r W^\varepsilon(r,\varphi,t) + \frac{1}{(r+\frac{t}{\varepsilon^2})^2}\partial_\varphi^2 W^\varepsilon(r,\varphi,t)\right)\\ &\quad = -\varepsilon^2\left(2\partial_t\partial_r W^\varepsilon(r,\varphi,t) + \frac{1}{\varepsilon^2 r+t}\partial_r W^\varepsilon(r,\varphi,t)\right)\\ &\qquad + \varepsilon^4\left(\partial_t^2 W^\varepsilon(r,\varphi,t) - \frac{1}{(\varepsilon^2 r+t)^2}\partial_\varphi^2 W^\varepsilon(r,\varphi,t)\right).\end{aligned}$$

We divide by ε^2 and take the distributional limit, exploiting that, by assumption, $W^\varepsilon \rightharpoonup W$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}\times(0,T))$. We obtain

$$\frac{1}{\varepsilon^2}\left(\partial_t^2 w^\varepsilon - AD^2 w^\varepsilon\right)\left(r+\tfrac{t}{\varepsilon^2},\varphi,\tfrac{t}{\varepsilon^2}\right) \rightharpoonup -2\partial_t\partial_r W - \frac{1}{t}\partial_r W \tag{3.15}$$

for $\varepsilon \to 0$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R} \times (0,T))$.

Step 2: The term $-\varepsilon^2 \partial_t^2 E D^2 w^\varepsilon + \varepsilon^2 F D^4 w^\varepsilon$*:* In view of the scaling in (3.15), we have to analyze the distributional limit of $-ED^2\partial_t^2 w^\varepsilon + FD^4 w^\varepsilon$.

We recall that, by our construction of E and F, the term can be written as $CD^4 w^\varepsilon + R^\varepsilon$ with some remainder R^ε that vanishes in the distributional limit as $\varepsilon \to 0$. Indeed, exploiting the solution property of w^ε and the decomposition (1.11) one obtains

$$-ED^2\partial_t^2 w^\varepsilon + FD^4 w^\varepsilon = CD^4 w^\varepsilon + \varepsilon^2 \left(-ED^2ED^2\partial_t^2 w^\varepsilon + ED^2FD^4 w^\varepsilon\right). \tag{3.16}$$

In particular, inserting an argument in scaled variables,

$$\begin{aligned} &-\left(ED^2\partial_t^2 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right) + \left(FD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right) \\ &\quad = \left(CD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right) + R^\varepsilon, \end{aligned} \tag{3.17}$$

where the remainder R^ε is given by

$$\begin{aligned} R^\varepsilon &= \varepsilon^2 \left[-\left(ED^2ED^2\partial_t^2 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right) + \left(ED^2FD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right)\right] \\ &= -\varepsilon^2 \left(\partial_r^2 - 2\varepsilon^2\partial_t\partial_r + \varepsilon^4\partial_t^2\right)\left(\left(ED^2ED^2 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right)\right) \\ &\quad + \varepsilon^2\left(ED^2FD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right). \end{aligned}$$

In the second equality we used relation (3.14). In this form we can apply Lemma 3.3 to R^ε. We obtain that $\left(ED^2ED^2 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right)$ and $\left(ED^2FD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right)$ have distributional limits. In particular, taking into account the ε^2-factor in the above formula, we conclude the convergence $R^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R} \times (0,T))$ as $\varepsilon \to 0$.

Regarding the term $\left(CD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right)$ in (3.17) one can directly apply Lemma 3.3 with $\gamma = 4$. Using the explicit form of C from (3.3), we find

$$\left(CD^4 w^\varepsilon\right)\left(r + \tfrac{t}{\varepsilon^2}, \varphi, \tfrac{t}{\varepsilon^2}\right) \rightharpoonup \left(6\beta\frac{\cos^2(\varphi)}{a_1}\frac{\sin^2(\varphi)}{a_2} + \alpha_1\frac{\cos^4(\varphi)}{a_1^2} + \alpha_2\frac{\sin^4(\varphi)}{a_2^2}\right)\partial_r^4 W$$

in $\mathcal{D}'(\mathbb{R} \times \mathbb{R} \times (0,T))$, as $\varepsilon \to 0$.

Step 3: Conclusion. Summing up the various terms, we find that the distribution W satisfies the equation

$$-2\partial_t\partial_r W - \frac{1}{t}\partial_r W + \left(6\beta\frac{\cos^2(\varphi)}{a_1}\frac{\sin^2(\varphi)}{a_2} + \alpha_1\frac{\cos^4(\varphi)}{a_1^2} + \alpha_2\frac{\sin^4(\varphi)}{a_2^2}\right)\partial_r^4 W = 0\,.$$

For $U := \partial_r W$ we obtain the equation

$$\partial_t U + \frac{1}{2t}U - \frac{1}{2}\left(6\beta\frac{\cos^2(\varphi)}{a_1}\frac{\sin^2(\varphi)}{a_2} + \alpha_1\frac{\cos^4(\varphi)}{a_1^2} + \alpha_2\frac{\sin^4(\varphi)}{a_2^2}\right)\partial_r^3 U = 0\,.$$

This concludes the proof of Proposition 3.1. □

4 Calculation of approximate solutions

In this section we discuss practical aspects of determining the coefficients in the effective dispersive equation (1.10). We present a numerical method and compare solutions u^ε of the original wave equation (1.1) with solutions w^ε of (1.10).

4.1 Cell problems

The decomposition lemmas 2.5 and 2.6 show that the coefficient tensors A, E and F of the weakly dispersive effective equation can be calculated from A and C, i.e. by the second and fourth derivatives of the Bloch eigenvalue $\mu_0(k)$ at $k = 0$. This fact makes the effective tensors computable in terms of cell-problems. In the following calculation we differentiate (1.5) with respect to k and integrate in y to obtain formulas for A and C. The result will be a practical algorithm to determine A and C. We will also transform Lemmas 2.5 and 2.6 into an algorithm that can be used to calculate E and F.

