$O(\alpha_s^3)$ Contributions to the Heavy Flavor Wilson Coefficients of the Structure Function $F_2(x, Q^2)$ at $Q^2 \gg m^2$

Dissertation

zur Erlangung des wissenschaftlichen Grades

Dr. rer. nat.

der Fakultät Physik der Technischen Universität Dortmund

vorgelegt von

Fabian Philipp Wißbrock *

geboren am 15.08.1983 in Hannover

Deutsches Elektronen–Synchrotron, DESY
Platanenallee 6, D–15738 Zeuthen

Betreuer: Prof. Dr. Johannes Blümlein

* *Present adress:*
Institut des Hautes Études Scientifiques, IHES
Route de Chartres 35, F–91440 Bures-Sur-Yvette, France

Contents

List of Tables

List of Figures

This Thesis contains contributions to the following publications:

i) Publications: Journals

1. **The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity**
   J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock,
   Nucl. Phys. **B844** (2010) [arXiv:hep-ph/1008.3347].

2. **Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements**
   J. Ablinger and J. Blümlein and A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock,
   Nucl. Phys. **B864** (2012) [arXiv:hep-ph/1206.2252].

3. **The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$**
   J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider and F. Wißbrock,
   Nucl. Phys. **B882** (2014) [arXiv:hep-ph/1402.0359].

4. **The logarithmic contributions to the $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering**
   A. Behring, I. Bierenbaum, J. Blümlein, A. De Freitas, S. Klein and F. Wißbrock,
   Eur.Phys.J. **C74** (2014) [arXiv:hep-ph/1403.6356].

5. **Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms**
   J. Ablinger, J. Blümlein, C. Raab, C. Schneider and F. Wißbrock,
   Nucl.Phys. **B885** (2014) [arXiv:hep-ph/1403.1137].

6. **The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity**
   J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider and F. Wißbrock,
   Nucl.Phys. **B886** (2014) [arXiv:hep-ph/1406.4654].

ii) Publications: Conference Proceedings

1. **Heavy Flavor DIS Wilson coefficients in the asymptotic regime**
   J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 10th DESY Workshop on Elementary Particle Theory: Loops and Legs in Quantum Field Theory, Wörlitz, Germany, 2010,
   Nucl.Phys.Proc.Suppl. **205-206** 2010 [arXiv:hep-ph/1007.0375].

2. **3-Loop Heavy Flavor Corrections to DIS with two Massive Fermion Lines**
   J. Ablinger,J. Blümlein, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 19th International Workshop on Deep-Inelastic Scattering and Related Subjects (DIS 2011), Newport News, Virginia, United States , 2011,
   [arXiv:hep-ph/1007.0375].

3. **New Heavy Flavor Contributions to the DIS Structure Function $F_2(x,Q^2)$ at $O(\alpha_s^3)$**
   J. Ablinger,J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 10th International Symposium on Radiative Corrections: Applications of Quantum Field Theory to Phenomenology (RADCOR 2011), Mamallapuram, Tamil Nadu, India , 2011,
   PoS RADCOR2011 (2011) [arXiv:hep-ph/1202.2700].

4. **New Results on the 3-Loop Heavy Flavor Wilson Coefcients in Deep-Inelastic Scattering**
   J. Ablinger,J. Blümlein, A. De Freitas, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 36th International Conference on High Energy (ICHEP 2012), Melbourne, Australia, 2012,
   PoS ICHEP2012 (2013) [arXiv:hep-ph/1212.5950].

5. **Recent Results on the 3-Loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering**
   J. Ablinger,J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Raab, C. Schneider and F. Wißbrock
   in: Proceedings of the 1st International Workshop on Deep-Inelastic Scattering and Related Subjects (DIS 2013) , Marseille, France, 2013,
   PoS DIS2013 (2013) [arXiv:hep-ph/1307.7548].

6. **New Results on the 3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering**
   J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, T. Pfoh, M. Round, C. Raab, C. Schneider and F. Wißbrock
   in: Proceedings of the 11th International Symposium on Radiative Corrections "Application of Quantum Field Theory to Phenomenology" (RADCOR 2013) , Durham, United Kingdom, 2013,
   PoS RADCOR2013 (2013) [arXiv:hep-ph/11312.0124].

7. **3-loop heavy flavor Wilson coefficients in deep-inelastic scattering**
   J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, T. Pfoh, M. Round, C. Raab, C. Schneider and F. Wißbrock
   in: Proceedings of the 37th International Conference on High Energy Physics (ICHEP 2014) , Valencia, Spain, 2014,
   [arXiv:hep-ph/1409.1804].

8. **Recent progress on the calculation of three-loop heavy flavor Wilson coefficients in deep-inelastic scattering**
   J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, T. Pfoh, M. Round, C. Raab, C. Schneider and F. Wißbrock
   in: Proceedings of the 12th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory (LL2014), Weimar, Germany, 2014,
   PoS LL2014 (2014) [arXiv:hep-ph/1407.3638].

9. **3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering with Two Heavy Quark Lines**
   J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider and F. Wißbrock
   in: Proceedings of the 12th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory (LL2014), Weimar, Germany, 2014,
   PoS LL2014 (2014) [arXiv:hep-ph/1407.2821].

Three further journal publications are in preparation.

1 Introduction

It has been observed experimentally that the the magnetic moment of the proton and the neutron derivates significantly from that of the electron. This has been the first hint that nucleons could be not point-like elementary particles, cf. [1, 2]. During the 1950s lepton-nucleon scattering experiments performed by Hofstadter and collaborators [3–6] showed furthermore that neutrons and protons possess extended charge distributions giving further evidence to the existence of a nucleonic substructure.
A series of newly discovered hadronic states at various particle accelerators, particularly during the late 1950s, led to the development of a deeper understanding of the hadronic composition. The various states were classified according to their spin as mesons (integer spin) and baryons (half–integer spin). The immense number of newly discovered states was calling for a systematic classification based on the properties charge, isospin and mass. Independent attempts by M. Gell–Mann and Y. Ne'eman to group the different hadrons into octet and decouplet–representations of the flavor group $SU(3)$ with similar properties proved to be very successful [7–9]. The prediction of the mass of the previously unknown Ω^-–baryon with strangeness -3 and electric charge -1 constituted a great success of this model. Based on these discoveries in 1964 M. Gell–Mann [10],and G. Zweig [11] proposed that baryons were bound states, being composed of three spin-$\frac{1}{2}$ particles named quarks and mesons composed of a quark–antiquark pair. To describe all hadron states known at that time the three quark flavors *up*(u), *down*(d) and *strange*(s) with electric charges $e_u = +2/3, e_d = e_s = -1/3$ proved to be sufficient. Despite its strong predictive power, the static quark–model also suffered from a fundamental problem. The Δ^{++} and Δ^-–resonances were predicted as spin– and flavor–symmetric wave functions of three quarkonic fermions thereby violating the well–established spin–statistic theorem [12–14]. This contradiction was finally resolved by introducing a new 3–valued quantum number named *color* [15–18] . Since no colored particles could be detected experimentally it was finally proposed, that physical hadrons must be color–singlets [16, 17, 19].
A systematic study of the nucleonic substructure was performed in the late 1960s at the Stanford Linear Accelerator SLAC [20, 21], by analyzing the differential deeply analytic cross section of unpolarized lepton–nucleon scattering which was parametrized by the non–perturbative structure functions F_2 and F_L [22, 23]. Kinematically this process is characterized by the energy transfer ν in the rest–frame of the nucleon and the virtual 4– momentum transfer from the lepton to the nucleon of $q^2 = -Q^2$. Studying the algebra of currents, Bjorken predicted [24] that in the limit $Q^2,\ \nu \to \infty$ with the ratio Q^2/ν fixed the structure functions do not depend on Q^2 and ν individually any longer, but become a function of the Bjorken variable $x = \frac{Q^2}{2M\nu}$ only, where M denotes the mass of the hadron and $0 < x < 1$. This behavior became known as Bjorken–scaling and has been experimentally confirmed in experiments of the SLAC and MIT groups [25–28], cf. also [29–31]. The scaling property constituted further evidence for the point–like substructure of hadrons and inspired Feynman to propose the *parton*–model [32–34]. Furthermore, the prediction that the structure Function F_L is much smaller than F_2, known as Callan–Gross relation [35], could be experimentally confirmed. This could only be explained if the scattering process was dominated by spin–1/2 partons.
The group theoretic approach by Gell-Mann and Ne'eman and Feynman's parton model were finally joint, when Bjorken and Paschos suggested, that Feynman's partons would be identical to the quarks proposed by Gell-Mann [36]. The ground for a full theoretical description of the force binding the different quarks together was laid in the 1950s when, inspired by the successful description of Quantum Electrodynamics as a quantum field theory based on the abelian group

$U(1)$, C.N. Yang and R.L. Mills [37] studied quantum field theories based on non–abelian gauge groups. The renomalizability of massless Yang–Mills theories was proven by G. 't Hooft [38]. These developments inspired M. Gell-Mann, H. Fritzsch and H. Leutwyler, [39], cf. also [17], to describe quark interactions by a $SU(3)_c$ Yang-Mills theory, thereby giving birth to *Quantum Chromodynamics* (QCD).
A short time before D. Gross and F. Wilczek, [40], and H. Politzer, [41] discovered *asymptotic freedom*, calculating the running coupling of QCD to one-loop order which decreases at ever shorter distances or larger momentum transfers. This explains why scaling is compatible with the fact that free quarks could never be observed and makes the use of perturbative computation methods for scattering processes at large enough momentum scales possible.
In 1974 the J/ψ meson has been discovered almost simultaneously at SLAC and at Brookhaven National Laboratory (BNL) [42, 43]. It was described as a meson of a fourth quark, the *charm* quark, which had been predicted previously [44–47]. A few years later as a first quark of the third family of elementary particles, the *bottom* quark (b) was experimentally found by discovering the Υ–resonance [48]. Another quark, the *top*–quark, had been predicted but due to its large mass it was difficult to be produced in particle accelerators for a longer period and it was observed in 1995 at TEVATRON, [49–52], finally.
Together with the electroweak $SU_L(2) \times U_Y(1)$ sector QCD forms the Standard Model of elementary particle physics. The theoretical description of the electroweak sector of the Standard Model has been developed by S. Glashow [53](1961) and S. Weinberg in 1967 [54], cf. also [55, 56]. G. 't Hooft and M. Veltman proved that this theory is renormalizable [57], see also [58–61].
One of the cleanest experimental approaches to study the substructure of hadrons are deep-inelastic scattering (DIS) experiments. On the theoretical side, applying QCD to practical calculations was problematic at first, as it is not perturbative in the complete kinematic region. Here the light cone expansions (LCE), the expansion of operator products around light like distances, emerged as an important tool to perform the corresponding computations [62–65]. Applying it to deeply inelastic scattering, one derives a factorization theorem, which separates hadronic bound state effects from the short distance behavior at leading twist [66–68] . The former are described by the *parton distribution functions* (PDFs) which can be extracted from experimental precision data or computed by using non-perturbative methods at a given scale Q_0^2. The short distance behavior, on the other side, is governed by the process dependent Wilson coefficients, which allow for a perturbative series representation in the strong coupling constant $\alpha_s = g_s^2/(4\pi)$. A first strong success for perturbative QCD computations was the prediction of logarithmic scaling violations in deeply inelastic scattering [69, 70]. These are due to parton–parton interactions and the fact that QCD is not conformally invariant [71]. They could experimentally be first observed in 1975 [72, 73].
As it constitutes the most direct way to probe the substructure of hadrons, numerous DIS experiments have been performed since the 1970s [74–79]. Currently the widest kinematic range for DIS cross-section studies has been covered by experiments at HERA at DESY [80–83]. Here the structure function F_2 has been measured over a large kinematic range with $6 \cdot 10^{-7} \leq x \leq 0.65$ and $0.045 \leq Q^2 \leq 30000 \mathrm{GeV}^2$ [84], while F_L has mainly been measured at fixed targets [85, 86]. For a theoretical description the case where only the massless quark flavors (u, d, s) contribute has been studied extensively. The massless Wilson coefficients for the F_2 and F_L were computed at $O(\alpha_s)$ in [87–89], at second order in [90–100], and at third order in [101–106].
The experimental data from HERA show, however, that especially in the small x–region the structure functions receive significant contributions due to charm quarks, cf. [107–110]. Furthermore the DIS-analyses constitute an excellent tool to extract the parton distribution functions

and QCD–parameters as the strong coupling constant $\alpha_s(M_Z)$ from experimental data. Very recently also the heavy quark masses m_b and m_c have been precisely measured using these processes [111, 112]. Here a missing precise next-to-next-to-leading order (NNLO) heavy flavor description is considered one of the dominant sources for the systematic differences between the PDF extractions of different groups [113–118]. A precise theoretical description of the heavy flavor contributions to DIS would be especially useful to constrain the low–x behavior of the gluon–distribution, which also constitutes an important input for the analysis of Higgs-Boson production at the LHC.

The complete analytic computation of the heavy flavor Wilson coefficients is much harder to obtain, however. The results at leading order were calculated in the late 1970s, [119–123]. At next–to–leading order (NLO) a semi-analytic result for the heavy flavor contributions has been computed in [124–126]. A precise numerical implementation in Mellin space has been given in Ref. [127].

For values of the momentum transfer Q^2 much larger than the mass of the heavy quark m^2, the heavy flavor Wilson coefficients factorize, cf. Ref. [128], into the process dependent light flavor Wilson coefficients $C_{(q,g),(2,L)}(x, Q^2/\mu^2)$ and the process independent massive operator matrix elements (OMEs) $A_{ij}(x, \mu^2/m^2)$. Here the OMEs contain all the mass dependence and are given as matrix elements of the different twist-2 local composite operators between the partonic states $\mid j\rangle$ $(i, j = q, g)$. For the structure function F_2 this representation holds from $Q^2 \gtrsim 10\ m^2$ cf. [128], which covers a very important kinematic region of the deep-inelastic scattering experiments at HERA being widely free of higher twist effects [129–132] . Using this representation the NNLO contributions to the structure function F_L have been obtained in Ref. [133]. However, in this case the factorized representation becomes only valid at values $Q^2 \gtrsim 800\ m^2$, [128], and thus does not describe the relevant kinematic region probed at HERA.

The NLO OMEs have been computed in Refs. [128, 134]. They were later re-computed using a different method in Refs. [133, 135–139] and extended up to order $O(\varepsilon)$ in the dimensional regularization parameter $\varepsilon = D - 4$. The latter contribution constitutes an important ingredient to the renormalization of the NNLO OMEs which has been worked out in [139]. Here as a first contribution to the NNLO description a number of fixed Mellin moments $N = 2 \dots 10$ and in some cases for $N = 2 \dots 14$ has been calculated. First results for general values of the Mellin-variable N at 3–loop order were obtained in [140, 141], where the $O(C_{A,F}T_F^2N_F)$ color-contributions to the OMEs contributing to F_2 have been computed. The calculation was based on integral representations in terms of (nested) sums over generalized hypergeometric functions [142–149]. These obey a sum representation which allows for a direct extraction of the singularities of the Laurent series expansion in the dimensional regularization parameter $\varepsilon = D - 4$. Afterwards, advanced summation algorithms encoded in the package `SIGMA` were applied to evaluate the nested sums. The same method has been used later to compute the gluonic OMEs $A_{gq,Q}$ and $A_{gg,Q}$ [150]. These are important to define the variable flavor number scheme (VFNS) for the PDFs. The method of representing integrals by hypergeometric functions is very efficient and allows for a direct computation of QCD Feynman diagrams without relying on integration by parts (IBP)-reductions. It is, however, restricted to a topological subclass of the Feynman integrals contributing to the NNLO OMEs. Generalizations thereof have been thought after and inspired the use of Appell–function representations to solve a large number of ladder topology diagrams [151] and approaches using intermediary Mellin-Barnes representations [152]. The complete $O(C_{A,F}T_F^2)$ contributions to the OME $A_{gg,Q}$ have been computed in Ref. [152]. Furthermore, recently a lot of progress in the calculation of massive operator matrix elements has been made using integration by parts reductions [153] and differential equation methods

[154, 155]. Additionally to the OMEs $A_{qq,Q}^{\rm PS}$ and $A_{qg,Q}$ which have been presented in [140] (see also Section 3 of this Thesis) now also the OMEs $A_{gq,Q}$ [156], $A_{qq,Q}^{\rm NS}$ [157] and $A_{Qq}^{\rm PS}$ [158] are completely known.
Previous OME–computations have been performed with N_F massless and one heavy quark flavor representing the c- or the b-quark, depending on the kinematic region one is interested in. There is, however, no strong hierarchy between the quark masses since $m_c^2/m_b^2 \sim 1/10$. The charm quark can thus not be considered massless at scales $\mu^2 \sim m_b^2$ and the sequential decoupling of the individual heavy quark flavors in the usual variable flavor number scheme has to be replaced by a variable flavor scheme which decouples both c- and b-quarks at the same time. One may, however, compute massive operator matrix elements depending on both heavy quark flavors and define heavy flavor Wilson coefficients containing heavy flavor contributions from both c and b-quarks. It is one of the main objectives of this Thesis to set up the framework for these computations.
Additionally to the physical importance it is one of the aims of this Thesis to explore the mathematical methods and structures which are best suited for this class of computations. From a technical viewpoint the computation of the massive OMEs is performed in Mellin space, which renders the factorization relation for the heavy flavor Wilson coefficients a pure product. The OMEs thus become functions of the Mellin variable N, which assumes positive integer values. In a final step one desires to find the analytic continuation in order to perform the inverse Mellin transform by a contour integration. One is thus faced by the discrete Mellin–N space and the continuous x–space, which are related to each other by the (inverse) Mellin transformation. As some algorithms (or their implementations) are better suited for a computation in a continuous space, e.g. [159], cf. [152, 156–158], and the inverse Mellin transform is hard to obtain for momentum integrals, a further continuous space may be introduced by the construction of appropriate generating functions. In both, the discrete and the continuous space, physical quantities are mostly described within distinctive sets of characteristic functions. For most massless single scale analyses and simple massive topologies at three loops [106, 160–165] these are typically nested harmonic sums [166, 167]. They obey algebraic, [168], and structural relations, [169–171]. At fixed values of N these results evaluate to multiple zeta values, [172,173]. A first generalization of the harmonic sums is obtained by adding additional weights into the nested sums. This yields the so-called generalized harmonic sums, which have been observed in [174]. Analytic and algorithmic aspects of (inverse) Mellin transforms, asymptotic expansions, etc. have been presented in Ref. [175]. Further extensions to cyclotomic sums [176] and additionally over letters involving binomials of the kind $\binom{2i}{i}$ [177] have been studied as well. The latter have been observed in Feynman diagrams with several massive fermionic lines [152] but also in other diagrams exhibiting a highly nested topology. They appear again in different parts of this Thesis. This includes the case of a diagram with a single massive line and 4–leg operator insertion and more general binomial summation kernels in the case of diagrams with massive lines of unequal mass. Algorithms performing many operations as (inverse) Mellin transforms, (asymptotic) expansions and reductions to a basis for most of these sum structures are available within the computer algebra package `HarmonicSums` [175, 177–179] . On the continuous side, each sum structure is associated to respective iterated integrals structures by an inverse Mellin transform. Here the most commonly observed functions were the harmonic polylogarithms [180] over the alphabet $\{0, \pm 1\}$. But generalizations thereof where soon required. Firstly the alphabets may be enlarged, but lately also more general classes of iterated integrals [176], for example over cyclotomic polynomials or square root-valued integration kernels [177] have been studied.
The outline of this Thesis is as follows. In Section 2 we describe the deeply-inelastic lepton-

nucleon scattering process and its kinematics. We define the structure functions, review the QCD-improved parton model and present the asymptotic structure of the heavy flavor Wilson coefficients. In Section 3 we use representations in terms of special functions and modern summation algorithms to compute the remaining $O(C_{A,F}T_F^2N_F)$ contribution given in terms of the OME $A_{qg,Q}$. Furthermore, a large number of diagrams with two massless loop insertions has been computed in an automatized way. The logarithmic contributions to the heavy flavor Wilson coefficients follow as a direct consequence of the renormalization prescription worked out in Ref. [139]. Their exact form has been determined and is presented together with the respective theoretical background in Chapter 4. The two heavy flavor Wilson coefficients $L_g^{\rm S}$ and $L_q^{\rm PS}$ are given in complete form. We also present all logarithmic contributions to the 3–loop Wilson coefficients $H_g^{\rm S}$ and $H_q^{\rm PS}$, which rely on the anomalous dimensions [69, 70, 100–103, 105, 161, 162, 181] up to three and the OMEs up to two loops [128, 133–139]. Section 5 presents the renormalization of the contributions with two distinct heavy quark flavors. Similar to the case of one heavy quark flavor [139] here the charge is renormalized in the MOM–scheme first using the background field method [182–184] to ensure that external gluons are strictly on–shell and the charge renormalization is covered by that of a propagator. A series of fixed Mellin moments $(N = 2, 4, 6)$ contributing to all the OMEs of the structure function F_2 and the variable flavor number scheme (VFNS) has been computed, see Section 6. Here the Feynman diagrams with operator insertions have been mapped to massive tadpoles, which could then be expanded and evaluated using the codes `Q2e/Exp` [185, 186]. First general-N results have been obtained for the non–singlet OME and all scalar topologies contributing to the OME $A_{gg,Q}$. In the non–singlet case the N–dependent contributions decouple completely from the infinite sums stemming from the mass distribution of the diagram. For the diagrams such a decoupling can be achieved in intermediate steps using a split of the integration domain and suitable mappings of integration variables. In Section 7 the application of a direct integration algorithm based on hyperlogarithms [159, 187] for convergent Feynman diagrams with operator insertions is explored. Since it relies on an integration kernel containing only rational or polylogarithmic functions, generating function representations for the Feynman diagrams with operator insertions have been introduced. The algorithm has been implemented in `MAPLE` [188] and provides a very efficient way to compute diagrams of nested topologies that were previously inaccessible with different methods. A linear reduction algorithm allows to predict which Feynman integrals are suitable for an integration via hyperlogarithms, where in some cases linear reducibility could be re-obtained by a suitable variable mapping. Appendix A presents the conventions used for the computations within this Thesis. The relevant Feynman rules for the computed QCD- and scalar diagrams are given in Appendix B. Appendix C summarizes the D-dimensional momentum integration and Appendix D shortly reviews the different special functions including nested sum structures and different forms of iterated integrals. The Mellin moments $N = 2, 4, 6$ for the contributions with both c- and b-quarks are presented in Appendix F and the contributions $\propto \ln(Q^2/m^2)$ for the heavy flavor Wilson coefficients $H_q^{\rm PS}$ and $H_g^{\rm S}$ are listed in Appendix E.

2 Deeply inelastic scattering

Deeply inelastic scattering denotes the scattering of highly energetic leptons off nucleons. This Thesis will focus on DIS via the exchange of a single photon. The electroweak corrections can be found in Ref. [189] to 1-loop and dominant higher order corrections can be found in Refs. [190–192, 192–194].

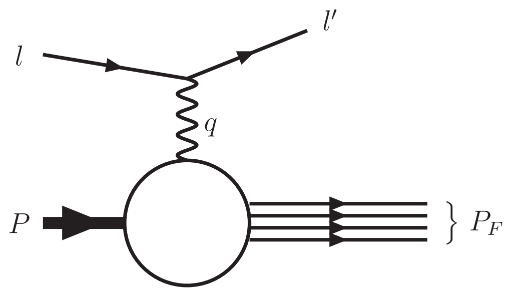


Figure 1: Schematic diagram of deeply inelastic scattering via single boson exchange

2.1 Kinematics

The basic kinematics of the deep-inelastic scattering process is depicted in Figure 1. A lepton with momentum l is scattered off a nucleon with momentum P via exchange of a virtual vector boson with 4-momentum q. The nucleon disintegrates and a new hadronic state F with a 4-momentum P_F is formed. The 4-momentum of the virtual photon is space–like and the virtuality Q^2 is defined by

$$Q^2 = -q^2 = -(l - l')^2 , \tag{1}$$

with l' denoting the 4-momentum of the outgoing lepton. The outgoing hadronic state is measured inclusively, thus only its total 4-momentum P_F is recorded. In this case the total cross-section depends only on two further independent kinematic variables. These are the total center of momentum energy squared

$$s \equiv (P + l)^2 , \tag{2}$$
$$\tag{3}$$

and the invariant mass of the final hadron state $\langle P_F \mid$,

$$W^2 \equiv (P + q)^2 = P_F^2 . \tag{4}$$

Here s denotes the total center of momentum energy squared and W is the invariant mass of the final hadron state $\langle P_F \mid$. The lepton mass is very small compared to the other mass scales and will be neglected from now on. The process is then described using the following Lorentz-invariant kinematic variables:

$$\nu \equiv \frac{P.q}{M} = \frac{W^2 + Q^2 - M^2}{2M} , \tag{5}$$
$$x \equiv \frac{-q^2}{2P.q} = \frac{Q^2}{2M\nu} = \frac{Q^2}{W^2 + Q^2 - M^2} , \tag{6}$$

$$y \equiv \frac{P.q}{P.l} = \frac{2M\nu}{s-M^2} = \frac{W^2+Q^2-M^2}{s-M^2} . \tag{7}$$

Thus ν is the total energy in the rest frame of the nucleon, x is the Bjorken variable and y is the inelasticity, cf. [195]. Combining the definitions (2, 6) and (7) one obtains

$$Q^2 = xys . \tag{8}$$

Deep-inelastic scattering occurs via the exchange of $\gamma, Z, W^{\pm}$–bosons. In this Thesis we study neutral current unpolarized lepton-nucleon scattering, which is dominated by single photon exchange for not too large virtualities, i.e. $Q^2 \lesssim 500$ GeV^2, cf. [196]. We will thus neglect weak boson contributions from Z–boson exchange in the following.
The physical region is constrained by several conditions. From Eqs. (4,6) one obtains

$$W^2 = (P+q)^2 = M^2 + Q^2\left(\frac{1}{x} - 1\right) \geq M^2 . \tag{9}$$

As the invariant hadronic mass obey the condition $W^2 \geq M^2$ this limits the Bjorken variable x to the region

$$0 \leq x \leq 1 . \tag{10}$$

$x = 1$ characterizes elastic scattering processes, while the inelastic region is described by $x < 1$ [197].
Further restrictions on the kinematic variables are obtained by demanding a positive energy-transfer in the rest-frame of the nucleon

$$\nu \geq 0 , \quad 0 \leq y \leq 1 , s \geq M^2 . \tag{11}$$

The differential cross section reads, cf. [198–201],

$$l_0' \frac{d\sigma}{d^3 l'} = \frac{1}{32(2\pi)^3 (l.P)} \sum_{\eta', \eta, \sigma, F} (2\pi)^4 \delta^4(P_F + l' - P - l) |M_{fi}|^2 . \tag{12}$$

Here the summation is over the the spin components of the leptons, quarks and gluons, $\eta(\eta')$ and σ, respectively. In the unpolarized case the cross section is obtained as an average over leptonic and hadronic spin degrees of freedom. M_{fi} denotes the transition matrix element for the electromagnetic current, which is given by

$$M_{fi} = e^2 \overline{u}(l', \eta') \gamma^{\mu} u(l, \eta) \frac{1}{q^2} \langle P_F \mid J_{\mu}^{em}(0) \mid P, \sigma \rangle , \tag{13}$$

for ep scattering at leading order, cf. e.g. [198, 200–202]. $\mid P, \sigma\rangle$ and $\langle P_F \mid$ denote the initial and final hadron state. $u(\overline{u})$ are the bi-spinors of the electron respectively its conjugate, and γ_{μ} denote the Dirac-matrices. Furthermore, e is the electric charge and $J_{\mu}^{em}(\xi)$ the quarkonic contribution to the electromagnetic current operator, which is self-adjoint:

$$J_{\mu}^{\dagger}(\xi) = J_{\mu}(\xi) . \tag{14}$$

For QCD it reads

$$J_\mu^{em}(\xi) = \sum_{f,f'} \overline{\Psi}_f(\xi)\gamma_\mu \lambda_{ff'}^{em} \Psi_{f'}(\xi) \ , \tag{15}$$

where the sum is over the different quark flavors f, f', $\Psi_f(\xi)$ denotes the respective quark field of flavor f and $\lambda_{ff'}^{em}$ is a matrix which describes the electromagnetic charges of the different quark flavors. For three quark flavors it reads

$$\lambda^{em} = \frac{1}{2}\Big(\lambda_{flavor}^3 + \frac{1}{\sqrt{3}}\lambda_{flavor}^8\Big) \ , \tag{16}$$

where the λ_{flavor}^i denote the Gell-Mann matrices , cf. [203, 204]. Using Eqs. (12) and (13) the differential cross section can be rewritten as a contraction of a tensor $L_{\mu\nu}$ which describes leptonic contributions only and a hadronic tensor $W_{\mu\nu}$

$$l_0' \frac{d\sigma}{d^3 l'} = \frac{1}{4P.l}\frac{\alpha^2}{Q^4} L^{\mu\nu} W_{\mu\nu} = \frac{1}{2(s-M^2)}\frac{\alpha^2}{Q^4} L^{\mu\nu} W_{\mu\nu} \ . \tag{17}$$

Here α denotes the fine structure constant and in the Born approximation the leptonic and hadronic tensors read

$$L_{\mu\nu}(l,l') = \sum_{\eta',\eta} \Big[\overline{u}(l',\eta')\gamma^\mu u(l,\eta)\Big]^* \Big[\overline{u}(l',\eta')\gamma^\nu u(l,\eta)\Big] \ , \tag{18}$$

$$W_{\mu\nu}(q,P) = \frac{1}{4\pi}\sum_{\sigma,F} (2\pi)^4 \delta^4(P_F - q - P)\langle P,\sigma \mid J_\mu^{em}(0) \mid P_F\rangle \langle P_F \mid J_\nu^{em}(0) \mid P,\sigma\rangle \ . \tag{19}$$

The decomposition into purely leptonic and hadronic tensors relies on general properties of this amplitude and holds for higher orders as well, cf. [205] The leptonic tensor may be easily evaluated within QED. One obtains

$$L_{\mu\nu}(l,l') = Tr[\not{l}\gamma^\mu \not{l'}\gamma^\nu] = 4\left(l_\mu l_\nu' + l_\mu' l_\nu - \frac{Q^2}{2} g_{\mu\nu}\right) \ . \tag{20}$$

The hadronic tensor cannot be computed using purely perturbative methods only due to the non-perturbative effects within the hadronic substructure. One may, however, separate perturbative and non-perturbative contributions and apply perturbative computation methods on the former part.

The hadronic tensor obeys various symmetry conditions, cf. e.g. [206]. These allow to extract the Lorentz structure and to parametrize the hadronic tensor by scalar functions, which contain all the information on the internal structure of the proton. In the general case there are 14 such structure functions, cf. Refs. [205, 207], while in the cases of unpolarized DIS via single photon exchange only two structure functions remain. Here the leptonic tensor (18) is symmetric and thus only the symmetric contribution of the hadronic tensor is physically significant. It follows that the hadronic tensor must be expressible as a linear combination of the following tensor structures

$$g_{\mu\nu} \ , \quad q_\mu q_\nu \ , \quad P_\mu P_\nu \ , \quad q_\mu P_\nu + q_\nu P_\mu \ . \tag{21}$$

From the conservation of the electromagnetic current, Lorentz- and time-reversal invariance one obtains the condition

$$q_\mu W^{\mu\nu} = 0 \ . \tag{22}$$

Imposing gauge invariance on the Ansatz (21) yields a representation of the hadronic tensor in terms of the structure functions $F_{1,2}$

$$\begin{aligned} W_{\mu\nu}(q,P) = & \quad \frac{1}{2x}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_1(x,Q^2) \\ & + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} + \frac{q_\mu q_\nu}{4x^2}\right) F_2(x,Q^2) \ . \end{aligned} \tag{23}$$

Frequently also the longitudinal structure function

$$F_L(x,Q^2) \ = \ F_2(x,Q^2) - 2xF_1(x,Q^2) \tag{24}$$

is used. The structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ are real functions. Their arguments are Bjorken-x and Q^2. In the elastic case $x = 1$ the cross section is determined by the total energy transfer only. Inserting (23) into (17) yields the differential cross section in terms of the structure functions F_2 and F_L:

$$\frac{d\sigma}{dxdy} = \frac{2\pi\alpha^2}{xyQ^2}\left\{\left[1+(1-y)^2\right]F_2(x,Q^2) - y^2 F_L(x,Q^2)\right\} \ . \tag{25}$$

The structure functions F_2 and F_L are extracted from the hadronic tensor (23) by applying the following D dimensional projection operators:

$$\begin{aligned} F_L(x,Q^2) \ &= \ \frac{8x^3}{Q^2} P^\mu P^\nu W_{\mu\nu}(q,P) \ , \\ F_2(x,Q^2) \ &= \ \frac{2x}{D-2}\left[(D-1)\frac{4x^2}{Q^2}P^\mu P^\nu - g^{\mu\nu}\right] W_{\mu\nu}(q,P) \ . \end{aligned} \tag{26}$$

2.2 The Parton Model

Studying the algebra of electromagnetic currents Bjorken observed [24] that in the limit $Q^2, \nu^2 \to \infty$ keeping the ratio Q^2/ν constant, the structure functions do not depend on the kinematic variables individually anymore, but on their ratio, i.e. on x, Eq. (16),

$$\lim_{\{Q^2,\ \nu\}\ \to\ \infty,\ x=const.} F_{(2,L)}(x,Q^2) = F_{(2,L)}(x) \ . \tag{27}$$

This effect became known as Bjorken scaling and has soon after been experimentally confirmed to hold approximately in electron–proton collisions performed at SLAC in 1968, [25–28]. Feynman has shown that this behavior may be explained by considering the proton as a composite object of point-like constituents at large enough scales. During the short interaction time these partons behave as quasi–free particles. The virtual photon scatters of one single parton while the other partons behave as "spectator" partons. The hadronic tensor is thus obtained as an incoherent

sum over the individual partonic contributions weighted by the probability to find a parton of the respective state i and with a momentum fraction z inside the proton. This probability is described by the parton distribution function $f_i(z)$.
In the collinear parton model, the momenta of the interacting parton p is taken to be collinear to the nucleons momentum P and thus

$$p = zP \ . \tag{28}$$

In Feynman's original model protons were always composed of two u and one d quarks. Radiative corrections were neglected and the *experimentally observed* strict correlation

$$\delta\left(\frac{q.p}{M} - \frac{Q^2}{2M}\right) \ , \tag{29}$$

which implies $z = x$, was enforced. The discovery of QCD allowed for a more sophisticated description which included contributions from virtual quark states and gluons. This QCD-improved parton model may be derived by applying the light-cone expansion.
The hadronic tensor is then derived from a partonic tensor $\mathcal{W}^i_{\mu\nu}(\tau, Q^2)$, which describes the photon-parton interaction, $\mathcal{W}^i_{\mu\nu}$:

$$W_{\mu\nu}(x, Q^2) = \frac{1}{4\pi}\sum_i \int_0^1 dz \int_0^1 d\tau \left[f_i(z) + f_{\bar{i}}(z)\right] \mathcal{W}^i_{\mu\nu}(\tau, Q^2)\delta(x - z\tau) \ . \tag{30}$$

Here $f(z)$ and $f_{\bar{i}}(z)$ denote the parton distribution function of the respective charged parton i and its anti-parton. The partonic tensor $\mathcal{W}^i_{\mu\nu}$ corresponds to the hadronic tensor (33), where the hadronic states $\langle P \mid$ are substituted by the corresponding partonic state $\langle p \mid$ of the interacting parton.
In the Born-approximation the electromagnetic current takes the form

$$\langle i \mid j^i_\mu(\tau) \mid i\rangle = -ie_i\overline{u}^i\gamma_\mu u^i \ , \tag{31}$$

where e_i is the charge of the respective parton i. This yields the partonic tensor

$$\mathcal{W}^i_{\mu\nu}(\tau, Q^2) = \frac{2\pi e_i^2}{q.p^i}\delta(1-\tau)\Big[2p^i_\mu p^i_\nu + p^i_\mu q_\nu + p^i_\nu q_\mu - g_{\mu\nu}q.p^i\Big] + O(\alpha_s) \ . \tag{32}$$

Using elementary quantum mechanical relations Eq. (19) is rewritten as, cf. [201, 208],

$$\begin{aligned} W_{\mu\nu}(q, P) &= \frac{1}{4\pi}\sum_\sigma \int d^4\xi \exp(iq\xi)\langle P \mid [J^{em}_\mu(\xi), J^{em}_\nu(0)] \mid P\rangle \\ &= \frac{1}{2\pi}\int d^4\xi \exp(iq\xi)\langle P \mid [J^{em}_\mu(\xi), J^{em}_\nu(0)] \mid P\rangle \ . \end{aligned} \tag{33}$$

By applying the optical theorem the hadronic tensor is expressed in terms of the forward Compton amplitude for virtual gauge boson nucleon scattering

$$W_{\mu\nu}(q, P) = \frac{1}{\pi}\mathsf{Im}\ T_{\mu\nu}(q, P) \ , \tag{34}$$

with the Compton amplitude

$$T_{\mu\nu}(q,P) \;=\; i\int d^4\xi\;\exp(iq\xi)\langle P \mid \mathsf{T} J_\mu(\xi) J_\nu(0) \mid P\rangle\;, \tag{35}$$

cf. [203].

In the Bjorken–limit the light cone expansion of the forward Compton amplitude (35) at twist $\tau = 2$ yields, cf. [88, 209],

$$\begin{aligned} T_{\mu\nu}(q,P) \;\to\; \sum_{i,N} \Biggl\{ & \Bigl[Q^2 g_{\mu\mu_1} g_{\nu\mu_2} + g_{\mu\mu_1} q_\nu q_{\mu_2} + g_{\nu\mu_2} q_\mu q_{\mu_1} - g_{\mu\nu} q_{\mu_1} q_{\mu_2}\Bigr] C_{i,2}\Bigl(N, \frac{Q^2}{\mu^2}\Bigr) \\ & + \Bigl[g_{\mu\nu} + \frac{q_\mu q_\mu}{Q^2}\Bigr] q_{\mu_1} q_{\mu_2} C_{i,L}\Bigl(N, \frac{Q^2}{\mu^2}\Bigr)\Biggr\} q_{\mu_3} \ldots q_{\mu_N} \Bigl(\frac{2}{Q^2}\Bigr)^N \langle P \mid O_i^{\mu_1 \ldots \mu_N}(\mu^2) \mid P\rangle\;. \end{aligned} \tag{36}$$

The light cone expansion of the products of electromagnetic currents in (35) yields different local operators, which in case of single photon exchange and at twist $\tau = 2$ are given by

$$O^{\mathsf{NS}}_{q,r;\mu_1,\ldots,\mu_N} \;=\; i^{N-1}\mathbf{S}[\overline{\psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \frac{\lambda_r}{2}\psi] - \text{trace terms}\;, \tag{37}$$

$$O^{\mathsf{S}}_{q;\mu_1,\ldots,\mu_N} \;=\; i^{N-1}\mathbf{S}[\overline{\psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N}\psi] - \text{trace terms}\;, \tag{38}$$

$$O^{\mathsf{S}}_{g;\mu_1,\ldots,\mu_N} \;=\; 2i^{N-2}\mathbf{SSp}[F^a_{\mu_1\alpha} D_{\mu_2} \ldots D_{\mu_{N-1}} F^{\alpha,a}_{\mu_N}] - \text{trace terms}\;. \tag{39}$$

In Eqs. (37-39) $\mathbf{S}$ denotes the symmetrization operator (332) of the Lorentz indices $\mu_1, \ldots, \mu_N$. D_μ is the covariant derivative, ψ and $\overline{\psi}$ the quark resp. anti–quark fields and $F^a_{\mu\nu}$ the gluonic field strength tensor with a the color index in the adjoint representation. Furthermore λ_r is the flavor matrix of $SU(N_F)$ with N_F fermion flavors. The labels q, g in Eqs. (37-39) distinguish quarkonic and gluonic operators.