For a wave parameter $k \in Z = (-1/2, 1/2)^n$ we consider the Bloch eigenvalue problem (1.5) for $m = 0$, i.e.

$$-(\nabla_y + \mathrm{i}k) \cdot (a_Y(y)(\nabla_y + \mathrm{i}k)\psi_0(y, k)) = \mu_0(k)\psi_0(y, k), \tag{4.1}$$

where $\mu_0(k) \in \mathbb{R}$ is the smallest eigenvalue for each $k \in Z$.

Remark 4.1. *For $k = 0$, the eigenvalue is $\mu_0(0) = 0$ and the eigenfunction $\psi_0(\cdot, 0)$ is a constant function. We normalize eigenfunctions so that*

$$\langle \psi_0(\cdot, k)\rangle_Y := \frac{1}{|Y|}\int_Y \psi_0(y, k)\, dy = 1\,, \tag{4.2}$$

in particular we obtain $\psi_0(\cdot, 0) \equiv 1$. The normalization is possible in a neighborhood of $k = 0$, since averages of the first eigenfunction $\psi_0(\cdot, k)$ do not vanish for small $|k|$. The eigenvalue map $k \mapsto \mu_0(k) \in \mathbb{R}$ and the eigenfunction map $k \mapsto \psi_0(\cdot, k) \in L^2(Y)$ are analytic, see [11].

Due to Remark 4.1, it is legitimate to determine derivatives of μ_0 by differentiating the eigenvalue problem (4.1) in k. We use standard multi-index notation with $\mathbb{N}_0 = \{0, 1, 2, ...\}$: a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ has length $|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_n$ and defines a differential operator ∂^α of order $|\alpha|$ (derivatives $\partial_j = \partial_{k_j}$ are with respect to $k \in \mathbb{R}^n$), $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}$. We define, for $\alpha \in \mathbb{N}_0^n$,

$$\mu_0^\alpha := \partial^\alpha \mu_0|_{k=0}\,, \qquad \psi_0^\alpha := \partial^\alpha \psi_0|_{k=0}\,.$$

Differentiating the normalization (4.2), we obtain that the averages of the higher order derivatives vanish,

$$\langle \psi_0^\alpha \rangle_Y = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^n,\ \alpha \neq 0.$$

We additionally define the differential operators

$$\begin{aligned} \mathcal{A}(k) &:= -(\nabla_y + \mathrm{i}k) \cdot (a_Y(y)(\nabla_y + \mathrm{i}k))\,, \\ \mathcal{A}^\alpha &:= \partial^\alpha \mathcal{A}|_{k=0}\,. \end{aligned}$$

Lemma 4.2 (The operators $\mathcal{A}^\alpha$)**.** *Let e_j denote the j-th Euclidean unit vector. The operators $\mathcal{A}^\alpha$ with $|\alpha| \leq 2$ are*

$$\begin{aligned} \mathcal{A}^0 f &= -\nabla_y \cdot (a_Y \nabla_y f), \\ \mathcal{A}^{e_j} f &= -\mathrm{i}\left[(a_Y \nabla_y f) \cdot e_j + \nabla_y \cdot (a_Y e_j f)\right] \\ \mathcal{A}^{e_i + e_j} f &= 2(a_Y)_{ij} f \end{aligned}$$

for $i, j = 1, ..., n$, where $f : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary smooth function. All operators $\mathcal{A}^\alpha$ with $|\alpha| \geq 3$ vanish identically.

Proof. The formula for $\mathcal{A}^0$ is obtained from the definition of $\mathcal{A}(k)$ by setting $k = 0$. Concerning the first order derivatives of $\mathcal{A}(k)$ we calculate

$$\begin{aligned}\partial_j \mathcal{A}(k) f &= -\mathrm{i} e_j \cdot (a_Y(\nabla_y + \mathrm{i}k)f) - \mathrm{i}(\nabla_y + \mathrm{i}k) \cdot (a_Y e_j f) \\ &= -\mathrm{i}\left[e_j \cdot (a_Y(\nabla_y + \mathrm{i}k)f) + (\nabla_y + \mathrm{i}k) \cdot (a_Y e_j f)\right] .\end{aligned}$$

Inserting $k = 0$ provides the claim about $\mathcal{A}^{e_j}$. For the second order derivatives of $\mathcal{A}(k)$ we calculate

$$\partial_i \partial_j \mathcal{A}(k) f = -\mathrm{i}^2 [e_j \cdot (a_Y e_i f) + e_i \cdot (a_Y e_j f)] = ((a_Y)_{ji} + (a_Y)_{ij}) f = 2(a_Y)_{ij} f,$$

where the last equality holds due to the symmetry of a_Y. □

We next want to obtain equations that characterize the functions ψ_0^α. In order to calculate derivatives of products, we will use the Leibniz formula in the following form: For every multi-index $\alpha \in \mathbb{N}_0^n$ and sufficiently smooth functions $f, g : \mathbb{R}^n \to \mathbb{C}$ there holds

$$\partial^\alpha (fg) = \sum_{\beta \in \mathbb{N}_0^n} \binom{\alpha}{\beta} \partial^\beta f \; \partial^{\alpha - \beta} g \, .$$

We use the binomial coefficient $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \dots \binom{\alpha_n}{\beta_n}$, the (partial) ordering $\beta \leq \alpha :\Leftrightarrow \beta_i \leq \alpha_i$ for every $i \in \{1, ..., n\}$, and note that $\binom{\alpha}{\beta} \neq 0$ is non-vanishing only for $\beta \leq \alpha$. In particular, in the following calculations, all sums over multi-indices are finite sums. Summation of multi-indices is performed in the standard way as $\alpha \pm \beta := (\alpha_1 \pm \beta_1, \dots, \alpha_n \pm \beta_n)$.

With the operator $\mathcal{A}(k)$, we can write equation (4.1) as $\mathcal{A}(k)\psi_0(y,k) = \mu_0(k)\psi_0(y,k)$. Taking partial derivatives with respect to k with the Leibniz formula, we find the following result.