2.3 The light flavor Wilson Coefficients

The structure functions F_i contains long-distance effects contributions from parton-parton interactions, which make it impossible to describe it using purely perturbative methods. However, for large enough virtualities Q^2 it obeys the factorization relation

$$\begin{aligned} F_i(x,Q^2) \;=\; & \frac{x}{N_F}\sum_q e_q^2 \Biggl[\Sigma(x,\mu^2) \otimes C^{\mathrm{S}}_{i,Q}\left(x, \frac{Q^2}{\mu^2}\right) + G\left(x,\mu^2\right) \otimes C_{i,g}\left(x, \frac{Q^2}{\mu^2}\right) \\ & + N_F \Delta_q(x,\mu^2) \otimes C^{\mathrm{NS}}_{i,q}\left(x, \frac{Q^2}{\mu^2}\right)\;, i = 2, L\;. \end{aligned} \tag{40}$$

Here N_F denotes the number of quark flavors, e_Q are the electric charges and the sum run over light quark flavors which typically are u, d, s. $G(x, Q^2/\mu^2)$ is the gluonic parton distribution function (PDF), $\Sigma(x,\mu^2)$ denotes the singlet combination of the quark and anti-quark PDFs, f_q and $f_{\bar{q}}$ respectively. It reads

$$\Sigma(N_F,N,\mu^2) \;=\; \sum_{l=1}^{N_F}\Big[f_q(N_F,N,\mu^2)+f_{\bar{q}}(N_F,N,\mu^2)\Big] \;. \tag{41}$$

Likewise the non-singlet combination of the quark PDFs is given by

$$\Delta_q(N_F,N,\mu^2) \;=\; f_q(N_F,N,\mu^2)+f_{\bar{q}}(N_F,N,\mu^2)-\frac{1}{N_F}\Sigma(N_F,N,\mu^2)\;. \tag{42}$$

In Eq. (40) the $C^{\mathsf{S,NS}}_{q,(2,L)}(N_F,N,Q^2/\mu^2)$ are the light flavor Wilson coefficients, which describe the hard scattering of a photon with a light quark. Here the singlet contribution is usually split into a non-singlet and a pure singlet contribution:

$$C^{\rm S}_{i,q} \;=\; C^{\rm NS}_{i,q}+C^{\rm PS}_{i,q}\;. \tag{43}$$

The symbol $\otimes$ denotes the Mellin convolution $f\otimes\ g$ of two functions $f,\ g$

$$[f\otimes g](z)=\int_0^1 dz_1\int_0^1 dz_2\ \ \delta(z-z_1z_2)\ f(z_1)g(z_2)\;. \tag{44}$$

The Mellin transformation $\mathbf{M}[f](N)$ of a function f is defined by the integral

$$\mathbf{M}[f](N)\equiv\int_0^1 dz\ z^{N-1}f(z)\;. \tag{45}$$

The Mellin transformation translates the convolution (44) into a simple product of two functions in Mellin–space,

$$\mathbf{M}[f\otimes g](N)=\mathbf{M}[f](N)\mathbf{M}[g](N)\;. \tag{46}$$

It is thus technically advantageous to perform computations of this kind in Mellin–space.
The renormalization prescription, see cf. Section 5, introduces the renormalization scale μ. As physical observables the structure functions are independent of the renormalization scale. This yields the renormalization group equation (RGE) [210–214]

$$\mathcal{D}(\mu^2)F_{(2,L)}(N,Q^2)=\mu^2\frac{d}{d\mu^2}F_{(2,L)}(N,Q^2)=0\;. \tag{47}$$

Here the total derivative operator $\mathcal{D}(\mu^2)$ is given by

$$\mathcal{D}(\mu^2) \;\equiv\; \mu^2\frac{\partial}{\partial\mu^2}+\beta(a_s(\mu^2))\frac{\partial}{\partial a_s(\mu^2)}-\sum_i\gamma_{m_i}(a_s(\mu^2))m_i(\mu^2)\frac{\partial}{\partial m_i(\mu^2)}, \tag{48}$$

where the sum runs over all the different masses. The β–function and the anomalous mass dimensions, γ_{m_i}, are given by

$$\beta(a_s(\mu^2)) \;\equiv\; \mu^2\frac{\partial a_s(\mu^2)}{\partial\mu^2}\;, \tag{49}$$

$$\gamma_{m_i}(a_s(\mu^2))) \equiv -\frac{\mu^2}{m_i(\mu^2)}\frac{\partial m_i(\mu^2)}{\partial \mu^2} . \tag{50}$$

From Eqs. (47) and (40) one may derive the RGEs for the parton distribution functions [209]

$$\frac{d}{d\ln\mu^2}\begin{pmatrix}\Sigma(N_F,N,\mu^2)\\ G(N_F,N,\mu^2)\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\gamma_{qq} & \gamma_{qg}\\ \gamma_{gq} & \gamma_{gg}\end{pmatrix}\begin{pmatrix}\Sigma(N_F,N,\mu^2)\\ G(N_F,N,\mu^2)\end{pmatrix} , \tag{51}$$

$$\frac{d}{d\ln\mu^2}\Delta_k(N_F,N,\mu^2) = -\frac{1}{2}\gamma_{qq}^{\mathsf{NS}}\Delta_k(N_F,N,\mu^2) , \tag{52}$$

and for the Wilson coefficients

$$\frac{d}{d\ln\mu^2}\begin{pmatrix}C_{q,i}^{\mathsf{PS}}(N_F,N,Q^2/\mu^2)\\ C_{g,i}(N_F,N,Q^2/\mu^2)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\gamma_{qq} & \gamma_{qg}\\ \gamma_{gq} & \gamma_{gg}\end{pmatrix}\begin{pmatrix}C_{q,i}^{\mathsf{PS}}(N_F,N,Q^2/\mu^2)\\ C_{g,i}(N_F,N,Q^2/\mu^2)\end{pmatrix} , \tag{53}$$

$$\frac{d}{d\ln\mu^2}C_{q,i}^{\mathsf{NS}}(N_F,N,Q^2/\mu^2) = \frac{1}{2}\gamma_{qq}^{\mathsf{NS}}C_{q,i}^{\mathsf{NS}}(N_F,N,Q^2/\mu^2) . \tag{54}$$

Eqs. (52-54) are the QCD evolution equations of the *massless* parton densities and Wilson coefficients. The explicit expressions for the anomalous dimensions and massless Wilson coefficients are found in Refs. [69, 70, 100–103, 105, 161, 162, 181] and [87–91, 93–103, 105, 106].

2.4 Heavy flavor corrections in the limit $Q^2 \gg m^2$

In the case of pure photon exchange[1] the heavy quark contribution to the structure functions $F_{(2,L)}(x, Q^2)$ for one heavy quark of mass m and N_F light flavors are given by, cf. [134],

$$\begin{aligned}
&\frac{1}{x}F_{(2,L)}^{Q\overline{Q}}(x, N_F+1, Q^2, m^2) = \\
&\qquad \sum_{k=1}^{N_F} e_k^2 \Biggl\{ L_{q,(2,L)}^{\mathsf{NS}}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Bigl[f_k(x,\mu^2,N_F) + f_{\overline{k}}(x,\mu^2,N_F)\Bigr] \\
&\qquad\qquad + \frac{1}{N_F} L_{q,(2,L)}^{\mathsf{PS}}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \\
&\qquad\qquad + \frac{1}{N_F} L_{g,(2,L)}^{\mathsf{S}}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \Biggr\} \\
&\quad + e_Q^2 \Biggl[H_{q,(2,L)}^{\mathsf{PS}}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \\
&\qquad\qquad + H_{g,(2,L)}^{\mathsf{S}}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \Biggr] ,
\end{aligned} \tag{55}$$

Here the argument (N_F+1) indicates the presence of N_F massless and one massive quark flavor. One distinguishes the heavy flavor Wilson coefficients $L_{i,q}^{\mathrm{S}}, L_{i,g}, L_{i,q}^{NS}$ and $H_{i,q}^{\mathrm{PS}}, H_{i,q}$, where the photon couples to a light (respectively heavy) quark line [119].
Generally the presence of one additional massive quark flavors introduces much more complex integrals and functions during the computation of the contributing Feynman diagrams when

[1] For heavy flavor corrections in case of $W^\pm$-boson exchange up to $O(\alpha_s^2)$ see [215–218].

compared to the purely massless case. Due to this we restrict ourselves to the kinematically very interesting region where the the virtuality Q^2 is much larger then mass m^2, such that power correction in m^2/Q^2 maybe be safely neglected, but not so large that logarithms of the kind $\ln(m^2/Q^2)$ become too large to allow for a perturbative series approximation.
This kinematic region the heavy flavor Wilson coefficients factor into the light flavor Wilson coefficients $C_{i,(2,L)}^{\mathsf{S,PS,NS}}(N, N_F+1, \frac{Q^2}{\mu^2})$ and the process-independent massive operator matrix elements $A_{ij}^{\mathsf{S,PS,NS}}$, cf. [139, 219],

$$
\begin{aligned}
\mathcal{C}_{j,(2,L)}^{\mathsf{S,PS,NS,asymp}}\Big(N, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\Big) \quad &= \\
\sum_i A_{ij}^{\mathsf{S,PS,NS}}\Big(N, N_F+1, \frac{m^2}{\mu^2}\Big) C_{i,(2,L)}^{\mathsf{S,PS,NS}}\Big(N, N_F+1, \frac{Q^2}{\mu^2}\Big) &+ O\Big(\frac{m^2}{Q^2}\Big) \ . \qquad (56)
\end{aligned}
$$

The explicit expressions for the massive operator matrix elements are obtained from the Green's functions

$$
\begin{aligned}
G_{Q,\mu\nu}^{ij} &= J_N \langle q, j \mid O_i \mid q, i \rangle_Q \ , && (57) \\
G_{Q,\mu\nu}^{ab} &= J_N \langle g, \nu, j \mid O_i \mid g\mu, i \rangle_Q \ . && (58)
\end{aligned}
$$

Here $\langle q, k \mid$, $\langle g, \nu, j \mid$ denote external quark and gluon states, respectively and the indices i,j (a,b) are the color indices of the fundamental(adjoint) representation. The operators O_i are defined in (37-39) and the subscript Q indicates that only contributions that contain at least one heavy quark line are taken into account. The local operators (37-39) are traceless and symmetric under the Lorentz group, whereas the Green's function contains trace terms which do not contribute to the final result. These terms may be projected out from the beginning by contracting with the external source term

$$
J_N \equiv \Delta_{\mu_1} ... \Delta_{\mu_N} \ , \qquad (59)
$$

where Δ_μ represents an arbitrary light-like vector, $\Delta^2 = 0$. The Green's functions still contains a Lorentz or spinor structure and color indices due to the external partonic states. These structure is mapped to 1 to define the massive operator matrix elements.
For the Green's functions with external gluonic states one may choose to include unphysical transverse gluon states in the summation over the indices μ, ν. In this case the respective projection reads [139]

$$
P_g^{(1)} \hat{G}_{l,(Q),\mu\nu}^{ab} \equiv -\frac{\delta_{ab}}{N_c^2-1} \frac{g^{\mu\nu}}{D-2} (\Delta \cdot p)^{-N} \hat{G}_{l,(Q),\mu\nu}^{ab} \ , \qquad (60)
$$

and diagrams with external ghosts have to be included to compensate the unphysical terms. If one uses the physical projector

$$
P_g^{(2)} \hat{G}_{l,(Q),\mu\nu}^{ab} \equiv \frac{\delta_{ab}}{N_c^2-1} \frac{1}{D-2} (\Delta \cdot p)^{-N} \Big(-g^{\mu\nu} + \frac{p^\mu \Delta^\nu + p^\nu \Delta^\mu}{\Delta \cdot p}\Big) \hat{G}_{l,(Q),\mu\nu}^{ab} \ , \qquad (61)
$$

additional ghost diagrams are not required.
For the Green's functions with external fermions one applies the projector

$$
P_q \hat{G}_{l,(Q)}^{ij} \equiv \frac{\delta^{ij}}{N_c} (\Delta \cdot p)^{-N} \frac{1}{4} \mathsf{Tr}[\, \not{p}\, \hat{G}_{l,(Q)}^{ij}] \ . \qquad (62)
$$

Here and in (60)–(61), N_c denotes the number of colors, cf. Appendix A.
The unrenormalized OMEs are then defined by the projections of the Green's functions [139]

$$\hat{\hat{A}}_{lg}\Bigl(\frac{\hat{m}^2}{\mu^2},\varepsilon,N\Bigr) = P_g^{(1,2)}\hat{G}^{ab}_{l,(Q),\mu\nu}\ , \tag{63}$$

$$\hat{\hat{A}}_{lq}\Bigl(\frac{\hat{m}^2}{\mu^2},\varepsilon,N\Bigr) = P_q\hat{G}^{ij}_{l,(Q)}\ . \tag{64}$$

The respective Feynman rules are given in Appendix B. The asymptotic heavy flavor Wilson coefficients are given by [219] [2]

$$\begin{aligned} C^{\mathsf{NS}}_{q,(2,L)}\Bigl(N,N_F,\frac{Q^2}{\mu^2}\Bigr)+L^{\mathsf{NS}}_{q,(2,L)}\Bigl(N,N_F+1,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\Bigr) &= \\ A^{\mathsf{NS}}_{qq,Q}\Bigl(N,N_F+1,\frac{m^2}{\mu^2}\Bigr)C^{\mathsf{NS}}_{q,(2,L)}\Bigl(N,N_F+1,\frac{Q^2}{\mu^2}\Bigr)\ , \end{aligned} \tag{65}$$

in the non–singlet case, and for the pure–singlet and singlet contribution

$$\begin{aligned} C^{\mathsf{PS}}_{q,(2,L)}(N_F)+L^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= \Bigl[A^{\mathsf{NS}}_{qq,Q}(N_F+1)+A^{\mathsf{PS}}_{qq,Q}(N_F+1)+A^{\mathsf{PS}}_{Qq}(N_F+1)\Bigr] \\ &\quad\times N_F\tilde{C}^{\mathsf{PS}}_{q,(2,L)}(N_F+1) \\ &\quad+A^{\mathsf{PS}}_{qq,Q}(N_F+1)C^{\mathsf{NS}}_{q,(2,L)}(N_F+1) \\ &\quad+A_{gq,Q}(N_F+1)N_F\tilde{C}_{g,(2,L)}(N_F+1)\ , \end{aligned} \tag{66}$$

$$\begin{aligned} C_{g,(2,L)}(N_F)+L_{g,(2,L)}(N_F+1) &= A_{gg,Q}(N_F+1)N_F\tilde{C}_{g,(2,L)}(N_F+1) \\ &\quad+A_{qg,Q}(N_F+1)C^{\mathsf{NS}}_{q,(2,L)}(N_F+1) \\ &\quad+\Bigl[A_{qg,Q}(N_F+1)+A_{Qg}(N_F+1)\Bigr] \\ &\quad\times N_F\tilde{C}^{\mathsf{PS}}_{q,(2,L)}(N_F+1)\ . \end{aligned} \tag{67}$$

The $H_{i,j}$s are decomposed into OMEs and light flavor Wilson coefficients by

$$\begin{aligned} H^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= A^{\mathsf{PS}}_{Qq}(N_F+1)\Bigl[C^{\mathsf{NS}}_{q,(2,L)}(N_F+1)+\tilde{C}^{\mathsf{PS}}_{q,(2,L)}(N_F+1)\Bigr] \\ &\quad+\Bigl[A^{\mathsf{NS}}_{qq,Q}(N_F+1)+A^{\mathsf{PS}}_{qq,Q}(N_F+1)\Bigr]\tilde{C}^{\mathsf{PS}}_{q,(2,L)}(N_F+1) \\ &\quad+A_{gq,Q}(N_F+1)\tilde{C}_{g,(2,L)}(N_F+1)\ , \end{aligned} \tag{68}$$

$$\begin{aligned} H_{g,(2,L)}(N_F+1) &= A_{gg,Q}(N_F+1)\tilde{C}_{g,(2,L)}(N_F+1)+A_{qg,Q}(N_F+1)\tilde{C}^{\mathsf{PS}}_{q,(2,L)}(N_F+1) \\ &\quad+A_{Qg}(N_F+1)\Bigl[C^{\mathsf{NS}}_{q,(2,L)}(N_F+1)+\tilde{C}^{\mathsf{PS}}_{q,(2,L)}(N_F+1)\Bigr]\ . \end{aligned} \tag{69}$$

Here and in the following, the index "asymp" to denote the asymptotic heavy flavor Wilson coefficients is omitted. Expanding the expressions (65–69) up to $O(\alpha_s^3)$ yields [139]

$$L^{\mathsf{NS}}_{q,(2,L)}(N_F+1) = a_s^2\Bigl[A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)\ \delta_2+\hat{C}^{(2),\mathsf{NS}}_{q,(2,L)}(N_F)\Bigr]$$

[2] Ref. [219] contains a few inconsistencies concerning the $\tilde{f}$ notation which have been corrected in Ref. [220]

$$
\begin{aligned}
&+\ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\\
&\qquad+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]\ , \qquad (70)
\end{aligned}
$$

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2+A_{gq,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&\qquad+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]\ , \qquad (71)\\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+\ a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\\
&\qquad+A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&\qquad+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] \qquad (73)\\
&+\ a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)\\
&\quad+A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&\quad+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ ,\\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2+\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]\\
&+\ a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\\
&\qquad+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]\\
&+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2+\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\\
&\qquad+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&\qquad+\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1)+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}\\
&\qquad+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]\ , \qquad (74)
\end{aligned}
$$

with $\delta_2 = 1$ for F_2 and $\delta_2 = 0$ for F_L. Here we have applied the notation

$$
\tilde{f}(N_F) = \frac{f(N_F)}{N_F}\ ,\ \ \hat{f}(N_F) = f(N_F+1) - f(N_F)\ . \qquad (75)
$$

2.5 The variable flavor number scheme (VFNS)

In an exact description only massless particles may be interpreted as partons in hard scattering processes since their lifetime must be large compared to the interaction time $\propto \frac{1}{Q^2}$ [221]. For large values of Q^2 the heavy quark flavors may, however, be be considered as becoming effectively massless and *variable flavor number scheme*- parton densities may defined to describe them.
In Mellin–space on obtains the following set of parton densities is obtained, [134] :

$$
f_k(N_F+1, N, \mu^2, m^2) + f_{\overline{k}}(N_F+1, N, \mu^2, m^2) =
$$

$$
\begin{aligned}
&A_{qq,Q}^{\mathsf{NS}}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot \left[f_k(N_F, N, \mu^2) + f_{\overline{k}}(N_F, N, \mu^2)\right] \\
&+\frac{1}{N_F} A_{qq,Q}^{\mathsf{PS}}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot \Sigma(N_F, N, \mu^2) \\
&+\frac{1}{N_F} A_{qg,Q}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot G(N_F, N, \mu^2), \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
&f_Q(N_F+1, N, \mu^2, m^2) + f_{\overline{Q}}(N_F+1, N, \mu^2, m^2) = \\
&\quad A_{Qq}^{\mathsf{PS}}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot \Sigma(N_F, N, \mu^2) \\
&\quad +A_{Qg}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot G(N_F, N, \mu^2)\ . \qquad (77)
\end{aligned}
$$

The flavor singlet, non–singlet and gluon densities for (N_F+1) flavors are

$$
\begin{aligned}
\Sigma(N_F+1, N, \mu^2, m^2) &= \Bigg[A_{qq,Q}^{\mathsf{NS}}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) + A_{qq,Q}^{\mathsf{PS}}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \\
&\quad +A_{Qq}^{\mathsf{PS}}\left(N, N_F+1, \frac{\mu^2}{m^2}\right)\Bigg] \cdot \Sigma(N_F, N, \mu^2) \\
&\quad + \left[A_{qg,Q}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) + A_{Qg}\left(N, N_F+1, \frac{\mu^2}{m^2}\right)\right] \cdot G(N_F, N, \mu^2)\ , \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_k(N_F+1, N, \mu^2, m^2) &= f_k(N_F+1, N, \mu^2, m^2) + f_{\overline{k}}(N_F+1, N, \mu^2, m^2) \\
&\quad -\frac{1}{N_F+1}\Sigma(N_F+1, N, \mu^2, m^2)\ , \qquad (79) \\
G(N_F+1, N, \mu^2, m^2) &= A_{gq,Q}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot \Sigma(N_F, N, \mu^2) \\
&\quad +A_{gg,Q}\left(N, N_F+1, \frac{\mu^2}{m^2}\right) \cdot G(N_F, N, \mu^2)\ . \qquad (80)
\end{aligned}
$$

These expressions, however, rely on the presence of a strong hierarchy among the masses of the heavy quark flavors m_{Q_i},

$$
m_{Q_1}^2 \ll m_{Q_2}^2 \ll \quad \cdots \ll m_{Q_l}\ . \qquad (81)
$$

This condition is not fulfilled when considering charm– and bottom–contributions, with $\eta = m_c^2/m_b^2 \sim \frac{1}{10}$. Up to two-loop order this does not cause an essential problem since no graphs with both c- and b-quarks contribute.This changes at $O(\alpha_s^3)$ and the respective power corrections have to be taken into account, see Sections 5 and 6.

3 Topologies with massless loops

Among the ~ 3000 diagrams contributing to the different massive OMEs at 3–loop order which describe the massive contributions to the structure function $F_2(x, Q^2)$ for $Q^2 \gg m^2$, there are many which contain at least one completely massless fermion loop. These massless loops can be integrated out first and yield new propagators with real exponents. Although these real exponents lead to a complication, methods inspired by those at 2-loop order can be applied to treat this class of diagrams. Using this method the complete OME $A_{qg,Q}$ has been computed [140, 222]. For all contributing diagrams with two massless bubble insertions this method has been completely automatized [223].

3.1 General approach

In [139] a database of fixed Mellin moments for all diagrams contributing to the OMEs $A_{qq,Q}^{\mathsf{NS}}$, $A_{qq,Q}^{\mathsf{PS}}$, A_{Qq}^{PS}, $A_{qg,Q}$, A_{Qg}, $A_{gq,Q}$ and $A_{gg,Q}$ has been given. Codes written in `FORM` [224] and `Maple` [188] have been set up to identify all diagrams with at least one massless loop. After applying the Feynman rules of Appendix B the occurring fermionic traces of γ-matrices were calculated using the built-in `FORM`–functions. The momentum integrals were performed in a loop–by–loop approach: All denominators contributing one loop were combined using the Feynman parametrization in Appendix (C.19). As a next step the momenta are shifted in order to symmetrize the D-dimensional momentum integral. As a byproduct this yields factors of the form

$$(\Delta.k_i + \Delta.(P_1(\{x_k\})k_{j_1} + \cdots + P_n(\{x_n\})k_{j_n}))^N \ , \tag{82}$$

with polynomials P_i in the Feynman parameters. Using the binomial theorem

$$(\Delta.k_i + \Delta.l)^N = \sum_{j=0}^{N} \binom{N}{j} (\Delta.l)^{N-j} (\Delta.k_i)^j \ , \tag{83}$$

only the first few terms contribute. This is due to the property that symmetric momentum integrals over odd powers of the momentum vanish and the occurrence of the contraction $\Delta.\Delta = 0$ for higher powers of $\Delta.k_i$. A maximum of the first three terms had to be considered in the present computation. The symmetric D-dimensional momentum integral was finally evaluated by applying the rules in Appendix C and the same steps were repeated for the remaining loop momenta. Here the order in which the momenta are integrated is not arbitrary. More simple special functions representation may be obtained in many cases by specific choices for the integration order of the loop momenta. One–scale one–loop subdiagrams were integrated out first, as in this case the scales factor out of the denominator functions and the Feynman parameter integrals reflect the integral definition of the Euler–B function

$$B(\alpha, \beta) = \int_0^1 dx_i \ x_i^{\alpha-1} (1 - x_i)^{\beta-1} = \Gamma \left[\begin{matrix} A, C \\ A + C \end{matrix} \right], \tag{84}$$

where the shorthand notation (D.27) has been applied. To avoid more involved special function representations it is furthermore advantageous to integrate out the loop consisting of all massive propagators as a next step if this does not yield too complicated polynomials P_i in the operator-insertion (82). After integrating out all the loop momenta using this prescription all Feynman parameter integrals obey the general form

$$I = \int_0^1 \cdots \int_0^1 \left(\prod_{i=1} \mathrm{d}x_i \; x_i^{\alpha_i} \left(1 - x_i\right)^{\beta_i} \right) \delta\left(1 - \sum_i x_j\right) \cdots \delta\left(1 - \sum_k x_k\right) \frac{OP\left(x_1, \cdots, x_n\right)}{Q^{\alpha_0}} \; . \tag{85}$$

Here $\delta\left(1 - \sum_i x_j\right)$ denotes the Dirac-δ distribution whereas the sums within are taken over all Feynman parameters attached to the different propagators within one loop-momentum integration. Furthermore, the α_i and β_i are either integer numbers or linear polynomials in the dimensional regularization parameter ε and Q is a polynomial in the Feynman parameters describing the mass distribution of the diagram. The precise form of the operator function $OP\left(x_1, \cdots, x_n\right)$ in Eq. (85) depends on the specific operator insertion (see Appendix B):

- operator insertion on a fermionic line

$$OP\left(x_1, \cdots, x_n\right) = P_1^N \tag{86}$$

- operator insertion on a qqg-vertex

$$OP\left(x_1, \cdots, x_n\right) = \sum_{j=0}^{N} P_1^j P_2^{N-j} = \frac{P_1^{N+1} - P_2^{N+1}}{P_1 - P_2} \tag{87}$$

- operator insertion on a $qqgg$-vertex

$$\begin{aligned} OP\left(x_1, \cdots, x_n\right) &= \sum_{j=0}^{N} \sum_{l=0}^{N-j} P_2^j P_1^{N-l-j} \left(P_1 + P_4\right)^l & (88) \\ &= \sum_{j=0}^{N} P_2^j \times \frac{\left(P_1 + P_4\right)^{N-j+1} - P_1^{N-j+1}}{P_4} & (89) \\ &= \frac{\left(P_1 + P_4\right)^{N+2} - \left(P_1 + P_4\right) P_2^{N+1}}{P_4 \left(P_1 + P_4 - P_2\right)} - \frac{P_1^{N+2} - P_1 P_2^{N+1}}{P_4 \left(P_1 - P_2\right)} & (90) \end{aligned}$$

with the P_i polynomials in the Feynman parameters x_i. Using the closed forms (87,89) or (90) if the respective denominators factor into Euler-B-like factors x_i^α or $\left(1 - x_i\right)^\beta$ in all Feynman parameters. In a next step the δ–distributions are integrated out by

$$\int_0^1 \mathrm{d}y \; \delta\left(1 - y \sum_{i=1}^{n} x_i\right) f\left(y, x_1, \cdots, x_n\right) = \theta\left(1 - \sum_{i=1}^{n} x_i\right) f\left(1 - \sum_{i=1}^{n} x_i, x_1, \cdots, x_n\right) \; , \tag{91}$$

where we have introduced the Heaviside function

$$\theta\left(x\right) \;=\; \begin{cases} 1, & \text{if } x > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{92}$$

The Feynman parameter integrals are then remapped to the hypercubus by applying

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \; \theta\left(1 - \sum_{i=1}^{n} x_i\right) \; f\left(x_1, x_2, \cdots, x_n\right) \\ &= \int_0^1 \cdots \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \; \theta\left(1 - \sum_{i=2}^{n} x_i\right) \; f\left(x_1 \left(1 - \sum_{i=2}^{n} x_i\right), x_2, \cdots, x_n\right) \left(1 - \sum_{i=2}^{n} x_i\right) \; . \end{aligned} \tag{93}$$

The way in which Eqs. (91) and (93) are applied may lead to a higher or lower entanglement of the integrals, specifically with respect to the parts stemming from the operator insertion. We will thus discuss strategies to obtain an integral representation with the least possible complexity.
If the remapping (93) of a specific parameter leads to a factorization of polynomials it is applied, e.g:

$$\begin{aligned}
&\int \cdots \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \theta \left(1 - \sum_{i=1}^n x_i\right) \left(x_1 + y_1 \left(1 - \sum_{i=2}^n x_i\right)\right)^\alpha \\
&= \int \cdots \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \theta \left(1 - \sum_{i=2}^n x_i\right) \, (x_1 + y_1)^\alpha \left(1 - \sum_{i=2}^n x_i\right)^\alpha
\end{aligned} \tag{94}$$

In a next step we map parameters that do not appear in the operator polynomials or other non-trivial factors. Here we always choose factors such that the corresponding mapping does not lead to additional terms with symbolic or negative exponents. If a Feynman parameter occurs only within one factor with negative or symbolic exponent, it may be integrated out directly. If these steps are not sufficient to obtain a factorization of the polynomials in the operator function into B–like terms, the binomial theorem is applied in order to impose this factorization.
After applying all these steps in an optimal way all diagrams of this class obey a representation of the form

$$I \propto \int_0^1 \cdots \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \prod_i x_i^{\alpha_i} \left(1 - x_i\right)^{\beta_i} \frac{OP'\left(x_1, \cdots, x_n\right)}{\left(1 - x_k x_l\right)^\gamma} \, . \tag{95}$$

Here the α_i and β_i denote linear functions in ε and the Mellin variable N, $x_k, x_l \in \{x_1, \cdots, x_n\}$. The exponent γ may be zero or a linear expression in ε and the operator function OP' may or may not contain finite sums stemming from respective operator Feynman rules. All integrals except for the ones over the Feynman parameters x_k and x_l are evaluated using the integral representation of the Euler–B function (84) and its representation in terms of Γ-functions (84). The nested integral over the two remaining Feynman parameters x_k and x_l reflects the integral definition of the hypergeometric ${}_3F_2$ function. For the generalized hypergeometric function ${}_{P+1}F_P$ it is given by, cf. [142, 143, 225],

$$\begin{aligned}
{}_{P+1}F_P\left[\begin{matrix} a_0, a_1, \ldots, a_P \\ b_1, \ldots, b_P \end{matrix}; z\right] &= \Gamma\left[\begin{matrix} b_1, \ldots, b_P \\ a_1, \ldots, a_P, b_1 - a_1, \ldots, b_P - a_P \end{matrix}\right] \\
&\times \int_0^1 \cdots \int_0^1 \prod_{i=1}^P \left(dx_i \;\; x_i^{a_i - 1}(1 - x_i)^{b_i - a_i - 1}\right) (1 - z x_1 \cdots x_P)^{-a_0} \, ,
\end{aligned} \tag{96}$$

It obeys the series representation

$${}_PF_Q\left[\begin{matrix} a_1, ..., a_P \\ b_1, ..., b_Q \end{matrix}; z\right] = \sum_{i=0}^\infty \frac{(a_1)_i ... (a_P)_i}{(b_1)_i ... (b_Q)_i} \frac{z^i}{\Gamma(i+1)} \, ., \tag{97}$$

with the radius of convergence

$$|z| < 1, \;\; \text{or} \;\; z = 1, \; \mathrm{Re}\left(\sum_{i=1}^P b_i - \sum_{i=1}^{P+1} a_i\right) > 0 \, . \tag{98}$$

The hypergeometric functions obey a class of contiguous relations [142, 225–228], which may be used in some cases to reduce the occurring ${}_3F_2$ functions to the Gauss hypergeometric function ${}_2F_1$. In this case Gauß' theorem

$$
{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \Gamma\left[\begin{matrix} c, c-a-b \\ c-a, c-b \end{matrix}\right] , \quad \mathsf{Re}(c-a-b) > 0 \; , \tag{99}
$$

is applied to obtain a sum–free representation. All diagrams are thus expressed in terms of nested sums over Γ–functions. This representation now allows to extract the ε-poles explicitly and to perform the Laurent series expansion using Eqns. (D.23-D.26). In a last step, the remaining sums are evaluated using summation algorithms based on difference fields [229–235] encoded in the packages `Sigma` [236, 237] and `EvaluateMultiSums` [238, 239] . In order to treat infinite summation bounds and to speed up the manipulations of expressions in terms of indefinite nested sums, the package `HarmonicSums` [175, 177–179] is utilized in addition.

3.2 The OME $A_{qg,Q}$

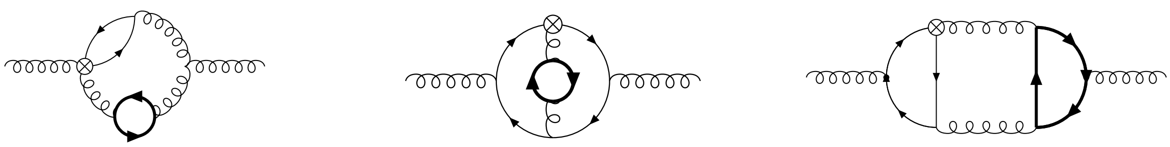


Figure 2: Some typical diagrams contributing to the OME $A_{qg,Q}$. Thick lines represent heavy quarks, whereas thin ones depict massless fermion flavors.

Most of the diagrams contributing to the operator matrix element $A_{qg,Q}$ contain at least one single–scale loop and do thereby belong to the class of diagrams discussed in the present Section. Furthermore a series of ladder graphs (as the rightmost graph in Figure 2) contributed. This class of graphs may be represented within the same set of special functions and is evaluated with the same methodology. The diagrams contributing to $A_{qg,Q}$ are very similar to the diagrams describing the $O\left(N_F T_F^2\right)$ contribution to the OME A_{Qg} with the only difference being the assignment of the masses to the different fermionic lines. All together there are 89 diagrams with external gluons and 8 ghost–diagrams contributing to the OME $A_{qg,Q}$. They can be grouped into 35 symmetry classes, where all members of a symmetry class yield the same result due to symmetry reasons. In this way we obtain for the constant term of the unrenormalized OME $\hat{\hat{A}}_{qg,Q}$ the expression

$$
\begin{aligned}
a_{qg,Q}^{(3),0} \;=\;& N_F T_F^2 \Biggl\{ C_F \Biggl[\frac{N^2+N+2}{N(N+1)(N+2)} \Bigl[-\frac{56}{9} S_4 + \frac{32}{27} S_3 S_1 + \frac{8}{9} S_2 S_1^2 + \frac{4}{9} S_2^2 + \frac{4}{27} S_1^4 \\
& + \frac{256}{9} S_1 \zeta_3 \Bigr] - \frac{16(10N^3+13N^2+29N+6)}{81N^2(1+N)(2+N)} \left[S_1^3 + 3 S_2 S_1 \right] \\
& + \frac{32(5N^3-16N^2+N-6)}{81N^2(1+N)(2+N)} S_3 + \frac{8(109N^4+291N^3+478N^2+324N+40)}{27N^2(1+N)^2(2+N)} S_2 \\
& + \frac{8(215N^4+481N^3+930N^2+748N+120)}{81N^2(1+N)^2(2+N)} S_1^2 - \frac{R_4}{243N^2(1+N)^3(2+N)} S_1
\end{aligned}
$$

$$
-\frac{64(N^2+N+2)R_5}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3+\frac{R_6}{243(N-1)N^6(1+N)^6(2+N)^5}\Bigg]
$$

$$
+C_A\Bigg[\frac{N^2+N+2}{N(N+1)(N+2)}\Bigg[-\frac{56}{9}S_4-\frac{128}{9}S_{-4}+\frac{160}{27}S_3S_1-\frac{4}{9}S_2^2+\frac{8}{9}S_2S_1^2
$$

$$
-\frac{4}{27}S_1^4-\frac{64}{9}S_{2,1}S_1-\frac{128}{9}S_{3,1}+\frac{64}{9}S_{2,1,1}-\frac{256}{9}\zeta_3S_1\Bigg]
$$

$$
+\frac{32(5N^4+20N^3+41N^2+49N+20)}{81N(1+N)^2(2+N)^2}\left[S_1^3+12S_{2,1}-3S_2S_1\right]
$$

$$
+\frac{64}{81}\frac{(5N^4+38N^3+59N^2+31N+20)}{N(1+N)^2(2+N)^2}S_3+\frac{128}{27}\frac{(5N^2+8N+10)}{N(1+N)(2+N)}S_{-3}
$$

$$
+\frac{512}{9}\frac{(N^2+N+1)(N^2+N+2)}{(N-1)N^2(1+N)^2(2+N)^2}\zeta_3-\frac{16R_7}{81N(1+N)^3(2+N)^3}S_2
$$

$$
-\frac{32(121N^3+293N^2+414N+224)}{81N(1+N)^2(2+N)}S_{-2}-\frac{R_8}{81N(1+N)^3(2+N)^3}S_1^2
$$

$$
+\frac{16R_9}{243(N-1)N^2(1+N)^4(2+N)^4}S_1+\frac{8R_{10}}{243(N-1)N^5(1+N)^5(2+N)^5}\Bigg]\Bigg\}, \tag{100}
$$

with the polynomials

$$
\begin{aligned}
R_4 &= 24368N^5+81984N^4+179200N^3+225232N^2+126880N+21504\ , && (101)\\
R_5 &= 3N^6+9N^5-N^4-17N^3-38N^2-28N-24\ , && (102)\\
R_6 &= 13923N^{17}+180999N^{16}+1064857N^{15}+3812487N^{14}+9348807N^{13}\\
&\quad +16391845N^{12}+20248499N^{11}+17070917N^{10}+11536274N^9+11303496N^8\\
&\quad +13846104N^7+16104128N^6+22643488N^5+29337472N^4\\
&\quad +26395008N^3+15388416N^2+5612544N+995328\ , && (103)\\
R_7 &= 139N^6+1093N^5+3438N^4+5776N^3+5724N^2+3220N+752\ , && (104)\\
R_8 &= 1648N^6+11104N^5+34368N^4+63856N^3+71904N^2+43264N+10880\ , && (105)\\
R_9 &= 1244N^{10}+10557N^9+40547N^8+90323N^7+114495N^6+49344N^5\\
&\quad -69902N^4-115200N^3-64352N^2-11264N+864\ , && (106)\\
R_{10} &= 3315N^{15}+39780N^{14}+194011N^{13}+471164N^{12}+416251N^{11}-860568N^{10}\\
&\quad -3525799N^9-6015120N^8-6333994N^7-4373672N^6-1907512N^5\\
&\quad -499824N^4-217952N^3-264192N^2-160128N-34560\ . && (107)
\end{aligned}
$$

Here C_F, C_A and T_F denote the color factors, cf. (A.20-A.23). $S_k \equiv S_k(N)$ denotes the single harmonic sum (D.44) and $\zeta_k = \sum_{k=1}^{\infty}$, $k \geq 2$, $k \in \mathbb{N}$ are values of the Riemann ζ-function. The heavy flavor Wilson coefficient L_g^{S} is thus completed.

3.3 Automatized computation[3]

Additionally to the $C_AT_F^2N_F$ and the $C_FT_F^2N_F$ contributions there is a much larger class of diagrams with similar topologies contributing to the relevant operator matrix elements. All

[3]In collaboration with Mark Round.

OMEs	#
A_{Qg} (including ghost diagrams)	100
$A_{qq,Q}^{\rm NS}$	29
$A_{qq,Q}^{\rm NS, Trans}$	29
$A_{Qq,Q}^{\rm PS}$	30
$A_{gg,Q}$ (including ghost diagrams)	89
$A_{gq,Q}$	27

Table 1: Number of diagrams that could be computed in an automated way via representations in special functions

diagrams with at least one massless internal two-point function are in principle computable with the same method. Due to the relatively large number of these diagrams it is advantageous to set up automatized procedures in order to allow for an efficient evaluation.
All contributing diagrams have been generated using `QGRAF` in Ref. [139]. We used the same diagram database to allow for easier comparison with the fixed Mellin moments computed in this reference. `FORM`–codes have been set up to implement the Feynman rules including those for the operator insertions, see Appendix B. A `Mathematica` code then analyzes the specific diagram topologies, selects those diagrams with at least one massless bubble subtopology and chooses an optimal integration order for the different loop-momenta. This integration order is determined by the following criteria

- Integrate all completely massless loops first
- Integrate all completely massive loops
- If in doubt integrate the loop without any operator insertion first
- Otherwise choose next integration momenta randomly.

.

The momenta are then integrated using a `FORM`–code and a Feynman–parameter representation is obtained. We apply the strategies outlined in Section 3.1 to map the integrals to the hypercubus in a way that avoids further entanglement among the different Feynman parameter integrals. All Feynman parameter integrals which may be performed without yielding a higher nesting among the remaining integration variables are performed at this step and the binomial theorem is applied until all remaining integrals correspond to the integral definitions of the Euler-B function or a generalized hypergeometric ${}_PF_Q$–function. The ε–expansion is performed using the sum representation of the hypergeometric functions and the nested sums are evaluated in a final step using the summation packages `Sigma` $[236, 237]$, `EvaluateMultiSums` $[238, 239]$ and ρ`-sum` $[240, 241]$. For fixed values of the Mellin variable N, all results were in agreement with the Mellin moments computed in Ref. [139].
All results of the individual diagrams are expressed within the class of nested harmonic sums up to weight $w = 5$, and the constants ζ_2 and ζ_3.

4 Logarithmic contributions to the heavy flavour Wilson coefficients

The computation of the complete heavy flavour Wilson coefficients at $O(\alpha_s^3)$ constitutes a very challenging task due to very large expressions and complex mathematical structures. Due to the renormalization procedure [139] we know, however, that all logarithmic contributions in (m^2/μ^2) can be expressed in terms of the 3–loop anomalous dimensions and different lower order quantities[4]. All 3–loop contributions to the two Wilson coefficients L_g^S and $L_q^{\rm PS}$ have been computed and they are now presented in a complete form, see also Ref. [220]. The complete Wilson coefficients $L_q^{\rm NS}$ and $H_q^{\rm PS}$ including logarithmic and non-logarithmic contributions can be found in Refs. [157, 158, 223, 244, 245]. Furthermore all logarithmic contributions to the operator matrix elements, which are needed vor the variable flavor number scheme have been computed [156].
The single-mass contributions to the operator matrix elements have the general form

$$\hat{A}_{ij}^{(3)}(\varepsilon) = \left(\frac{m^2}{\mu^2}\right)^{3/2\varepsilon/2} \left[\frac{1}{\varepsilon^3}\hat{a}_{ij}^{(3),3} + \frac{1}{\varepsilon^2}\hat{a}_{ij}^{(3),2} + \frac{1}{\varepsilon}\hat{a}_{ij}^{(3),1} + a_{ij}^{(3),0}\right] , \tag{108}$$

where the $\hat{a}_{ij}^{(3),k}$ are functions of the Mellin variable N only. After renormalizing the mass, the coupling, the operator and applying mass factorization, the renormalized expressions for the massive operator matrix elements A_{ij} obey the general structure [139, 246]

$$A_{ij}^{(3)}\left(\frac{m^2}{\mu_R^2}\right) = \hat{a}_{ij}^{(3),3}\ln^3\left(\frac{m^2}{\mu_R^2}\right) + \hat{a}_{ij}^{(3),2}\ln^2\left(\frac{m^2}{\mu_R^2}\right) + \hat{a}_{ij}^{(3),1}\ln\left(\frac{m^2}{\mu_R^2}\right) + \hat{a}_{ij}^{(3),0} , \tag{109}$$

where μ_R denotes the renormalization scale. For simplicity we will identify the renormalization scale with the factorization scale from now on, $\mu_R = \mu_F = \mu$ and use the shorthand notation

$$L_M = \ln\left(\frac{m^2}{\mu^2}\right) . \tag{110}$$

Due to the form of the mass-dependence in (108) the coefficients $\hat{a}^{(3),1}$, $\hat{a}^{(3),2}$ and $\hat{a}^{(3),3}$ depend on the pole terms of the unrenormalized OME only. The renormalization procedure introduces terms which cancel these pole terms and all logarithmic coefficients of the renormalized OMEs can thus be expressed in terms of anomalous dimensions and lower order quantities. The only contributions that have not been known yet are the constant parts of the 3–loop OMEs a_{ij}. The analytic expression for $a_{qq}^{\sf PS}$ has been published in [140] as well as $a_{qg,Q}$ the computation of which is outlined in Section 3. In the $\overline{\sf MS}$–scheme the renormalized OMEs $A_{qq,Q}^{\sf PS}$ and $A_{qg,Q}$ read [139]

$$\begin{aligned} A_{qq,Q}^{(3),{\sf PS},\overline{\sf MS}} &= N_F\Biggl\{\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}}{12}L_M^3 + \frac{1}{8}\Bigl(4\hat{\gamma}_{qq}^{(1),{\sf PS}}\beta_{0,Q} + \hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)}\Bigr)L_M^2 \\ &\quad +\frac{1}{4}\Bigl(2\hat{\tilde{\gamma}}_{qq}^{(2),{\sf PS}} + \hat{\gamma}_{qg}^{(0)}\Bigl\{2a_{gq,Q}^{(2)} - \gamma_{gq}^{(0)}\beta_{0,Q}\zeta_2\Bigr\}\Bigr)L_M \end{aligned}$$

[4] For heavy flavour Wilson coefficients depending on both m_b and m_c there are further logarithms in the mass ratio $\eta = m_c^2/m_b^2$ which can not be determined by the renormalization procedure only, [242, 243]

$$-\hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} + \frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\zeta_3}{12} - \frac{\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\beta_{0,Q}\zeta_2}{4}\Bigg\} + a_{qq,Q}^{(3),\mathsf{PS}} \;, \tag{111}$$

and

$$\begin{aligned}
A_{qg,Q}^{(3),\overline{\mathsf{MS}}} &= N_F\Bigg[\frac{\hat{\gamma}_{qg}^{(0)}}{48}\Bigg\{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + 2\beta_{0,Q}\Big(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0\Big)\Bigg\}L_M^3 + \frac{1}{8}\Bigg\{2\hat{\gamma}_{qg}^{(1)}\beta_{0,Q} \\
&+\hat{\gamma}_{qg}^{(0)}\Big(\hat{\gamma}_{qq}^{(1),\mathsf{PS}} - \hat{\gamma}_{qq}^{(1),\mathsf{NS}} + \hat{\gamma}_{gg}^{(1)} + 2\beta_{1,Q}\Big)\Bigg\}L_M^2 + \frac{1}{2}\Bigg\{\hat{\hat{\gamma}}_{qg}^{(2)} + \hat{\gamma}_{qg}^{(0)}\Big(a_{gg,Q}^{(2)} \\
&-a_{qq,Q}^{(2),\mathsf{NS}} + \beta_{1,Q}^{(1)}\Big) - \frac{\hat{\gamma}_{qg}^{(0)}}{8}\zeta_2\Big(\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + 2\beta_{0,Q}\Big[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0\Big]\Big)\Bigg\}L_M \\
&+\hat{\gamma}_{qg}^{(0)}\Big(\overline{a}_{qq,Q}^{(2),\mathsf{NS}} - \overline{a}_{gg,Q}^{(2)} - \beta_{1,Q}^{(2)}\Big) + \frac{\hat{\gamma}_{qg}^{(0)}}{48}\zeta_3\Big(\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + 2\beta_{0,Q}\Big[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0\Big]\Big) \\
&-\frac{\zeta_2}{16}\Big(\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{(1),\mathsf{PS}} + 2\hat{\gamma}_{qg}^{(1)}\beta_{0,Q}\Big) + \frac{a_{qg,Q}^{(3)}}{N_F}\Bigg] \;.
\end{aligned} \tag{112}$$

In (111,112) the γ_{ij} denote the anomalous dimensions, and the β_i and $\beta_{i,Q}^j$ coefficients of the QCD-β function, see Eqs. (195-198). The light flavor Wilson coefficients obey the general form

$$C_i\left(N, a_s, Q^2/\mu^2\right) = \sum_{l=0}^{\infty} a_s^l \sum_{j=0}^{l} c_i^{l,j} L_Q^j \;, \tag{113}$$

with the logarithm

$$L_Q = \ln\left(\frac{Q^2}{\mu^2}\right) \;. \tag{114}$$

We use the matrix notation

$$c^{j,k} = \left(c_q^{j,k}, c_g^{j,k}\right) \;, \tag{115}$$

and

$$\gamma^{(k)} = \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \;. \end{pmatrix} \tag{116}$$

Here $c^{0,0} = (1,0)$ and the first order contributions $c^{1,0}$ are given in Ref. [89]. The second order contributions have been computed in Refs. [95–97]. The coefficients $c_i^{l,j}$ with $j > 1$ follow from the evolution equation obtained by resolving the RGE for the Wilson coefficients. Up to $O(a_s^3)$ they are given by [247]

$$c^{1,1} = c^{0,0}\gamma^{(0)} \tag{117}$$

$$c^{2,1} = c^{0,0}\gamma^{(1)} + c^{1,0}\left(\gamma^{(0)} - \beta_0 E\right) \tag{118}$$

$$c^{2,2} = \frac{1}{2}c^{1,1}\left(\gamma^{(0)} - \beta_0 E\right) \tag{119}$$

$$c^{3,1} = c^{0,0}\gamma^{(2)} + c^{1,0}\left(\gamma^{(1)} - \beta_1 E\right) + c^{2,0}\left(\gamma^{(0)} - 2\beta_0 E\right) \tag{120}$$

$$c^{3,2} = \frac{1}{2}\left(c^{1,1}\left(\gamma^{(1)} - \beta_1 E\right) + c^{2,1}\left(\gamma^{(0)} - 2\beta_0 E\right)\right) \quad (121)$$

$$c^{3,3} = \frac{1}{3}c^{2,2}\left(\gamma^{(0)} - 2\beta_0 E\right) \ , \quad (122)$$

with E the unity matrix.
For $Q^2 = \mu^2$ the light flavor Wilson coefficients are given in [87–91, 93–103, 105, 106, 248]. Renormalizing the OMEs using Eqs. (283,112) and inserting the result into (71,72) yields the complete expression for the respective heavy flavour Wilson coefficients. The leading order contribution to the Wilson coefficient $L_{q,2}^{\mathsf{PS}}$ starts at $O(a_s^3)$. In Mellin N-space it reads

$$\begin{aligned}
L_{q,2}^{\mathsf{PS}} = &\tfrac{1}{2}\left[1+(-1)^N\right] \\
&\times a_s^3 \Biggl\{ C_F N_F T_F^2 \Biggl[-\frac{32 P_4 L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_Q \Biggl[\frac{64 P_6}{27(N-1)N^4(N+1)^4(N+2)^3} \\
&- \frac{256 P_1 (-1)^N}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{2(\hat{\gamma}_{qg}^0)^2 (N+2) L_M^2}{3(N-1)} \\
&+ \Biggl[\frac{64\left(N^2+N+2\right)\left(8N^3+13N^2+27N+16\right)}{9(N-1)N^2(N+1)^3(N+2)} - \frac{64\left(N^2+N+2\right)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \Biggr] L_M \\
&+ \frac{512 S_{-2}}{3(N-1)N(N+1)(N+2)} \Biggr] - \frac{32 P_4 L_M^2}{9(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \Biggl[-\frac{32 P_7}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64 P_2 S_1}{3(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)} \frac{32}{3}\left(S_1^2 - S_2\right) \Biggr] L_M \\
&- \frac{32 P_9}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{32 P_8 S_1}{81(N-1)N^4(N+1)^4(N+2)^3} \\
&- \frac{16 P_3 S_1^2}{27(N-1)N^3(N+1)^3(N+2)^2} - \frac{16 P_5 S_2}{27(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{32 L_Q^3 \left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)} - \frac{32\left(N^2+N+2\right)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} \\
&+ \frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{64}{27} S_1^3 + \frac{32}{9} S_2 S_1 + \frac{160 S_3}{27} + \frac{256\zeta_3}{9} \right] \Biggr] \\
&+ N_F \hat{\tilde{C}}_{2,q}^{\mathsf{PS},(3)}(N_F) \Biggr\} \ ,
\end{aligned} \quad (123)$$

with the polynomials

$$P_1 = 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \quad (124)$$

$$P_2 = N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \quad (125)$$

$$P_3 = N^7 - 37N^6 - 248N^5 - 799N^4 - 1183N^3 - 970N^2 - 580N - 168 \quad (126)$$

$$P_4 = 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \quad (127)$$

$$P_5 = 49N^7 + 185N^6 + 340N^5 + 287N^4 + 65N^3 + 62N^2 - 196N - 168 \quad (128)$$

$$\begin{aligned} P_6 = &85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 \\ &+1864N^2 + 1248N + 432 \end{aligned} \quad (129)$$

$$
\begin{aligned}
P_7 &= 143N^{10}+838N^9+1995N^8+1833N^7-1609N^6-5961N^5-7503N^4-6928N^3\\
&\quad -4024N^2-816N+144 && (130)\\
P_8 &= 176N^{10}+973N^9+1824N^8-948N^7-10192N^6-19173N^5-20424N^4-16036N^3\\
&\quad -7816N^2-1248N+288 && (131)\\
P_9 &= 1717N^{13}+16037N^{12}+66983N^{11}+161797N^{10}+241447N^9+216696N^8+86480N^7\\
&\quad -67484N^6-170003N^5-165454N^4-81976N^3-15792N^2-1008N-864\,. && (132)
\end{aligned}
$$

For brevity, here and in the following the massless 3-loop Wilson coefficients $C_{i,j}^k$ have been left symbolically. Their analytic expressions were given in Ref. [106]. Furthermore we have used the shorthand notation

$$
\tilde{\gamma}_{qg}^0 = -4\frac{N^2+N+2}{N(N+1)(N+2)}\ . \tag{133}
$$

The gluonic Wilson coefficient $L_{g,2}^S$ reads

$$
\begin{aligned}
L_{g,2}^{\mathsf{S}} &= \tfrac{1}{2}\left[1+(-1)^N\right]\Biggl\{a_s^2 T_F^2 N_F \Biggl\{L_M\left[\frac{4}{3}\tilde{\gamma}_{qg}^0 S_1 - \frac{16\bigl(N^3-4N^2-N-2\bigr)}{3N^2(N+1)(N+2)}\right]\\
&\quad -\frac{4}{3}\tilde{\gamma}_{qg}^0 L_Q L_M\Biggr\} + a_s^3\Biggl\{N_F T_F^3\left[L_M^2\left[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 - \frac{64\bigl(N^3-4N^2-N-2\bigr)}{9N^2(N+1)(N+2)}\right] - \frac{16}{9}\tilde{\gamma}_{qg}^0 L_Q L_M^2\right]\\
&\quad +C_A N_F T_F^2\Biggl[\left[\left[\frac{64\bigl(N^2+N+1\bigr)\bigl(N^2+N+2\bigr)}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\right]L_Q^3\right.\\
&\quad +\Biggl[-\frac{64(-1)^N\bigl(N^3+4N^2+7N+5\bigr)}{3(N+1)^3(N+2)^3}+\frac{8P_{25}}{9(N-1)N^3(N+1)^3(N+2)^3}\\
&\quad +\frac{32\bigl(8N^4-7N^3+5N^2-17N-13\bigr)S_1}{9(N-1)N(N+1)^2(N+2)}+L_M\Biggl[\frac{64\bigl(N^2+N+1\bigr)\bigl(N^2+N+2\bigr)}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&\quad +\frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\Biggr]+\tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^2+\frac{4S_2}{3}+\frac{8}{3}S_{-2}\right]\Biggr]L_Q^2\\
&\quad +\Biggl[-\frac{32\bigl(8N^4-7N^3+5N^2-17N-13\bigr)S_1^2}{9(N-1)N(N+1)^2(N+2)}+\frac{128(-1)^N\bigl(N^3+4N^2+7N+5\bigr)S_1}{3(N+1)^3(N+2)^3}\\
&\quad -\frac{32P_{24}S_1}{27(N-1)N^2(N+1)^3(N+2)^3}+\frac{64(-1)^N P_{18}}{9(N-1)N^2(N+1)^4(N+2)^4}\\
&\quad -\frac{16P_{32}}{27(N-1)N^3(N+1)^4(N+2)^4}+L_M^2\Biggl[\frac{64\bigl(N^2+N+1\bigr)\bigl(N^2+N+2\bigr)}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&\quad +\frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\Biggr]+\frac{32\bigl(8N^4+13N^3-22N^2-9N-26\bigr)S_2}{9(N-1)N(N+1)(N+2)^2}+\frac{128\bigl(N^2+N-1\bigr)S_3}{9N(N+1)(N+2)}\\
&\quad +\frac{64\bigl(8N^5+15N^4+6N^3+11N^2+16N+16\bigr)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}+L_M\Biggl[\frac{32P_{26}}{9(N-1)N^3(N+1)^3(N+2)^3}\\
&\quad -\frac{128(-1)^N\bigl(N^3+4N^2+7N+5\bigr)}{3(N+1)^3(N+2)^3}-\frac{64(2N-1)\bigl(N^3+9N^2+7N+7\bigr)S_1}{9(N-1)N(N+1)^2(N+2)}\\
&\quad +\tilde{\gamma}_{qg}^0\left[-\frac{8}{3}S_1^2+\frac{8S_2}{3}+\frac{16}{3}S_{-2}\right]\Biggr]-\frac{128\bigl(N^2+N+3\bigr)S_{-3}}{3N(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[\frac{8}{9}S_1^3-8S_2S_1+\frac{32}{3}S_{2,1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{256S_{-2,1}}{3N(N+1)(N+2)}+\frac{(N-1)\Bigl[\frac{64}{3}S_{-2}S_1-32\zeta_3\Bigr]}{N(N+1)}\Biggr]L_Q+\frac{16P_{12}S_1^2}{81N(N+1)^3(N+2)^3}\\
&+\frac{8P_{39}}{243(N-1)N^5(N+1)^5(N+2)^5}+\frac{512}{9}\frac{\bigl(N^2+N+1\bigr)\bigl(N^2+N+2\bigr)}{(N-1)N^2(N+1)^2(N+2)^2}\zeta_3\\
&+\frac{8P_{36}S_1}{243(N-1)N^4(N+1)^4(N+2)^4}+L_M^3\Biggl[-\frac{64\bigl(N^2+N+1\bigr)\bigl(N^2+N+2\bigr)}{9(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\Biggr]-\frac{16P_{13}S_2}{81N(N+1)^3(N+2)^3}+\frac{64\bigl(5N^4+38N^3+59N^2+31N+20\bigr)S_3}{81N(N+1)^2(N+2)^2}\\
&-\frac{32\bigl(121N^3+293N^2+414N+224\bigr)S_{-2}}{81N(N+1)^2(N+2)}+L_M^2\Biggl[-\frac{64(-1)^N\bigl(N^3+4N^2+7N+5\bigr)}{3(N+1)^3(N+2)^3}\\
&+\frac{8P_{25}}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32\bigl(8N^4-7N^3+5N^2-17N-13\bigr)S_1}{9(N-1)N(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Bigl[-\frac{4}{3}S_1^2\\
&+\frac{4S_2}{3}+\frac{8}{3}S_{-2}\Bigr]\Biggr]+\frac{128\bigl(5N^2+8N+10\bigr)S_{-3}}{27N(N+1)(N+2)}\\
&+\frac{\bigl(5N^4+20N^3+41N^2+49N+20\bigr)\Bigl[\frac{32}{81}S_1^3-\frac{32}{27}S_2S_1+\frac{128}{27}S_{2,1}\Bigr]}{N(N+1)^2(N+2)^2}\\
&+L_M\Biggl[\frac{32\bigl(2N^5+21N^4+27N^3+11N^2+25N-14\bigr)S_1^2}{9(N-1)N(N+1)^2(N+2)^2}+\frac{16P_{27}S_1}{27(N-1)N^3(N+1)^3(N+2)^3}\\
&+\frac{128(-1)^N\bigl(N^3+4N^2+7N+5\bigr)S_1}{3(N+1)^3(N+2)^3}-\frac{16}{3}\tilde{\gamma}_{qg}^0S_2S_1-\frac{64(-1)^NP_{14}}{9(N-1)N^2(N+1)^3(N+2)^4}\\
&+\frac{16P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4}-\frac{32\bigl(2N^5+21N^4+51N^3+23N^2-11N-14\bigr)S_2}{9(N-1)N(N+1)^2(N+2)^2}\\
&+\frac{64S_3}{3(N+2)}-\frac{64\bigl(2N^5+21N^4+36N^3-7N^2-68N-56\bigr)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}+\frac{\frac{256}{3}S_{-2,1}-\frac{128}{3}S_{-3}}{N(N+1)(N+2)}\\
&+\frac{(N-1)\bigl(\frac{64}{3}S_{-2}S_1-32\zeta_3\bigr)}{N(N+1)}\Biggr]+\tilde{\gamma}_{qg}^0\Bigl[\frac{1}{27}S_1^4-\frac{2}{9}S_2S_1^2+\Bigl[\frac{16}{9}S_{2,1}-\frac{40S_3}{27}\Bigr]S_1\\
&+\frac{64}{9}\zeta_3S_1+\frac{1}{9}S_2^2+\frac{14S_4}{9}+\frac{32}{9}S_{-4}+\frac{32}{9}S_{3,1}-\frac{16}{9}S_{2,1,1}\Bigr]\Biggr]\\
&+C_FN_FT_F^2\Biggl[\Biggl[\frac{16\bigl(N^2+N+1\bigr)\bigl(N^2+N+2\bigr)\bigl(3N^4+6N^3-N^2-4N+12\bigr)}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{16}{9}\tilde{\gamma}_{qg}^0S_1\Biggr]L_Q^3\\
&+\Biggl[-\frac{4P_{31}}{9(N-1)N^4(N+1)^4(N+2)^3}+\frac{16P_{21}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\Bigl[\frac{20S_2}{3}-4S_1^2\Bigr]\\
&+L_M\Biggl[\frac{8\bigl(N^2+N+2\bigr)P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2}+\frac{8}{3}\tilde{\gamma}_{qg}^0S_1\Biggr]\Biggr]L_Q^2+\Biggl[\frac{16L_M^2\bigl(N^2+N+2\bigr)^3}{(N-1)N^3(N+1)^3(N+2)^2}\\
&-\frac{16P_{22}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{64(-1)^NP_{37}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}\\
&+\frac{4P_{42}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}-\frac{8P_{30}S_1}{9(N-1)N^4(N+1)^4(N+2)^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{16P_{23}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}+L_M\Biggl[-\frac{16P_{28}}{3(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{16P_{17}S_1}{3(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\Bigl(\frac{16S_2}{3}-\frac{16}{3}S_1^2\Bigr)\Biggr]-\frac{256\bigl(N^2+N+1\bigr)S_3}{3N(N+1)(N+2)}\\
&+\frac{64P_{16}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}+\tilde{\gamma}_{qg}^0\Bigl[\frac{8}{3}S_1^3-8S_2S_1-\frac{32}{3}S_{2,1}\Bigr]\\
&+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}+\frac{64(N-1)\zeta_3}{N(N+1)}\Biggr]L_Q-\frac{64}{9}\frac{\bigl(N^2+N+2\bigr)P_{10}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}\\
&+\frac{8\bigl(215N^4+481N^3+930N^2+748N+120\bigr)S_1^2}{81N^2(N+1)^2(N+2)}+\frac{P_{40}}{243(N-1)N^6(N+1)^6(N+2)^5}\\
&-\frac{4P_{35}S_1}{243(N-1)N^5(N+1)^5(N+2)^2}+L_M^3\Biggl[\frac{8\bigl(N^2+N+2\bigr)P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\Biggr]\\
&+L_M^2\Biggl[\frac{4P_{29}}{9(N-1)N^4(N+1)^4(N+2)^3}-\frac{16P_{20}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Bigl[-\frac{4}{3}S_1^2-\frac{4S_2}{3}\Bigr]\Biggr]+\frac{8\bigl(109N^4+291N^3+478N^2+324N+40\bigr)S_2}{27N^2(N+1)^2(N+2)}\\
&+\frac{\bigl(10N^3+13N^2+29N+6\bigr)\Bigl[-\frac{16}{81}S_1^3-\frac{16}{27}S_2S_1\Bigr]}{N^2(N+1)(N+2)}+\frac{32\bigl(5N^3-16N^2+N-6\bigr)S_3}{81N^2(N+1)(N+2)}\\
&+\tilde{\gamma}_{qg}^0\Bigl[-\frac{1}{27}S_1^4-\frac{2}{9}S_2S_1^2-\frac{8}{27}S_3S_1-\frac{64}{9}\zeta_3S_1-\frac{1}{9}S_2^2+\frac{14S_4}{9}\Bigr]\\
&+L_M\Biggl[-\frac{8P_{19}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}-\frac{16P_{33}S_1}{27(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{64(-1)^NP_{38}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}+\frac{8\bigl(N^2+N+2\bigr)P_{11}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\frac{4P_{41}}{135(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}+\tilde{\gamma}_{qg}^0\Bigl[\frac{8}{3}S_1^3-\frac{8}{3}S_2S_1-\frac{16}{3}S_{2,1}\Bigr]\\
&+\frac{64P_{15}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}\\
&+\frac{(N-1)\bigl(64\zeta_3-\frac{64S_3}{3}\bigr)}{N(N+1)}\Biggr]\Biggr]\\
&+N_F\hat{\tilde{C}}_{2,g}^{\mathsf{S},(3)}(N_F)\Biggr\}\Biggr\}\,, \qquad (134)
\end{aligned}
$$

where

$$
\begin{aligned}
P_{10} &= 3N^6+9N^5-N^4-17N^3-38N^2-28N-24 && (135)\\
P_{11} &= 47N^6+141N^5+59N^4-117N^3+2N^2+84N+72 && (136)\\
P_{12} &= 65N^6+455N^5+1218N^4+1820N^3+1968N^2+1460N+448 && (137)\\
P_{13} &= 139N^6+1093N^5+3438N^4+5776N^3+5724N^2+3220N+752 && (138)\\
P_{14} &= 9N^7+71N^6+214N^5+320N^4+275N^3+215N^2+160N+32 && (139)\\
P_{15} &= N^8+8N^7-2N^6-60N^5-23N^4+108N^3+96N^2+16N+48 && (140)\\
P_{16} &= N^8+8N^7-2N^6-60N^5+N^4+156N^3+24N^2-80N-240 && (141)
\end{aligned}
$$

$$
\begin{aligned}
P_{17} &= 3N^8+8N^7-2N^6-24N^5+15N^4+88N^3+152N^2+96N+48 && (142)\\
P_{18} &= 5N^8-8N^7-137N^6-436N^5-713N^4-672N^3-407N^2-192N-32 && (143)\\
P_{19} &= 7N^8+4N^7-90N^6-224N^5-21N^4+388N^3+608N^2+336N+144 && (144)\\
P_{20} &= 10N^8+46N^7+105N^6+139N^5+87N^4-17N^3+50N^2+84N+72 && (145)\\
P_{21} &= 19N^8+70N^7+63N^6-41N^5-192N^4-221N^3-142N^2-60N-72 && (146)\\
P_{22} &= 38N^8+146N^7+177N^6+35N^5-249N^4-373N^3-218N^2-60N-72 && (147)\\
P_{23} &= 56N^8+194N^7+213N^6+83N^5-231N^4-469N^3-290N^2-60N-72 && (148)\\
P_{24} &= 113N^8+348N^7+109N^6-289N^5-272N^4-859N^3-778N^2-172N+72 && (149)\\
P_{25} &= 9N^9+54N^8+56N^7-110N^6-381N^5-568N^4-364N^3-72N^2+128N+96 && (150)\\
P_{26} &= 9N^9+54N^8+167N^7+397N^6+780N^5+1241N^4+1448N^3+1200N^2\\
&\quad +608N+144 && (151)\\
P_{27} &= 55N^9+336N^8+218N^7-2180N^6-6529N^5-9764N^4-9368N^3-6032N^2\\
&\quad -2448N-576 && (152)\\
P_{28} &= N^{11}-56N^9-236N^8-373N^7+82N^6+1244N^5+2330N^4+2560N^3+1712N^2\\
&\quad +896N+288 && (153)\\
P_{29} &= 33N^{11}+231N^{10}+662N^9+1254N^8+1801N^7+2759N^6+5440N^5+9884N^4\\
&\quad +12512N^3+9200N^2+5184N+1728 && (154)\\
P_{30} &= 45N^{11}+383N^{10}+958N^9+526N^8-763N^7+1375N^6+7808N^5+13028N^4\\
&\quad +12976N^3+8016N^2+4608N+1728 && (155)\\
P_{31} &= 81N^{11}+483N^{10}+1142N^9+1086N^8-767N^7-4645N^6-8936N^5-11980N^4\\
&\quad -12352N^3-8272N^2-4800N-1728 && (156)\\
P_{32} &= 120N^{11}+1017N^{10}+2737N^9+1292N^8-8086N^7-20743N^6-24563N^5\\
&\quad -16702N^4-6840N^3+120N^2+2432N+960 && (157)\\
P_{33} &= 121N^{11}+988N^{10}+3554N^9+6972N^8+7131N^7-846N^6-14806N^5-25354N^4\\
&\quad -26096N^3-16752N^2-8352N-2592 && (158)\\
P_{34} &= 27N^{12}+441N^{11}+2206N^{10}+5360N^9+7445N^8+8555N^7+18766N^6+44852N^5\\
&\quad +67572N^4+63960N^3+39632N^2+15648N+2880 && (159)\\
P_{35} &= 2447N^{12}+16902N^{11}+59649N^{10}+125860N^9+128761N^8-36530N^7-248341N^6\\
&\quad -304460N^5-162188N^4-11724N^3+29160N^2+19440N+7776 && (160)\\
P_{36} &= 3361N^{12}+23769N^{11}+62338N^{10}+59992N^9-63303N^8-317823N^7-585520N^6\\
&\quad -640602N^5-430132N^4-167536N^3-27648N^2+9504N+5184 && (161)\\
P_{37} &= 76N^{14}+802N^{13}+2979N^{12}+1847N^{11}-19377N^{10}-58253N^9-26543N^8\\
&\quad +170601N^7+362177N^6+225119N^5-103240N^4-193092N^3-137160N^2\\
&\quad -117072N-25920 && (162)\\
P_{38} &= 76N^{14}+1042N^{13}+5979N^{12}+16367N^{11}+11883N^{10}-47693N^9-125723N^8\\
&\quad -86079N^7+36437N^6+22559N^5-51700N^4+24828N^3+132840N^2+116208N\\
&\quad +25920 && (163)\\
P_{39} &= 3180N^{15}+38835N^{14}+188728N^{13}+456665N^{12}+460954N^{11}-406761N^{10}\\
&\quad -1972948N^9-2827653N^8-1857970N^7+109786N^6+1302824N^5+1092456N^4\\
&\quad +265888N^3-227616N^2-194688N-44928 && (164)\\
P_{40} &= 28503N^{17}+297639N^{16}+1232041N^{15}+2461407N^{14}+2169615N^{13}+662941N^{12}
\end{aligned}
$$

$$
\begin{aligned}
& +2110979N^{11}+5346653N^{10}+2021366N^{9}-7290864N^{8}-11721384N^{7} \\
& -3689680N^{6}+15676192N^{5}+32276800N^{4}+31869312N^{3}+18809856N^{2} \\
& +6856704N+1244160 \qquad (165) \\
P_{41} = \; & 75N^{18}+3330N^{17}+35497N^{16}+175010N^{15}+486862N^{14}+966996N^{13} \\
& +2037362N^{12}+3604404N^{11}-1625689N^{10}-29506022N^{9}-78753403N^{8} \\
& -107977014N^{7}-71548880N^{6}+18344016N^{5}+89016048N^{4}+92657952N^{3} \\
& +58942080N^{2}+25505280N+5598720 \qquad (166) \\
P_{42} = \; & 325N^{18}+4280N^{17}+17759N^{16}-14880N^{15}-412326N^{14}-1696848N^{13} \\
& -3216546N^{12}-1169232N^{11}+8956857N^{10}+23914216N^{9}+31536899N^{8} \\
& +25361392N^{7}+9982840N^{6}-10154128N^{5}-26098704N^{4}-26761536N^{3} \\
& -17642880N^{2}-8087040N-1866240\,. \qquad (167)
\end{aligned}
$$

The expressions in Eqs. (134),(123) are given in an irreducible basis representation. Apart from those in the 3–loop massless Wilson coefficients the Wilson coefficients $L_{q,2}^{\mathsf{PS}}$ and $L_{g,2}^{S}$ are given in terms of the harmonic sums (D.44)

$$S_1, S_{-2}, S_2, S_{-3}, S_3, S_{-4}, S_4, S_{-2,1}, S_{2,1}, S_{3,1}, S_{2,1,1}. \qquad (168)$$

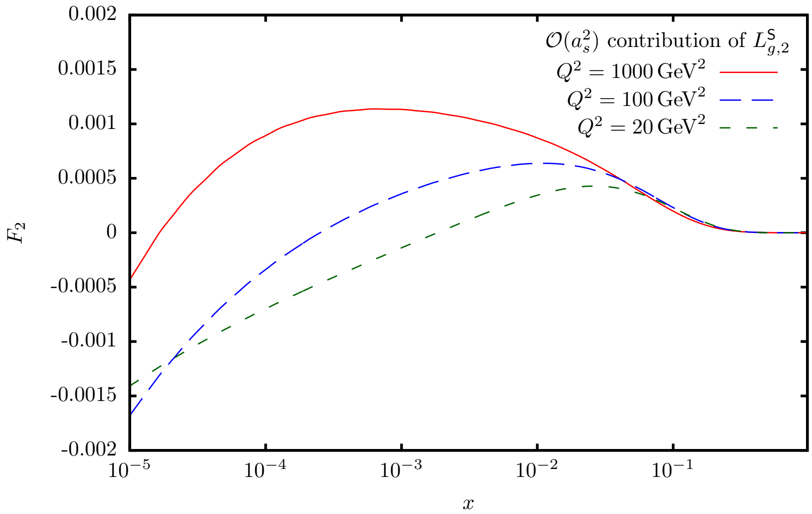


Figure 3: The $O(a_s^2)$ contribution by $L_{g,2}^{\mathsf{S}}$ to the structure function $F_2(x,Q^2)$.

The Mellin transforms of the functions (168) are known [166, 180] and the z–space expressions for $L_{q,2}^{\mathsf{PS}}$ and $L_{g,2}^{S}$ have been published in Ref. [220][5] together with the logarithmic contributions to the other heavy flavour Wilson coefficients contributing to the structure function F_2, whereas the non–singlet Wilson coefficient $L_{q,2}^{\mathsf{NS}}$ has been published separately and including the non-logarithmic contributions in Ref. [157].

[5]I would like to thank A. Behring for discussions.

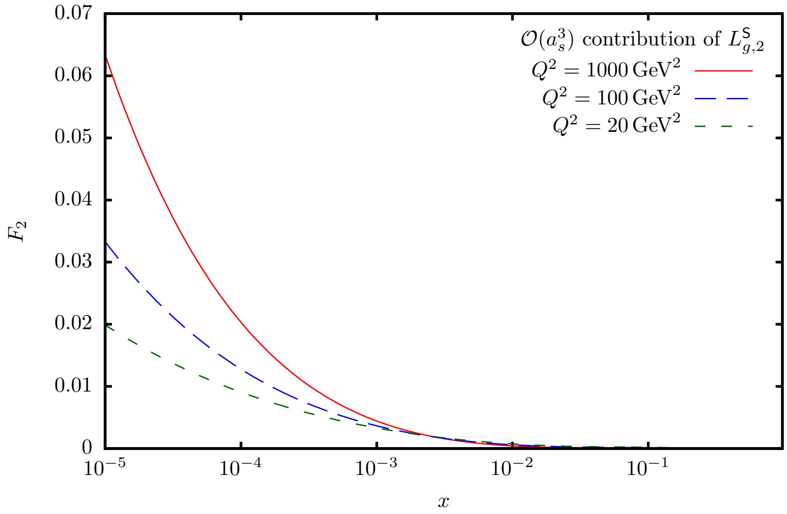


Figure 4: The $O(a_s^3)$ contribution by $L_{g,2}^{\mathsf{S}}$ to the structure function $F_2(x, Q^2)$.

Figure 5 displays the contribution of the Wilson coefficient $L_{q,2}^{\mathsf{PS}}$ to the structure function $F_2(x, Q^2)$ up to $O(\alpha_s^3)$. Here the PDFs of Ref. [249], cf. Eq. (11), have been used. The Figures 3 and 4 illustrate the corresponding two- and three-loop contributions to the Wilson coefficient $L_{g,2}^{S}$. The $O(\alpha_s^2)$ contribution is smaller than the the $O(\alpha_s^3)$-term. This is due to terms $\propto 1/z$ in the three loop contribution which do not contribute at 2-loop order. Generally the contributions at two- and three-loop order turn out to be minor compared to other contributions to the structure function $F_2(x, Q^2)$ [220].

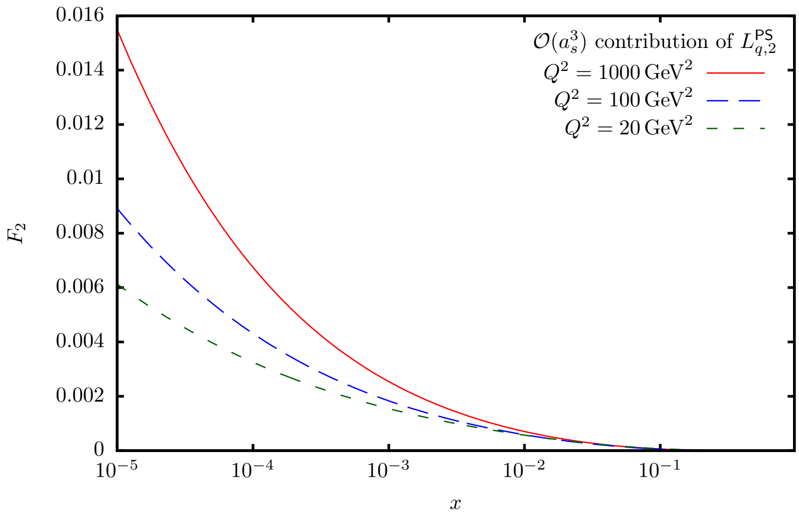


Figure 5: The $O(a_s^3)$ contribution by $L_{q,2}^{\mathsf{PS}}$ to the structure function $F_2(x, Q^2)$.

All logarithmic contributions to the massive operator matrix elements which are needed for the variable flavor number scheme have been published together with the logarithmic contributions to the heavy flavor Wilson coefficients in Ref. [220].