Lemma 4.3 (Cell Problems for ψ_0^α)**.** *Let $\alpha \in \mathbb{N}_0^n$ be a multi-index. Then the function ψ_0^α satisfies the relation*

$$\mathcal{A}^0 \psi_0^\alpha + \sum_{j=1}^n \alpha_j \mathcal{A}^{e_j} \psi_0^{\alpha - e_j} + \sum_{i \leq j = 1}^n \binom{\alpha}{e_i + e_j} \mathcal{A}^{e_i + e_j} \psi_0^{\alpha - e_i - e_j} = \sum_{\beta \in \mathbb{N}_0^n} \binom{\alpha}{\beta} \mu_0^\beta \psi_0^{\alpha - \beta} .$$

Inserting the operators from Lemma 4.2 and using $\mu_0^0 = \mu_0|_{k=0} = 0$, this equation reads

$$\begin{aligned}-\nabla_y \cdot [a_Y \nabla_y \psi_0^\alpha] = \mathrm{i} \sum_{j=1}^n \alpha_j &\left[(a_Y \nabla_y \psi_0^{\alpha - e_j}) \cdot e_j + \nabla_y \cdot \left(\psi_0^{\alpha - e_j} a_Y e_j \right) \right] \\ &- 2 \sum_{i \leq j = 1}^n \binom{\alpha}{e_i + e_j} (a_Y)_{ij} \psi_0^{\alpha - e_i - e_j} + \sum_{0 \neq \beta \in \mathbb{N}_0^n} \binom{\alpha}{\beta} \mu_0^\beta \psi_0^{\alpha - \beta} .\end{aligned} \tag{4.3}$$

In our next step we obtain formulas for μ_0^α in terms of ψ_0^β and μ_0^β with $\beta < \alpha$.

Lemma 4.4 (Formulas for μ_0^α)**.** *Let $\alpha \in \mathbb{N}_0^n$ be a multi-index. In the case $\alpha = 0$ we have $\mu_0^\alpha = \mu_0^0 = 0$. For $|\alpha| \geq 1$ there holds*

$$\mu_0^\alpha = 0 \qquad \text{if } |\alpha| \text{ is odd,} \tag{4.4}$$

$$\mu_0^\alpha = 2 \sum_{i \leq j = 1}^{n} \binom{\alpha}{e_i + e_j} \left\langle (a_Y)_{ij} \psi_0^{\alpha - e_i - e_j} \right\rangle_Y - \mathrm{i} \sum_{j=1}^{n} \alpha_j \left\langle a_Y \nabla_y \psi_0^{\alpha - e_j} \right\rangle_Y \cdot e_j \qquad \text{if } |\alpha| \text{ is even.} \tag{4.5}$$

Proof. The statement for μ_0^0 has already been observed, cf. Remark 4.1. For odd $|\alpha|$, the symmetry $\mu_0(-k) = \mu(k)$ for all $k \in Z$ implies $\mu_0^\alpha = 0$ (compare e.g. Remark 2.7 in [12]). For even $|\alpha|$, the formula is a consequence of relation (4.3). Indeed, we can reorganize (4.3), writing the term with $\beta = \alpha$ in the last sum explicitely, and find

$$\begin{aligned} \mu_0^\alpha \psi_0^0 = &- \nabla_y \cdot [a_Y \nabla_y \psi_0^\alpha] - \mathrm{i} \sum_{j=1}^{n} \alpha_j \left[(a_Y \nabla_y \psi_0^{\alpha - e_j}) \cdot e_j + \nabla_y \cdot \left(\psi_0^{\alpha - e_j} a_Y e_j \right) \right] \\ &+ 2 \sum_{i \leq j = 1}^{n} \binom{\alpha}{e_i + e_j} (a_Y)_{ij} \psi_0^{\alpha - e_i - e_j} - \sum_{1 \leq |\beta| \leq |\alpha| - 1} \binom{\alpha}{\beta} \mu_0^\beta \psi_0^{\alpha - \beta}. \end{aligned}$$

We integrate this relation over the periodicity cell Y, exploiting $\langle \psi_0^0 \rangle_Y = 1$ and $\langle \psi_0^\alpha \rangle_Y = 0$ for all $\alpha \neq 0$. Furthermore, we use that the integral over the two terms in divergence form vanishes by periodicity, and obtain (4.5). $\square$

The above formulas allow to calculate all unknowns ψ_0^α and μ_0^α in a recursive scheme. From Lemmas 4.3 and 4.4, we extract the following algorithm for the computation of the tensors A and C.

Algorithm 1. *(Computation of A and C) The tensors A and C can be computed in five steps.*

1. *For $j \in \{1, \ldots, n\}$, solve* (4.3) *for $\alpha = e_j$, i.e.*

$$-\nabla_y \cdot [a_Y \nabla_y \psi_0^{e_j}] = \mathrm{i} \nabla_y \cdot (a_Y e_j) .$$

2. *For $i, j \in \{1, \ldots, n\}$, evaluate* (4.5) *for $\alpha = e_i + e_j$, i.e.*

$$A_{ij} = \frac{1}{2} \mu_0^{e_i + e_j} = \left\langle (a_Y)_{ij} - \frac{\mathrm{i}}{2} \left((a_Y \nabla_y \psi_0^{e_j}) \cdot e_i + (a_Y \nabla_y \psi_0^{e_i}) \cdot e_j \right) \right\rangle_Y .$$

3. *For $i, j \in \{1, \ldots, n\}$, solve* (4.3) *for $\alpha = e_i + e_j$, i.e.*

$$\begin{aligned} -\nabla_y \cdot [a_Y \nabla \psi_0^{e_i + e_j}] = &\mathrm{i} \left[(a_Y \nabla_y \psi_0^{e_j}) \cdot e_i + (a_Y \nabla_y \psi_0^{e_i}) \cdot e_j \right. \\ &\left. + \nabla_y \cdot (\psi_0^{e_j} a_Y e_i) + \nabla_y \cdot (\psi_0^{e_i} a_Y e_j) \right] - 2 (a_Y)_{ij} + \mu_0^{e_i + e_j} . \end{aligned}$$

4. *For $i, j, k \in \{1, \dots, n\}$, solve* (4.3) *for $\alpha = e_i + e_j + e_k$,*

$$\begin{aligned}
&- \nabla_y \cdot [a_Y \nabla_y \psi_0^{e_i+e_j+e_k}] = \\
&\quad = \mathrm{i} \Big[(a_Y \nabla_y \psi_0^{e_j+e_k}) \cdot e_i + (a_Y \nabla_y \psi_0^{e_i+e_k}) \cdot e_j + (a_Y \nabla_y \psi_0^{e_i+e_j}) \cdot e_k \\
&\qquad + \nabla_y \cdot (\psi_0^{e_j+e_k} a_Y e_i) + \nabla_y \cdot (\psi_0^{e_i+e_k} a_Y e_j) + \nabla_y \cdot (\psi_0^{e_i+e_j} a_Y e_k) \Big] \\
&\quad - 2 \left((a_Y)_{ij} \psi_0^{e_k} + (a_Y)_{ik} \psi_0^{e_j} + (a_Y)_{jk} \psi_0^{e_i} \right) \\
&\quad + \mu_0^{e_i+e_j} \psi_0^{e_k} + \mu_0^{e_i+e_k} \psi_0^{e_j} + \mu_0^{e_j+e_k} \psi_0^{e_i} .
\end{aligned}$$

5. *For $i, j, k, l \in \{1, \dots, n\}$, evaluate* (4.5) *for $\alpha = e_i + e_j + e_k + e_l$ to find*

$$\begin{aligned}
C_{ijkl} &= \frac{1}{24} \mu_0^{e_i+e_j+e_k+e_l} \\
&= \frac{1}{24} \Bigg\langle 2 \Big((a_Y)_{ij} \psi_0^{e_k+e_l} + (a_Y)_{ik} \psi_0^{e_j+e_l} + (a_Y)_{il} \psi_0^{e_j+e_k} \\
&\qquad + (a_Y)_{jk} \psi_0^{e_i+e_l} + (a_Y)_{jl} \psi_0^{e_i+e_k} + (a_Y)_{kl} \psi_0^{e_i+e_j} \Big) \\
&\qquad - \mathrm{i} \Big((a_Y \nabla_y \psi_0^{e_j+e_k+e_l}) \cdot e_i + (a_Y \nabla_y \psi_0^{e_i+e_k+e_l}) \cdot e_j \\
&\qquad + (a_Y \nabla_y \psi_0^{e_i+e_j+e_l}) \cdot e_k + (a_Y \nabla_y \psi_0^{e_i+e_j+e_k}) \cdot e_l \Big) \Bigg\rangle_Y .
\end{aligned}$$

The elliptic problems in Steps 1,3, and 4 are posed on Y with periodic boundary conditions and with the condition of zero mean, i.e.

$$\langle \psi_0^{e_j} \rangle_Y = \langle \psi_0^{e_i+e_j} \rangle_Y = \langle \psi_0^{e_i+e_j+e_k} \rangle_Y = 0.$$

In the above algorithm we have used the fact that $\mu_0^\alpha = 0$ for all α with $|\alpha| = 1$ and $|\alpha| = 3$. The above cell-problems are complex valued, but the resulting tensors A and C are real. The fact that values of ψ_0^α and μ_0^α do not change upon permutations of the entries $\alpha_1, \alpha_2, \dots, \alpha_n$, allows to reduce the number of problems to be solved in Steps 3 and 4: The relevant number is $\binom{n+1}{2}$ and $\binom{n+2}{3}$ rather than n^2 and n^3. Moreover, spatial symmetries of $a_Y(.)$ can reduce the number of problems further: If $a_Y(.)$ is even in both variables, $a_Y(y_1, y_2) = a_Y(-y_1, y_2) = a_Y(y_1, -y_2)$ for all $y \in Y$, then all derivatives of μ_0 at $k = 0$ involving an odd number of derivatives in one direction vanish, and only A_{ii}, C_{iiii}, and $C_{iijj} = C_{ijij} = C_{jiij}$ for $i, j \in \{1, \dots, n\}$ are potentially nonzero.

At this point, we have presented a scheme to compute the effective tensors A and C with the help of a sequence of cell-problems. We conclude this section with the outline of an algorithm (based on the proofs of Lemmas 2.5 and 2.6) that provides formulas for the effective coefficient tensors E and F.

Algorithm 2. *(Computation of E and F) The tensors E and F can be computed in four steps.*

1. *Determine $S \in SO(n)$ such that $A = S^T \tilde{A} S$ with $\tilde{A} = \mathrm{diag}(a_1, a_2, ..., a_n)$. Define $\tilde{C}$ by* (2.8).

2. *Loop over all indices* $1 \le \alpha, \beta, \gamma, \delta \le n$ *such that no two indices coincide (an empty set in dimension* $n \le 3$*). Use* (2.7) *to compile* $\hat{F}$ *and set* $\bar{C} := C - \hat{F}$.

3. *Loop over all remaining indices* $1 \le \alpha, \beta, \gamma, \delta \le n$. *Corresponding to* $\tilde{A}$ *and* $\bar{C}$, *compile* $\bar{E}$ *and* $\bar{F}$ *from the explicit formulas of Cases 1-3 in the proof of Lemma 2.6.*

4. *Set* $\tilde{E} := \bar{E}$ *and* $\tilde{F} = \bar{F} + \hat{F}$. *Obtain* $E := S^T \tilde{E} S$ *and* F *from* (2.11).

4.2 Numerical results in 2D

In the following, we present numerical results for all the three parts of the homogenization problem: The computation of the original ε-problem, the computation of effective coefficients with the help of cell-problems, and the computation of the weakly dispersive effective problem. The methods vary, we use finite element schemes and finite difference schemes, see e.g. [20]. We refer also to [1] for a recent analysis of numerical methods and further references. Our results show an excellent agreement between solutions u^ε of (1.1) and solutions w^ε of (1.10).