5 Renormalization of massive OMEs with two masses

The Feynman integrals contributing to the various operator matrix elements contain ultraviolet and collinear divergences. We choose to regularize both by applying dimensional regularization [250] in $D = 4+\varepsilon$ dimensions. This additionally requires the extension of the Lorentz-metric $g_{\mu\nu}$ and the Clifford algebra of the Dirac matrices to D dimensions. The divergences then constitute themselves as poles in a Laurent series expansion around $\varepsilon = 0$.
At one and two loop order the two–mass massive operator matrix elements A_{ij} are given in terms of the known single mass contributions since they do not contain more than one internal massive fermion line. The first genuine diagrams with two different masses emerge at $O(\alpha_s^3)$. The two–mass OMEs can be decomposed into contributions depending on just a single mass each and contributions stemming from diagrams with both masses

$$\hat{\hat{A}}_{ij}^{(l)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) = \left[\left(\frac{m_1^2}{\mu^2}\right)^{l/2\varepsilon} + \left(\frac{m_2^2}{\mu^2}\right)^{l/2\varepsilon}\right] \hat{\hat{A}}_{ij}^{(l)} + \tilde{\hat{\hat{A}}}_{ij}^{(l)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right). \tag{169}$$

Here the parts $\hat{\hat{A}}_{ij}^{(l)}$ correspond the single-mass OMEs the renormalization of which has been derived in Ref. [139] and μ^2 is the renormalization scale, which we choose equal to the factorization scale.It would be sufficient to cover the renormalization of the mixed contribution $\tilde{\hat{\hat{A}}}_{ij}^{(l)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)$ only. However, it is technically advantageous to construct the renormalization prescription for the complete two–mass OMEs. Furthermore, a change in the renormalization scheme as in Eqs. (207,208,241,240) generally introduces a mixing between the different components of Eq. (169). We thus present the renormalization for the complete two–mass OME. In a fixed renormalization scheme. The renormalized mixed-mass contribution is then obtained by subtracting the respective single–mass terms.
To renormalize the operator matrix elements with two masses we follow the renormalization procedure outlined in Ref. [139] before. At various steps adjustments have to be made to account for the second mass which is why we present the different steps here again in complete form. We study the case of N_F massless and two massive quark flavors as this covers the physically interesting case of contributions due to c– and b–quarks and the respective power corrections in

$$\eta = \frac{m_c^2}{m_b^2}\ , \tag{170}$$

which are not suppressed by a strong mass hierarchy. Up to $O(\alpha_s^3)$ no diagrams with more than two massive fermions contribute.

5.1 Mass Renormalization

The most frequently used schemes for the mass renormalization schemes are the $\overline{\mathsf{MS}}$– and the on–mass shell (OMS) scheme. We renormalize the mass in the OMS scheme first and provide the finite renormalization to switch to the $\overline{\mathsf{MS}}$-mass, cf. Eq. (241).
The bare masses $\hat{m}_i,\ i \in \{1,2\}$ are expressed by the renormalized on–shell masses m_i via

$$\hat{m}_i = Z_{m,i}(m_1, m_2)\ m_i = m_i\Big[1 + \hat{a}_s\Big(\frac{m_i^2}{\mu^2}\Big)^{\varepsilon/2}\delta m_1 + \hat{a}_s^2\Big(\frac{m_i^2}{\mu^2}\Big)^{\varepsilon}\delta m_{2,i}\,(m_1, m_2)\Big] + O(\hat{a}_s^3)\ , \tag{171}$$

where $\hat{a}_s$ denotes the bare coupling $\hat{g}_s^2/(4\pi^2)^2$ and

$$\delta m_2^i\,(m_1, m_2) = \delta m_2^0 + \tilde{\delta}_{m_2}^i(m_1, m_2)\ . \tag{172}$$

In Eq. (172) δm_2^0 is the single mass-contribution, whereas $\tilde{\delta}^i_{m_2}$ denotes the additional contribution emerging in the case of two different massive flavors. Note that from order $O(\hat{a}_s^2)$ onward the Z-factor renormalizing $\hat{m}_1$ depends on m_2 and vice versa. For the massive operator matrix elements this is observed at 3–loop order for the first time. The coefficients δm_1 and δm_2 have been derived in [251, 252] up to $O(\varepsilon^0)$ and $O(\varepsilon^{-1})$ respectively. The constant part of δm_2 was given in [253, 254] and the $O(\varepsilon)$-term of δm_1 in [139],

$$\begin{aligned}
\delta m_0 &= C_F\left[\frac{6}{\varepsilon} - 4 + \left(4 + \frac{3}{4}\zeta_2\right)\varepsilon\right] && (173)\\
&\equiv \frac{\delta m_1^{(-1)}}{\varepsilon} + \delta m_1^{(0)} + \delta m_1^{(1)}\varepsilon\ , && (174)\\
\delta m_2^0 &= C_F\Biggl[\frac{1}{\varepsilon^2}\left(18C_F - 22C_A + 8T_F(N_F+1)\right) + \frac{1}{\varepsilon}\Biggl(-\frac{45}{2}C_F + \frac{91}{2}C_A\\
&\quad -14T_F(N_F+1)\Biggr) + C_F\left(\frac{199}{8} - \frac{51}{2}\zeta_2 + 48\ln(2)\zeta_2 - 12\zeta_3\right) + C_A\Biggl(-\frac{605}{8}\\
&\quad +\frac{5}{2}\zeta_2 - 24\ln(2)\zeta_2 + 6\zeta_3\Biggr) + T_F\left[N_F\left(\frac{45}{2} + 10\zeta_2\right) + 1\left(\frac{69}{2} - 14\zeta_2\right)\right]\Biggr] && (175)\\
&\equiv \frac{\delta m_2^{0,(-2)}}{\varepsilon^2} + \frac{\delta m_2^{0,(-1)}}{\varepsilon} + \delta m_2^{0,(0)}\ , && (176)
\end{aligned}$$

$$\begin{aligned}
\tilde{\delta m_2}^i(m_1, m_2) &= C_F T_F\Biggl\{\frac{8}{\varepsilon^2} - \frac{14}{\varepsilon} + 8r_i^4 H_0^2(r_i) - 8(r_i+1)^2\left(r_i^2 - r_i + 1\right)H_{-1,0}(r_i)\\
&\quad +8(r_i-1)^2\left(r_i^2 + r_i + 1\right)H_{1,0}(r_i) + 8r_i^2 H_0(r_i) + \frac{3}{2}\left(8r_i^2 + 15\right)\\
&\quad +2\Bigl[4r_i^4 - 12r_i^3 - 12r_i + 5\Bigr]\zeta_2\Biggr\} && (177)\\
&\equiv \frac{\tilde{\delta m_2}^{(-2)}}{\varepsilon^2} + \frac{\tilde{\delta m_2}^{(-1)}}{\varepsilon} + \tilde{\delta m_2}^{i,(0)}\ , && (178)
\end{aligned}$$

with $i \in \{1, 2\}$,

$$r_1 = \sqrt{\eta} \text{ and } r_2 = \frac{1}{\sqrt{\eta}}\ . \tag{179}$$

The superscript i for the coefficients $\tilde{\delta m}_2^{(-2)}$ and $\tilde{\delta m}_2^{(-2)}$ has been dropped as they are independent of the renormalized mass m_i. Here $H_{\vec{a}}(\zeta)$ denotes harmonic polylogarithms [180] (see Appendix (D.2.1)) with $H_0(\zeta) = \ln(\zeta)$, $H_{-1,0}(\zeta) = \mathrm{Li}_2(-\zeta) + \ln(\zeta)\ln(1+\zeta)$, $H_{1,0}(\zeta) = \mathrm{Li}_2(1-\zeta) - \zeta_2$ and $\mathrm{Li}_2(x)$ the dilogarithm (D.33).

Eq. (178) states the complete analytic form of the contribution of the respective other massive flavor to the renormalization of the bare masses. In the present analysis we will focus on m_1, m_2 being the masses of the charm and bottom quark respectively. Due to the small ratio

$$\eta \sim 0.1 \tag{180}$$

we may restrict ourselves to the expansions up to $O\left(\eta^3 \ln^3(\eta)\right)$

$$\begin{aligned}
\tilde{\delta} {m_2}^{1,(0)}(m_1,m_2) &= C_F T_F \Biggl(+10\zeta_2 + \frac{45}{2} - 24\zeta_2 \eta^{1/2} + 24\eta - 24\zeta_2 \eta^{3/2} \\
&+32\left(-\frac{13}{48}\ln(\eta) + \frac{151}{288} + \frac{1}{16}\ln^2(\eta) + \frac{1}{4}\zeta_2\right)\eta^2 \\
&+32\left(\frac{1}{30}\ln(\eta) - \frac{19}{300}\right)\eta^3 \Biggr) + O\left(\eta^3 \ln(\eta)^2\right) \qquad (181) \\
\tilde{\delta} {m_2}^{2,(0)}(m_1,m_2) &= C_F T_F \Biggl(-2\ln^2(\eta) + \frac{26}{3}\ln(\eta) + 2\zeta_2 + \frac{103}{18} \\
&+32\left(-\frac{1}{30}\ln(\eta) + \frac{19}{300}\right)\eta + 32\left(-\frac{9}{1120}\ln(\eta) + \frac{1389}{156800}\right)\eta^2 \\
&+32\left(-\frac{1}{315}\ln(\eta) + \frac{997}{396900}\right)\eta^3 \Biggr) + O\left(\eta^3 \ln(\eta)^2\right) \ . \qquad (182)
\end{aligned}$$

Applying Eq. (171) we obtain the mass renormalized operator matrix elements by

$$\begin{aligned}
\hat{\hat{A}}_{ij}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) &= \delta_{ij} + \hat{a}_s \, \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) + \hat{a}_s^2 \Biggl\{ \hat{\hat{A}}_{ij}^{(2)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \\
&+\delta m_1 \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{\mathrm{d}}{\mathrm{d}m_1} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_2 \frac{\mathrm{d}}{\mathrm{d}m_2}\right] \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \Biggr\} \\
&+\hat{a}_s^3 \Biggl\{ \hat{\hat{A}}_{ij}^{(3)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \\
&+\delta m_1 \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{\mathrm{d}}{\mathrm{d}m_1} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_2 \frac{\mathrm{d}}{\mathrm{d}m_2}\right] \hat{\hat{A}}_{ij}^{(2)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \\
&+\left[\delta m_2^1(m_1,m_2) \frac{m_1 \mathrm{d}}{\mathrm{d}m_1} + \delta m_2^2(m_1,m_2) \frac{m_2 \mathrm{d}}{\mathrm{d}m_2}\right] \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \\
&+\frac{\delta m_1^2}{2}\left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} \frac{m_1^2 \mathrm{d}^2}{\mathrm{d}m_1^2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon} \frac{m_2^2 \mathrm{d}^2}{\mathrm{d}m_2^2}\right] \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \\
&+{\delta m_1}^2 \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} \frac{m_1 \mathrm{d}}{\mathrm{d}m_1} \frac{m_2 \mathrm{d}}{\mathrm{d}m_2} \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N\right) \Biggr\} \ . \qquad (183)
\end{aligned}$$

They are symmetric under the interchange of the masses m_1 and m_2.

5.2 Renormalization of the Coupling Constant

When renormalizing the coupling constant it is important to note that the factorization relation (56) strictly requires the external massless partonic legs of the operator matrix elements to be on–shell

$$p^2 = 0 \ , \tag{184}$$

with p the external momentum of the OME. This condition would be violated by massive loop corrections to the gluon propagator. We follow [139] and absorb these corrections uniquely into the coupling constant by using the background field method [182–184] to maintain the Slavnov–Taylor-identities of QCD. We thereby adopt a MOM–scheme for the coupling constant. A finite renormalization relates the MOM coupling constants to the $\overline{\mathsf{MS}}$–scheme and is applied later. For this scheme transformation we assume the decoupling of the two heavy quark flavors.
Concerning the light fermionic flavors, the coupling constant is renormalized as in the $\overline{\mathsf{MS}}$–scheme, with

$$\begin{aligned} \hat{a}_s &= Z_g^{\overline{\mathsf{MS}}^2}(\varepsilon, N_F) a_s^{\overline{\mathsf{MS}}}(\mu^2) \\ &= a_s^{\overline{\mathsf{MS}}}(\mu^2)\left[1 + \delta a_{s,1}^{\overline{\mathsf{MS}}}(N_F) a_s^{\overline{\mathsf{MS}}}(\mu^2) + \delta a_{s,2}^{\overline{\mathsf{MS}}}(N_F) a_s^{\overline{\mathsf{MS}}^2}(\mu)\right] + O(a_s^{\overline{\mathsf{MS}}^3}) \ . \end{aligned} \tag{185}$$

The coefficients $\delta a_{s,i}^{\overline{\mathsf{MS}}}(N_F)$ are given by

$$\delta a_{s,1}^{\overline{\mathsf{MS}}}(N_F) = \frac{2}{\varepsilon}\beta_0(N_F) \ , \tag{186}$$

$$\delta a_{s,2}^{\overline{\mathsf{MS}}}(N_F) = \frac{4}{\varepsilon^2}\beta_0^2(N_F) + \frac{1}{\varepsilon}\beta_1(N_F) \ . \tag{187}$$

From the renormalization prescription (185-187) it follows directly that that $\beta_0(nf)$, $\beta_1(nf)$ are the coefficients of the QCD β–function for N_F massless quark flavors

$$\beta^{\overline{\mathsf{MS}}}(N_F) = -\beta_0(N_F) a_s^{\overline{\mathsf{MS}}^2} - \beta_1(N_F) a_s^{\overline{\mathsf{MS}}^3} + O\left(a_s^{\overline{\mathsf{MS}}^4}\right) \ . \tag{188}$$

They are given by [40, 41, 255–258]

$$\beta_0(N_F) = \frac{11}{3}C_A - \frac{4}{3}T_F N_F \ , \tag{189}$$

$$\beta_1(N_F) = \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right) T_F N_F \ . \tag{190}$$

We split the renormalized gluon self energy Π into purely light and the remaining heavy flavor contributions Π_L and Π_H

$$\Pi\left(p^2, m_1^2, m_2^2\right) = \Pi_L\left(p^2\right) + \Pi_H\left(p^2, m_1^2, m_2^2\right) \ . \tag{191}$$

The heavy quarks are required to decouple from the running coupling constant and the renormalized OMEs for $\mu^2 < m_1^2, m_2^2$ which implies [128]

$$\Pi_H(0, m_1^2, m_2^2) = 0 \ . \tag{192}$$

We apply the background field method which has the advantage of producing gauge-invariant results also for unphysical quantities as e.g. off–shell Green's functions to compute the heavy flavor contributions to the unrenormalized gluon self–energy [182, 259]. Applying the respective Feynman rules [204] we obtain the result

$$
\begin{aligned}
\hat{\Pi}^{\mu\nu}_{H,ab,\mathrm{BF}}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) &= i(-p^2 g^{\mu\nu} + p^\mu p^\nu)\delta_{ab}\hat{\Pi}_{H,\mathrm{BF}}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) , \\
\hat{\Pi}_{H,\mathrm{BF}}(0, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) &= \hat{a}_s \frac{2\beta_{0,Q}}{\varepsilon} \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2}\right] \exp\Bigl(\sum_{i=2}^{\infty} \frac{\zeta_i}{i}\Bigl(\frac{\varepsilon}{2}\Bigr)^i\Bigr) \\
&\quad + \hat{a}_s^2 \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon}\right] \Biggl[\frac{1}{\varepsilon}\Bigl(-\frac{20}{3}T_F C_A - 4T_F C_F\Bigr) \\
&\quad - \frac{32}{9}T_F C_A + 15 T_F C_F \\
&\quad + \varepsilon\Bigl(-\frac{86}{27}T_F C_A - \frac{31}{4}T_F C_F - \frac{5}{3}\zeta_2 T_F C_A - \zeta_2 T_F C_F\Bigr) \\
&\quad + 2\left(\frac{2\beta_{0,Q}}{\varepsilon}\right)^2 \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} \exp\Bigl(2\sum_{i=2}^{\infty} \frac{\zeta_i}{i}\Bigl(\frac{\varepsilon}{2}\Bigr)^i\Bigr)\Biggr] \\
&\quad + O(\hat{a}_s^3) ,
\end{aligned}
\tag{193}
$$

where the masses m_1 and m_2 have been renormalized in the on–shell scheme given in Eq. (171). In order to write (193) more compactly we used the notation

$$
f(\varepsilon) \equiv \left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2}\right) \exp\Bigl(\sum_{i=2}^{\infty} \frac{\zeta_i}{i}\Bigl(\frac{\varepsilon}{2}\Bigr)^i\Bigr) , \tag{194}
$$

and keep this factors unexpanded in the dimensional regularization parameter ε. Furthermore we denote the contributions to the QCD-β function coefficients by $\beta_{i,Q}^{(j)}$ [40, 41, 128, 139, 255–258]

$$
\begin{aligned}
\beta_{0,Q} &= -\frac{4}{3}T_F , && (195) \\
\beta_{1,Q} &= -4\left(\frac{5}{3}C_A + C_F\right) T_F , && (196) \\
\beta_{1,Q}^{(1)} &= -\frac{32}{9}T_F C_A + 15 T_F C_F , && (197) \\
\beta_{1,Q}^{(2)} &= -\frac{86}{27}T_F C_A - \frac{31}{4}T_F C_F - \zeta_2\left(\frac{5}{3}T_F C_A + T_F C_F\right) . && (198)
\end{aligned}
$$

Eq. (193) differs from the sum of the two individual single–mass contributions [139] by the last term only. This term is due to additional reducible Feynman diagrams in the cases of two heavy quark flavors of different mass.

The background field is renormalized using the Z–factor Z_A which is split into light and heavy quark contributions $Z_{A,L}$ and $Z_{A,H}$. It is related to the Z-factor renormalizing the coupling constant g via

$$
Z_g = Z_A^{-\frac{1}{2}} = \frac{1}{(Z_{A,L} + Z_{A,H})^{1/2}} . \tag{199}
$$

Concerning the light flavors we require the renormalization to correspond to the $\overline{\mathsf{MS}}$–scheme with N_F light flavors

$$Z_{A,l}(N_F) \;=\; Z_g^{\overline{\mathsf{MS}}^{1/2}} \;. \tag{200}$$

The heavy flavor contribution is fixed by condition (192) which implies

$$\Pi_{H,\mathrm{BF}}(0,\mu^2,a_s,m_1^2,m_2^2) + Z_{A,H} \equiv 0 \;. \tag{201}$$

The Z–factor in the MOM–scheme is read off by combining (199,(192), (193) and (201)

$$Z_g^{\mathsf{MOM}}(\varepsilon, N_F+2,\mu,m) \equiv \frac{1}{(Z_{A,l}+Z_{A,H})^{1/2}} \;. \tag{202}$$

Up to $O(a_s^{\mathsf{MOM}^2})$ we obtain the renormalization constant

$$\begin{aligned}
Z_g^{\mathsf{MOM}^2}(\varepsilon,m,\mu,N_F+2) \;=\;& 1 + a_s^{\mathsf{MOM}}(\mu^2)\Bigl[\frac{2}{\varepsilon}(\beta_0(N_F)+\beta_{0,Q}f(\varepsilon))\Bigr] \\
&+ a_s^{\mathsf{MOM}^2}(\mu^2)\Bigl[\frac{\beta_1(N_F)}{\varepsilon} + \frac{4}{\varepsilon^2}(\beta_0(N_F)+\beta_{0,Q}f(\varepsilon))^2 \\
&+\frac{1}{\varepsilon}\left(\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon} + \Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon}\right)\left(\beta_{1,Q}+\varepsilon\beta_{1,Q}^{(1)}+\varepsilon^2\beta_{1,Q}^{(2)}\right)\Bigr] \\
&+ O(\varepsilon^2, a_s^{\mathsf{MOM}^3}) \;.
\end{aligned} \tag{203}$$

.

We define the coefficients of the MOM–scheme Z-factor, $\delta a_{s,1}^{\mathsf{MOM}}$ and $\delta a_{s,2}^{\mathsf{MOM}}$, analogously to the $\overline{\mathsf{MS}}$–coefficients in (185)

$$\delta a_{s,1}^{\mathsf{MOM}} \;=\; \Bigl[\frac{2\beta_0(N_F)}{\varepsilon} + \frac{2\beta_{0,Q}}{\varepsilon}f(\varepsilon)\Bigr] \;, \tag{204}$$

$$\begin{aligned}
\delta a_{s,2}^{\mathsf{MOM}} \;=\;& \Bigl[\frac{\beta_1(N_F)}{\varepsilon} + \Bigl\{\frac{2\beta_0(N_F)}{\varepsilon} + \frac{2\beta_{0,Q}}{\varepsilon}f(\varepsilon)\Bigr\}^2 \\
&+\frac{1}{\varepsilon}\left(\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon} + \Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon}\right)\left(\beta_{1,Q}+\varepsilon\beta_{1,Q}^{(1)}+\varepsilon^2\beta_{1,Q}^{(2)}\right)\Bigr] + O(\varepsilon^2) \;.
\end{aligned} \tag{205}$$

Finally we express our results in the $\overline{\mathsf{MS}}$–scheme. For this transition we assume the decoupling of the heavy quark flavors. The renormalization–scheme transition then follows from the equality of the unrenormalized coupling constant.

$$Z_g^{\overline{\mathsf{MS}}^2}(\varepsilon,N_F+2)a_s^{\overline{\mathsf{MS}}}(\mu^2) = Z_g^{\mathsf{MOM}^2}(\varepsilon,m,\mu,N_F+2)a_s^{\mathsf{MOM}}(\mu^2) \;. \tag{206}$$

Solving (206) perturbatively we obtain

$$\begin{aligned}
a_s^{\mathsf{MOM}} \;=\;& a_s^{\overline{\mathsf{MS}}} - \beta_{0,Q}\left(\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr) + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\right)a_s^{\overline{\mathsf{MS}}^2} + \left[\beta_{0,Q}^2\left(\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr) + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\right)^2\right. \\
&\left. -\beta_{1,Q}\left(\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr) + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\right) - 2\beta_{1,Q}^{(1)}\right]a_s^{\overline{\mathsf{MS}}^3} + O\left(a_s^{\overline{\mathsf{MS}}^4}\right) \;,
\end{aligned} \tag{207}$$

or,

$$
\begin{aligned}
a_s^{\overline{\mathsf{MS}}} &= a_s^{\mathsf{MOM}} + a_s^{\mathsf{MOM}^2}\Biggl(\delta a_{s,1}^{\mathsf{MOM}} - \delta a_{s,1}^{\overline{\mathsf{MS}}}(N_F+2)\Biggr) + a_s^{\mathsf{MOM}^3}\Biggl(\delta a_{s,2}^{\mathsf{MOM}} - \delta a_{s,2}^{\overline{\mathsf{MS}}}(N_F+2) \\
&\quad -2\delta a_{s,1}^{\overline{\mathsf{MS}}}(N_F+2)\Bigl[\delta a_{s,1}^{\mathsf{MOM}} - \delta a_{s,1}^{\overline{\mathsf{MS}}}(N_F+2)\Bigr]\Biggr) + O(a_s^{\mathsf{MOM}^4}) \ , \qquad (208)
\end{aligned}
$$

as finite renormalization prescriptions describing the scheme–change $\overline{\mathsf{MS}} \leftrightarrow \mathsf{MOM}$. Note that unlike in Eq. (185) in Eq. (207) and (208) $a_s^{\overline{\mathsf{MS}}} = a_s^{\overline{\mathsf{MS}}}(N_F+2)$. Applying the coupling renormalization (203) to (183) we obtain as combined formula for the combined mass– and coupling–renormalization up to $O\left(a_s^{\mathsf{MOM}^3}\right)$

$$
\begin{aligned}
\hat{A}_{ij} &= \delta_{ij} + a_s^{\mathsf{MOM}}\ \hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) \\
&\quad + a_s^{\mathsf{MOM}^2}\Biggl[\hat{\hat{A}}_{ij}^{(2)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) + \delta a_{s,1}^{\mathsf{MOM}}\hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) \\
&\quad + \delta m_1\left(\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon/2} m_1\frac{d}{dm_1} + \Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon/2} m_2\frac{d}{dm_2}\right)\hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr)\Biggr] \\
&\quad + a_s^{\mathsf{MOM}^3}\Biggl[\hat{\hat{A}}_{ij}^{(3)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) + \delta a_{s,2}^{\mathsf{MOM}}\hat{\hat{A}}_{ij}^{(1)} + 2\delta a_{s,1}^{\mathsf{MOM}}\Biggl[\hat{\hat{A}}_{ij}^{(2)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) \\
&\quad + \delta m_1\left(\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon/2} m_1\frac{d}{dm_1} + \Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon/2} m_2\frac{d}{dm_2}\right)\hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr)\Biggr] \\
&\quad + \delta m_1\left(\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon/2} m_1\frac{d}{dm_1} + \Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon/2} m_2\frac{d}{dm_2}\right)\hat{\hat{A}}_{ij}^{(2)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) \\
&\quad + \left(\delta m_{2,1}(m_1,m_2) m_1\frac{d}{dm_1} + \delta m_{2,2}(m_1,m_2) m_2\frac{d}{dm_2}\right)\hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) \\
&\quad + \frac{\delta m_1^2}{2}\left(\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon} m_1^2 + \frac{d^2}{{dm_1}^2}\Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon} m_2^2\frac{d^2}{{dm_2}^2}\right)\hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr) \\
&\quad + {\delta m_1}^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\varepsilon/2}\Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\varepsilon/2} m_1\frac{d}{dm_1} m_2\frac{d}{dm_2}\hat{\hat{A}}_{ij}^{(1)}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\Bigr)\Biggr] \ , \qquad (209)
\end{aligned}
$$

where the ε– and N–dependence of the OMEs has been suppressed.

5.3 Operator Renormalization

We now remove the UV–divergences by introducing the respective Z–factors for the different operators defined in Eqs. (37-39)

$$
O_{q,r;\mu_1,\ldots,\mu_N}^{\mathsf{NS}} = Z^{\mathsf{NS}}(\mu^2)\hat{O}_{q,r;\mu_1,\ldots,\mu_N}^{\mathsf{NS}} \ , \qquad (210)
$$

$$
O_{i;\mu_1,\ldots,\mu_N}^{\mathsf{S}} = Z_{ij}^{\mathsf{S}}(\mu^2)\hat{O}_{j;\mu_1,\ldots,\mu_N}^{\mathsf{S}} \ , \qquad i = q, g \ . \qquad (211)
$$

In the singlet case the operator renormalization introduces a mixing between the different operators since they carry the same quantum numbers. Analogously to the OMEs here the Z-factors were split into flavor pure singlet(PS) and flavor non-singlet (NS) contributions

$$Z_{qq}^{-1} = Z_{qq}^{-1,\mathsf{PS}} + Z_{qq}^{-1,\mathsf{NS}} . \tag{212}$$

Each Z-factor is associated with an anomalous dimension γ_{ij} via

$$\gamma_{qq}^{\mathsf{NS}}(a_s^{\overline{\mathsf{MS}}}, N_F, N) = \mu \frac{d}{d\mu} \ln \left(Z_{qq}^{\mathsf{NS}}(a_s^{\overline{\mathsf{MS}}}, N_F, \varepsilon, N) \right) , \tag{213}$$

$$\gamma_{ij}(a_s^{\overline{\mathsf{MS}}}, N_F, N) = \mu \frac{d}{d\mu} \ln \left(Z_{lj}(a_s^{\overline{\mathsf{MS}}}, N_F, \varepsilon, N) \right) . \tag{214}$$

Here both the anomalous dimensions and the operator Z-factors obey perturbative series expansions in the coupling constant

$$\gamma_{ij}^{\mathsf{S,\ PS,\ NS}}(a_s^{\overline{\mathsf{MS}}}, N_F, N) = \sum_{l=1}^{\infty} a_s^{\overline{\mathsf{MS}}^l} \gamma_{ij}^{(l-1),\mathsf{S,\ PS,\ NS}}(N_F, N) \tag{215}$$

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} a_s^k Z_{ij}^{(k)} \tag{216}$$

$$Z_{ij}^{-1} = \delta_{ij} + \sum_{k=1}^{\infty} a_s^k Z_{ij}^{-1,(k)} . \tag{217}$$

In order to renormalize the respective operators we consider operator matrix elements with off-shell external legs as a sum of massive and massless contributions:

$$\begin{aligned} \hat{A}_{ij}\left(p^2, m_1^2, m_2^2, \mu^2, a_s^{\mathsf{MOM}}, N_F + 2\right) &= \hat{A}_{ij}\left(\frac{-p^2}{\mu^2}, \alpha_s^{\overline{\mathsf{MS}}}, N_F\right) \\ &\quad + \hat{A}_{ij}^{Q}\left(p^2, m_1^2, m_2^2, \mu^2, a_s^{\mathsf{MOM}}, N_F + 2\right) . \end{aligned} \tag{218}$$

Here the massless contribution depends on $a_s^{\overline{\mathsf{MS}}}$ since the MOM–scheme in Section 5.2 was constructed in such a way, that it corresponds to the $\overline{\mathsf{MS}}$–scheme concerning the renormalization of the light quark flavor and gluon contributions. The term δ_{ij} does not have any mass-dependence and is considered a part of the light flavor part $\hat{A}_{ij}\left(\frac{-p^2}{\mu^2}, \alpha_s^{\overline{\mathsf{MS}}}, N_F\right)$.

We first consider the renormalization of the purely massless contribution in the $\overline{\mathsf{MS}}$–scheme [260]

$$A_{qq}^{\mathsf{NS}}\Big(\frac{-p^2}{\mu^2}, a_s^{\overline{\mathsf{MS}}}, N_F, N\Big) = Z_{qq}^{-1,\mathsf{NS}}(a_s^{\overline{\mathsf{MS}}}, N_F, \varepsilon, N) \hat{A}_{qq}^{\mathsf{NS}}\Big(\frac{-p^2}{\mu^2}, a_s^{\overline{\mathsf{MS}}}, N_F, \varepsilon, N\Big) \tag{219}$$

$$A_{ij}\Big(\frac{-p^2}{\mu^2}, a_s^{\overline{\mathsf{MS}}}, N_F, N\Big) = Z_{il}^{-1}(a_s^{\overline{\mathsf{MS}}}, N_F, \varepsilon, N) \hat{A}_{lj}\Big(\frac{-p^2}{\mu^2}, a_s^{\overline{\mathsf{MS}}}, N_F, \varepsilon, N\Big) , \ i, j, l = q, g . \tag{220}$$

Solving (213–214) yields the Z-factors

$$Z_{ij}(a_s^{\overline{\mathsf{MS}}}, N_F) = \delta_{ij} + a_s^{\overline{\mathsf{MS}}} \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a_s^{\overline{\mathsf{MS}}^2} \left\{ \frac{1}{\varepsilon^2} \Big(\frac{1}{2} \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \Big) + \frac{1}{2\varepsilon} \gamma_{ij}^{(1)} \right\}$$

$$+a_s^{\overline{\mathsf{MS}}^3}\Biggl\{\frac{1}{\varepsilon^3}\Bigl(\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lk}^{(0)}\gamma_{kj}^{(0)}+\beta_0\gamma_{il}^{(0)}\gamma_{lj}^{(0)}+\frac{4}{3}\beta_0^2\gamma_{ij}^{(0)}\Bigr)$$

$$+\frac{1}{\varepsilon^2}\Bigl(\frac{1}{6}\gamma_{il}^{(1)}\gamma_{lj}^{(0)}+\frac{1}{3}\gamma_{il}^{(0)}\gamma_{lj}^{(1)}+\frac{2}{3}\beta_0\gamma_{ij}^{(1)}+\frac{2}{3}\beta_1\gamma_{ij}^{(0)}\Bigr)+\frac{\gamma_{ij}^{(2)}}{3\varepsilon}\Biggr\} \qquad (221)$$

and for the NS– and PS–case

$$Z_{qq}^{\mathsf{NS}}(a_s^{\overline{\mathsf{MS}}},N_F) = 1+a_s^{\overline{\mathsf{MS}}}\frac{\gamma_{qq}^{(0),\mathsf{NS}}}{\varepsilon}+a_s^{\overline{\mathsf{MS}}^2}\Biggl\{\frac{1}{\varepsilon^2}\Bigl(\frac{1}{2}\gamma_{qq}^{(0),\mathsf{NS}^2}+\beta_0\gamma_{qq}^{(0),\mathsf{NS}}\Bigr)+\frac{1}{2\varepsilon}\gamma_{qq}^{(1),\mathsf{NS}}\Biggr\}$$

$$+a_s^{\overline{\mathsf{MS}}^3}\Biggl\{\frac{1}{\varepsilon^3}\Bigl(\frac{1}{6}\gamma_{qq}^{(0),\mathsf{NS}^3}+\beta_0\gamma_{qq}^{(0),\mathsf{NS}^2}+\frac{4}{3}\beta_0^2\gamma_{qq}^{(0),\mathsf{NS}}\Bigr) \qquad (222)$$

$$Z_{qq}^{\mathsf{PS}}(a_s^{\overline{\mathsf{MS}}},N_F) = a_s^{\overline{\mathsf{MS}}^2}\Biggl\{\frac{1}{2\varepsilon^2}\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{2\varepsilon}\gamma_{qq}^{(1),\mathsf{PS}}\Biggr\}+a_s^{\overline{\mathsf{MS}}^3}\Biggl\{\frac{1}{\varepsilon^3}\Bigl(\frac{1}{3}\gamma_{qq}^{(0)}\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}$$

$$+\frac{1}{6}\gamma_{qg}^{(0)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\beta_0\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}\Bigr)+\frac{1}{\varepsilon^2}\Bigl(\frac{1}{3}\gamma_{qg}^{(0)}\gamma_{gq}^{(1)}$$

$$+\frac{1}{6}\gamma_{qg}^{(1)}\gamma_{gq}^{(0)}+\frac{1}{2}\gamma_{qq}^{(0)}\gamma_{qq}^{(1),\mathsf{PS}}+\frac{2}{3}\beta_0\gamma_{qq}^{(1),\mathsf{PS}}\Bigr)+\frac{\gamma_{qq}^{(2),\mathsf{PS}}}{3\varepsilon}\Biggr\}. \qquad (223)$$

The Z-factors describing the UV renormalization of the complete operator matrix elements $\hat{A}_{ij}\left(p^2,m_1^2,m_2^2,\mu^2,a_s^{\mathsf{MOM}},N_F+2\right)$ are obtained by inverting (221-223), replacing $N_F\to N_F+2$ and by applying the scheme transformation (208). The resulting operator Z-factors are

$$Z_{ij}^{-1}(a_s^{\mathsf{MOM}},N_F+2,\mu) = \delta_{ij}-a_s^{\mathsf{MOM}}\frac{\gamma_{ij}^{(0)}}{\varepsilon}+a_s^{\mathsf{MOM}^2}\Biggl[\frac{1}{\varepsilon}\Bigl(-\frac{1}{2}\gamma_{ij}^{(1)}-\delta a_{s,1}^{\mathsf{MOM}}\gamma_{ij}^{(0)}\Bigr)$$

$$+\frac{1}{\varepsilon^2}\Bigl(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)}+\beta_0\gamma_{ij}^{(0)}\Bigr)\Biggr]+a_s^{\mathsf{MOM}^3}\Biggl[\frac{1}{\varepsilon}\Bigl(-\frac{1}{3}\gamma_{ij}^{(2)}-\delta a_{s,1}^{\mathsf{MOM}}\gamma_{ij}^{(1)}$$

$$-\delta a_{s,2}^{\mathsf{MOM}}\gamma_{ij}^{(0)}\Bigr)+\frac{1}{\varepsilon^2}\Bigl(\frac{4}{3}\beta_0\gamma_{ij}^{(1)}+2\delta a_{s,1}^{\mathsf{MOM}}\beta_0\gamma_{ij}^{(0)}+\frac{1}{3}\beta_1\gamma_{ij}^{(0)}$$

$$+\delta a_{s,1}^{\mathsf{MOM}}\gamma_{il}^{(0)}\gamma_{lj}^{(0)}+\frac{1}{3}\gamma_{il}^{(1)}\gamma_{lj}^{(0)}+\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lj}^{(1)}\Bigr)+\frac{1}{\varepsilon^3}\Bigl(-\frac{4}{3}\beta_0^2\gamma_{ij}^{(0)}$$

$$-\beta_0\gamma_{il}^{(0)}\gamma_{lj}^{(0)}-\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lk}^{(0)}\gamma_{kj}^{(0)}\Bigr)\Biggr], \qquad (224)$$

and

$$Z_{qq}^{-1,\mathsf{NS}}(a_s^{\mathsf{MOM}},N_F+2) = 1-a_s^{\mathsf{MOM}}\frac{\gamma_{qq}^{(0),\mathsf{NS}}}{\varepsilon}+a_s^{\mathsf{MOM}^2}\Biggl[\frac{1}{\varepsilon}\Bigl(-\frac{1}{2}\gamma_{qq}^{(1),\mathsf{NS}}-\delta a_{s,1}^{\mathsf{MOM}}\gamma_{qq}^{(0),\mathsf{NS}}\Bigr)$$

$$+\frac{1}{\varepsilon^2}\Bigl(\beta_0\gamma_{qq}^{(0),\mathsf{NS}}+\frac{1}{2}\gamma_{qq}^{(0),\mathsf{NS}^2}\Bigr)\Biggr]+a_s^{\mathsf{MOM}^3}\Biggl[\frac{1}{\varepsilon}\Bigl(-\frac{1}{3}\gamma_{qq}^{(2),\mathsf{NS}}-\delta a_{s,1}^{\mathsf{MOM}}\gamma_{qq}^{(1),\mathsf{NS}}$$

$$-\delta a_{s,2}^{\mathsf{MOM}}\gamma_{qq}^{(0),\mathsf{NS}}\Bigr)+\frac{1}{\varepsilon^2}\Bigl(\frac{4}{3}\beta_0\gamma_{qq}^{(1),\mathsf{NS}}+2\delta a_{s,1}^{\mathsf{MOM}}\beta_0\gamma_{qq}^{(0),\mathsf{NS}}+\frac{1}{3}\beta_1\gamma_{qq}^{(0),\mathsf{NS}}$$

$$+\frac{1}{2}\gamma_{qq}^{(0),\mathsf{NS}}\gamma_{qq}^{(1),\mathsf{NS}}+\delta a_{s,1}^{\mathsf{MOM}}\gamma_{qq}^{(0),\mathsf{NS}^2}\Bigr)+\frac{1}{\varepsilon^3}\Bigl(-\frac{4}{3}\beta_0^2\gamma_{qq}^{(0),\mathsf{NS}}-\beta_0\gamma_{qq}^{(0),\mathsf{NS}^2}$$

$$
\left.\left. -\frac{1}{6}\gamma_{qq}^{(0),\mathsf{NS}^3}\right)\right] ~, \tag{225}
$$

$$
\begin{aligned}
Z_{qq}^{-1,\mathsf{PS}}(a_s^{\mathsf{MOM}}, N_F+2) \;=\;& a_s^{\mathsf{MOM}^2}\left[\frac{1}{\varepsilon}\Bigl(-\frac{1}{2}\gamma_{qq}^{(1),\mathsf{PS}}\Bigr) + \frac{1}{\varepsilon^2}\Bigl(\frac{1}{2}\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}\Bigr)\right] + a_s^{\mathsf{MOM}^3}\left[\frac{1}{\varepsilon}\Bigl(-\frac{1}{3}\gamma_{qq}^{(2),\mathsf{PS}}\right.\\
& -\delta a_{s,1}^{\mathsf{MOM}}\gamma_{qq}^{(1),\mathsf{PS}}\Bigr) + \frac{1}{\varepsilon^2}\Bigl(\frac{1}{6}\gamma_{qg}^{(0)}\gamma_{gq}^{(1)} + \frac{1}{3}\gamma_{gq}^{(0)}\gamma_{qg}^{(1)} + \frac{1}{2}\gamma_{qq}^{(0)}\gamma_{qq}^{(1),\mathsf{PS}}\\
& +\frac{4}{3}\beta_0\gamma_{qq}^{(1),\mathsf{PS}} + \delta a_{s,1}^{\mathsf{MOM}}\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}\Bigr) + \frac{1}{\varepsilon^3}\Bigl(-\frac{1}{3}\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}\gamma_{qq}^{(0)} - \frac{1}{6}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\gamma_{gg}^{(0)}\\
& \left.-\beta_0\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}\Bigr)\right] ~.
\end{aligned}\tag{226}
$$

Here and in Eqs. (221-223) we have dropped the N_F dependence of the anomalous dimensions γ_{ij} and the coefficients β_i for brevity. The inverse Z-factors for the purely light situation correspond to (224-226) after substituting $N_F+2 \rightarrow N_F$ and $\delta a_{s,i}^{\mathsf{MOM}} \rightarrow \delta a_{s,i}^{\overline{\mathsf{MS}}}$.
We are only interested to perform the UV–renormalization for the massive contribution to the operator matrix element in (218) and thus subtract the contributions stemming from purely light parts again

$$
\begin{aligned}
\tilde{\hat{A}}_{ij}^{Q}(p^2, m_1^2, m_2^2, \mu^2, a_s^{\mathsf{MOM}}, N_F+2) \;=\;& Z_{il}^{-1}(a_s^{\mathsf{MOM}}, N_F+2, \mu)\hat{A}_{ij}^{Q}(p^2, m_1^2, m_2^2, \mu^2, a_s^{\mathsf{MOM}}, N_F+2)\\
& +Z_{il}^{-1}(a_s^{\mathsf{MOM}}, N_F+2, \mu)\hat{A}_{ij}\Bigl(\frac{-p^2}{\mu^2}, a_s^{\overline{\mathsf{MS}}}, N_F\Bigr)\\
& -Z_{il}^{-1}(a_s^{\overline{\mathsf{MS}}}, N_F, \mu)\hat{A}_{ij}\Bigl(\frac{-p^2}{\mu^2}, a_s^{\overline{\mathsf{MS}}}, N_F\Bigr) ~.
\end{aligned}\tag{227}
$$

Finally the limit $p^2 \rightarrow 0$ is performed. Since scale-less diagrams vanish when computing them in dimensional regularization only the Born piece of the massless OME contributes

$$
\hat{A}_{ij}\Bigl(0, \alpha_s^{\overline{\mathsf{MS}}}, N_F\Bigr) = \delta_{ij} ~. \tag{228}
$$

One obtains the UV–renormalization prescription

$$
\begin{aligned}
\tilde{\hat{A}}_{ij}^{Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, a_s^{\mathsf{MOM}}, N_F+2\Bigr) \;=\;& a_s^{\mathsf{MOM}}\Biggl(\hat{A}_{ij}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\Bigr) + Z_{ij}^{-1,(1)}(N_F+2, \mu) - Z_{ij}^{-1,(1)}(N_F)\Biggr)\\
& +a_s^{\mathsf{MOM}^2}\Biggl(\hat{A}_{ij}^{(2),Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\Bigr) + Z_{ij}^{-1,(2)}(N_F+2, \mu)\\
& -Z_{ij}^{-1,(2)}(N_F) + Z_{ik}^{-1,(1)}(N_F+2, \mu)\hat{A}_{kj}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\Bigr)\Biggr)\\
& +a_s^{\mathsf{MOM}^3}\Biggl(\hat{A}_{ij}^{(3),Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\Bigr) + Z_{ij}^{-1,(3)}(N_F+2, \mu)\\
& -Z_{ij}^{-1,(3)}(N_F) + Z_{ik}^{-1,(1)}(N_F+2, \mu)\hat{A}_{kj}^{(2),Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\Bigr)\\
& + Z_{ik}^{-1,(2)}(N_F+2, \mu)\hat{A}_{kj}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\Bigr)\Biggr) ~.
\end{aligned}\tag{229}
$$

Here Z–factors at N_F+2 flavors describe the massive case (224-226) while those with argument N_F denote the Z–factors for the massless case.