The initial conditions for all the tests below are

$$u^\varepsilon(x,0) = w^\varepsilon(x,0) = e^{-4(x_1^2+x_2^2)}, \quad \partial_t u^\varepsilon(x,0) = \partial_t w^\varepsilon(x,0) = 0. \tag{4.6}$$

The periodicity cell is $Y = [-\pi,\pi]^2$ in the $y-$variables and εY in the $x-$variables. All the tests are carried out for the case of even symmetry in a_Y, i.e. $a_Y(-y_1, y_2) = a_Y(y_1, -y_2) = a_Y(y_1, y_2)$ for all $y \in Y$, so that for the even initial data in (4.6) problems (1.1) and (1.10) can be reduced to one quadrant with homogeneous Neumann boundary conditions along the coordinate axes $x_1 = 0, x_2 = 0$. The computational domain Ω is rectangular with two sides coinciding with the negative x_1 and x_2 axes. At the remaining two sides of the rectangle we use homogeneous Dirichlet boundary conditions. The size of Ω is chosen so that the solution remains localized in Ω until the final computation time. The function a_Y is chosen piecewise constant in all tests; the observed error convergence agrees with (2.1) although Theorem 2.2 treats only differentiable fields a_Y.

The even symmetry of a_Y in both y_1 and y_2 causes that A is diagonal and C has only eight nonzero entries, see (3.2)–(3.3):

$$\begin{aligned} &A_{ii} =: a_i\,, \qquad C_{iiii} =: \alpha_i\,, \qquad i \in \{1,2\}\,,\\ &C_{1122} = C_{2211} = C_{1212} = C_{2121} = C_{1221} = C_{2112} =: \beta\,, \end{aligned}$$

with all other entries of A and C zero. A choice of E and F according to Algorithm 2 is

$$\begin{aligned} &E_{11} = \frac{\{-\alpha_1\}_+}{a_1} + 3\frac{\{-\beta\}_+}{a_2},\ E_{22} = \frac{\{-\alpha_2\}_+}{a_2} + 3\frac{\{-\beta\}_+}{a_1},\ E_{12} = E_{21} = 0,\\ &F_{1111} = \{\alpha_1\}_+ + 3\frac{a_1}{a_2}\{-\beta\}_+,\ F_{2222} = \{\alpha_2\}_+ + 3\frac{a_2}{a_1}\{-\beta\}_+,\\ &F_{2121} = \frac{a_1}{a_2}\{-\alpha_2\}_+ + 3\{\beta\}_+,\ F_{1212} = \frac{a_2}{a_1}\{-\alpha_1\}_+ + 3\{\beta\}_+ \end{aligned} \tag{4.7}$$

with all other entries of F being zero. We use these tensors in the numerical tests below. Note that $\alpha_i \le 0$ holds for $i \in \{1,2\}$, hence $\{-\alpha_i\}_+ = -\alpha_i$ and $\{\alpha_i\}_+ = 0$.

Numerical method

The values a_j, α_j, and β for $j = 1, 2$ are computed via Algorithm 1, where the elliptic equations in Steps 1, 3, and 4 are discretized by linear finite elements using the PDE-Toolbox of Matlab. The periodic boundary conditions are implemented by modifying the stiffness matrix and the load vector corresponding to homogeneous Neumann boundary conditions. In all tests a uniform discretization conforming to the material geometry is generated with the Matlab function `poimesh`, i.e. the elements are all right angled, have equal size and the discontinuity lines of a_Y intersect no elements. The element size is given by specifying the spacings h_1 and h_2, i.e. the lengths of the triangle legs in `poimesh`.

The original wave equation (1.1) is discretized in space also via linear finite elements using the PDE-Toolbox of Matlab with uniform elements (generated by `poimesh`) conforming to the geometry. The values of h_1 and h_2 are given for each example below.

For the weakly dispersive problem (1.10), which has constant coefficients, we use the fourth order centered finite difference discretization in space for both the second order and fourth order derivatives. In all tests we use the spacing $dx_1 = dx_2 = 0.2$ for (1.10).

The time discretization of both (1.1) and (1.10) is done via the second order Leap-Frog method in two step formulation, e.g. for (1.1) the semi-discrete problem is thus

$$u^{(n+1)} = 2u^{(n)} - u^{(n-1)} + (dt)^2\nabla\cdot(a^\varepsilon\nabla u^{(n)}),$$

where $u^{(n)} \approx u^\varepsilon(t = n\cdot dt)$. For the initialization of the scheme we use the second order Taylor expansion

$$u^{(1)} = u^{(0)} + (dt)^2\nabla\cdot(a^\varepsilon\nabla u^{(0)}),$$

according to the initial condition $\partial_t u^\varepsilon(x, 0) = 0$. For (1.1) we use the time step $dt = \min\{0.01, h_1/4, h_2/4\}$, for (1.10) we use $dt = 0.02$.

ε-Convergence of the error

Here, our aim is to determine experimentally the convergence rate of the approximation error $\|u^\varepsilon(t) - w^\varepsilon(t)\|_{L^2(\Omega)}$. To this end we fix a periodic coefficient matrix field a_Y. With the identity matrix $I := \mathrm{id}_{\mathbb{R}^2}$ we set $a_Y(y) = \tilde a_Y(y)I$, where $\tilde a_Y$ is defined through a rectangular geometry,

$$\tilde a_Y(y) = \frac{1}{2} + b(y) - \frac{1}{|Y|}\int_Y b(y)dy, \quad b(y) = \begin{cases} 1.6 & \text{for } y \in [-\frac{11\pi}{13}, \frac{11\pi}{13}] \times [-\frac{\pi}{3}, \frac{\pi}{3}], \\ 0.2 & \text{otherwise}, \end{cases} \tag{4.8}$$

which is illustrated in Fig. 3 (a).

Our best numerical approximation of the effective coefficients is

$$a_1 \approx 0.281,\ a_2 \approx 0.179,\ \alpha_1 \approx -0.273,\ \alpha_2 \approx -0.044,\ \beta \approx 0.024.$$

These values have been computed with $h_1 = 2\pi/208$ and $h_2 = 2\pi/192$. Within rounding to three decimal places the values do not change with a further mesh refinement.