5.4 Mass Factorization

At this stage, only collinear singularities remain. They arise in massless subgraphs only and are therefore independent of the additional heavy quark flavor considered in this analysis. We thus follow [139] directly and remove the collinear singularities via mass factorization

$$A_{ij}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},a_s^{\mathsf{MOM}},N_F+2\Bigr) \;=\; \tilde{\tilde{A}}_{il}^{Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},a_s^{\mathsf{MOM}},N_F+2\Bigr)\Gamma_{lj}^{-1}\ . \tag{230}$$

In a fully massless scenario the transition functions Γ_{ij} would be related to the light flavor renormalization constant via the identity

$$\Gamma_{ij}\left(N_F\right)=Z_{ij}^{-1}\left(N_F\right)\ . \tag{231}$$

However in the presence of one or more heavy quark flavors the transition functions apply to massless subgraphs only. Due to this and the subtraction of the δ_{ij}–term in the UV-renormalized OMEs $\tilde{\tilde{A}}_{ij}^{Q}$ the transition functions contribute up to $O(\alpha_s^2)$ only and finally the combined renormalization formula

$$\begin{aligned}
&A_{ij}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},a_s^{\mathsf{MOM}},N_F+2\Bigr)=\\
&\quad a_s^{\mathsf{MOM}}\left(\hat{A}_{ij}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{ij}^{-1,(1)}(N_F+2)-Z_{ij}^{-1,(1)}(N_F)\right)\\
&+a_s^{\mathsf{MOM}2}\Biggl(\hat{A}_{ij}^{(2),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{ij}^{-1,(2)}(N_F+2)-Z_{ij}^{-1,(2)}(N_F)\\
&\qquad+Z_{ik}^{-1,(1)}(N_F+2)\hat{A}_{kj}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+\Bigl[\hat{A}_{il}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{il}^{-1,(1)}(N_F+2)\\
&\qquad-Z_{il}^{-1,(1)}(N_F)\Bigr]\Gamma_{lj}^{-1,(1)}(N_F)\Biggr)\\
&+a_s^{\mathsf{MOM}3}\Biggl(\hat{A}_{ij}^{(3),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{ij}^{-1,(3)}(N_F+2)-Z_{ij}^{-1,(3)}(N_F)\\
&\qquad+Z_{ik}^{-1,(1)}(N_F+2)\hat{A}_{kj}^{(2),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{ik}^{-1,(2)}(N_F+2)\hat{A}_{kj}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)\\
&\qquad+\Bigl[\hat{A}_{il}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{il}^{-1,(1)}(N_F+2)-Z_{il}^{-1,(1)}(N_F)\Bigr]\Gamma_{lj}^{-1,(2)}(N_F)\\
&\qquad+\Bigl[\hat{A}_{il}^{(2),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)+Z_{il}^{-1,(2)}(N_F+2)-Z_{il}^{-1,(2)}(N_F)\\
&\qquad+Z_{ik}^{-1,(1)}(N_F+2)\hat{A}_{kl}^{(1),Q}\Bigl(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\Bigr)\Bigr]\Gamma_{lj}^{-1,(1)}(N_F)\Biggr)
\end{aligned} \tag{232}$$

is obtained. Eq. (232) differs from the corresponding renormalization prescription for one heavy quark flavor [139] by the definition of the renormalization constants $Z_{ij}^{-1,(k)}(N_F+2)$ only. Now the term δ_{ij} is added back to the massive OME. In a final step the coupling constant is transformed to the $\overline{\mathsf{MS}}$–scheme via Eq. (207).

5.5 Mass renormalization schemes

For the renormalization of the quark masses the on-shell and the $\overline{\mathsf{MS}}$–scheme are frequently used. The $\overline{\mathsf{MS}}$–renormalization constant is given by

$$\hat{m} = Z_m^{\overline{\mathsf{MS}}}\overline{m} \;=\; \overline{m}\Big[1 + \hat{a}_s\delta\overline{m}_1 + \hat{a}_s^2\delta\overline{m}_2\Big] + O(\hat{a}_s^3) \;, \tag{233}$$

with the coefficients [251],

$$\begin{aligned}
\delta\overline{m}_1 &= \frac{6}{\varepsilon}C_F \equiv \frac{\delta\overline{m}_1^{(-1)}}{\varepsilon} \;, &(234)\\
\delta\overline{m}_2 &= \frac{C_F}{\varepsilon^2}\left(18C_F - 22C_A + 8T_F(N_F+2)\right) + \frac{C_F}{\varepsilon}\left(\frac{3}{2}C_F + \frac{97}{6}C_A - \frac{10}{3}T_F(N_F+2)\right)\\
&\equiv \frac{\delta\overline{m}_2^{(-2)}}{\varepsilon^2} + \frac{\delta\overline{m}_2^{(-1)}}{\varepsilon} \;. &(235)
\end{aligned}$$

The following relations between the coefficients in ε of the on-shell renormalization constant and the $\overline{\mathsf{MS}}$–renormalization constant are observed:

$$\begin{aligned}
\delta\overline{m}_1^{(-1)} &= \delta m_1^{(-1)} \;, &(236)\\
\delta\overline{m}_2^{(-2)} &= \delta m_2^{(-2)} \;, &(237)\\
\delta\overline{m}_2^{(-1)} &= \delta m_2^{(-1)} - \delta m_1^{(-1)}\delta m_1^{(0)} + 2\delta m_1^{(0)}(\beta_0 + N_h\beta_{0,Q}) \;. &(238)
\end{aligned}$$

The scheme transformation are derived from the fact that the bare mass does not depend on the renormalization scheme:

$$\hat{m} = Z_m^{\overline{\mathsf{MS}}}\overline{m} = Z_m m \tag{239}$$

After renormalizing the coupling to the $\overline{\mathsf{MS}}$–scheme we obtain the following scheme transformations for the masses $m_i, \quad i \in 1,2$:

$$\begin{aligned}
\overline{m_i} \;=\; m_i\Bigg(1 &+ \left\{\frac{1}{2}\ln\left(\frac{m_i^2}{\mu^2}\right)\delta m_1^{(1)} + \delta m_1^{(0)}\right\}a_s^{\overline{\mathsf{MS}}} + \Bigg\{\delta m_2^{(0),i}(m_1,m_2) - \delta m_1^{(1)}\delta m_1^{(1)}\\
&+2\delta m_1^{(1)}\beta_0(N_F) + 4\delta m_1^{(1)}\beta_{0,Q} + \Bigg(-\frac{1}{2}\delta m_1^{(1)}\delta m_1^{(0)} + \delta m_2^{(-1)} + 2\delta m_1^{(0)}\beta_{0,Q}\\
&+\delta m_1^{(0)}\beta_0(N_F)\Bigg)\ln\left(\frac{m_i^2}{\mu^2}\right) + \left(-\frac{1}{4}\delta m_1^{(1)}\beta_0(N_F) + \frac{1}{8}\delta m_1^{(1)^2} - \frac{1}{2}\delta m_1^{(1)}\beta_{0,Q}\right)\\
&\ln\left(\frac{m_i^2}{\mu^2}\right)^2\Bigg\}a_s^{\overline{\mathsf{MS}}2}\Bigg) \;.
\end{aligned} \tag{240}$$

Inverting this relation yields

$$
\begin{aligned}
m_i &= \overline{m_i}\Biggl(1-\Biggl\{\delta m_1^{(0)}-\frac{1}{2}\delta m_1^{(-1)}\ln\left(\frac{\overline{m_i}^2}{\mu^2}\right)\Biggr\}a_s^{\overline{\mathsf{MS}}}+\Biggl\{-\delta m_2^{i,(0)}(\overline{m_1},\overline{m_2})+\delta m_1^{(-1)}\delta m_1^{(0)} \\
&+\left(\delta m_1^{(0)}\right)^2+\delta m_1^{(-1)}\delta m_1^{(1)}-2\delta m_1^{(1)}\beta_0(N_F)-4\delta m_1^{(1)}\beta_{0,Q}+\Biggl(\frac{1}{2}\left(\delta m_1^{(-1)}\right)^2 \\
&-2\delta m_1^{(0)}\beta_{0,Q}+\frac{3}{2}\delta m_1^{(-1)}\delta m_1^{(0)}-\delta m_1^{(0)}\beta_0(N_F)-\delta m_2^{(-1)}\Biggr)\ln\left(\frac{\overline{m_i}^2}{\mu^2}\right) \\
&+\Biggl(\frac{1}{8}\left(\delta m_1^{(-1)}\right)^2+\frac{1}{4}\delta m_1^{(-1)}\beta_0(N_F)+\frac{1}{2}\delta m_1^{(-1)}\beta_{0,Q}\Biggr)\ln\left(\frac{\overline{m_i}^2}{\mu^2}\right)^2\Biggr\}a_s^{\overline{\mathsf{MS}}^2}\Biggr) . \qquad (241)
\end{aligned}
$$

We will present all our results in the on–shell scheme. The transformation to the $\overline{\mathsf{MS}}$–scheme can then easily be performed applying Eq. (241).

5.6 One–particle reducible contributions

Starting from $O(\alpha_s^2)$ the OMEs also contain one-particle reducible contributions. Up to now only the irreducible diagrams were computed. The renormalization acts on the complete OME. Adding the one–particle reducible contributions requires the knowledge of the corresponding quark– and gluon self–energies as well as lower order OMEs.

5.6.1 Self–energy contributions

The scalar self energies are obtained by projecting out the Lorentz–structure

$$\hat{\Pi}^{ab}_{\mu\nu}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2,\hat{a}_s) = i\delta^{ab}\left[-g_{\mu\nu}p^2+p_\mu p_\nu\right]\hat{\Pi}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2,\hat{a}_s) , \qquad (242)$$

$$\hat{\Pi}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2,\hat{a}_s) = \sum_{k=1}^{\infty}\hat{a}_s^k\hat{\Pi}^{(k)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) , \qquad (243)$$

$$\hat{\Sigma}_{ij}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2,\hat{a}_s) = i\ \delta_{ij}\ \not{p}\ \hat{\Sigma}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2,\hat{a_s}) , \qquad (244)$$

$$\hat{\Sigma}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2,\hat{a}_s) = \sum_{k=2}^{\infty}\hat{a}_s^k\hat{\Sigma}^{(k)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) . \qquad (245)$$

We decompose the irreducible two–mass self energies into contributions which depend on one mass only and an additional part stemming from diagrams with both heavy quark flavors

$$\hat{\Pi}^{(k)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) = \hat{\Pi}^{(k)}(p^2,\frac{\hat{m_1}^2}{\mu^2})+\hat{\Pi}^{(k)}(p^2,\frac{\hat{m_2}^2}{\mu^2})+\hat{\tilde{\Pi}}^{(k)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) \qquad (246)$$

and for the quark self–energy

$$\hat{\Sigma}^{(j)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) = \hat{\Sigma}^{(j)}(p^2,\frac{\hat{m_1}^2}{\mu^2})+\hat{\Sigma}^{(j)}(p^2,\frac{\hat{m_2}^2}{\mu^2})+\hat{\tilde{\Sigma}}^{(j)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) . \qquad (247)$$

Up to two–loop order no diagrams with two heavy flavors contribute

$$\hat{\tilde{\Pi}}^{(k)}(p^2,\hat{m_1}^2,\hat{m_2}^2,\mu^2) = 0 \text{ for } k\in\{1,2\} , \qquad (248)$$

$$\hat{\hat{\Sigma}}^{(2)}(p^2, \hat{m}_1{}^2, \hat{m}_2{}^2, \mu^2) \;=\; 0\,. \tag{249}$$

The single-mass contributions are known from [139, 261–263]

$$\hat{\Pi}^{(1)}\Bigl(0, \frac{\hat{m}^2}{\mu^2}\Bigr) \;=\; T_F\Bigl(\frac{\hat{m}^2}{\mu^2}\Bigr)^{\varepsilon/2}\Biggl\{-\frac{8}{3\varepsilon}\exp\Bigl(\sum_{i=2}^{\infty}\frac{\zeta_i}{i}\Bigl(\frac{\varepsilon}{2}\Bigr)^i\Bigr)\Biggr\}\,, \tag{250}$$

$$\begin{aligned}
\hat{\Pi}^{(2)}\Bigl(0, \frac{\hat{m}^2}{\mu^2}\Bigr) \;=\;& T_F\Bigl(\frac{\hat{m}^2}{\mu^2}\Bigr)^{\varepsilon}\Biggl\{-\frac{4}{\varepsilon^2}C_A + \frac{1}{\varepsilon}\Bigl\{-12C_F + 5C_A\Bigr\} + \\
& C_A\Bigl(\frac{13}{12} - \zeta_2\Bigr) - \frac{13}{3}C_F + \varepsilon\,\Biggl\{C_A\Bigl(\frac{169}{144} + \frac{5}{4}\zeta_2 - \frac{\zeta_3}{3}\Bigr) + C_F\Bigl(-\frac{35}{12} - 3\zeta_2\Bigr)\Biggr\}\Biggr\} \\
& + O(\varepsilon^2)\,,
\end{aligned} \tag{251}$$

$$\begin{aligned}
\hat{\Pi}^{(3)}\Bigl(0, \frac{\hat{m}^2}{\mu^2}\Bigr) \;=\;& T_F\Bigl(\frac{\hat{m}^2}{\mu^2}\Bigr)^{3\varepsilon/2}\Biggl(\frac{1}{\varepsilon^3}\Biggl\{-\frac{32}{9}T_F C_A\Bigl(2N_F + 1\Bigr) + C_A^2\frac{164}{9}\Biggr\} \\
& + \frac{1}{\varepsilon^2}\Biggl\{\frac{80}{27}\Bigl(C_A - 6C_F\Bigr)N_F T_F + \frac{8}{27}\Bigl(35C_A - 48C_F\Bigr)T_F - C_A^2\frac{781}{27} \\
& + \frac{712}{9}C_A C_F\Biggr\} + \frac{1}{\varepsilon}\Biggl\{\frac{4}{27}\Bigl(C_A(-101 - 18\zeta_2) - 62C_F\Bigr)N_F T_F \\
& + \frac{2}{27}\Bigl(C_A(-37 - 18\zeta_2) - 80C_F\Bigr)T_F + C_A^2\Bigl(-12\zeta_3 + \frac{41}{6}\zeta_2 + \frac{3181}{108}\Bigr) \\
& + C_A C_F\Bigl(16\zeta_3 - \frac{1570}{27}\Bigr) + \frac{272}{3}C_F^2\Biggr\} + N_F T_F\Biggl\{C_A\Bigl(\frac{56}{9}\zeta_3 + \frac{10}{9}\zeta_2 \\
& - \frac{3203}{243}\Bigr) + C_F\Bigl(-\frac{20}{3}\zeta_2 - \frac{1942}{81}\Bigr)\Biggr\} + T_F\Biggl\{C_A\Bigl(-\frac{295}{18}\zeta_3 + \frac{35}{9}\zeta_2 + \frac{6361}{486}\Bigr) \\
& + C_F\Bigl(-7\zeta_3 - \frac{16}{3}\zeta_2 - \frac{218}{81}\Bigr)\Biggr\} + C_A^2\Biggl\{4\mathsf{B}_4 - 27\zeta_4 + \frac{1969}{72}\zeta_3 - \frac{781}{72}\zeta_2 \\
& + \frac{42799}{3888}\Biggr\} + C_A C_F\Biggl\{-8\mathsf{B}_4 + 36\zeta_4 - \frac{1957}{12}\zeta_3 + \frac{89}{3}\zeta_2 + \frac{10633}{81}\Biggr\} \\
& \quad + C_F^2\Biggl\{\frac{95}{3}\zeta_3 + \frac{274}{9}\Biggr\}\Biggr) + O(\varepsilon)\,,
\end{aligned} \tag{252}$$

and for the quark self–energy,

$$\hat{\Sigma}^{(2)}(0, \frac{\hat{m}^2}{\mu^2}) \;=\; T_F C_F\Biggl(\Bigl(\frac{\hat{m}_1{}^2}{\mu^2}\Bigr)^{\varepsilon} + \Bigl(\frac{\hat{m}_2{}^2}{\mu^2}\Bigr)^{\varepsilon}\Biggr)\Biggl\{\frac{2}{\varepsilon} + \frac{5}{6} + \Biggl[\frac{89}{72} + \frac{\zeta_2}{2}\Biggr]\varepsilon\Biggr\} + O(\varepsilon^2)\,. \tag{253}$$

$$\begin{aligned}
\hat{\Sigma}^{(3)}(0, \frac{\hat{m}^2}{\mu^2}) \;=\;& T_F C_F\Bigl(\frac{\hat{m}^2}{\mu^2}\Bigr)^{3\varepsilon/2}\Biggl(\frac{8}{3\varepsilon^3}C_A + \frac{1}{\varepsilon^2}\Bigl\{\frac{32}{9}T_F(N_F + 2) - C_A\frac{40}{9} \\
& - \frac{8}{3}C_F\Bigr\} + \frac{1}{\varepsilon}\Biggl\{\frac{40}{27}T_F(N_F + 2) + C_A\Bigl\{\zeta_2 + \frac{454}{27}\Bigr\} - 26C_F\Biggr\}
\end{aligned}$$

$$+N_FT_F\Big\{\frac{4}{3}\zeta_2+\frac{674}{81}\Big\}+T_F\Big\{\frac{8}{3}\zeta_2+\frac{604}{81}\Big\}+C_A\Big\{\frac{17}{3}\zeta_3-\frac{5}{3}\zeta_2+\frac{1879}{162}\Big\}$$
$$+C_F\Big\{-8\zeta_3-\zeta_2-\frac{335}{18}\Big\}\Bigg)+O(\varepsilon)\,. \tag{254}$$

Similarly to other massive processes [264–268], the constant

$$\mathsf{B}_4 \;=\; -4\zeta_2\ln^2(2)+\frac{2}{3}\ln^4(2)-\frac{13}{2}\zeta_4+16\mathrm{Li}_4\Big(\frac{1}{2}\Big) \;\approx\; -1.762800093... \tag{255}$$

emerges in Eq (252). At $O(\alpha_s^3)$ irreducible diagrams with two different masses contribute for the first time. For the gluonic case we computed the respective diagrams up to $O(\eta^3)$ using the codes `Q2e/Exp` [185, 186]

$$\begin{aligned}
\hat{\hat{\Pi}}^{(3)}\Big(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\Big) \;=\; & \Bigg\{-\frac{1}{\varepsilon^2}\frac{256}{9}+\frac{1}{\varepsilon}\Bigg(-\frac{320}{27}-\frac{64}{3}\Bigg(\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)+\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\Bigg)\Bigg)\\
&+\Bigg(-\frac{128}{315}\eta^3-\frac{40}{3}-\frac{32}{35}\eta^2\Bigg)\Bigg(\ln^2\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)+\ln^2\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\Bigg)\\
&+\Bigg(-16/3+\frac{64}{35}\eta^2\frac{256}{315}\eta^3\Bigg)\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\\
&+\Bigg(-\frac{32}{3}-\frac{64}{15}\eta-\frac{10208}{3675}\eta^2-\frac{39616}{99225}\eta^3\Bigg)\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\\
&+\Bigg(-\frac{64}{9}+\frac{64}{15}\eta+\frac{10208}{3675}\eta^2+\frac{39616}{99225}\eta^3\Bigg)\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\\
&-\frac{1504}{81}-\frac{32}{3}\zeta_2+\frac{416}{225}\eta-\frac{1987136}{385875}\eta^2-\frac{7026016}{31255875}\eta^3\Bigg\}T_F^2C_F\\
&+\Bigg\{-\frac{1}{\varepsilon^3}\frac{64}{9}+\frac{1}{\varepsilon^2}\Bigg[\frac{560}{27}-16/3\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)-16/3\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\Bigg]\\
&+\frac{1}{\varepsilon}\Bigg[-8/3\zeta_2-4\Bigg(\ln^2\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)+\ln^2\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\Bigg)-\frac{148}{27}\\
&+\frac{140}{9}\Bigg(\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)+\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\Bigg)\Bigg]-\frac{22}{9}\ln^3\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\\
&+4/3\Bigg(\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\ln^2\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)-\ln^2\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\Bigg)\\
&+\Bigg(\frac{65}{9}-2/15\,\eta-\frac{16}{21}\,\eta^2-\frac{50}{189}\,\eta^3\Bigg)\ln^2\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\\
&+\Bigg(\frac{80}{9}+\frac{4}{15}\eta+\frac{32}{21}\eta^2+\frac{100}{189}\eta^3\Bigg)\ln\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\\
&+\Bigg(-\frac{389}{27}-2\zeta_2-\frac{1924}{225}\eta-\frac{6392}{2205}\eta^2-\frac{20284}{59535}\eta^3\Bigg)\ln\Bigg(\frac{\hat{m}_1{}^2}{\mu^2}\Bigg)\\
&-\frac{14}{9}\ln^3\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)+\Bigg(\frac{65}{9}-2/15\eta-\frac{16}{21}\eta^2-\frac{50}{189}\eta^3\Bigg)\ln^2\Bigg(\frac{\hat{m}_2{}^2}{\mu^2}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{167}{27}-2\zeta_2+\frac{1924}{225}\eta+\frac{6392}{2205}\eta^2+\frac{20284}{59535}\eta^3\right)\ln\left(\frac{\hat{m}_2{}^2}{\mu^2}\right)\\
&-\frac{1139}{243}+\frac{70}{9}\zeta_2+\frac{56}{9}\zeta_3-\frac{34144}{3375}\eta-\frac{1292594}{231525}\eta^2\\
&\left.\left.-\frac{4231264}{18753525}\eta^3\right\}T_F^2C_A+O\left(\eta^3\right)\right\}+O\left(\varepsilon\right)\ . \qquad (256)
\end{aligned}
$$

The quarkonic self-energy contributions have been calculated analytically in η using the methods described in Section 6.3

$$
\hat{\hat{\Sigma}}^{(3)}(p^2,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2) = C_FT_F^2\left\{-\frac{128}{9\varepsilon^2}+\frac{80}{27\varepsilon}+\frac{2\ln^2(\eta)}{3}+\frac{8\zeta_2}{3}+\frac{604}{81}\right\}+O(\varepsilon)\ . \qquad (257)
$$

5.6.2 The Operator matrix elements

As in Eqs. (246-247) we define the two–mass OMEs at one–loop order and the irreducible OMEs at $O(\alpha_s^2)$ by

$$
\begin{aligned}
\hat{\hat{A}}_{ij}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_1^2}{\mu^2}\right) &= \hat{\hat{A}}_{ij}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)+\hat{\hat{A}}_{ij}^{(1)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\ , \qquad (258)\\
\hat{\hat{A}}_{ij}^{(2),\mathrm{irr}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_1^2}{\mu^2}\right) &= \hat{\hat{A}}_{ij}^{(2),\mathrm{irr}}\left(\frac{\hat{m}_1^2}{\mu^2}\right)+\hat{\hat{A}}_{ij}^{(2),\mathrm{irr}}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\ , \qquad (259)
\end{aligned}
$$

where the A_{ij} functions with one argument denote the usual single–mass OMEs. Using the definitions (246-247) and (258-259) we compose the reducible massive operator matrix elements at $O(\alpha_2^2)$ by

$$
\begin{aligned}
\hat{\hat{A}}_{qq}^{(2),\mathrm{NS}}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right) &= \hat{\hat{A}}_{qq}^{(2),\mathrm{NS,irr}}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)+\hat{\Sigma}^{(2)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\ , \qquad (260)\\
\hat{\hat{A}}_{Qg}^{(2)}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right) &= \hat{\hat{A}}_{Qg}^{(2),\mathrm{irr}}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\\
&+\hat{\hat{A}}_{Qg}^{(1)}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\hat{\Pi}^{(1)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\ , \qquad (261)\\
\hat{\hat{A}}_{gg}^{(2)}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right) &= \hat{\hat{A}}_{gg}^{(2),\mathrm{irr}}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)+\hat{\Pi}^{(2)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\\
&+\hat{\Pi}^{(1)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)^2\ , \qquad (262)
\end{aligned}
$$

and at $O(\alpha_s^3)$ by

$$
\begin{aligned}
\hat{\hat{A}}_{qq}^{(3),\mathrm{NS}}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right) &= \hat{\hat{A}}_{qq}^{(3),\mathrm{NS,irr}}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)+\hat{\Sigma}^{(3)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right) \qquad (263)\\
\hat{\hat{A}}_{Qg}^{(3)} &= \hat{\hat{A}}_{Qg}^{(3),\mathrm{irr}}+\hat{\hat{A}}_{Qg}^{(2)}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\hat{\Pi}^{(1)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\\
&+\hat{\hat{A}}_{Qg}^{(1)}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\hat{\Pi}^{(2)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right) \qquad (264)\\
\hat{\hat{A}}_{gg}^{(3)} &= \hat{\hat{A}}_{gg}^{(3),\mathrm{irr}}+\hat{\hat{A}}_{gg}^{(2)}\left(\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\hat{\Pi}^{(1)}\left(0,\hat{m}_1{}^2,\hat{m}_2{}^2,\mu^2\right)\ . \qquad (265)
\end{aligned}
$$

5.7 General Structure of the Massive Operator Matrix Elements

In the case of one heavy quark flavor with mass m, the mass dependence of the unrenormalized massive operator matrix element at order α_s^l is given by

$$\hat{\hat{A}}_{ij}^{(l)}\Bigl(\frac{\hat{m}^2}{\mu^2},\varepsilon,N\Bigr) = \Bigl(\frac{m^2}{\mu^2}\Bigr)^{\frac{l\varepsilon}{2}} \hat{\hat{A}}_{ij}^{(l)}\Bigl(\varepsilon,N\Bigr) . \tag{266}$$

Here $\hat{\hat{A}}_{ij}^{(l)}\Bigl(\varepsilon,N\Bigr)$ does not depend on the mass anymore. It exhibits poles in the dimensional regularization parameter ε up to ε^{-l}

$$\hat{\hat{A}}_{ij}^{(l)}\Bigl(\varepsilon,N\Bigr) = \sum_{k=0}^{\infty} \frac{a_{ij}^{(l,k)}}{\varepsilon^{l-k}} . \tag{267}$$

Furthermore, we adopt the notation of Ref. [139] and denote

$$a^{(l,l)} \equiv a^{(l)} , \quad a^{(l,l+1)} \equiv \overline{a}^{(l)}. \tag{268}$$

The unrenormalized operator matrix elements with two massive fermion flavors with masses $m_1 \neq m_2$ are split into the respective single-mass terms (266,267) and a part $\hat{\hat{\tilde{A}}}_{ij}^{(l)}\left[\frac{\hat{m_1}^2}{\mu^2},\frac{\hat{m_2}^2}{\mu^2},\varepsilon,N\right]$ depending on both masses

$$\hat{\hat{A}}_{ij}^{(l)}\Bigl(\frac{\hat{m_1}^2}{\mu^2},\frac{\hat{m_2}^2}{\mu^2},\varepsilon,N\Bigr) = \left[\Bigl(\frac{m_1^2}{\mu^2}\Bigr)^{\frac{l\varepsilon}{2}} + \Bigl(\frac{m_2^2}{\mu^2}\Bigr)^{\frac{l\varepsilon}{2}}\right] \hat{\hat{A}}_{ij}^{(l)}\Bigl(\varepsilon,N\Bigr) + \hat{\hat{\tilde{A}}}_{ij}^{(l)}\Bigl(\frac{\hat{m_1}^2}{\mu^2},\frac{\hat{m_2}^2}{\mu^2},\varepsilon,N\Bigr) . \tag{269}$$

The two flavor contributions $\hat{\hat{\tilde{A}}}_{ij}^{(l)}\Bigl(\frac{\hat{m_1}^2}{\mu^2},\frac{\hat{m_2}^2}{\mu^2},\varepsilon,N\Bigr)$, $m_1 \neq m_2$, to the massive OMEs do not obey a factorization relation as (266) and the mass dependence is pulled into the coefficients of the Laurent expansion

$$\hat{\hat{\tilde{A}}}_{ij}^{(l)}\Bigl(\varepsilon,N\Bigr) = \sum_{k=0}^{\infty} \frac{a_{ij}^{(l,k)}\left(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)}{\varepsilon^{l-k}} . \tag{270}$$

Analogously to (268) we define

$$\tilde{a}^{(l,l)}\left(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right) \equiv \tilde{a}^{(l)}\left(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right). \tag{271}$$

In the following $a^{(l,k)}$, $a^{(l)}$, $\overline{a}^{(l)}$ without argument will denote the single mass–quantities corresponding to the definitions in (267,268), while $\tilde{a}^{(l,l)}\left(\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)$ refers to the two-mass contribution. From Eq. (232) it is obvious that the renormalization of the 3–loop OMEs requires the knowledge of the one–loop OMEs $A_{ij}^{(1)}(m_1,m_2)$ up to $O(\varepsilon^2)$ and the two–loop OMEs $A_{ij}^{(2)}(m_1,m_2)$

up to $O(\varepsilon)$. Up to $O(\alpha_s^2)$ these two mass quantities can be traced back to the corresponding single–mass quantities by Eqs. (258-259) and (260-262).
It is technically advantageous to perform the renormalization on the complete two-flavor OMEs $\hat{\hat{A}}_{ij}^{(l)}\left(\frac{\hat{m_1}^2}{\mu^2}, \frac{\hat{m_2}^2}{\mu^2}, \varepsilon, N\right)$. For brevity we will present the renormalization formulas for the two-mass contribution $\hat{\hat{\tilde{A}}}_{ij}^{(l)}\left(\frac{\hat{m_1}^2}{\mu^2}, \frac{\hat{m_2}^2}{\mu^2}, \varepsilon, N\right)$ only, which is obtained after subtracting the respective single-mass contributions, cf. Refs. [139, 246].

5.7.1 $A_{qq,Q}^{\mathsf{NS}}$

The lowest non–trivial flavor non-singlet(NS)–contribution is of $O(a_s^2)$,

$$A_{qq,Q}^{\mathsf{NS}} = 1 + a_s^2 A_{qq,Q}^{(2),\mathsf{NS}} + a_s^3 A_{qq,Q}^{(3),\mathsf{NS}} + O(a_s^4) \ . \tag{272}$$

Starting from a_s^3 it exhibits non-trivial two–mass contributions

$$\tilde{A}_{qq,Q}^{\mathsf{NS}} = 1 + a_s^3 \tilde{A}_{qq,Q}^{(3),\mathsf{NS}} + O(a_s^4) \ . \tag{273}$$

The renormalized two-mass OME in the MOM–scheme is obtained from the bare quantities combining Eqs. (209, 232). It is given by

$$\begin{aligned} A_{qq,Q}^{(3),\mathsf{NS},\mathsf{MOM}}(N_F+2) &= \hat{A}_{qq,Q}^{(3),\mathsf{NS},\mathsf{MOM}} + Z_{qq}^{-1,(3),\mathsf{NS}}(N_F+N_h) - Z_{qq}^{-1,(3),\mathsf{NS}}(N_F) \\ &\quad + Z_{qq}^{-1,(1),\mathsf{NS}}(N_F+N_h)\hat{A}_{qq,Q}^{(2),\mathsf{NS},\mathsf{MOM}} + \Big[\hat{A}_{qq,Q}^{(2),\mathsf{NS},\mathsf{MOM}} \\ &\quad + Z_{qq}^{-1,(2),\mathsf{NS}}(N_F+N_h) - Z_{qq}^{-1,(2),\mathsf{NS}}(N_F)\Big]\Gamma_{qq}^{-1,(1)}(N_F) \ . \end{aligned} \tag{274}$$

After a finite renormalization to the $\overline{\mathsf{MS}}$–scheme and the subtraction of the single-mass contributions one obtains the the pole-structure of the two-flavor piece

$$\begin{aligned} \hat{\hat{\tilde{A}}}_{qq,Q}^{(3),\text{NS}} &= -\frac{16}{3}\frac{1}{\varepsilon^3}\gamma_{qq}^{(0)}\beta_{0,Q}^2 + \frac{1}{\varepsilon^2}\left[-\frac{8}{3}\beta_{0,Q}\hat{\gamma}_{qq}^{\text{NS},(1)} - 4\gamma_{qq}^{(0)}\beta_{0,Q}^2\left(L_2+L_1\right)\right] \\ &\quad + \frac{1}{\varepsilon}\left[-2\beta_{0,Q}\hat{\gamma}_{qq}^{\text{NS},(1)}\left(L_2+L_1\right) - 2\gamma_{qq}^{(0)}\beta_{0,Q}^2\left(L_2^2+L_2L_1+L_1^2\right) - 8a_{qq}^{\text{NS},(2)}\beta_{0,Q} \right. \\ &\quad \left. + \frac{2}{3}\hat{\tilde{\gamma}}_{qq}^{(2),\text{NS}}\right] + \tilde{a}_{qq,Q}^{(3),\text{NS}}\left(m_1^2, m_2^2, \mu^2\right) \ , \end{aligned} \tag{275}$$

with

$$L_1 = \ln\left(\frac{m_1^2}{\mu^2}\right) \ , \ L_2 = \ln\left(\frac{m_2^2}{\mu^2}\right) \ . \tag{276}$$

The renormalized expression in the $\overline{\mathsf{MS}}$–scheme is given by

$$\begin{aligned} \tilde{A}_{qq,Q}^{\overline{\mathsf{MS}},\text{NS}} &= \gamma_{qq}^{(0)}\beta_{0,Q}^2\left(\frac{2}{3}L_2^3 + \frac{2}{3}L_1^3 + \frac{1}{2}L_2^2L_1 + \frac{1}{2}L_1^2L_2\right) + \beta_{0,Q}\hat{\gamma}_{qq}^{\text{NS},(1)}\left(L_2^2+L_1^2\right) \\ &\quad + \left\{4a_{qq}^{\text{NS},(2)}\beta_{0,Q} + \frac{1}{2}\beta_{0,Q}^2\gamma_{qq}^{(0)}\zeta_2\right\}\left(L_2+L_1\right) + 8\overline{a}_{qq}^{\text{NS},(2)}\beta_{0,Q} \end{aligned}$$

$$+\tilde{a}_{qq,Q}^{(3),\rm NS}\left(m_1^2,m_2^2,\mu^2\right)~. \tag{277}$$

For the explicit expressions for the anomalous dimensions γ_{ij} see Refs. [161, 162] and for the 2–loop constant and $O(\varepsilon)$ part of the OMEs see [135, 137] and references in these papers.