We solve (1.1) and (1.10) for the values $\varepsilon = 0.2, 0.17, 0.15, 0.12, 0.1, 0.07$ up to times $t = 12.5, 17.3, 22.5, 35, 50, 100 \approx \varepsilon^{-2}/2$, respectively. To keep the computational expense within limits, we use the discretization $h_1 = 2\pi\varepsilon/13, h_2 = 2\pi\varepsilon/12$ in the simulations of (1.1). Using the same number of uniform elements in the cell problems as in each periodic cell in (1.1), i.e. discretizing Y by $2 \times 13 \times 12 = 312$ uniform elements, we obtain the values

$$a_1 \approx 0.2784,\ a_2 \approx 0.1506,\ \alpha_1 \approx -0.369,\ \alpha_2 \approx -0.034,\ \beta \approx 0.032. \tag{4.9}$$

We calculate solutions to the weakly dispersive equation (1.10) with the values (4.9) rather than with the converged coefficient values; this is justified by the fact that (4.9) are the effective coefficients for the particular discretization. Formulas (4.7) and the values in (4.9) produce the effective coefficients

$$E_{11} = 1.3256,\ E_{22} = 0.2257,\ F_{1111} = F_{2222} = 0,\ F_{2121} = 0.1588,\ F_{1212} = 0.2957.$$

We can now investigate the convergence of the error in ε: The $L^2(\Omega)$-error $\|u^\varepsilon(t) - w^\varepsilon(t)\|_{L^2(\Omega)}$ at the final time $t = \varepsilon^{-2}/2$ is plotted in Fig. 2. As Fig. 2 (b) shows, the convergence is close to linear in accordance with estimate (2.1).

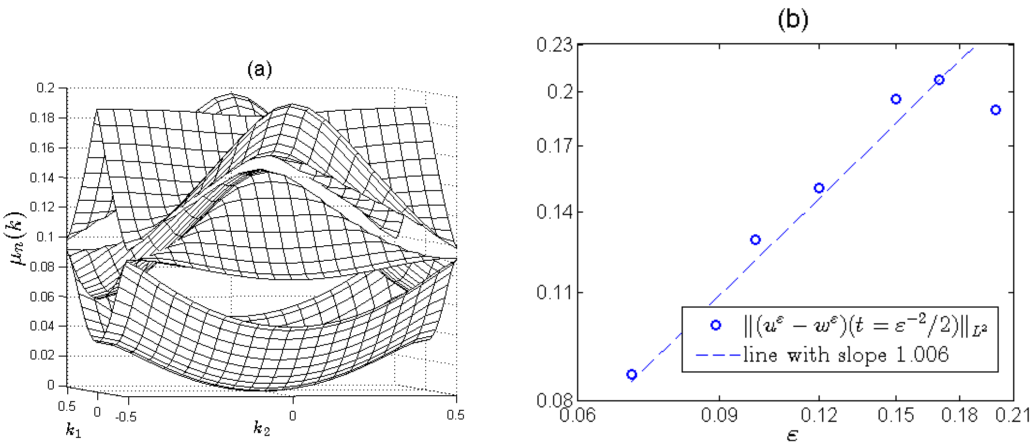

Figure 2: (a) The first three eigenvalues $\mu_0(k), \mu_1(k), \mu_2(k)$ of (1.5) for a_Y from (4.8). (b) Experimental convergence in ε of the error $e_\varepsilon := \|(u^\varepsilon - w^\varepsilon)(t = \varepsilon^{-2}/2)\|_{L^2(\Omega)}$. A logarithmic scale is used on both axes. The line with slope 1.1 was obtained by a linear interpolation of the error in the logarithmic scale for the 5 smallest ε-values. The experimental rate is thus $e_\varepsilon \sim \varepsilon^{1.006}$.

In Fig. 3 the solutions u^ε and w^ε are plotted for $\varepsilon = 0.07$ at $t = 100$. In both plots we see clearly the main pulse located along an ellipse, we see the dispersive oscillations behind the main pulse, and we see that the dispersion is weakest along a ray that has an approximate angle $\pi/4$. Due to the weak dispersion along this ray, the main pulse has its maximal amplitude in this direction (compare also Fig. 6). An excellent agreement of the two calculations is observed.

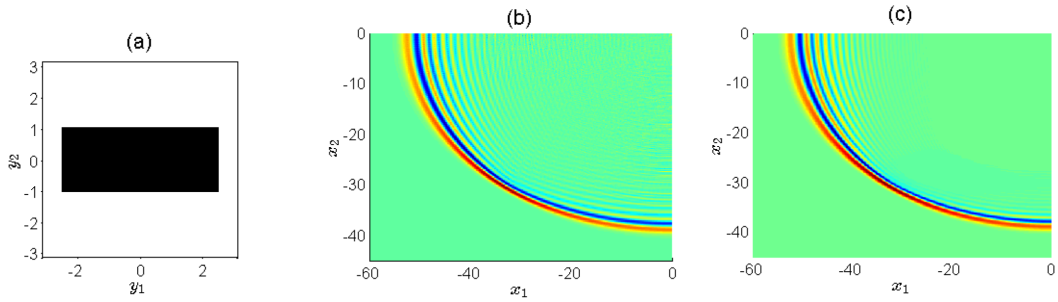

Figure 3: (a) Illustration of the geometry and of the function $a_Y(.)$ from (4.8). (b) The solution u^ε of the wave equation in a highly oscillatory medium. (c) The solution w^ε of the weakly dispersive wave equation with constant coefficients. The geometry is as in (4.8), the plots are for $\varepsilon = 0.07$ and $t = 100$.

Further numerical examples

We consider two more geometries. For a cross-shaped geometry as illustrated in Fig. 4, (a), we set $a_Y(y) := \tilde{a}_Y(y)I$ with

$$\tilde{a}_Y(y) = \begin{cases} 2 & \text{for } y \in [-\frac{7\pi}{9}, \frac{7\pi}{9}] \times [-\frac{2\pi}{9}, \frac{2\pi}{9}] \cup [-\frac{2\pi}{9}, \frac{2\pi}{9}] \times [-\frac{7\pi}{9}, \frac{7\pi}{9}], \\ 0.2 & \text{otherwise.} \end{cases} \tag{4.10}$$

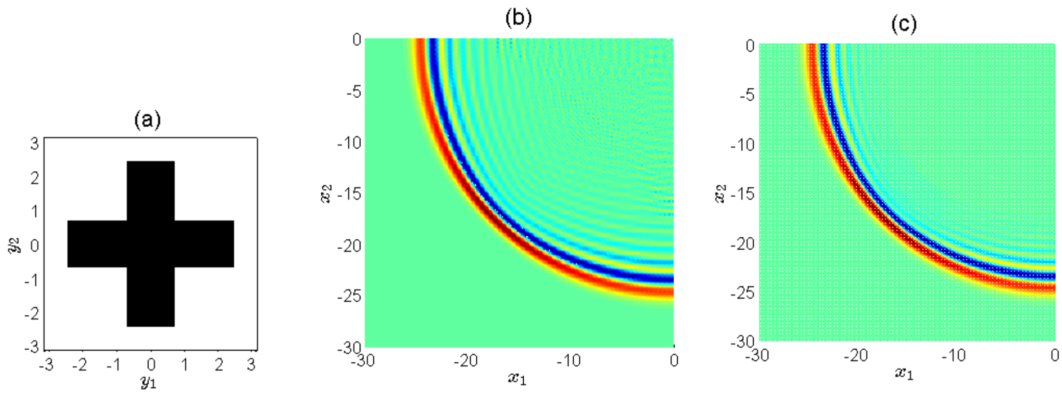

Figure 4: (a) Illustration of the geometry that defines the function $a_Y(.)$ in (4.10). (b) The solution $u^\varepsilon(x, t = 40)$ for $\varepsilon = 0.07$. (c) The solution $w^\varepsilon(x, t = 40)$ for $\varepsilon = 0.07$.

The additional symmetry $a_Y(y_1, y_2) = a_Y(y_2, y_1)$ for all $y \in Y$ implies the relations $a_1 = a_2$ and $\alpha_1 = \alpha_2$, see Lemma 2.6 in [12]. Algorithm 1 provides, with Y discretized by $2 \times 18^2 = 648$ uniform elements of size $h_1 = h_2 = 2\pi/18$, the values

$$a_1 = a_2 \approx 0.3816, \ \ \alpha_1 = \alpha_2 \approx -0.1970, \ \ \beta \approx 0.0394. \tag{4.11}$$

To check the accuracy, we calculated the values also using a fine resolution with $2 \times 360 \times 360$ uniform elements. The fine resolution provides $a_1 = a_2 \approx 0.406$, $\alpha_1 = \alpha_2 \approx -0.235, \beta \approx 0.044$, and the first three decimal places do not change upon further refinement. Similarly to the example in Subsection 4.2, the discretization error in (4.11) is quite large. Nevertheless, we use these coefficients for the calculation of the solution w^ε of (1.10). The solution is compared to u^ε, which is computed with the corresponding spatial discretization $h_1 = h_2 = 2\pi\varepsilon/18$. The results at time $t = 40$ for $\varepsilon = 0.07$ are plotted in Fig. 4, (b) and (c).

We finally consider a laminated structure $a_Y(y) = \tilde{a}_Y(y)I$ with

$$\tilde{a}_Y(y) = \begin{cases} 2 & \text{for } y \in [-\pi, \pi] \times [-\frac{2\pi}{5}, \frac{2\pi}{5}], \\ 0.2 & \text{otherwise,} \end{cases} \tag{4.12}$$

compare Fig. 5 (a). Effective coefficients are obtained by Algorithm 1, Y is dis-

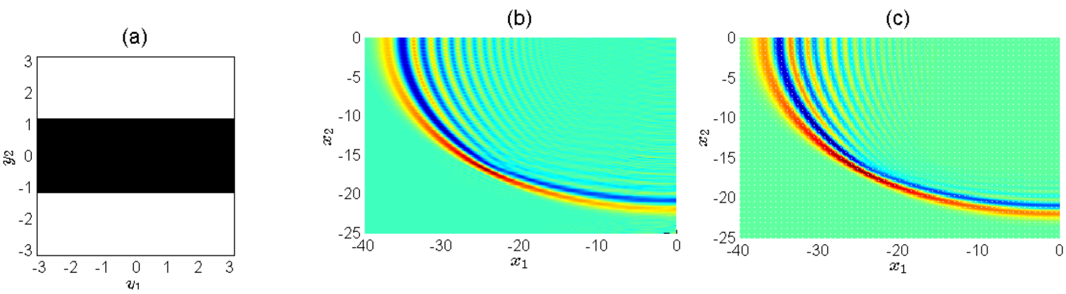

Figure 5: (a) Illustration of the function $a_Y(.)$ of (4.12). (b) The solution $u^\varepsilon(x, t = 40)$ for $\varepsilon = 0.07$. (c) The solution $w^\varepsilon(x, t = 40)$ for $\varepsilon = 0.07$.

cretized with $2 \times 12 \times 16 = 384$ uniform elements ($h_1 = 2\pi/12$, $h_2 = 2\pi/16$):

$$a_1 \approx 0.8750,\ a_2 \approx 0.3019,\ \alpha_1 \approx -1.9185,\ \alpha_2 \approx -0.0933,\ \beta \approx 0.1448. \tag{4.13}$$

The converged values (with four reliable digits, computed with $2\times140\times180$ uniform elements) are: $a_1 \approx 0.9200, a_2 \approx 0.3125$, $\alpha_1 \approx -1.9645$, $\alpha_2 \approx -0.1170$, $\beta \approx 0.1599$. The effective coefficients determined using (4.7) and (4.13) are

$$E_{11} = 2.1925,\ E_{22} = 0.3091,\ F_{1111} = F_{2222} = 0,\ F_{2121} = 0.7050,\ F_{1212} = 1.0964.$$

Equation (1.1) was discretized in space using $h_1 = 2\pi\varepsilon/12$ and $h_2 = 2\pi\varepsilon/16$, the solutions u^ε and w^ε are computed using the above coefficients; they are plotted for $t = 40$ and $\varepsilon = 0.07$ in Fig. 5, (b) and (c).