5.7.2 $A_{Qq}^{\sf PS}$

Depending on whether the operator couples to a heavy or a light fermion, there are two pure–singlet contributions [139]

$$A_{Qq}^{\sf PS} = a_s^2 A_{Qq}^{(2),\sf PS} + a_s^3 A_{Qq}^{(3),\sf PS} + O(a_s^4)~, \tag{278}$$

$$A_{qq,Q}^{\sf PS} = a_s^3 A_{qq,Q}^{(3),\sf PS} + O(a_s^4)~. \tag{279}$$

Up to $O(a_s^3)$ only the OME A_{Qq} contains a generic two–mass contribution, since $A_{qq,Q}^{\sf PS}$ emerges only at $O(a_s^3)$ and contains one internal massless fermion line. One has

$$\tilde{A}_{Qq}^{\sf PS} = a_s^3 \tilde{A}_{Qq}^{(3),\sf PS} + O(a_s^4)~. \tag{280}$$

The combined renormalization relation at third order is given by

$$\begin{aligned}
A_{Qq}^{(3),\sf PS,MOM} &+ A_{qq,Q}^{(3),\sf PS,MOM} = \hat{A}_{Qq}^{(3),\sf PS,MOM} + \hat{A}_{qq,Q}^{(3),\sf PS,MOM} + Z_{qq}^{-1,(3),\sf PS}(N_F+N_h) \\
&- Z_{qq}^{-1,(3),\sf PS}(N_F) + Z_{qq}^{-1,(1)}(N_F+N_h)\hat{A}_{Qq}^{(2),\sf PS,MOM} + Z_{qg}^{-1,(1)}(N_F+N_h)\hat{A}_{gq,Q}^{(2),\sf MOM} \\
&+ \Big[\hat{A}_{Qg}^{(1),\sf MOM} + Z_{qg}^{-1,(1)}(N_F+N_h) - Z_{qg}^{-1,(1)}(N_F)\Big]\Gamma_{gq}^{-1,(2)}(N_F) + \Big[\hat{A}_{Qq}^{(2),\sf PS,MOM} \\
&+ Z_{qq}^{-1,(2),\sf PS}(N_F+N_h) - Z_{qq}^{-1,(2),\sf PS}(N_F)\Big]\Gamma_{qq}^{-1,(1)}(N_F) + \Big[\hat{A}_{Qg}^{(2),\sf MOM} + Z_{qg}^{-1,(2)}(N_F+N_h) \\
&- Z_{qg}^{-1,(2)}(N_F) + Z_{qq}^{-1,(1)}(N_F+N_h)A_{Qg}^{(1),\sf MOM} + Z_{qg}^{-1,(1)}(N_F+N_h)A_{gg,Q}^{(1),\sf MOM}\Big]\Gamma_{gq}^{-1,(1)}(N_F)~.
\end{aligned} \tag{281}$$

This yields the generic pole structure for the $\sf PS$ two–mass contribution

$$\begin{aligned}
\hat{\hat{\tilde{A}}}_{Qq}^{(3),\rm PS} &= \frac{16}{3}\frac{1}{\varepsilon^3}\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q} + \frac{1}{\varepsilon^2}\Bigg[4\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\left(L_2+L_1\right) + \frac{2}{3}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)} - \frac{8}{3}\beta_{0,Q}\hat{\gamma}_{qq}^{\rm PS,(1)}\Bigg] \\
&+\frac{1}{\varepsilon}\Bigg[2\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\left(L_2^2+L_1L_2+L_1^2\right) + \left\{\frac{1}{2}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)} - 2\beta_{0,Q}\hat{\gamma}_{qq}^{\rm PS,(1)}\right\}\left(L_2+L_1\right) \\
&+\frac{2}{3}\hat{\tilde{\gamma}}_{qq}^{(2),\rm PS} - 8a_{Qq}^{(2),\rm PS}\beta_{0,Q} + 2\hat{\gamma}_{qg}^{(0)}a_{gq}^{(2)}\Bigg] + \tilde{a}_{Qq}^{(3),\rm PS}\left(m_1^2,m_2^2,\mu^2\right)~.
\end{aligned} \tag{282}$$

In the $\overline{\sf MS}$–scheme one obtains the renormalized expression

$$\begin{aligned}
\tilde{A}_{Qq}^{\overline{\sf MS},\rm PS} &= -\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\left(\frac{1}{2}L_2^2L_1 + \frac{1}{2}L_1^2L_2 + \frac{2}{3}L_2^3 + \frac{2}{3}L_1^3\right) + \left\{-\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)} + \beta_{0,Q}\hat{\gamma}_{qq}^{\rm PS,(1)}\right\} \\
&\times\left(L_2^2+L_1^2\right) + \left\{4a_{Qq}^{(2),\rm PS}\beta_{0,Q} - \hat{\gamma}_{qg}^{(0)}a_{gq}^{(2)} - \frac{1}{2}\beta_{0,Q}\zeta_2\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\right\}\left(L_2+L_1\right) + 8\overline{a}_{Qq}^{(2),\rm PS}\beta_{0,Q} \\
&-2\hat{\gamma}_{qg}^{(0)}\overline{a}_{gq}^{(2)} + \tilde{a}_{Qq}^{(3),\rm PS}\left(m_1^2,m_2^2,\mu^2\right)~.
\end{aligned} \tag{283}$$

5.7.3 A_{Qg}

Like in the PS case, there are two different contributions to the OME A_{Qg}

$$
\begin{aligned}
A_{Qg} &= a_s A_{Qg}^{(1)} + a_s^2 A_{Qg}^{(2)} + a_s^3 A_{Qg}^{(3)} + O(a_s^4)\ . && (284)\\
A_{qg,Q} &= a_s^3 A_{qg,Q}^{(3)} + O(a_s^4)\ . && (285)
\end{aligned}
$$

Of these OMEs only A_{Qg} contains two flavor contributions starting from $O(a_s^2)$

$$
\begin{aligned}
\tilde{A}_{Qg} &= a_s^2 \tilde{A}_{Qg}^{(2)} + a_s^3 \tilde{A}_{Qg}^{(3)} + O(a_s^4)\ . && (286)\\
& && (287)
\end{aligned}
$$

In Eq. (286) the a_s^2 contribution consists of one–particle reducible diagrams only, see Eq. (261). As a consequence the flavor dependence factorizes in the $O(a_s^2)$ terms.
The renormalized MOM–scheme two–loop contribution is obtained by

$$
\begin{aligned}
A_{Qg}^{(2),\mathsf{MOM}} &= \hat{A}_{Qg}^{(2),\mathsf{MOM}} + Z_{qg}^{-1,(2)}(N_F+N_h) - Z_{qg}^{-1,(2)}(N_F) + Z_{qg}^{-1,(1)}(N_F+N_h)\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}\\
&+ Z_{qq}^{-1,(1)}(N_F+N_h)\hat{A}_{Qg}^{(1),\mathsf{MOM}} + \Big[\hat{A}_{Qg}^{(1),\mathsf{MOM}} + Z_{qg}^{-1,(1)}(N_F+N_h)\\
&- Z_{qg}^{-1,(1)}(N_F)\Big]\Gamma_{gg}^{-1,(1)}(N_F)\ . && (288)
\end{aligned}
$$

The unrenormalized terms are given by

$$
\begin{aligned}
\hat{\hat{\tilde{A}}}_{Qg}^{(2)} &= -\frac{1}{\varepsilon^2}4\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} - \frac{1}{\varepsilon}2\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left(\ln\left(\frac{m_b^2}{\mu^2}\right) + \ln\left(\frac{m_c^2}{\mu^2}\right)\right) + \tilde{a}_{Qg}^{(2)}\left(\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)\\
&+ \varepsilon\overline{\tilde{a}}_{Qg}^{(2)}\left(\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)\ . && (289)
\end{aligned}
$$

The coefficients $\tilde{a}_{Qg}^{(2)}$ and $\overline{\tilde{a}}_{Qg}^{(2)}\left(\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)$ are read off Eqn. (261)

$$
\begin{aligned}
a_{Qg}^{(2)} &= -\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left\{\frac{1}{2}(L_1+L_2)^2 + \zeta_2\right\}, && (290)\\
\overline{a}_{Qg}^{(2)} &= \beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left\{-\frac{1}{12}(L_2+L_1)^3 - \frac{1}{2}\zeta_2(L_2+L_1) - \frac{1}{3}\zeta_3\right\}\ . && (291)
\end{aligned}
$$

The renormalized expression at 2 loops then reads

$$
\tilde{A}_{Qg}^{(2),\overline{\mathsf{MS}}} = \frac{1}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left(L_1^2+L_2^2\right) + \zeta_2\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} + \tilde{a}_{Qg}^{(2)}\ . \qquad (292)
$$

The renormalized 3–loop OMEs in the MOM–scheme are obtained from the charge– and mass–renormalized OMEs by

$$
\begin{aligned}
&A_{Qg}^{(3),\mathsf{MOM}} + A_{qg,Q}^{(3),\mathsf{MOM}} = \hat{A}_{Qg}^{(3),\mathsf{MOM}} + \hat{A}_{qg,Q}^{(3),\mathsf{MOM}} + Z_{qg}^{-1,(3)}(N_F+N_h) - Z_{qg}^{-1,(3)}(N_F)\\
&\quad + Z_{qg}^{-1,(2)}(N_F+N_h)\hat{A}_{gg,Q}^{(1),\mathsf{MOM}} + Z_{qg}^{-1,(1)}(N_F+N_h)\hat{A}_{gg,Q}^{(2),\mathsf{MOM}} + Z_{qq}^{-1,(2)}(N_F+N_h)\hat{A}_{Qg}^{(1),\mathsf{MOM}}\\
&\quad + Z_{qq}^{-1,(1)}(N_F+N_h)\hat{A}_{Qg}^{(2),\mathsf{MOM}} + \Big[\hat{A}_{Qg}^{(1),\mathsf{MOM}} + Z_{qg}^{-1,(1)}(N_F+N_h)
\end{aligned}
$$

$$
\begin{aligned}
&- Z_{qq}^{-1,(1)}(N_F)\Big]\Gamma_{gg}^{-1,(2)}(N_F) + \Big[\hat{A}_{Qg}^{(2),\mathsf{MOM}} + Z_{qg}^{-1,(2)}(N_F+N_h) - Z_{qg}^{-1,(2)}(N_F) \\
&+ Z_{qq}^{-1,(1)}(N_F+N_h)A_{Qg}^{(1),\mathsf{MOM}} + Z_{qg}^{-1,(1)}(N_F+N_h)A_{gg,Q}^{(1),\mathsf{MOM}}\Big]\Gamma_{gg}^{-1,(1)}(N_F) \\
&+ \Big[\hat{A}_{Qq}^{(2),\mathsf{PS,MOM}} + Z_{qq}^{-1,(2),\mathsf{PS}}(N_F+N_h) - Z_{qq}^{-1,(2),\mathsf{PS}}(N_F)\Big]\Gamma_{qg}^{-1,(1)}(N_F) \\
&+ \Big[\hat{A}_{qq,Q}^{(2),\mathsf{NS,MOM}} + Z_{qq}^{-1,(2),\mathsf{NS}}(N_F+N_h) - Z_{qq}^{-1,(2),\mathsf{NS}}(N_F)\Big]\Gamma_{qg}^{-1,(1)}(N_F)\ . \qquad (293)
\end{aligned}
$$

The structure of the unrenormalized OME is more complex than in the NS– or PS case. It is given by

$$
\begin{aligned}
\hat{\hat{A}}_{Qg}^{(3)} &= \frac{1}{\varepsilon^3}\Bigg[\frac{28}{3}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} - \frac{8}{3}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q} + \frac{14}{3}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)} + 24\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)} + \frac{1}{3}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\Bigg] \\
&+\frac{1}{\varepsilon^2}\Bigg[\Bigg\{\frac{1}{4}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2 + 18\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)} + 7\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} - 2\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q} + \frac{7}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}\Bigg\} \\
&\times(L_2+L_1) + \frac{1}{3}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{PS},(1)} + \frac{1}{3}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)} - \frac{10}{3}\beta_{0,Q}\hat{\gamma}_{qg}^{(1)} + \frac{2}{3}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gg}^{(1)} - \frac{2}{3}\hat{\gamma}_{qg}^{(0)}\beta_{1,Q} \\
&+10\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)}\Bigg] + \frac{1}{\varepsilon}\Bigg[\Bigg\{\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)} + \frac{15}{2}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)} + \frac{1}{4}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{PS},(1)} - \frac{5}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(1)} \\
&+\frac{1}{2}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gg}^{(1)} - \frac{1}{2}\hat{\gamma}_{qg}^{(0)}\beta_{1,Q}\Bigg\}(L_2+L_1) + \Bigg\{\frac{13}{4}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} + \frac{13}{8}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)} + \frac{15}{2}\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)} \\
&+\frac{3}{16}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2 - \hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q}\Bigg\}\left(L_2^2+L_1^2\right) + \Bigg\{-\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q} + 4\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} + 12\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)} \\
&+2\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}\Bigg\}L_2L_1 + \frac{2}{3}\hat{\hat{\gamma}}_{qg}^{(2)} - 8\beta_{0,Q}a_{Qg}^{(2)} - \frac{1}{8}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\zeta_2\gamma_{gq}^{(0)} + 2\hat{\gamma}_{qg}^{(0)}a_{gg,Q}^{(2)} - 2\hat{\gamma}_{qg}^{(0)}\tilde{\delta}m_2^{(-1)} \\
&+9\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}^2 + \frac{1}{4}\beta_{0,Q}\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)} + \frac{1}{2}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\beta_0 + 8\delta m_1^{(0)}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\Bigg] \\
&+\tilde{a}_{Qg}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ . \qquad (294)
\end{aligned}
$$

After subtracting the respective one-mass OMEs for both masses m_1 and m_2 the two–flavor contribution is obtained. It reads

$$
\begin{aligned}
\tilde{A}_{Qg}^{(3),\overline{\mathsf{MS}}} &= \Bigg\{-\frac{9}{4}\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)} - \frac{7}{96}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2 + \frac{1}{3}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q} - \frac{25}{48}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)} - \frac{25}{24}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\Bigg\} \\
&\times\left(L_2^3+L_1^3\right) + \Bigg\{\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q} - \beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} - \frac{1}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)} - 3\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)}\Bigg\} \\
&\times\left(L_2^2L_1+L_1^2L_2\right) + \Bigg\{-\frac{1}{16}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{PS},(1)} - \frac{1}{16}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)} + \frac{9}{8}\beta_{0,Q}\hat{\gamma}_{qg}^{(1)} - \frac{1}{4}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gg}^{(1)} \\
&-\frac{29}{8}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)} + \frac{1}{8}\hat{\gamma}_{qg}^{(0)}\beta_{1,Q}\Bigg\}\left(L_2^2+L_1^2\right) - 4L_2L_1\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)} + \Bigg\{\frac{3}{2}\hat{\gamma}_{qg}^{(0)}\tilde{\delta}m_2^{(-1)}
\end{aligned}
$$

$$
+\frac{1}{32}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\zeta_2\gamma_{gq}^{(0)}+\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\gamma_{qq}^{(0)}-6\delta m_1^{(0)}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}-\hat{\gamma}_{qg}^{(0)}a_{gg,Q}^{(2)}-\frac{9}{16}\beta_{0,Q}\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}
-\frac{27}{4}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}^2-\frac{9}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\beta_0+4\beta_{0,Q}a_{Qg}^{(2)}\Big\}(L_2+L_1)+8\overline{a}_{Qg}^{(2)}\beta_{0,Q}-\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\hat{\gamma}_{qq}^{\mathrm{PS},(1)}
-\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\hat{\gamma}_{qq}^{\mathrm{NS},(1)}+\frac{1}{24}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\zeta_3\gamma_{gq}^{(0)}-3\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}^2\zeta_3+\frac{1}{4}\hat{\gamma}_{qg}^{(1)}\beta_{0,Q}\zeta_2-8\delta m_1^{(1)}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}
+\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{1,Q}-2\hat{\gamma}_{qg}^{(0)}\overline{a}_{gg,Q}^{(2)}-\frac{1}{6}\hat{\gamma}_{qg}^{(0)}\beta_0\beta_{0,Q}\zeta_3-\frac{1}{12}\beta_{0,Q}\zeta_3\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}+\hat{\gamma}_{qg}^{(0)}\Big(\tilde{\delta}m_2{}^{1,(0)}
+\tilde{\delta}m_2{}^{2,(0)}\Big)-\frac{9}{4}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\delta m_1^{(-1)}+\tilde{a}_{Qg}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ . \qquad (295)
$$

5.7.4 $A_{gq,Q}$

The matrix element $A_{gq,Q}$ contains contributions starting at $O(a_s^2)$,

$$
A_{gq,Q} = a_s^2A_{gq,Q}^{(2)}+a_s^3A_{gq,Q}^{(3)}+O(a_s^4)\ . \qquad (296)
$$

Diagrams with two different masses however contribute only from $O(a_s^3)$

$$
\tilde{A}_{gq,Q} = a_s^3\tilde{A}_{gq,Q}^{(3)}+O(a_s^4)\ . \qquad (297)
$$

The renormalization in the MOM–scheme is performed using

$$
\begin{aligned}
A_{gq,Q}^{(2),\mathsf{MOM}} &= \hat{A}_{gq,Q}^{(2),\mathsf{MOM}}+Z_{gq}^{-1,(2)}(N_F+N_h)-Z_{gq}^{-1,(2)}(N_F)\\
&\quad+\Big(\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}+Z_{gg}^{-1,(1)}(N_F+N_h)-Z_{gg}^{-1,(1)}(N_F)\Big)\Gamma_{gq}^{-1,(1)}\ , \qquad (298)\\
A_{gq,Q}^{(3),\mathsf{MOM}} &= \hat{A}_{gq,Q}^{(3),\mathsf{MOM}}+Z_{gq}^{-1,(3)}(N_F+N_h)-Z_{gq}^{-1,(3)}(N_F)+Z_{gg}^{-1,(1)}(N_F+N_h)\hat{A}_{gq,Q}^{(2),\mathsf{MOM}}\\
&\quad+Z_{gq}^{-1,(1)}(N_F+N_h)\hat{A}_{qq}^{(2),\mathsf{MOM}}+\Big[\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}+Z_{gg}^{-1,(1)}(N_F+N_h)\\
&\quad-Z_{gg}^{-1,(1)}(N_F)\Big]\Gamma_{gq}^{-1,(2)}(N_F)+\Big[\hat{A}_{gq,Q}^{(2),\mathsf{MOM}}+Z_{gq}^{-1,(2)}(N_F+N_h)\\
&\quad-Z_{gq}^{-1,(2)}(N_F)\Big]\Gamma_{qq}^{-1,(1)}(N_F)+\Big[\hat{A}_{gg,Q}^{(2),\mathsf{MOM}}+Z_{gg}^{-1,(2)}(N_F+N_h)\\
&\quad-Z_{gg}^{-1,(2)}(N_F)+Z_{gg}^{-1,(1)}(N_F+N_h)\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}\\
&\quad+Z_{gq}^{-1,(1)}(N_F+N_h)\hat{A}_{Qg}^{(1),\mathsf{MOM}}\Big]\Gamma_{gq}^{-1,(1)}(N_F)\ . \qquad (299)
\end{aligned}
$$

Applying Eq. (299) yields the unrenormalized expression

$$
\begin{aligned}
\hat{\hat{\tilde{A}}}_{gq,Q}^{(3)} &= -16\frac{1}{\varepsilon^3}\gamma_{gq}^{(0)}\beta_{0,Q}^2+\frac{1}{\varepsilon^2}\Big[-12\gamma_{gq}^{(0)}\beta_{0,Q}^2(L_2+L_1)-4\beta_{0,Q}\hat{\gamma}_{gq}^{(1)}\Big]+\frac{1}{\varepsilon}\Big[-6\gamma_{gq}^{(0)}\beta_{0,Q}^2\\
&\quad\times\left(L_2^2+L_1L_2+L_1^2\right)-3\beta_{0,Q}\hat{\gamma}_{gq}^{(1)}(L_2+L_1)+\frac{2}{3}\hat{\tilde{\gamma}}_{gq}^{(2)}-12a_{gq}^{(2)}\beta_{0,Q}\Big]\\
&\quad+\tilde{a}_{gq,Q}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ , \qquad (300)
\end{aligned}
$$

and the renormalized operator matrix element

$$
\tilde{A}_{gq,Q}^{(3),\overline{\mathsf{MS}}} = \gamma_{gq}^{(0)}\beta_{0,Q}^2\left(2L_2^3+2L_1^3+\frac{3}{2}L_2^2L_1+\frac{3}{2}L_1^2L_2\right)+\frac{3}{2}\beta_{0,Q}\hat{\gamma}_{gq}^{(1)}\left(L_2^2+L_1^2\right)
$$

$$+\left\{6a_{gq}^{(2)}\beta_{0,Q}+\frac{3}{2}\gamma_{gq}^{(0)}\beta_{0,Q}^2\zeta_2\right\}(L_2+L_1)+12\overline{a}_{gq}^{(2)}\beta_{0,Q}+\tilde{a}_{gq,Q}^{(3)}\left(m_1^2,m_2^2,\mu^2\right) \ .(301)$$

5.7.5 $A_{gg,Q}$

Finally, the matrix element $A_{gg,Q}$ obeys the expansion

$$A_{gg,Q} \;=\; 1+a_sA_{gg,Q}^{(1)}+a_s^2A_{gg,Q}^{(2)}+a_s^3A_{gg,Q}^{(3)}+O(a_s^4) \ , \tag{302}$$

with two–mass contributions starting at $O(a_s^2)$,

$$\tilde{A}_{gg,Q} \;=\; +a_s^2\tilde{A}_{gg,Q}^{(2)}+a_s^3\tilde{A}_{gg,Q}^{(3)}+O(a_s^4) \ . \tag{303}$$

The renormalization formulas in the MOM–scheme read

$$\begin{aligned}
A_{gg,Q}^{(2),\mathsf{MOM}} \;=\;& \hat{A}_{gg,Q}^{(2),\mathsf{MOM}}+Z_{gg}^{-1,(2)}(N_F+N_h)-Z_{gg}^{-1,(2)}(N_F)\\
&+Z_{gg}^{-1,(1)}(N_F+N_h)\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}+Z_{gq}^{-1,(1)}(N_F+N_h)\hat{A}_{Qg}^{(1),\mathsf{MOM}}\\
&+\Big[\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}+Z_{gg}^{-1,(1)}(N_F+N_h)-Z_{gg}^{-1,(1)}(N_F)\Big]\Gamma_{gg}^{-1,(1)}(N_F) \ , \qquad (304)\\
A_{gg,Q}^{(3),\mathsf{MOM}} \;=\;& \hat{A}_{gg,Q}^{(3),\mathsf{MOM}}+Z_{gg}^{-1,(3)}(N_F+N_h)-Z_{gg}^{-1,(3)}(N_F)+Z_{gg}^{-1,(2)}(N_F+N_h)\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}\\
&+Z_{gg}^{-1,(1)}(N_F+N_h)\hat{A}_{gg,Q}^{(2),\mathsf{MOM}}+Z_{gq}^{-1,(2)}(N_F+N_h)\hat{A}_{Qg}^{(1),\mathsf{MOM}}\\
&+Z_{gq}^{-1,(1)}(N_F+N_h)\hat{A}_{Qg}^{(2),\mathsf{MOM}}+\Big[\hat{A}_{gg,Q}^{(1),\mathsf{MOM}}+Z_{gg}^{-1,(1)}(N_F+N_h)\\
&-Z_{gg}^{-1,(1)}(N_F)\Big]\Gamma_{gg}^{-1,(2)}(N_F)+\Big[\hat{A}_{gg,Q}^{(2),\mathsf{MOM}}+Z_{gg}^{-1,(2)}(N_F+N_h)\\
&-Z_{gg}^{-1,(2)}(N_F)+Z_{gq}^{-1,(1)}(N_F+N_h)A_{Qg}^{(1),\mathsf{MOM}}\\
&+Z_{gg}^{-1,(1)}(N_F+N_h)A_{gg,Q}^{(1),\mathsf{MOM}}\Big]\Gamma_{gg}^{-1,(1)}(N_F)\\
&+\Big[\hat{A}_{gq,Q}^{(2),\mathsf{MOM}}+Z_{gq}^{-1,(2)}(N_F+N_h)-Z_{gq}^{-1,(2)}(N_F)\Big]\Gamma_{qg}^{-1,(1)}(N_F) \ . \qquad (305)
\end{aligned}$$

After subtracting all single–mass contributions we obtain the unrenormalized pure two-flavor contribution at 2 loops

$$\hat{\hat{\tilde{A}}}_{gg,Q}^{(2)} \;=\; \frac{8\beta_{0,Q}}{\varepsilon^2}+\frac{4\beta_{0,Q}^2}{\varepsilon}(L_1+L_2)+\tilde{a}_{gg,Q}\left(m_1^2,m_2^2,\mu^2\right)+\varepsilon\overline{\tilde{a}}_{gg,Q}\left(m_1^2,m_2^2,\mu^2\right) \ . \tag{306}$$

and the renormalized expression

$$\tilde{A}_{gg,Q}^{(2),\overline{\mathsf{MS}}} \;=\; -\beta_{0,Q}^2\left(\ln^2\left(\frac{m_b^2}{\mu^2}\right)+\ln^2\left(\frac{m_b^2}{\mu^2}\right)\right)-2\beta_{0,Q}^2\zeta_2+\tilde{a}_{gg,Q}\left(m_1^2,m_2^2,\mu^2\right) \ . \tag{307}$$

The $O(a_s^2)$ contribution consists of one particle reducible contributions only and the coefficients follow from Eq. (262)

$$a_{gg,Q}^{(2)} \;=\; \beta_{0,Q}^2(L_2+L_1)^2+2\beta_{0,Q}^2\zeta_2 \ , \tag{308}$$

$$\overline{a}_{gg,Q}^{(2)} \;=\; \frac{1}{6}\beta_{0,Q}^2(L_1+L_2)^3+\beta_{0,Q}^2\zeta_2(L_2+L_1)+\frac{2}{3}\beta_{0,Q}^2\zeta_3 \ . \tag{309}$$

The unrenormalized 3-loop contribution from two masses reads

$$\hat{\hat{A}}_{gg,Q}^{(3)} = \frac{1}{\varepsilon^3}\Biggl[-\frac{10}{3}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)} - \frac{56}{3}\beta_0\beta_{0,Q}^2 - \frac{28}{3}\beta_{0,Q}^2\gamma_{gg}^{(0)} - 48\beta_{0,Q}^3\Biggr] + \frac{1}{\varepsilon^2}\Biggl[\Biggl\{-7\beta_{0,Q}^2\gamma_{gg}^{(0)}$$
$$-14\beta_0\beta_{0,Q}^2 - \frac{5}{2}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)} - 36\beta_{0,Q}^3\Biggr\}(L_2+L_1) + \frac{1}{3}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)} - \frac{14}{3}\beta_{0,Q}\hat{\gamma}_{gg}^{(1)}$$
$$+\frac{4}{3}\beta_{1,Q}\beta_{0,Q} - 20\delta m_1^{(-1)}\beta_{0,Q}^2\Biggr] + \frac{1}{\varepsilon}\Biggl[\Biggl\{\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)} - 15\delta m_1^{(-1)}\beta_{0,Q}^2 - \frac{7}{2}\beta_{0,Q}\hat{\gamma}_{gg}^{(1)}$$
$$+\beta_{1,Q}\beta_{0,Q}\Biggr\}(L_2+L_1) + \Biggl\{-15\beta_{0,Q}^3 - \frac{11}{8}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)} - \frac{13}{2}\beta_0\beta_{0,Q}^2 - \frac{13}{4}\beta_{0,Q}^2\gamma_{gg}^{(0)}\Biggr\}$$
$$\times\left(L_2^2+L_1^2\right) + \Biggl\{-4\beta_{0,Q}^2\gamma_{gg}^{(0)} - 24\beta_{0,Q}^3 - 8\beta_0\beta_{0,Q}^2 - \hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)}\Biggr\}L_2L_1 - \frac{1}{2}\beta_{0,Q}^2\zeta_2\gamma_{gg}^{(0)}$$
$$+\frac{2}{3}\hat{\tilde{\gamma}}_{gg}^{(2)} - 12\beta_{0,Q}a_{gg,Q}^{(2)} - 18\beta_{0,Q}^3\zeta_2 + \frac{1}{4}\beta_{0,Q}\zeta_2\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} - \beta_0\beta_{0,Q}^2\zeta_2 - 16\delta m_1^{(0)}\beta_{0,Q}^2$$
$$+4\beta_{0,Q}\tilde{\delta}m_2^{(-1)}\Biggr] + \tilde{a}_{gg,Q}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ . \qquad (310)$$

The renormalized result in the $\overline{\mathsf{MS}}$–scheme is given by

$$\tilde{A}_{gg,Q}^{(3),\overline{\mathsf{MS}}} = +\Biggl\{\frac{25}{24}\beta_{0,Q}^2\gamma_{gg}^{(0)} + \frac{25}{12}\beta_0\beta_{0,Q}^2 + \frac{9}{2}\beta_{0,Q}^3 + \frac{23}{48}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)}\Biggr\}\left(L_2^3+L_1^3\right) + \Biggl\{\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)}$$
$$+\beta_{0,Q}^2\gamma_{gg}^{(0)} + 2\beta_0\beta_{0,Q}^2 + 6\beta_{0,Q}^3\Biggr\}\left(L_2^2L_1 + L_1^2L_2\right) + \Biggl\{-\frac{1}{4}\beta_{1,Q}\beta_{0,Q} + \frac{13}{8}\beta_{0,Q}\hat{\gamma}_{gg}^{(1)}$$
$$+\frac{29}{4}\delta m_1^{(-1)}\beta_{0,Q}^2 - \frac{1}{16}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)}\Biggr\}\left(L_2^2+L_1^2\right) + 8L_2L_1\delta m_1^{(-1)}\beta_{0,Q}^2 + \Biggl\{\frac{9}{4}\beta_0\beta_{0,Q}^2\zeta_2$$
$$+\frac{27}{2}\beta_{0,Q}^3\zeta_2 - 3\beta_{0,Q}\tilde{\delta}m_2^{(-1)} + \frac{9}{8}\beta_{0,Q}^2\zeta_2\gamma_{gg}^{(0)} + 12\delta m_1^{(0)}\beta_{0,Q}^2 + \frac{3}{16}\beta_{0,Q}\zeta_2\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}$$
$$+6\beta_{0,Q}a_{gg,Q}^{(2)}\Biggr\}(L_2+L_1) - \frac{1}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\hat{\gamma}_{gq}^{(1)} + \frac{1}{4}\beta_{0,Q}\zeta_2\hat{\gamma}_{gg}^{(1)} + \frac{1}{3}\beta_0\beta_{0,Q}^2\zeta_3 + 12\beta_{0,Q}\overline{a}_{gg,Q}^{(2)}$$
$$+6\beta_{0,Q}^3\zeta_3 + 16\delta m_1^{(1)}\beta_{0,Q}^2 + \frac{1}{6}\beta_{0,Q}^2\zeta_3\gamma_{gg}^{(0)} - 2\beta_{0,Q}\left(\tilde{\delta}_{m2}^{1,(0)} + \tilde{\delta}_{m2}^{2,(0)}\right)$$
$$+\frac{9}{2}\delta m_1^{(-1)}\beta_{0,Q}^2\zeta_2 - \frac{1}{12}\beta_{0,Q}\zeta_3\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} - \frac{1}{2}\beta_{0,Q}\zeta_2\beta_{1,Q}$$
$$+\tilde{a}_{gg,Q}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ . \qquad (311)$$

6 Massive OMEs with two masses

Starting at 3–loop order Feynman diagrams, which carry internal fermion lines of different mass contribute to the OMEs. Contributions of this type are dealt with here for the first time. In Section 6.1 we discuss the decoupling relations for the kinematic region where one has to consider two massive quark flavors. For all OMEs we are calculating a series of Mellin moments $N = 2, 4, 6$ [243,269], which are presented in the OMS–scheme in Section 6.2. Results at generals values of the Mellin variable N are given for the flavor non-singlet case for the vector and axial-vector current and in the case of transversity in Section 6.3 [243]. The equal mass case for the non-singlet and pure singlet terms is dealt with in Section 6.4. In Section 6.5 we devise a method to calculate the diagrams contributing to the OME $A_{gg,Q}$ and compute all scalar topologies both in x- and in N-space [242, 270].

6.1 Decoupling two massive quark flavours

In Section 2.2 the decoupling of the heavy flavor Wilson coefficient into light flavor Wilson coefficients and massive operator matrix elements has been discussed for the case of one massive quark flavor. However, the relevant Eqs. (65-69) maybe be generalized to allow for the simultaneous decoupling of more than one heavy quark flavor. This is advised because $m_c^2/m_b^2 \sim 0.1$. One obtains

$$\begin{aligned} C_{q,(2,L)}^{\mathsf{NS}}\left(N, N_F, \frac{Q^2}{\mu^2}\right) + L_{q,(2,L)}^{\mathsf{NS}}\left(N, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \quad = \\ A_{qq,Q}^{\mathsf{NS}}\left(N, N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) C_{q,(2,L)}^{\mathsf{NS}}\left(N, N_F+2, \frac{Q^2}{\mu^2}\right) \ , \end{aligned} \tag{312}$$

in the non–singlet case. For the pure–singlet and singlet contribution the relations read

$$\begin{aligned} C_{q,(2,L)}^{\mathsf{PS}}(N_F) + L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \Big[A_{qq,Q}^{\mathsf{NS}}(N_F+2) + A_{qq,Q}^{\mathsf{PS}}(N_F+2) + A_{Qq}^{\mathsf{PS}}(N_F+2)\Big] \\ &\quad \times N_F \tilde{C}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) \\ &\quad + A_{qq,Q}^{\mathsf{PS}}(N_F+2) C_{q,(2,L)}^{\mathsf{NS}}(N_F+2) \\ &\quad + A_{gq,Q}(N_F+2) N_F \tilde{C}_{g,(2,L)}(N_F+2) \ , \end{aligned} \tag{313}$$

$$\begin{aligned} C_{g,(2,L)}(N_F) + L_{g,(2,L)}(N_F+2) &= A_{gg,Q}(N_F+2) N_F \tilde{C}_{g,(2,L)}(N_F+2) \\ &\quad + A_{qg,Q}(N_F+2) C_{q,(2,L)}^{\mathsf{NS}}(N_F+2) \\ &\quad + \Big[A_{qg,Q}(N_F+2) + A_{Qg}(N_F+2)\Big] \\ &\quad \times N_F \tilde{C}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) \ . \end{aligned} \tag{314}$$

Here and in the following the mass- and Q^2-dependence of the Wilson coefficients and operator matrix elements have been suppressed for brevity. For the $H_{i,j}$ functions the factorization relations into light flavor Wilson coefficients and massive OMEs take the following form in the case of two different masses

$$
\begin{aligned}
H_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= A_{Qq}^{\mathsf{PS}}(N_F+2)\Big[C_{q,(2,L)}^{\mathsf{NS}}(N_F+2)+\tilde{C}_{q,(2,L)}^{\mathsf{PS}}(N_F+2)\Big] \\
&\quad +\Big[A_{qq,Q}^{\mathsf{NS}}(N_F+2)+A_{qq,Q}^{\mathsf{PS}}(N_F+2)\Big]\tilde{C}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) \\
&\quad +A_{gq,Q}(N_F+2)\tilde{C}_{g,(2,L)}(N_F+2)\ , \qquad (315) \\
H_{g,(2,L)}(N_F+2) &= A_{gg,Q}(N_F+2)\tilde{C}_{g,(2,L)}(N_F+2)+A_{qg,Q}(N_F+2)\tilde{C}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) \\
&\quad +A_{Qg}(N_F+2)\Big[C_{q,(2,L)}^{\mathsf{NS}}(N_F+2)+\tilde{C}_{q,(2,L)}^{\mathsf{PS}}(N_F+2)\Big]\ . \qquad (316)
\end{aligned}
$$

Expanding the expressions (312–316) up to $O(\alpha_s^3)$ then yields

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2+\hat{\hat{C}}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+\ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\qquad +\hat{\hat{C}}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]\ , \qquad (317) \\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2+A_{gq,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\qquad +N_F\hat{\hat{\tilde{C}}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]\ , \qquad (318) \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2A_{gg,Q}^{(1)}(N_F+2)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2+A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) \\
&\qquad +A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\qquad +\ A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)+N_F\hat{\hat{\tilde{C}}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \qquad (319)
\end{aligned}
$$

$$
\begin{aligned}
H_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] \qquad (320) \\
&+\ a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) \\
&\quad +A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad +A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2+\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] \\
&+\ a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2+\ A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\qquad +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big] \\
&+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2+\ A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\qquad +\ A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\qquad +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2)+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} \\
&\qquad +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)+\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\ , \qquad (321)
\end{aligned}
$$

with $\delta_2 = 1$ for the structure functions F_2 and $\delta_2 = 0$ for F_L, respectively. In Eqs (312-321) the following notation has been applied

$$\tilde{f}(x) = \frac{f(x)}{x} , \tag{322}$$

$$\hat{\hat{f}}(x) = f(x+2) - f(x) . \tag{323}$$

The presence of diagrams with c- and b-quarks at 3–loop order yields power corrections in $\eta = m_c^2/m_b^2$ to the massive operator matrix elements, which can not be absorbed into the charm– or bottom-densities of the variable flavor number scheme directly. It is therefore advantageous to extend Eqs. (324-328) to allow for a simultaneous decoupling of both heavy quark flavors in the variable flavour number scheme:

$$\begin{aligned}
&f_k(N_F+2,N,\mu^2,m_1^2,m_2^2) + f_{\overline{k}}(N_F+2,N,\mu^2,m_1^2,m_2^2) = \\
&\quad A_{qq,Q}^{\mathsf{NS}}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F,N,\mu^2)+f_{\overline{k}}(N_F,N,\mu^2)\right] \\
&\quad +\frac{1}{N_F}A_{qq,Q}^{\mathsf{PS}}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F,N,\mu^2) \\
&\quad +\frac{1}{N_F}A_{qg,Q}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F,N,\mu^2),
\end{aligned} \tag{324}$$

$$\begin{aligned}
&f_Q(N_F+2,N,\mu^2,m^2) + f_{\overline{Q}}(N_F+2,N,\mu^2,m^2) = \\
&\quad A_{Qq}^{\mathsf{PS}}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F,N,\mu^2) \\
&\quad +A_{Qg}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F,N,\mu^2) .
\end{aligned} \tag{325}$$

The flavor singlet, non–singlet and gluon densities for (N_F+2) flavors are given by

$$\begin{aligned}
\Sigma(N_F+2,N,\mu^2,m^2) &= \Bigg[A_{qq,Q}^{\mathsf{NS}}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{\mathsf{PS}}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right) \\
&\quad +A_{Qq}^{\mathsf{PS}}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\Bigg]\cdot\Sigma(N_F,N,\mu^2) \\
&\quad +\left[A_{qg,Q}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F,N,\mu^2) ,
\end{aligned} \tag{326}$$

$$\begin{aligned}
\Delta_k(N_F+2,N,\mu^2,m^2) &= f_k(N_F+2,N,\mu^2,m^2)+f_{\overline{k}}(N_F+2,N,\mu^2,m^2) \\
&\quad -\frac{1}{N_F+2}\Sigma(N_F+2,N,\mu^2,m^2) ,
\end{aligned} \tag{327}$$

$$\begin{aligned}
G(N_F+2,N,\mu^2,m^2) &= A_{gq,Q}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F,N,\mu^2) \\
&\quad +A_{gg,Q}\left(N,N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F,N,\mu^2) .
\end{aligned} \tag{328}$$

Here $f_{k(\overline{k})}$ denote the quarkonic parton densities, Σ is the singlet distribution $\Sigma = \sum_{k=1}^{N_F}(f_k + f_{\overline{k}})$ and G is the gluon density.

6.2 Fixed Mellin Moments

While not yet sufficient for a phenomenological analysis the study of fixed Mellin moments is still interesting for various reasons. They provide a thorough check on the renormalization prescription given in Section 5, and the general N results given later in Sections 6.3 and 6.5. In theory the knowledge of a sufficiently large number of fixed Mellin moments also allows to construct the general N-expression [271]. In many cases it is, however, very difficult to a compute such number of fixed Mellin moments with the usual single-moment methods.

In Ref. [139, 246] fixed Mellin moments for the single mass case were computed by projecting the respective integrals onto massive tadpoles and evaluating them using the code `MATAD` [272]. In the present study we deal with one additional mass. The Feynman integrals are therefore first expanded in the mass ratio $\eta = \frac{m_c^2}{m_b^2}$ by an expansion in subgraphs [273–276] using the codes `Q2e/Exp` [185, 186] which also rely on `MATAD` to evaluate the single-mass tadpole diagrams.

The Feynman diagrams are generated using `QGRAF` [277]. In order to take account for the local operator insertions we introduce new additional propagators which either carry an operator insertion or which generate an operator on an attached vertex. In case of operator insertions on a gauge boson, this method leads to a double counting of some vertex diagrams which has to be removed.

After applying the Feynman rules (cf. Appendix B.2) and the projection operators, Eqs. (60-62) the momentum integrals take the form

$$I^{(l)}(p, m_1, m_2, n_1 \dots n_j) \equiv \int \frac{d^D k_1}{(2\pi)^D} \dots \int \frac{d^D k_l}{(2\pi)^D} (\Delta.q_1)^{n_1} \dots (\Delta.q_j)^{n_j} f(k_1 \dots k_l, p, m_1, m_2) . \tag{329}$$

Here p denotes the external momentum, $p^2 = 0$, Δ is an arbitrary light–like vector $\Delta^2 = 0$ and q_i are linear combinations of the loop momenta k_j and the external momentum p. The exponents n_i are integer-valued and obey $\sum n_i = N$, while the function $f(k_1 \dots k_l, p, m_1, m_2)$ contains the remaining numerator structure and denominators.

The light-like vector Δ has been introduced in Eq. (59) to project out trace terms. Removing it again yields the tensor $\tilde{I}^{(l)}_{\mu_1,\dots,\mu_N}$

$$I^{(l)}\left(p, m_1, m_2, n_1 \dots n_j\right) = \prod_{j=1}^{N} \Delta^{\mu_j} \tilde{I}^{(l)}_{\mu_1,\dots,\mu_N}\left(p, m_1, m_2, n_1 \dots n_j\right) . \tag{330}$$

Since $\prod_{j=1}^{N} \Delta^{\mu_j}$ constitutes a completely symmetric tensor only the purely symmetric part of $\tilde{I}^{(l)}_{\mu_1,\dots,\mu_N}$ contributes. We thus symmetrize it by shuffling the indices, [166–168, 172, 174, 180], and normalize it by dividing by the number of terms. For the general integral (329) the symmetrized tensor is given by

$$I^{(l)}_{\mu_1,\dots,\mu_M}\left(p, m_1, m_2, n_1 \dots n_j\right) = \mathbf{S} \ \tilde{I}^{(l)}_{\mu_1,\dots,\mu_M}\left(p, m_1, m_2, n_1 \dots n_j\right) . \tag{331}$$

Here the symmetrization operator $\mathbf{S}$ is defined by

$$\mathbf{S} f_{\mu_1,\dots,\mu_M} = \frac{1}{M!} \sum_{w} f_w , \tag{332}$$

where the sum is over the words w given by the different orderings of the Lorentz indices $\{\mu_1, \dots, \mu_M\}$.

For example for $M = 3$ one obtains

$$\mathbf{S}f_{\mu_1,\mu_2,\mu_3} = \frac{1}{6}\left[f_{\mu_1,\mu_2,\mu_3} + f_{\mu_1,\mu_3,\mu_2} + f_{\mu_2,\mu_1,\mu_3} + f_{\mu_2,\mu_3,\mu_1} + f_{\mu_3,\mu_1,\mu_2} + f_{\mu_3,\mu_2,\mu_1}\right] . \tag{333}$$

The result of the original integral (329) may then be re-obtained by applying the projection operator [139]

$$\Pi_{\mu_1\ldots\mu_N} = F(N)\sum_{i=1}^{[N/2]+1} C(i,N)\Big(\prod_{l=1}^{[N/2]-i+1}\frac{g_{\mu_{2l-1}\mu_{2l}}}{p^2}\Big)\Big(\prod_{k=2[N/2]-2i+3}^{N}\frac{p_{\mu_k}}{p^2}\Big) . \tag{334}$$

The prefactors $F(N)$ and the combinatorial factors $C(i,N)$ are given by

$$F^{odd}(N) = \frac{2^{3/2-N/2}\Gamma(D/2+1/2)}{(D-1)\Gamma(N/2+D/2-1)} , \tag{335}$$

$$C^{odd}(k,N) = (-1)^{N/2+k+1/2}\frac{2^{2k-N/2-3/2}\Gamma(N+1)\Gamma(D/2+N/2+k-3/2)}{\Gamma(N/2-k+3/2)\Gamma(2k)\Gamma(D/2+N/2-1/2)} , \tag{336}$$

for odd values and

$$C^{even}(k,N) = (-1)^{N/2+k+1}\frac{2^{2k-N/2-2}\Gamma(N+1)\Gamma(D/2+N/2-2+k)}{\Gamma(N/2-k+2)\Gamma(2k-1)\Gamma(D/2+N/2-1)} , \tag{337}$$

$$F^{even}(N) = \frac{2^{1-N/2}\Gamma(D/2+1/2)}{(D-1)\Gamma(N/2+D/2-1/2)} , \tag{338}$$

for even values of the Mellin variable N. The prefactors $F^{odd}(N)$, $F^{even}(N)$ are chosen such, that the projector (334) obeys the normalization condition

$$\Pi_{\mu_1\ldots\mu_N}p^{\mu_1}\ldots p^{\mu_N} = 1 . \tag{339}$$

The integrals with local operator insertion for fixed values of N are thus represented in terms of tadpole diagrams with a modified numerator structure. The projection operators (334) become very large for large values of N, which leads to an exponential increase in the computation time. In the case of two heavy quarks of different mass the computation of the Mellin moment $N = 6$ of the two–mass contributions to the OME A_{Qg} took already about one year of CPU–time.
The pole structure of the unrenormalized OMEs corresponds to the one which was deduced from the renormalization prescription in Section 5. It constitutes an important check on our results.
In the following we present the moment $N = 2$ for the two-flavor contributions to the constant parts of the various operator matrix elements as defined in Eq. (169). For $N = 2$ the following non-singlet contribution 2-mass contribution is obtained

$$\begin{aligned}\tilde{a}_{qq,Q}^{\mathsf{NS},(3)}(N=2) = C_F T_F^2\Biggl\{&\left(-\frac{1024}{8505}L_\eta^2 - \frac{190500608}{843908625} - \frac{747008}{2679075}L_\eta\right)\eta^3 \\ &+\left(-\frac{7176352}{1157625} - \frac{64}{105}L_\eta^2 - \frac{33856}{11025}L_\eta\right)\eta^2 + \left(-\frac{12032}{675} - \frac{512}{45}L_\eta\right)\eta \\ &+\left(-\frac{1024}{81} - \frac{128}{9}\left(L_2+L_1\right)\right)\zeta_2 - \frac{153856}{2187} + \frac{512}{81}\zeta_3 - \frac{14080}{243}L_1 - \frac{2048}{81}L_2\end{aligned}$$

$$
\begin{aligned}
&-\frac{1024}{81}L_2L_1-\frac{256}{27}L_1L_2^2-\frac{512}{81}L_2^3-\frac{1024}{81}\left(L_2^2+L_1^2\right)-\frac{640}{81}L_1^3-\frac{128}{27}L_1^2L_2\Bigg\}\\
&+O(\eta^4L_\eta^3) \qquad (340)
\end{aligned}
$$

Here and in the following we apply the notation

$$
L_1=\ln\left(\frac{m_1^2}{\mu^2}\right),\ L_2=\ln\left(\frac{m_2^2}{\mu^2}\right),\ L_\eta=\ln(\eta)\equiv\ln\left(\frac{m_2^2}{m_1^2}\right). \qquad (341)
$$

Note that here as well as in all other computed QCD-results within this thesis power corrections $\propto\eta^i$ do not appear in the pole-terms in the dimensional regularization parameter ε.
The constant contribution to the OME $A_{Qq}^{\mathsf{PS},(3)}$ is given by

$$
\begin{aligned}
\tilde{a}_{Qq}^{\mathsf{PS},(3)}(N=2) \;=\;& C_FT_F^2\Bigg\{\left(-\frac{381001216}{843908625}-\frac{1494016}{2679075}L_\eta-\frac{2048}{8505}L_\eta^2\right)\eta^3+\Bigg(-\frac{14352704}{1157625}\\
&-\frac{128}{105}L_\eta^2-\frac{67712}{11025}L_\eta\Bigg)\eta^2+\left(-\frac{24064}{675}-\frac{1024}{45}L_\eta\right)\eta+\Bigg(-\frac{1472}{81}\\
&-\frac{256}{9}\left(L_2+L_1\right)\Bigg)\zeta_2-\frac{1280}{81}L_1^3-\frac{256}{27}L_1^2L_2+\frac{1024}{81}\zeta_3-\frac{26720}{243}L_1-\frac{3616}{81}L_2\\
&-\frac{1472}{81}L_2L_1-\frac{512}{27}L_1L_2^2-\frac{266528}{2187}-\frac{1024}{81}L_2^3-\frac{1472}{81}\left(L_2^2+L_1^2\right)\Bigg\}\\
&+O\left(\eta^4L_\eta^3\right)\ . \qquad (342)
\end{aligned}
$$

For $\tilde{a}_{Qg}^{(3)}$ one obtains

$$
\begin{aligned}
\tilde{a}_{Qg}^{(3)}(N=2) \;=\;& C_AT_F^2\Bigg\{\left(\frac{56086736}{843908625}-\frac{164464}{2679075}L_\eta-\frac{2552}{8505}L_\eta^2\right)\eta^3+\Bigg(\frac{6008}{4725}L_\eta+\frac{1565036}{496125}\\
&-\frac{8}{45}L_\eta^2\Bigg)\eta^2+\left(\frac{256304}{10125}+\frac{7184}{675}L_\eta-\frac{8}{45}L_\eta^2\right)\eta+\left(-\frac{74}{81}+\frac{140}{9}\left(L_2+L_1\right)\right)\zeta_2\\
&-\frac{5}{3}L_2+\frac{772}{81}L_1^3-\frac{848}{81}\zeta_3+\frac{9355}{243}L_1+\frac{280}{27}L_1L_2^2-\frac{35}{81}\left(L_2^2+L_1^2\right)+\frac{104}{27}L_1^2L_2\\
&-\frac{152}{81}L_2L_1+\frac{596}{81}L_2^3\\
&+\frac{78229}{2187}\Bigg\}\\
&+C_FT_F^2\Bigg\{\left(\frac{826805984}{843908625}+\frac{5893184}{2679075}L_\eta+\frac{23872}{8505}L_\eta^2\right)\eta^3+\Bigg(\frac{1028192}{99225}L_\eta\\
&+\frac{169892864}{10418625}+\frac{4768}{945}L_\eta^2\Bigg)\eta^2+\left(-\frac{758944}{30375}+\frac{22976}{2025}L_\eta+\frac{448}{135}L_\eta^2\right)\eta\\
&+\left(\frac{320}{27}-\frac{64}{9}\left(L_2+L_1\right)\right)\zeta_2+\frac{6752}{243}L_2-\frac{704}{81}L_1^3+\frac{1792}{81}\zeta_3+\frac{128}{81}L_1-\frac{128}{27}L_1L_2^2
\end{aligned}
$$

$$
+\frac{968}{81}\left(L_2^2+L_1^2\right)+\frac{128}{27}L_1^2L_2+\frac{944}{81}L_2L_1-\frac{448}{81}L_2^3+\frac{64}{243}\Biggr\}+O\left(\eta^4L_\eta^3\right) . \quad (343)
$$

Finally the gluonic contributions to the OMEs $A_{gq,Q}^{(3)}$ and $A_{gg,Q}^{(3)}$ are given by

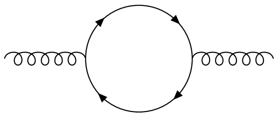
$$
\begin{aligned}
\tilde{a}_{gq,Q}^{(3)}(N=2) \;=\; & C_FT_F^2\Biggl\{\left(\frac{190500608}{281302875}+\frac{747008}{893025}L_\eta+\frac{1024}{2835}L_\eta^2\right)\eta^3+\Biggl(\frac{7176352}{385875}+\frac{64}{35}L_\eta^2 \\
&+\frac{33856}{3675}L_\eta\Biggr)\eta^2+\left(\frac{12032}{225}+\frac{512}{15}L_\eta\right)\eta+\left(\frac{832}{27}+\frac{128}{3}\left(L_2+L_1\right)\right)\zeta_2 \\
&+\frac{128}{9}L_1^2L_2-\frac{512}{27}\zeta_3+\frac{13600}{81}L_1+\frac{1888}{27}L_2+\frac{832}{27}L_2L_1+\frac{256}{9}L_1L_2^2 \\
&+\frac{512}{27}L_2^3+\frac{832}{27}\left(L_2^2+L_1^2\right)+\frac{640}{27}L_1^3+\frac{140128}{729}\Biggr\}+O\left(\eta^4L_\eta^3\right) , \qquad (344)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3)}(N=2) \;=\; & C_AT_F^2\Biggl\{\left(\frac{19188592}{120558375}+\frac{153892}{382725}L_\eta+\frac{686}{1215}L_\eta^2\right)\eta^3+\Biggl(\frac{53824}{33075}L_\eta+\frac{8433658}{3472875} \\
&+\frac{296}{315}L_\eta^2\Biggr)\eta^2+\left(-\frac{153872}{10125}-\frac{1412}{675}L_\eta+\frac{14}{45}L_\eta^2\right)\eta+\Biggl(-\frac{556}{81} \\
&-\frac{140}{9}\left(L_2+L_1\right)\Biggr)\zeta_2-\frac{214}{27}L_2-\frac{628}{81}L_1^3+\frac{272}{81}\zeta_3-\frac{6682}{243}L_1-\frac{280}{27}L_1L_2^2 \\
&-\frac{550}{81}\left(L_2^2+L_1^2\right)-\frac{176}{27}L_1^2L_2-\frac{568}{81}L_2L_1-\frac{524}{81}L_2^3-\frac{71578}{2187}\Biggr\} \\
&+C_FT_F^2\Biggl\{\left(-\frac{637103552}{843908625}-\frac{4823552}{2679075}L_\eta-\frac{20416}{8505}L_\eta^2\right)\eta^3+\Biggl(-\frac{752576}{99225}L_\eta \\
&-\frac{116240192}{10418625}-\frac{3904}{945}L_\eta^2\Biggr)\eta^2+\left(\frac{702784}{30375}-\frac{14336}{2025}L_\eta-\frac{448}{135}L_\eta^2\right)\eta+\Biggl(-\frac{32}{27} \\
&+\frac{64}{9}\left(L_2+L_1\right)\Biggr)\zeta_2-\frac{5024}{243}L_2+\frac{704}{81}L_1^3-\frac{1792}{81}\zeta_3+\frac{736}{81}L_1+\frac{128}{27}L_1L_2^2 \\
&+\frac{112}{81}\left(L_2^2+L_1^2\right)-\frac{128}{27}L_1^2L_2-\frac{512}{81}L_2L_1+\frac{448}{81}L_2^3+\frac{4448}{243}\Biggr\}+O\left(\eta^4L_\eta^3\right) . \\
&\qquad (345)
\end{aligned}
$$

The results for the constant parts in the Laurent expansion around $\varepsilon=0$ for the further Mellin moments $N=4$ and $N=6$ are listed in Appendix F.

6.3 The Non–Singlet Contributions

All non–singlet diagrams at 3–loop order contain two massive fermion loops. One of these may be stripped of its mass by using the Mellin–Barnes representation [278–281], see Figure 6. This yields similar integrals as in the case with one massive and one massless fermionic line [140].



$$
\begin{aligned}
&\hat{=}\, a_s T_F \frac{4}{\pi} (4\pi)^{-\varepsilon/2} \left(k_\mu k_\nu - k^2 g_{\mu\nu}\right) \\
&\times \int_{-i\infty}^{+i\infty} \mathrm{d}\sigma \left(\frac{m^2}{\mu^2}\right)^{\sigma} \left(-k^2\right)^{1/2\varepsilon-\sigma} \frac{\Gamma(\sigma-\varepsilon/2)\Gamma^2(2-\sigma+\varepsilon/2)\Gamma(-\sigma)}{\Gamma(4-2\sigma+\varepsilon)}
\end{aligned}
$$

Figure 6: One of the massive fermion loop insertion is effectively rendered massless via a Mellin–Barnes representation.

One may now introduce a Feynman parameter representation, integrate the momenta and perform the Feynman parameter integrals in terms of Euler B–functions (D.29) analogously to the procedure described in Section 3. The remaining contour integral is then of the general form

$$
I \propto \Gamma\begin{bmatrix} f_1(\varepsilon,N),\dots,f_i(\varepsilon,N) \\ f_{i+1}(\varepsilon,N),\dots,f_I(\varepsilon,N)\end{bmatrix} \int_{-i\infty}^{+i\infty} \mathrm{d}\xi\, \Gamma\begin{bmatrix} g_1(\varepsilon)+\xi, g_2(\varepsilon)+\xi, g_3(\varepsilon)+\xi, g_4(\varepsilon)-\xi, g_5(\varepsilon)-\xi \\ g_6(\varepsilon)+\xi, g_7(\varepsilon)-\xi \end{bmatrix} \eta^{\xi}\ , \tag{346}
$$

where the f_j and the g_j are linear functions. Furthermore, the notation

$$
\Gamma\begin{bmatrix} a_1,\dots,a_i \\ b_1,\dots,b_j \end{bmatrix} = \frac{\Gamma(a_1)\cdots\Gamma(a_i)}{\Gamma(b_1)\cdots\Gamma(b_j)} \tag{347}
$$

is applied. After closing the contour in (346) and collecting the residues a linear combination of generalized hypergeometric ${}_4F_3$–functions [142, 143] is obtained

$$
I = \sum_j C_j\left(\varepsilon, N\right) {}_4F_3\begin{bmatrix} a_1(\varepsilon), a_2(\varepsilon), a_3(\varepsilon), a4_(\varepsilon) \\ b_1(\varepsilon), b_2(\varepsilon), b_3(\varepsilon) \end{bmatrix}, \eta \Bigg]\ . \tag{348}
$$

For the NS–contributions the arguments of the hypergeometric ${}_PF_Q$ are completely independent of the Mellin variable N and each term factors into contributions that describe the operator insertions and the generalized hypergeometric functions covering the mass structure of the diagrams. Due to the fact that the arguments of the hypergeometric functions depend only on the dimensional regularization parameter ε their respective expansion may be performed with the code `HypExp 2` [282]. The results of these expansions are then given in terms of the following (poly)logarithmic functions [283–286], Eq. (D.33),

$$
\left\{ \ln(\eta),\ \ln\left(\frac{1-\eta_1}{1+\eta_1}\right),\ \mathrm{Li}_2\left(\sqrt{\eta}\right),\ \mathrm{Li}_2\left(\eta\right),\ \mathrm{Li}_3\left(\sqrt{\eta}\right) \right\}\ , \tag{349}
$$

with $\eta_1 = \sqrt{\eta}$. The prefactor $C_j\left(\varepsilon, N\right)$ may contain a sum stemming from the operator insertion on the vertex, see Appendix B. This sum is easily evaluated in terms of single harmonic sums using the summation package `Sigma` [236, 237] .
The general pole structure for the unrenormalized two-mass contribution to the OME $A_{qq,Q}^{\mathsf{NS}}$ is given in Eq. (275). The only contribution which is not determined by the renormalization prescription is the constant part, for which we obtain

$$
\begin{aligned}
&\tilde{a}_{qq,Q}^{(3),\mathrm{NS}} = \\
&C_F T_F^2 \Biggl\{ \left(\frac{32}{27} S_1 - \frac{8\left(3N^2+3N+2\right)}{27N(N+1)} \right) \ln^3(\eta) + \Biggl[- \frac{R_1}{18N^2(N+1)^2\eta} \\
&+ \Bigl[\frac{\left(3N^2+3N+2\right)(\eta+1)\left(5\eta^2+22\eta+5\right)}{36N(N+1)\eta^{3/2}} - \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}} S_1 \Bigr] \ln\left(\frac{1+\eta_1}{1-\eta_1}\right) \\
&+ \frac{2\left(5\eta^2+2\eta+5\right)}{9\eta} S_1 + \ln(1-\eta) \left(\frac{16\left(3N^2+3N+2\right)}{9N(N+1)} - \frac{64}{9} S_1 \right) + \frac{32}{9} S_2 \Biggr] \ln^2(\eta) \\
&+ \Biggl[\frac{40(\eta-1)(\eta+1)}{9\eta} S_1 - \frac{10\left(3N^2+3N+2\right)(\eta-1)(\eta+1)}{9N(N+1)\eta} + \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}} \\
&\times \Bigl[8S_1 - \frac{2\left(3N^2+3N+2\right)}{N(N+1)} \Bigr] \mathrm{Li}_2\left(\eta_1\right) \\
&+ \frac{\left(\eta_1+1\right)^2\left(-10\eta^{3/2}+5\eta^2+42\eta-10\eta_1+5\right)}{9\eta^{3/2}} \left[\frac{\left(3N^2+3N+2\right)}{2N(N+1)} - 2S_1 \right] \mathrm{Li}_2(\eta) \Biggr] \ln(\eta) \\
&+ \frac{16\left(3N^4+6N^3+47N^2+20N-12\right)\zeta_2}{27N^2(N+1)^2} + \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}} \Bigl[\frac{4\left(3N^2+3N+2\right)}{N(N+1)} \\
&- 16S_1 \Bigr] \mathrm{Li}_3\left(\eta_1\right) + \frac{\left(\eta_1+1\right)^2\left(-10\eta^{3/2}+5\eta^2+42\eta-10\eta_1+5\right)}{9\eta^{3/2}} \Bigl[2S_1 \\
&- \frac{\left(3N^2+3N+2\right)}{2N(N+1)} \Bigr] \mathrm{Li}_3(\eta) + \left[\frac{16\left(405\eta^2-3238\eta+405\right)}{729\eta} + \frac{256\zeta_3}{27} - \frac{640\zeta_2}{27} \right] S_1 \\
&+ \left[\frac{128\zeta_2}{9} + \frac{3712}{81} \right] S_2 - \frac{1280}{81} S_3 + \frac{256}{27} S_4 - \frac{64\left(3N^2+3N+2\right)\zeta_3}{27N(N+1)} \\
&- \frac{4R_2}{729N^4(N+1)^4\eta} \Biggr\} . \qquad (350)
\end{aligned}
$$

Here we have used the notation $\eta_1 = \sqrt{\eta}$ and the polynomials R_i read

$$
\begin{aligned}
R_1 &= 15\eta^2N^4 + 78\eta N^4 + 15N^4 + 30\eta^2N^3 + 156\eta N^3 + 30N^3 + 25\eta^2N^2 + 18\eta N^2 + 25N^2 \\
&\quad + 10\eta^2 N + 4\eta N + 10N + 32\eta , \qquad (351) \\
R_2 &= 1215\eta^2N^8 - 1596\eta N^8 + 1215N^8 + 4860\eta^2N^7 - 6384\eta N^7 + 4860N^7 + 8100\eta^2N^6 \\
&\quad - 25844\eta N^6 + 8100N^6 + 7290\eta^2N^5 - 39348\eta N^5 + 7290N^5 + 3645\eta^2N^4 - 20304\eta N^4 \\
&\quad + 3645N^4 + 810\eta^2N^3 - 140\eta N^3 + 810N^3 + 432\eta N^2 + 288\eta N + 864\eta . \qquad (352)
\end{aligned}
$$

The pole structure of the unrenormalized transversity OME corresponds to the one in Eq. (275) after substituting the anomalous dimensions $\gamma_{qq}^{\mathsf{NS}} \to \gamma_{qq}^{\mathsf{NS,trans}}$. The constant contribution is given by

$$
\begin{aligned}
&\tilde{a}_{qq,Q}^{(3),\mathrm{NS,TR}} = \\
&C_F T_F^2 \Biggl\{ \left[\frac{32}{27} S_1 - \frac{8}{9} \right] \ln^3(\eta) + \Biggl[- \frac{(\eta+5)(5\eta+1)}{6\eta} + \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{36\eta^{3/2}} \left[4S_1 - 3\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \ln\left(\frac{1-\eta_1}{1+\eta_1}\right) + \frac{2\left(5\eta^2+2\eta+5\right)}{9\eta}S_1 + \ln(1-\eta)\left(\frac{16}{3}-\frac{64}{9}S_1\right)+\frac{32}{9}S_2\Bigg]\ln^2(\eta) \\
&+\Bigg[\frac{40(\eta-1)(\eta+1)}{9\eta}S_1 - \frac{10(\eta-1)(\eta+1)}{3\eta} + \frac{2(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}}\left[4S_1-3\right] \\
&\times \mathrm{Li}_2\left(\eta_1\right) + \frac{\left(\eta_1+1\right)^2\left(-10\eta^{3/2}+5\eta^2+42\eta-10\eta_1+5\right)}{18\eta^{3/2}}\left[3-4S_1\right]\mathrm{Li}_2(\eta)\Bigg]\ln(\eta) \\
&+\frac{16\zeta_2}{9}+\frac{2(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}}\left[6-8S_1\right]\mathrm{Li}_3\left(\eta_1\right) \\
&+\frac{\left(\eta_1+1\right)^2\left(-10\eta^{3/2}+5\eta^2+42\eta-10\eta_1+5\right)}{18\eta^{3/2}}\left[4S_1-3\right]\mathrm{Li}_3(\eta) \\
&+\left(2\frac{8\left(405\eta^2-3238\eta+405\right)}{729\eta}+\frac{128\zeta_3}{27}-\frac{320\zeta_2}{27}\right)S_1+\left(\frac{128\zeta_2}{9}+\frac{3712}{81}\right)S_2 \\
&-\frac{1280}{81}S_3+\frac{256}{27}S_4-\frac{64\zeta_3}{9}-\frac{4R_3}{243N^2(N+1)^2\eta}\Bigg\}, \qquad (353)
\end{aligned}
$$

with

$$
\begin{aligned}
R_3 \;=\;& 405\eta^2N^4-532\eta N^4+405N^4+810\eta^2N^3-1064\eta N^3+810N^3+405\eta^2N^2 \\
&-1012\eta N^2+405N^2+96\eta N+288\eta\ . \qquad (354)
\end{aligned}
$$

6.4 The contributions with $m_1 = m_2$

Additionally to the case of two fermionic lines of unequal mass all diagrams of this class emerge also with both fermionic lines of the same mass. For the OME $A_{gg,Q}$ these contributions have been given in Ref. [287].
For the non-singlet contribution one may obtain the equal mass result by taking the limit $\eta \to 1$ in Eq. (350) and multiplying by a factor of $\frac{1}{2}$ to avoid double counting of diagrams. One obtains

$$
\begin{aligned}
\hat{a}_{qq,Q}^{(3),NS} \;=\;& T_F^2C_F\Bigg\{\frac{128}{27}S_4-\frac{1024}{27}\zeta_3S_1+\frac{64}{9}\zeta_2S_2+\frac{256\left(3N^2+3N+2\right)}{27N(N+1)}\zeta_3-\frac{320}{27}\zeta_2S_1 \\
&-\frac{640}{81}S_3+\frac{8\left(3N^4+6N^3+47N^2+20N-12\right)}{27N^2(N+1)^2}\zeta_2+\frac{1856}{81}S_2-\frac{19424}{729}S_1 \\
&-\frac{4R_4}{729N^4(N+1)^4}\Bigg\}, \qquad (355)
\end{aligned}
$$

with

$$
\begin{aligned}
R_4 \;=\;& 417N^8+1668N^7-4822N^6-12384N^5-6507N^4+740N^3+216N^2+144N \\
&+432\ . \qquad (356)
\end{aligned}
$$

For the pure-single OME A_{Qq}^{PS} the method described in Section 6.3 yields generalized hypergeometric functions of argument η. In the general case the resulting infinite sums were not accessible with the present summation technology, but the limit $\eta = 1$ could still be obtained. The the single-mass contribution to the constant term of this OME is given by

$$
\begin{aligned}
\hat{a}_{Qq}^{(3),PS} &= \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Biggl\{\left(N^2+N+2\right)^2\Bigl(\frac{32}{27}S_1^3 - \frac{512}{27}S_3 \\
&\quad +\frac{128}{3}S_{2,1} - \frac{1024}{9}\zeta_3 - \frac{160}{9}S_2S_1 + \frac{32}{3}\zeta_2 S_1\Bigr) - \frac{32P_1(N)}{9N(N+2)}\zeta_2 \\
&\quad +\frac{32P_2(N)}{27N(N+2)(N+3)(N+4)(N+5)}S_2 \\
&\quad -\frac{32P_3(N)}{27N(N+1)(N+2)(N+3)(N+4)(N+5)}S_1^2 \\
&\quad +\frac{64P_4(N)}{81N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)}S_1 \\
&\quad -\frac{64P_5(N)}{243N^3(N+1)^2(N+2)^3(N+3)(N+4)(N+5)}\Biggr\}\,, \qquad (357)
\end{aligned}
$$

where the polynomials P_i are given by

$$
\begin{aligned}
P_1 &= 8N^6+29N^5+84N^4+193N^3+162N^2+124N+24\,, &(358)\\
P_2 &= 40N^9+625N^8+3284N^7+5392N^6-7014N^5-33693N^4-47454N^3 \\
&\quad -46100N^2-26280N+7200\,, &(359)\\
P_3 &= 8N^{10}+133N^9+1095N^8+5724N^7+18410N^6+34749N^5+40683N^4 \\
&\quad +37370N^3+22748N^2-3960N-7200\,, &(360)\\
P_4 &= 52N^{13}+746N^{12}+4658N^{11}+20431N^{10}+79990N^9+251778N^8+553796N^7 \\
&\quad +837697N^6+886552N^5+599060N^4+155864N^3-82368N^2-76896N \\
&\quad -17280\,, &(361)\\
P_5 &= 293N^{15}+4670N^{14}+32280N^{13}+145948N^{12}+559575N^{11}+1871440N^{10} \\
&\quad +4877344N^9+9333994N^8+12958212N^7+12693884N^6+8472792N^5 \\
&\quad +4514336N^4+3109248N^3+2192832N^2+1026432N+207360\,. &(362)
\end{aligned}
$$

6.5 Scalar $A_{gg,Q}$ diagrams with $m_1 \neq m_2$

The factorization into parts depending purely on the Mellin variable N and contributions depending only on the mass ratio $\eta = \frac{m_2^2}{m_1^2}$ which has been observed for the non–singlet diagrams, constitutes a very special case. In general a mixing between both variables occurs and more advanced methods are required to perform the calculation. Since the complexity of the mathematical structures contributing to a Feynman diagram depends on the denominator functions mainly and in the present case also on the form of the operator insertion, we first study the scalar topologies contributing to the OME $A_{gg,Q}$. Due to the nesting between the Mellin variable and the mass ratio, novel η–dependent sums and integrals contribute.

6.5.1 Strategy

As we expect new functions to appear in the results and since the construction of the inverse Mellin transforms for these functions is a non-trivial task we opt for an approach were we derive

the z-space representation of the respective diagrams directly. The N-space representation[6] is then obtained in a final step by using a generating function representation, constructing a difference equation and solving it using `Sigma` [236, 237] .
First we introduce Feynman parameters and perform the momentum integration for all closed fermion lines. Therefore, for each massive fermion loop we obtain one effective propagator and each mass is now only contained in one of these.
We detach the mass of one of these effective propagators by using the Mellin-Barnes representation [278–281, 288]

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}\xi \frac{B^\xi}{A^{\lambda+\xi}}\Gamma(\lambda+\xi)\Gamma(-\xi) \tag{363}$$

and perform the last momentum integration to obtain expressions like

$$\begin{aligned} I_5 \;=\;& \left(m_1^2\right)^{d/2-\nu_{56}} \left(m_2^2\right)^{d-\nu_{12347}} \int_{-i\infty}^{+i\infty} \mathrm{d}\xi \; \Gamma\begin{bmatrix} -\xi_0, \nu_5+\xi_0, \nu_6+\xi, \nu_{56}-d/2+\xi_0, \nu_{12347}-d-\xi_0 \\ \nu_1,\nu_2,\nu_3,\nu_5,\nu_6,\nu_7,\nu_4-\xi_0,\nu_{56}+2\xi_0 \end{bmatrix} \\ & \int_0^1 \cdots \int_0^1 \mathrm{d}y_0\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3\,\mathrm{d}z_0\,\mathrm{d}z_1\,\delta\left(1-y_0-y_1-y_2-y_3\right)\delta\left(1-z_0-z_1\right) y_1^{\nu_1-1} y_2^{\nu_3-1} y_3^{\nu_7-1} \\ & \times \left(1-y_1-y_2-y_3\right)^{\nu_2-1}\left(1-y_2-y_3\right)^{-N+\nu_4-d/2+\xi_0}\left(y_2+y_3\right)^{-N+\nu_4-d/2-\xi_0} z_1^{N+d/2-1-\nu_4+\xi_0} \\ & \times \left(1-z_1\right)^{\nu 4-d/2-\xi_0}\left((y_1+y_3)(y_2+y_3)-y_3\right)^N \;. \end{aligned} \tag{364}$$

Except for the additional Mellin-Barnes integral the Feynman parameter integrals are now of a similar form as the integrals in Section 3.1 and the appropriate application of the same techniques allows to disentangle all Feynman parameter integrals of this Section into Beta-function integrals of which only one depends on both the Mellin variable N and the Mellin Barnes variable ξ. Usually one may rewrite the integrals in terms of Γ-functions. If this is not the case, we obtain representations as for example

$$\begin{aligned} I_{5a} \;=\;& C_1(m_1,m_2,\varepsilon)\frac{1}{2\pi i}\frac{1}{N+1}\int_{-i\infty}^{+i\infty}\mathrm{d}\xi \; \left(\Gamma\begin{bmatrix}1+N+\nu_3 \\ \nu_3\end{bmatrix} - (-1)^N\,\Gamma\begin{bmatrix}1+N+\nu_7 \\ \nu_7\end{bmatrix}\right) \\ & \times\Gamma\begin{bmatrix} -\frac{\varepsilon}{2}+\nu_4-\xi, -2-\varepsilon+\nu_{347}-\xi, -\xi, \nu_5+\xi, \nu_6+\xi, -2-\frac{\varepsilon}{2}+\nu_{56}+\xi \\ \nu_5\nu_6, 1+N+\nu_{37}, -2-\varepsilon+\nu_3+2\nu_4+\nu_7-2\xi, \nu_4-\xi, \nu_{56}+2\xi \end{bmatrix} \\ & \times\int_0^1 \mathrm{d}X \eta^\xi X^{1+\varepsilon/2+N-\nu_4+\xi}\left(1-X\right)^{\nu_4-1-\xi} \;, \end{aligned} \tag{365}$$

where C_1 is a function of the masses m_1, m_2 and ε. Next, convergent sum representations are derived. Mellin Barnes integrals of the form

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma\begin{bmatrix} a_1+\xi,\cdots,a_i+\xi,b_1-\xi,\cdots,b_jc_1+\xi,\cdots c_k+\xi,d_1-\xi,\cdots,d_l-\xi \\ \end{bmatrix} Z^\xi \tag{366}$$

are usually solved by closing the contour either to the left or to the right and applying Cauchy's theorem

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_i res_{z_i} f \;. \tag{367}$$

[6]The steps to compute these Mellin transforms are included in the computer algebra package `HarmonicSums` [175, 177–179] .

If we close the integration contour in (366) to the left(right) the residue sum only converges for $Z > 1$ ($Z < 1$), respectively. In (365) we have $Z = \frac{\eta X}{1-X}$ which covers both, values below and above 1. We thus follow the method that has been applied in the equal mass case Ref. [289] and split the integration contour and remap the individual parts to the integration domain $[0, 1]$:

$$
\begin{aligned}
\int_{-i\infty}^{+i\infty} \mathrm{d}\xi \int_0^1 \mathrm{d}X f(\xi, X) \left(\frac{\eta X}{1-X}\right)^{\xi} &= \left(\int_0^{\frac{1}{1+\eta}} \mathrm{d}X + \int_{\frac{1}{1+\eta}}^1 \mathrm{d}X\right) f(\xi, X) \left(\frac{\eta X}{1-X}\right)^{\xi} \\
&= \int_{-i\infty}^{+i\infty} d\xi \int_0^1 \mathrm{d}T \left[\frac{\eta}{(\eta+T)^2} f\left(\xi, \frac{T}{\eta+T}\right) T^{\xi}\right. \\
&\quad \left. + \frac{\xi, \eta}{(1+\eta T)^2} f\left(\frac{1}{1+\eta T}\right) T^{-\xi}\right] \\
&= \int_{-i\infty}^{+i\infty} d\xi \int_0^1 \mathrm{d}T\; T^{\xi} \left[\frac{\eta}{(\eta+T)^2} f\left(\xi, \frac{T}{\eta+T}\right)\right. \\
&\quad \left. + \frac{\xi, \eta}{(1+\eta T)^2} f\left(\frac{1}{1+\eta T}\right)\right]. \qquad (368)
\end{aligned}
$$

A further advantage of this procedure is that the contour integration practically decouples the η-dependence which now only enters indirectly through the T-integration.
We now follow the well known procedure of deforming the contour integral in order to separate the ascending from the descending poles [7] and apply Cauchy's theorem to evaluate it. As we are left with only one integration step at this point there are no overlapping singularities anymore. If necessary we map $T \to 1 - T$ in order to have singularities which are regulated by ε only at $T = 0$. These singularities are then written as ε-poles explicitly by applying the following integration by parts relation:

$$
\int_0^1 \mathrm{d}T\; T^{-a} f(T) = \frac{1}{1-a} T^{-a+1} f(T)\Big|_0^1 - \frac{1}{1-a} \int_0^1 \mathrm{d}T\; T^{-a+1} f'(T) \qquad (369)
$$

Being left with a sum representation and an integration which is regular for $\varepsilon \to 0$ we may now perform the Laurent series expansion around $\varepsilon = 0$.

Rewriting the sums. In order to do this and to perform the infinite sums [8] we apply the package `Sigma` [236, 237] . The sums are then expressed in terms of generalized harmonic sums (D.46) at infinity which have to be rewritten in terms of generalized harmonic polylogarithms at argument $x = 1$ using `HarmonicSums` [175, 177–179] . These generalized harmonic polylogarithms are iterated integrals over the following alphabet:

$$
\left\{\frac{\mathrm{d}\tau}{\tau}, \frac{\mathrm{d}\tau}{\tau+T}, \frac{\mathrm{d}\tau}{1+T\tau^2}\right\}. \qquad (370)
$$

In order to process them we want the remaining integration variable T to only appear in the argument of the HPLs. Because of the emergence of letters with non-linear denominators we

[7] In some cases an additional regularization parameter was introduced in order to separate overlapping poles.
[8] Due to the integral transformation (368) these infinite sums are independent of the mass ratio η which renders them much easier to solve.

cannot apply the methods given in Section 7.3 directly, although extensions as described in Section 6.5.1 should suffice to transform these HPLs. However, due to the relatively simple structure of the letters in Eq. (370) there is a way based on applying the shuffle relations and rescaling the internal integration variables to rewrite the corresponding iterated integrals in the desired form.