Dependence of the dispersion on the propagation angle

The rate of dispersion depends on the angle of propagation. Here, we have to distinguish the angle φ of the elliptic coordinates and the corresponding angle ϕ in polar coordinates (describing the observable angle of the ray). We measure the angles such that $\phi = 0$ corresponds to the negative x_1-axis. The two angles are related by $\tan(\phi) = \sqrt{a_2/a_1}\tan(\varphi)$. The dependence of the dispersion on the angle can be observed in the numerical results: we see few oscillations in a propagation angle of approximately $\phi = \pi/4$, more oscillations along rays that are aligned with the coordinate axes, i.e. at angles $\phi = 0$ and $\phi = \pi/2$. The angle of minimal dispersion can be obtained by minimizing $\kappa = \kappa(\varphi)$ of (3.7).

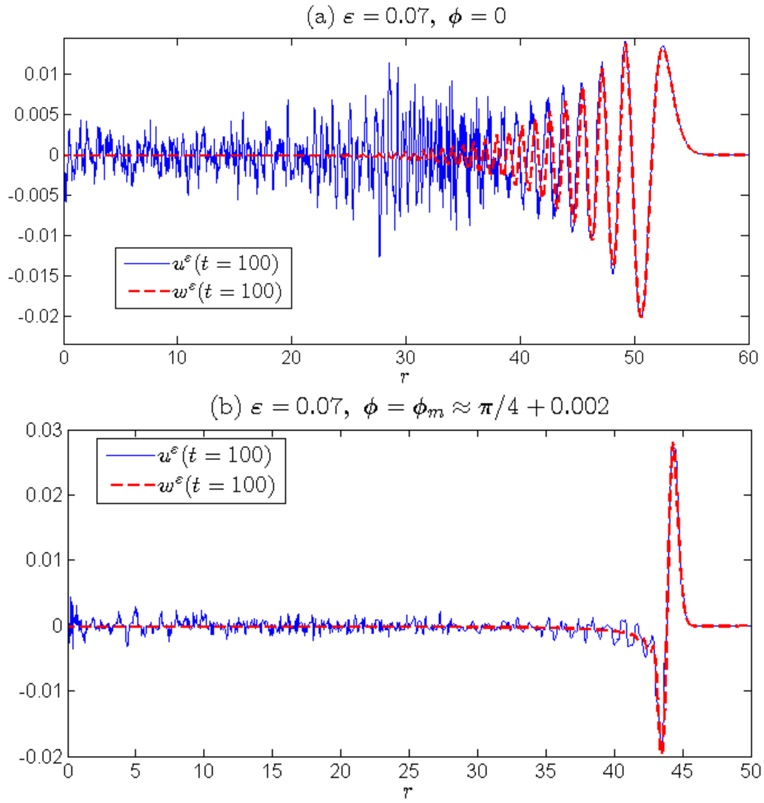

Figure 6: Solutions u^ε and w^ε for $\varepsilon = 0.07$ at $t = 100$ for the rectangle geometry (4.8) along two rays: (a) $\phi = 0$; (b) $\phi = \phi_m \approx \pi/4 + 0.002$.

To illustrate the angular dependence, we consider the rectangular geometry (4.8) and the two rays corresponding to $\varphi = 0$ and $\varphi = \varphi_m$, the minimizer of $\kappa(.)$. We plot u^ε and w^ε for $\varepsilon = 0.07$ and $t = 100$ in Fig. 6. One can see clearly a much smaller dispersion at the angle $\phi_m \approx \pi/4 + 0.002$ corresponding to φ_m. Additionally, we observe a much larger error in the aligned direction $\varphi = 0$. The values of the dispersion coefficient are $\kappa(0) \approx -4.762$ and $\kappa(\varphi_m) \approx -0.175$.

For the laminate structure (4.12) we compare the solutions along three directions in Fig. 7, namely the horizontal direction $\phi = 0$ along which the structure is constant, the vertical direction $\phi = \pi/2$, orthogonal to the laminates, and an intermediate direction ϕ_m that corresponds to the minimizer $\varphi_m \approx \pi/4 - 0.153$ of κ. The values of the dispersion coefficient are $\kappa(0) \approx -2.506$, $\kappa(\pi/2) \approx -1.024$, and $\kappa(\varphi_m) \approx 0.02$. The analytical values of κ are always non-positive, hence the positive value $\kappa(\varphi_m)$ is caused by discretization errors.

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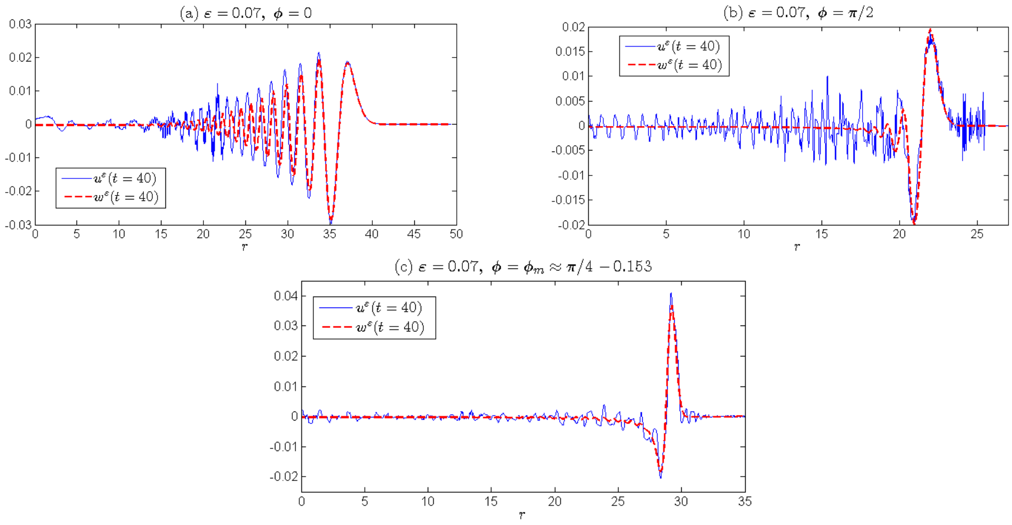

Figure 7: Solutions u^ε and w^ε for $\varepsilon = 0.07$ at $t = 40$ for the laminated geometry (4.12) along three rays: (a) $\phi = 0$. (b) $\phi = \pi/2$. (c) $\phi = \phi_m \approx \pi/4 - 0.153$.

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