Absorbing rational, N-dependent factors into the integral. There might be some rational prefactors depending on N left stemming from the integration of the Feynman parameters. These are now pulled into the T-integration by performing a partial fraction decomposition and then applying the following partial integration identities:

$$N^i g(x)^N f(x) = N^{i-1} g(x)^N \frac{f(x)}{g(x)} \frac{1}{\frac{\mathrm{d}g(x)}{\mathrm{d}x}}\Bigg|_0^1 - \int_0^1 \mathrm{d}x\, (g(x))^N \frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} \frac{1}{\frac{\mathrm{d}g(x)}{\mathrm{d}x}} \tag{371}$$

$$\begin{aligned}\frac{1}{(N+a)^i} g(x)^N f(x) &= \frac{1}{(N+a)^i} g(x)^{N+a} \left(\int \mathrm{d}x \frac{f(x)}{g(x)^a}\right)\Bigg|_{x=0}^1 \\ &\quad - \int_0^1 \mathrm{d}x \frac{1}{(N+a)^{i-1}} g(x)^{N+a-1} \frac{\mathrm{d}g(x)}{\mathrm{d}x} \left(\int \mathrm{d}x \frac{f(x)}{g(x)^a}\right) .\end{aligned} \tag{372}$$

Especially relation (371) has to be handled with care, as its application may introduce new divergences in each term. This issue is solved by regularizing the remaining integral in (371) by a +-type distribution which cancels these additional singularities, e.g.

$$\begin{aligned} N\int_0^1 \mathrm{d}x\; x H_0(x) \left(\frac{\eta}{\eta+x^2}\right)^N &= \frac{\eta}{2} H_0(x)\Big|_{x=0}^1 \\ &\quad + \int_0^1 \mathrm{d}x \left(\frac{\eta+x^2}{2x} + x H_0(x)\right)\left(\frac{\eta}{\eta+x^2}\right)^N \qquad (373)\\ &:= \frac{\eta}{2} H_0(x)\Big|_{x=0}^1 + \int_0^1 \mathrm{d}x\; x\left(\frac{1}{2} + H_0(x)\right)\left(\frac{\eta}{\eta+x^2}\right)^N \\ &\quad + \frac{\eta}{2}\int_0^1 \mathrm{d}x\; \frac{1}{x}\left[\left(\frac{\eta}{\eta+x^2}\right)^N - 1\right] + \frac{\eta}{2} H_0(x)\Big|_{x=0} \qquad (374)\\ &= \int_0^1 \mathrm{d}x\; x\left(\frac{1}{2} + H_0(x)\right)\left(\frac{\eta}{\eta+x^2}\right)^N \qquad (375)\\ &\quad + \frac{\eta}{2}\int_0^1 \mathrm{d}x\; \frac{1}{x}\left[\left(\frac{\eta}{\eta+x^2}\right)^N - 1\right] . \qquad (376)\end{aligned}$$

We now rewrite the remaining integral as a Mellin transform. Considering these expressions one obtains

$$I = \int_0^1 \mathrm{d}x f(x,\eta)\, (g(x,\eta))^N$$

$$
= \begin{cases} \int\limits_{g(0)}^{g(1)} \mathrm{d}X f(g^{-1}(X,\eta),\eta) X^N \left|\frac{\mathrm{d}g^{-1}(X,\eta)}{\mathrm{d}X}\right| & , \ g(x,\eta) > 0 \text{ for } 0 < x < 1, \eta > 0 \\ (-1)^N \int\limits_{-g(1)}^{-g(0)} \mathrm{d}X f(-g^{-1}(X,\eta),\eta) X^N \left|\frac{\mathrm{d}g^{-1}(X,\eta)}{\mathrm{d}X}\right| & , \ g(x,\eta) < 0 \text{ for } 0 < x < 1, \eta < 0 \ . \end{cases}
$$

To do this the class of harmonic polylogarithms is not sufficient and generalizations thereof are required. We thus extend the class of harmonic polylogarithms by introducing addition letters with quadratic polynomials in the respective denominators. These are given by

$$
\{4, i\} \ \rightarrow \ \frac{d\tau \ \tau^i}{\Phi_4(\tau)}, \tag{377}
$$

or more generally

$$
\{\{a, b, c\}, i\} \ \rightarrow \ \frac{d\tau \tau^i}{a + b\tau + c\tau^2} \ , \tag{378}
$$

where $\Phi_4(\tau) = \tau^2 + 1$ denotes the forth cyclotomic polynomial and $d\tau$ indicates that the iteration proceeds over τ. Generalizations of HPLs over the cyclotomic polynomials are also known as cyclotomic HPLs [176]. Thus $H_{0,\{4,1\}}(x)$ represents the iterated integral

$$
H_{0,\{4,1\}}(x) = \int_0^x \frac{d\tau_1}{\tau_1} \int_0^{\tau_1} \frac{d\tau_2 \ \tau_2}{\tau_2^2 + 1} \ . \tag{379}
$$

By using this larger functional space, diagram 8_a, Figure 14 below, is rewritten as

$$
\begin{aligned}
I = & \int_0^1 \mathrm{d}x \ \frac{x H_{\{\{1,0,\eta\},1\},\{\{1,0,\eta\},0\},0}(x)}{(1+\eta x^2)^2} \left(\frac{\eta x^2}{1+\eta x^2}\right)^N \\
= & \frac{\eta}{2} \int_0^{\eta/(1+\eta)} \mathrm{d}x \ H_{\{\{1,0,\eta\},1\},\{\{1,0,\eta\},0\},0} \left(\frac{\sqrt{x}}{\sqrt{1-x}\sqrt{\eta}}\right) x^N \\
= & \frac{1}{32\eta^3} \int_0^{\eta/(1+\eta)} \Bigg\{ \left[H_0(x) - H_0(\eta)\right] H_1^2(x) + H_1^3(x) \\
& - 16 \left(H_{0,\{4,1\},\{4,1\}} \left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) + H_{\{4,1\},0,\{4,1\}} \left(\frac{\sqrt{x}}{\sqrt{1-x}}\right)\right) \Bigg\} x^N \ ,
\end{aligned} \tag{380}
$$

where in the last step we removed the η-dependence of the argument by again applying a rescaling of the inner integration variables. At this stage it is desirable to remove the square roots in the arguments of the HPLs and to obtain iterated integrals with the argument x only. In order to obtain this representation we once again exploit the property that taking the derivative reduces the transcendental weight of a hyperlogarithms and use a method similar to the one given in Section 7.3 below, e.g.:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} H_{\{4,1\},0} \left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) & = \frac{1}{2} \frac{H_0\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right)}{1-x} \\
& = \frac{1}{3} \frac{1}{1-x} \left[H_0(x) + H_1(x)\right]
\end{aligned} \tag{381}
$$

$$H_{\{4,1\},0}\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) = \frac{1}{4}\left(H_{1,0}(x)+H_{1,1}(x)\right) \ . \tag{382}$$

However, not all the occurring HPLs can be expressed in terms of generalized HPLs of the previous kind and new, root-valued letters have to be introduced. We thus introduce a new, more general class of iterated integrals which we define recursively by

$$G\left(\{f_1(\tau), f_2(\tau), \cdots, f_n(\tau)\}, z\right) \;=\; \int_0^z d\tau_1 \; f_1(\tau_1) G\left(\{f_2(\tau), \cdots, f_n(\tau)\}, \tau_1\right) \ , \tag{383}$$

with the special cases

$$G\left(\{\}, z\right) \;=\; 1 \ , \tag{384}$$

and

$$G\left(\Bigg\{\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \cdots, \frac{1}{\tau}}_{\text{n times}}\Bigg\}, z\right) \;=\; \frac{1}{n!}\ln^n(z) \ . \tag{385}$$

Using these generalized iterated integrals we rewrite rewrite the HPLs with root-valued functions in the argument. For example one has

$$\begin{aligned}
H_{\{4,0\},\{\{\eta,0,1\},0\}}\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) = & \frac{(3-6x+3\eta x+3\eta^2 x+7x^2-2\eta x^2-5\eta^2 x^2-3x^3+3\eta^2 x^3)}{3\,(\eta-1)\,\eta} \\
& + \frac{2\,(1+\eta)\,\sqrt{1-x}\sqrt{x}\,(-1+2x)}{\eta} G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, x\right) \\
& - \frac{(-1+\eta)^2\sqrt{1-x}\sqrt{x}(-1+2x)}{2\eta} G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\}, x\right) \\
& + \frac{8\,(1+\eta)}{\eta} G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}\right\}, x\right) \\
& + \frac{2\,(\eta-1)^2}{\eta} G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\}, x\right) \\
& + \frac{1+\eta}{2(\eta-1)} G\left(\left\{\frac{1}{-\eta-\tau+\eta\tau}\right\}, x\right) \ .
\end{aligned} \tag{386}$$

In the present computation similar HPLs up to weight $w=3$ had to be transformed. Due to the size of the expressions and the necessity to cancel spurious terms all relations obeyed by these quantities have to be used. These are:

- shuffle relations
- integration by parts relations, such that only factors with exponents $\in \{1/2, -1\}$ contribute to the different letters
- shuffling of single square root terms to the end and performing the integrals e.g.:

$$G\left(\left\{\sqrt{\tau}, \frac{1}{\tau+1}\right\}, x\right) = \frac{2}{3}\left[G\left(\left\{\sqrt{\tau}\right\}, x\right) + x^{3/2} G\left(\left\{\frac{1}{1+\tau}\right\}, x\right) + G\left(\left\{\frac{\sqrt{\tau}}{1+\tau}\right\}, x\right)\right] \ . \tag{387}$$

These identities have now been implemented in `HarmonicSums` [175, 177–179] and allow a significant simplification of expressions with iterated integrals of this type. Finally the integrals are merged. After the mapping of the integration variables (380) we are left with integrals of the form $\int_0^{f(\eta)} \mathrm{d}x$ or $\int_{f(\eta)}^1 \mathrm{d}x$. This is due to the splitting of the X-integration in Eq. (368) and can be undone by re-merging the integrals

$$\int_{f(\eta)}^{1} \mathrm{d}x\ G(x) = \int_0^1 \mathrm{d}x\ G(x) - \int_0^{f(\eta)} \mathrm{d}x\ G(x)\ . \tag{388}$$

As would have been expected, the integrals $\int_0^{f(\eta)} \mathrm{d}x\ G(x)$ completely cancel already existing integrals and only trivial integrals of the form

$$\int_0^{f(\eta)} \mathrm{d}x\ x^\alpha x^N = \frac{1}{\alpha+N+1} x^{N+\alpha}\bigg|_{x=0}^{f(\eta)} \tag{389}$$

remain. We now use `HarmonicSums` [175, 177–179] to perform the inverse Mellin transform for terms that do not contain any x-integration . They usually stem from integration by parts applied in steps (369), (371) or (372). We are left with a z-space representation for our diagram. This representation usually also includes a part proportional to a δ-distribution and a part proportional to a $+$-distribution.

Mellin transform and rewriting generalized HPLs. As a last step we want to generate a N-space representation for our result, which corresponds to performing the last remaining integration. This is done by using a generating function representation performing the integral into generalized HPLs and then generating a recurrence relation for the Nth coefficient of this result as described in Section 7.4. All this is automatized in the package `HarmonicSums` [175, 177–179] and the resulting recurrences were solved with `Sigma` [236, 237] . The result contains many generalized HPLs at argument 1 stemming from the upper integration limit. In case their letters are free of the mass ratio η they can be evaluated in terms of mathematical constants like $\pi, \ln 2$, the Catalan number $\mathbf{C}$, ζ_2, ζ_3 by using standard integration methods or applying the internal integration algorithms of computer algebra packages like Mathematica or Maple. In case these generalized HPLs are not entirely free of η it is desirable to rewrite them as iterated integrals with argument η in order to obtain linear independence and an easier access to series representations. Rewriting these generalized HPLs cannot be done by rescaling integration variables or by just applying (7.3) as due the root valued letters the derivative with respect to an inner variable in general does not lead to a weight reduction in this case. There is, however, an extension to the idea of (7.3): Taking the derivative with respect to inner variables we observe, that only generalized HPLs of a lower weight, GHPLs independent of this variable and the original GHPL itself contribute, e.g.:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta} G\left(\left\{\sqrt{\tau}\sqrt{1-\tau}, \frac{\sqrt{\tau}\sqrt{1-\tau}}{-\eta-\tau-\eta\tau}\right\}, 1\right) = {} & \frac{(1+\eta)\left(1-8\eta+\eta^2\right)}{12\,(\eta-1)^4\,\eta} \\ & - \frac{\eta}{(\eta-1)^4} G\left(\left\{\frac{1}{-\eta-\tau+\eta\tau}\right\}, 1\right) \\ & - \frac{2}{(\eta-1)\,\eta} G\left(\left\{\sqrt{\tau}\sqrt{1-\tau}, \sqrt{\tau}\sqrt{1-\tau}\right\}, 1\right)\end{aligned}$$

$$- \frac{1+3\eta}{2(\eta-1)\eta} G\left(\left\{\sqrt{\tau}\sqrt{1-\tau}, \frac{\sqrt{\tau}\sqrt{1-\tau}}{-\eta-\tau-\eta\tau}\right\}, 1\right) . \tag{390}$$

Therefore, the linear first order differential operator

$$\frac{\mathrm{d}}{\mathrm{d}\eta} + \frac{1+3\eta}{2(\eta-1)\eta} \tag{391}$$

does lead to a weight reduced expression when applied to the GHPL

$$G\left(\left\{\sqrt{\tau}\sqrt{1-\tau}, \frac{\sqrt{\tau}\sqrt{1-\tau}}{-\eta-\tau-\eta\tau}\right\}, 1\right). \tag{392}$$

The weight reduced expression can be rewritten with the same method and we have to undo the effect of the differential operator by using the general solution for linear first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) + p(x) f(x) = q(x) \tag{393}$$

$$\to f(x) = \int_a^x \left(f(a)\delta(\tau-a) + q(\tau)\right) \exp\left(-\int p(x) \mathrm{d}\, x\right) \mathrm{d}\tau \, . \tag{394}$$

Applying this method to the GHPL considered above we obtain

$$\begin{aligned} G\left(\left\{\sqrt{\tau}\sqrt{1-\tau}, \frac{\sqrt{\tau}\sqrt{1-\tau}}{-\eta-\tau-\eta\tau}\right\}, 1\right) &= \frac{1+4\eta-2\eta^2}{6(-1+\eta)^3} - \frac{3\left(1-4\sqrt{\eta}+\eta\right)}{16(\eta-1)^2}\zeta_2 \\ &+ \frac{\sqrt{\eta}}{8(\eta-1)^2} G\left(\left\{\frac{\sqrt{\tau}}{1-\tau}, \frac{1}{\tau}\right\}, \eta\right) + \frac{(\eta-3)\eta^2}{4(-1+\eta)^4} \ln\eta \, . \end{aligned} \tag{395}$$

For all the GHPLs considered in this section it is always possible to construct a linear first order differential operator [9], which does yield a weight reduced expression when applied to the respective generalized HPL and all the GHPLs could thus be rewritten in terms of GHPLs with argument η.

6.5.2 Results

Up to a global prefactor all results are expressed as functions of the mass ratio

$$\eta = \frac{m_2^2}{m_1^2} \tag{396}$$

only. We furthermore define the function $L_1(\eta)$ which appears frequently

$$\begin{aligned} L_1(\eta) &= \frac{1}{2}\int_0^\eta dx \frac{\sqrt{x}}{1-x} \ln^2(x) \qquad (397) \\ &= 4H_{-1,0,0}\left(\sqrt{\eta}\right) + 4H_{1,0,0}\left(\sqrt{\eta}\right) - 8\sqrt{\eta} - \sqrt{\eta}\ln^2(\eta) + 4\sqrt{\eta}\ln(\eta) \, , \qquad (398) \end{aligned}$$

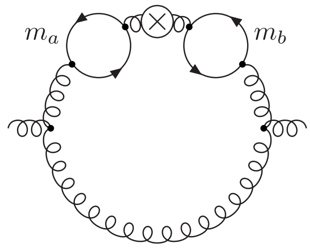


Figure 7: Diagram 1. Here both mass assignments $m_a = m_1$, $m_b = m_2$ and $m_a = m_2$, $m_b = m_1$ yield identical results.

and present the results for all scalar two-mass topologies contributing to $A_{gg,Q}$ both in z- and in N-space. Each term in the z-space result of Diagram 1 factors completely into z and η-dependent contributions. No iterated integrals involving both, η and z, contribute. The z-space result is completely expressible within the class of harmonic polylogarithms

$$
\begin{aligned}
D_1(z) \quad = \quad & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{\varepsilon-3}\Biggl\{ \\
& \frac{1}{\varepsilon^2}\frac{(1+\eta^3)}{105}+\frac{1}{\varepsilon}\Biggl[\frac{(74-245\eta-245\eta^2+74\eta^3)}{44100}-\frac{1}{210}\eta^3 H_0(\eta) \\
& -\frac{1}{210}\left(1+\eta^3\right)H_1(z)\Biggr]+\frac{(5520349-7928445\eta-7928445\eta^2+5520349\eta^3)}{1185408000} \\
& +\frac{1}{280}\left(1+\eta^3\right)\zeta_2-\frac{(74-245\eta-245\eta^2+74\eta^3)H_1(z)}{88200} \\
& +H_0(\eta)\Biggl[\frac{(-55125+37975\eta+24745\eta^2+36181\eta^3)}{22579200}+\frac{1}{420}\eta^3 H_1(z)\Biggr] \\
& +\frac{(525-245\eta-245\eta^2+1549\eta^3)H_{0,0}(\eta)}{430080}+\frac{1}{420}\left(1+\eta^3\right)H_{1,1}(z) \\
& -\frac{(-1+\eta)^2(5+6\eta+5\eta^2)}{2048\sqrt{\eta}}\left[H_{-1,0,0}(\sqrt{\eta})+H_{1,0,0}(\sqrt{\eta})\right]\Biggr\}\,. \qquad (399)
\end{aligned}
$$

Due to the z-space structure, in N-space only single harmonic sums contribute. The Mellin-space result is given by

$$
\begin{aligned}
D_1(N) \quad = \quad & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{\varepsilon-3}\left[\frac{1+(-1)^N}{2}\right]\Biggl\{\frac{\eta^3+1}{105\varepsilon^2(N+1)}+\frac{1}{\varepsilon}\Biggl[-\frac{\left(\eta^3+1\right)S_1(N)}{210(N+1)} \\
& -\frac{\eta^3\ln(\eta)}{210(N+1)}+\frac{(\eta+1)\left(2\eta^2(37N-68)-\eta(319N+109)+74N-136\right)}{44100(N+1)^2}\Biggr]
\end{aligned}
$$

[9] Using first order linear differential operators instead of pure differentiation could be used to extend the parametric integration method of Section 7 . However, remapping parameters as in Section 7.5 might be a more suitable method to integrate Feynman parameter integrals which are not a priori reducible. Both methods break down when genuine elliptic functions appear.

$$
-\frac{\left(5\eta^2+6\eta+5\right)(\eta-1)^2}{2048\sqrt{\eta}(N+1)}\left[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\right]+\ln(\eta)\left[\frac{P_1}{22579200(N+1)^2}\right.
$$
$$
\left.+\frac{\eta^3 S_1(N)}{420(N+1)}\right]+\frac{1549\eta^3-245\eta^2-245\eta+525}{860160(N+1)}\ln^2(\eta)+\frac{P_2(\eta+1)}{1185408000(N+1)^3}
$$
$$
+\frac{\left(\eta^3+1\right)\left[S_1^2(N)+S_2(N)\right]}{840\,(N+1)}+\frac{\left(\eta^3+1\right)\zeta_2}{280(N+1)}
$$
$$
\left.-\frac{(\eta+1)\left(2\eta^2(37N-68)-\eta(319N+109)+74N-136\right)S_1(N)}{88200(N+1)^2}\right\}, \tag{400}
$$

with the polynomials $P_i(\eta, N)$

$$
\begin{aligned}
P_1 &= 36181\eta^3 N+89941\eta^3+24745\eta^2 N+24745\eta^2+37975\eta N+37975\eta \\
&\quad -55125N-55125 \qquad (401)\\
P_2 &= 5520349\eta^2 N^2+10046138\eta^2 N+7348189\eta^2-13448794\eta N^2 \\
&\quad -22610228\eta N-11983834\eta \\
&\quad +5520349N^2+10046138N+7348189\,. \qquad (402)
\end{aligned}
$$

Although topologically very similar to Diagrams D_1, diagrams D_{2a} and D_{2b} exhibit much more

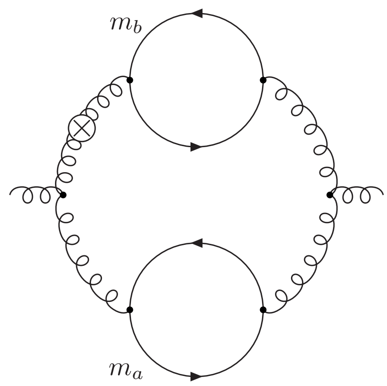


Figure 8: Topology 2. D_{2a} represents the mass assignment $m_a = m_2$, $m_b = m_1$ and D_{2b} $(m_1 \leftrightarrow m_2)$.

evolved mathematical structures. As we restrict ourselves to a representation within the class of iterated integrals of argument z, additional root-valued integration kernels had to be introduced. Furthermore, iterated integrals depending on both variables η and z contribute.
In z-space diagram D_{2a} consists of contribution $D_{2a}^{\rm Reg}$, which is regular as $z \to 1$ and a contribution D_{2a}^{+},

$$
D_{2a}(z) = D_{2a}^{\rm Reg}(z)+D_{2a}^{(+)}(z)\,. \tag{403}
$$

The latter contains terms $\propto 1/(1-z)$ or $\propto 1/\sqrt{1-z}^3$ and requires a regularization via +-type distribution when performing the Mellin transform. With a function $f^{(+)}(x)$ of the general form

$$f^{(+)}(x) = \sum_{k=0}^{m} (\ln(1-x))^k \frac{p_k(x)}{q_k(x)} \ , \tag{404}$$

we define

$$\mathbf{M}\left[f^{(+)}\right](N) = \int_0^1 dx \sum_{k=0}^{m} \frac{x^n p_k(x) - p_k(1)}{q_k(x)} (\ln(1-x))^k \ . \tag{405}$$

This regularization is required for the Mellin transform of the diagrams D_{2a}, D_{2b}, D_{8a} and D_{8b}. For D_{2a} the +-part is given by

$$\begin{aligned} D_{2a}^{(+)} &= \left(m_1^2\right)^{\varepsilon/2} \left(m_2^2\right)^{-3+\varepsilon} \Biggl\{ \\ & -\frac{1}{\varepsilon} \frac{\eta^3}{210(-1+z)} - \frac{37\eta^3}{44100(-1+z)} + \frac{\eta^3}{420(-1+z)} G\left[\left\{\frac{1}{1-\tau}\right\}, z\right] \\ & + \frac{5(-1+\eta)\eta^3\sqrt{z}}{1536(1-z)^{3/2}} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] + G\left[\left\{\frac{1}{\tau}\right\}, \eta\right] \Biggl[\frac{\eta^3}{420(-1+z)} \\ & - \frac{5\eta^3(1+\eta)\sqrt{z}}{3072(1-z)^{3/2}} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] + \frac{5(-1+\eta)^2\eta^3\sqrt{z}}{12288(1-z)^{3/2}} G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\}, z\right]\Biggr] \\ & - \frac{5\eta^3(1+\eta)\sqrt{z}}{3072(1-z)^{3/2}} \left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau}\right\}, z\right] + G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{\tau}\right\}, z\right]\right] \\ & + \frac{5(-1+\eta)^2\eta^3\sqrt{z}}{12288(1-z)^{3/2}} \Biggl[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{1-\tau}\right\}, z\right] \\ & + G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{\tau}\right\}, z\right]\Biggr] \Biggr\} , \end{aligned} \tag{406}$$

and the regular contribution to Diagram $2a$ is

$$\begin{aligned} D_{2a}^{\mathrm{Reg}}(z) &= \left(m_1^2\right)^{\varepsilon/2} \left(m_2^2\right)^{-3+\varepsilon} \Biggl\{ \\ & \delta(1-z) \Biggl[\frac{1}{\varepsilon^2}\frac{\eta^3}{105} + \frac{1}{\varepsilon}\left[\frac{37\eta^3}{22050} - \frac{1}{210}\eta^3 G\left[\left\{\frac{1}{\tau}\right\}, \eta\right]\right] - \frac{523\eta^3}{2315250} + \frac{\eta^3\zeta_2}{280} \\ & - \frac{37\eta^3 G\left[\left\{\frac{1}{\tau}\right\}, \eta\right]}{44100} + \frac{1}{420}\eta^3 G\left[\left\{\frac{1}{\tau}, \frac{1}{\tau}\right\}, \eta\right]\Biggr] \\ & - \frac{1}{\varepsilon}\frac{3 - 6z + \left(3 + 7\eta^2 + 6\eta^3\right) z^2}{1260 z^2} + \frac{Q_1}{11289600\eta z^2} + \frac{Q_2}{645120\eta z} G\left[\left\{\frac{1}{1-\tau}\right\}, z\right] \\ & - \frac{Q_4}{645120\eta z^2} G\left[\left\{\frac{1}{\tau}\right\}, z\right] - \frac{(\eta-1)\,Q_3}{1536\eta\sqrt{1-z}z^{5/2}} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] \\ & + G\left[\left\{\frac{1}{\tau}\right\}, \eta\right] \Biggl[\frac{Q_2}{645120\eta z} + \frac{(1+\eta)\,Q_3}{3072\eta\sqrt{1-z}z^{5/2}} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] \end{aligned}$$

$$
-\frac{(-1+\eta)^2 Q_3}{12288\eta\sqrt{1-z}z^{5/2}} G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\}, z\right]\Bigg]
$$
$$
+\frac{(1+\eta)Q_3}{3072\eta\sqrt{1-z}z^{5/2}}\left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau}\right\}, z\right] + G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{\tau}\right\}, z\right]\right]
$$
$$
-\frac{(-1+\eta)^2 Q_3}{12288\eta\sqrt{1-z}z^{5/2}}\Bigg[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{1-\tau}\right\}, z\right]
$$
$$
+G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{\tau}\right\}, z\right]\Bigg]\Bigg\}, \tag{407}
$$

with the polynomials

$$
\begin{aligned}
Q_1 &= 11025(-1+z)^3 z + 18375\eta^5 z^4 + \eta^4 z^2\left(-9472 + 25725z - 62475z^2\right) \\
&\quad +49\eta^3 z^2\left(-1091 - 900z + 1350z^2\right) - \eta(-1+z)^2\left(-8704 - 22050z + 25725z^2\right) \\
&\quad -245\eta^2 z\left(133 - 253z + 90z^2 + 30z^3\right), \qquad (408)\\
Q_2 &= 315(-1+z)^3 - 525\eta^5 z^3 - 105\eta(-1+z)^2(6+z) + 7\eta^3 z\left(11 + 180z + 90z^2\right) \\
&\quad +3\eta^4 z\left(512 - 595z + 245z^2\right) - 105\eta^2\left(-1 + 9z - 18z^2 + 10z^3\right), \qquad (409)\\
Q_3 &= 3 + (-9+4\eta)z + \left(9 - 8\eta - 6\eta^2\right)z^2 + (-1+\eta)^3(3+5\eta)z^3, \qquad (410)\\
Q_4 &= 105(\eta-1)^2\left(17\eta^2 + 22\eta + 9\right)z^3 - 3\left(35\eta^2 + 302\eta - 105\right)z + \big(1715\eta^3 \\
&\quad +945\eta^2 - 387\eta - 945\big)z^2 + 105(\eta-1)^3(\eta+1)(5\eta+3)z^4 + 768\eta. \qquad (411)
\end{aligned}
$$

Performing the Mellin transform by using the regularization (405) yields

$$
\begin{aligned}
D_{2a}(N) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{\frac{\eta^3}{105\varepsilon^2} + \frac{1}{\varepsilon}\Bigg[-\frac{1}{210}S_1(N)\eta^3 - \frac{1}{210}\ln(\eta)\eta^3 \\
&\quad +\frac{2N\left(37N^2 - 105N + 68\right)\eta^3 - 245(N-1)N\eta^2 - 210}{44100(N-1)N(N+1)}\Bigg] + \Big[S_1^2(N) \\
&\quad +S_2(N)\Big]\frac{\eta^3}{840} + \frac{\zeta_2\eta^3}{280} + \Bigg[\frac{\eta^3}{840} - \frac{(-1)^N(\eta-1)^{-N-1}}{(2N-3)(2N-1)(2N+1)}\frac{P_2}{12288} \\
&\quad +\frac{2^{-2N-13}\binom{2N}{N}P_4}{3(\eta-1)(N+1)(2N-3)(2N-1)} \\
&\quad +\frac{2^{-2N-13}\binom{2N}{N}P_4}{3(\eta-1)(N+1)(2N-3)(2N-1)}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}2^{2i_1}(-1+\eta)^{-i_1}}{\binom{2i_1}{i_1}\left(1+2i_1\right)}\Bigg]\ln^2(\eta) \\
&\quad +\frac{P_1}{1185408000(N-1)^2N^2(N+1)^2(2N-3)(2N-1)} \\
&\quad -\frac{2^{-2N-11}\binom{2N}{N}P_4}{3\sqrt{\eta}(N+1)(2N-3)(2N-1)}\Big[H_{-1,0,0}\left(\sqrt{\eta}\right) + H_{1,0,0}\left(\sqrt{\eta}\right)\Big] \\
&\quad +\frac{2^{-2N}\binom{2N}{N}P_4}{12288(\eta-1)(N+1)(2N-3)(2N-1)} \\
&\quad \times\sum_{i_1=1}^{N}\frac{(-1)^{i_1}2^{2i_1}(-1+\eta)^{-i_1}\Big[S_{1,1}\left(1-\eta,1,i_1\right) - S_2\left(1-\eta,i_1\right)\Big]}{\binom{2i_1}{i_1}\left(1+2i_1\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(-1)^N(\eta-1)^{-N-1}P_2}{6144(2N-3)(2N-1)(2N+1)}\Big[S_2(1-\eta,N)-S_{1,1}(1-\eta,1,N)\Big]\\
&+\frac{P_3S_1(N)}{22579200(N+1)(2N-3)(2N-1)}+\ln(\eta)\Bigg[\frac{1}{420}S_1(N)\eta^3\\
&+\frac{P_3}{22579200(N+1)(2N-3)(2N-1)}\\
&+\frac{2^{-2N-12}\binom{2N}{N}P_4}{3(\eta-1)(N+1)(2N-3)(2N-1)}\sum_{i_1=1}^N\frac{(-1)^{i_1}2^{2i_1}(-1+\eta)^{-i_1}S_1(1-\eta,i_1)}{\binom{2i_1}{i_1}(1+2i_1)}\\
&-\frac{(-1)^N(\eta-1)^{-N-1}}{(2N-3)(2N-1)(2N+1)}\frac{P_2S_1(1-\eta,N)}{6144}\Bigg]\Bigg\}\,, \qquad (412)
\end{aligned}
$$

with the polynomials

$$
\begin{aligned}
P_1 &= 14363896\eta^3N^8-4\eta^2(6247133\eta+7928445)N^7\\
&-10\eta\left(1788305\eta^2-10831254\eta+519645\right)N^6\\
&+\left(18840889\eta^3-108183915\eta^2+18290475\eta+24675735\right)N^5\\
&+\left(66146587\eta^3+4378395\eta^2-17775975\eta+1881705\right)N^4\\
&-\left(78524357\eta^3-41113695\eta^2+8412075\eta+86929815\right)N^3\\
&+3\left(7348189\eta^3-4635645\eta^2+7657475\eta+15366365\right)N^2\\
&-40320(245\eta-424)N-8467200 \qquad (413)\\
P_2 &= 5\eta^3\left(8N^3-12N^2-2N+3\right)+\eta^2\left(-28N^2+64N-9\right)-3\eta(2N+17)+45 \qquad (414)\\
P_3 &= \eta^3\left(71224N^3+217316N^2-666110N+269823\right)+24745\eta^2\left(4N^2-8N+3\right)\\
&-3675\eta(14N-31)-165375 \qquad (415)\\
P_4 &= 5\eta^4\left(16N^4-40N^2+9\right)-12\eta^3\left(8N^3-12N^2-2N+3\right)-6\eta^2\left(4N^2-8N+3\right)\\
&+12\eta(2N-3)+45\,. \qquad (416)
\end{aligned}
$$

Diagram $2b$ exhibits a very similar structure and is related to diagram $2a$ by the interchange $m_1 \leftrightarrow m_2,\ \eta \to 1/\eta$. Its z-space contributions consists of a part which requires regularization via the $+$-distribution,

$$
\begin{aligned}
D_{2b}^{(+)}(z) &= \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Bigg\{\\
&-\frac{1}{\varepsilon}\frac{1}{210\eta^3(-1+z)}-\frac{37}{44100\eta^3(-1+z)}+\frac{1}{420\eta^3(-1+z)}G\left[\left\{\frac{1}{1-\tau}\right\},z\right]\\
&-\frac{5(-1+\eta)\sqrt{z}}{1536\eta^4(1-z)^{3/2}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]+G\left[\left\{\frac{1}{\tau}\right\},\eta\right]\Bigg[-\frac{1}{420\eta^3(-1+z)}\\
&+\frac{5(1+\eta)\sqrt{z}}{3072\eta^4(1-z)^{3/2}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]\\
&+\frac{5(-1+\eta)^2\sqrt{z}}{12288\eta^4(1-z)^{3/2}}G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\},z\right]\Bigg]\\
&-\frac{5(1+\eta)\sqrt{z}}{3072\eta^4(1-z)^{3/2}}\left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{5(-1+\eta)^2\sqrt{z}}{12288\eta^4(1-z)^{3/2}}\Biggl[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right] \\
& +G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Biggr]\Biggr\}, \qquad (417)
\end{aligned}
$$

and a remainder contribution

$$
\begin{aligned}
D_{2b}^{\mathrm{Reg}}(z) &= \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Biggl\{ \\
& \delta(1-z)\Biggl[\frac{1}{105\varepsilon^2\eta^3}+\frac{1}{\varepsilon}\left[\frac{37}{22050\eta^3}+\frac{G\left(\{\frac{1}{\tau}\},\eta\right)}{210\eta^3}\right]-\frac{523}{2315250\eta^3}+\frac{37G\left(\{\frac{1}{\tau}\},\eta\right)}{44100\eta^3} \\
& +\frac{G\left(\{\frac{1}{\tau},\frac{1}{\tau}\},\eta\right)}{420\eta^3}+\frac{\zeta_2}{280\eta^3}\Biggr]-\frac{1}{\varepsilon}\frac{3\eta^3(z-1)^2+6z^2+7\eta z^2}{1260\eta^3z^2}+\frac{G\left(\{\frac{1}{\tau}\},z\right)Q_3}{645120\eta^4z^2} \\
& +\frac{Q_4}{11289600\eta^4z^2}+\frac{1}{\eta^4z}\frac{G\left(\{\frac{1}{1-\tau}\},z\right)Q_1}{645120}-\frac{(\eta+1)Q_5}{3072\eta^4\sqrt{1-z}z^{5/2}} \\
& \times\left[G\left(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\right)+G\left(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\right)\right]-\frac{(\eta-1)^2Q_5}{12288\eta^4\sqrt{1-z}z^{5/2}} \\
& \times\left[G\left(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\right)+G\left(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\right)\right]+\Biggl[\frac{Q_2}{645120\eta^4z} \\
& +\frac{1}{\eta^4\sqrt{1-z}z^{5/2}}\frac{(\eta+1)Q_5}{3072}G\left(\{\sqrt{1-\tau}\sqrt{\tau}\},z\right)+\frac{1}{\eta^4\sqrt{1-z}z^{5/2}}\frac{(\eta-1)^2Q_5}{12288} \\
& \times G\left(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\right)\Biggr]G\left(\{\frac{1}{\tau}\},\eta\right)-\frac{(\eta-1)Q_5}{1536\eta^4\sqrt{1-z}z^{5/2}}G\left(\{\sqrt{1-\tau}\sqrt{\tau}\},z\right)\Biggr\}, \qquad (418)
\end{aligned}
$$

with

$$
\begin{aligned}
Q_1 &= 315\eta^5(z-1)^3-105\eta^4(z-1)^2(z+6)-105\eta^3\bigl(10z^3-18z^2+9z-1\bigr) \\
& +7\eta^2z\bigl(90z^2+180z+11\bigr)+3\eta z\bigl(245z^2-595z+512\bigr)-525z^3, \qquad (419) \\
Q_2 &= -315\eta^5(z-1)^3+105\eta^4(z-1)^2(z+6)+105\eta^3\bigl(10z^3-18z^2+9z-1\bigr) \\
& -7\eta^2z\bigl(90z^2+180z+11\bigr)-3\eta z\bigl(245z^2-595z+512\bigr)+525z^3, \qquad (420) \\
Q_3 &= 315\eta^5(z-1)^3z-3\eta^4(z-1)^2\bigl(35z^2+210z+256\bigr)-105\eta^3z\bigl(10z^3-18z^2 \\
& +9z-1\bigr)+35\eta^2z^2\bigl(18z^2+36z-49\bigr)+105\eta z^3(7z-17)-525z^4, \qquad (421) \\
Q_4 &= 11025\eta^5(z-1)^3z-\eta^4(z-1)^2\bigl(25725z^2-22050z-8704\bigr)-245\eta^3z\bigl(30z^3 \\
& +90z^2-253z+133\bigr)+49\eta^2z^2\bigl(1350z^2-900z-1091\bigr)+\eta z^2\bigl(-62475z^2 \\
& +25725z-9472\bigr)+18375z^4, \qquad (422) \\
Q_5 &= 3\eta^4(z-1)^3-4\eta^3(z-1)^2z-6\eta^2(z-1)z^2-5z^3+12\eta z^3. \qquad (423)
\end{aligned}
$$

In Mellin N-space, this yields

$$
D_{2b}(N) = \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\left[\frac{1+(-1)^N}{2}\right]\Biggl\{\frac{1}{105\varepsilon^2\eta^3}
$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\Biggl[-\frac{210\eta^3+245(N-1)N\eta-2N\bigl(37N^2-105N+68\bigr)}{44100\eta^3(N-1)N(N+1)}-\frac{S_1\left(N\right)}{210\eta^3}+\frac{\ln(\eta)}{210\eta^3}\Biggr]\\
&+\Biggl[\frac{(\eta-1)^{-N-1}\eta^{N-2}P_2}{12288(2N-3)(2N-1)(2N+1)}-\frac{2^{-2N-13}\binom{2N}{N}P_3}{3(\eta-1)\eta^3(N+1)(2N-3)(2N-1)}\\
&-\frac{2^{-2N-13}\binom{2N}{N}P_3}{3(\eta-1)\eta^3(N+1)(2N-3)(2N-1)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}}{\binom{2i_1}{i_1}\bigl(1+2i_1\bigr)}\\
&+\frac{1}{840\eta^3}\Biggr]\ln^2(\eta)+\Biggl[\frac{P_1}{22579200\eta^3(N+1)(2N-3)(2N-1)}\\
&-\frac{(\eta-1)^{-N-1}\eta^{N-2}P_2}{6144(2N-3)(2N-1)(2N+1)}S_1\left(\frac{\eta-1}{\eta},N\right)\\
&+\frac{2^{-2N-12}\binom{2N}{N}P_3}{3(\eta-1)\eta^3(N+1)(2N-3)(2N-1)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}\bigl(1+2i_1\bigr)}\\
&-\frac{S_1\left(N\right)}{420\eta^3}\Biggr]\ln(\eta)+\frac{P_4}{1185408000\eta^3(N-1)^2N^2(N+1)^2(2N-3)(2N-1)}\\
&-\frac{2^{-2N-11}\binom{2N}{N}P_3}{3\eta^{7/2}(N+1)(2N-3)(2N-1)}\Biggl[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\Biggr]\\
&+\frac{2^{-2N}\binom{2N}{N}P_3}{12288(\eta-1)\eta^3(N+1)(2N-3)(2N-1)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}\Bigl[S_2\left(\frac{-1+\eta}{\eta},i_1\right)-S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)\Bigr]}{\binom{2i_1}{i_1}\bigl(1+2i_1\bigr)}\\
&+\frac{(\eta-1)^{-N-1}\eta^{N-2}P_2}{6144(2N-3)(2N-1)(2N+1)}\Bigl[S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)-S_2\left(\frac{\eta-1}{\eta},N\right)\Bigr]\\
&-\frac{P_1}{22579200\eta^3(N+1)(2N-3)(2N-1)}S_1\left(N\right)+\frac{S_1^2(N)+S_2\left(N\right)}{840\eta^3}\\
&+\frac{\zeta_2}{280\eta^3}\Biggr\}\,, \qquad (424)
\end{aligned}
$$

with polynomials

$$
\begin{aligned}
P_1 &= 165375\eta^3+3675\eta^2(14N-31)-24745\eta\left(4N^2-8N+3\right)-71224N^3\\
&\quad -217316N^2+666110N-269823 \qquad (425)\\
P_2 &= 45\eta^3-3\eta^2(2N+17)+\eta\left(-28N^2+64N-9\right)+5\left(8N^3-12N^2-2N+3\right) \qquad (426)\\
P_3 &= 45\eta^4+12\eta^3(2N-3)-6\eta^2\left(4N^2-8N+3\right)-12\eta\left(8N^3-12N^2-2N+3\right)\\
&\quad +5\left(16N^4-40N^2+9\right) \qquad (427)\\
P_4 &= 105\eta^3\left(235007N^5+17921N^4-827903N^3+439039N^2+162816N-80640\right)\\
&\quad -25725\eta^2(N-1)^2N\left(202N^3-307N^2-125N+384\right)\\
&\quad -5145\eta(N-1)^2N^2\left(6164N^3-8724N^2-2585N+2703\right)\\
&\quad +(N-1)^2N^2\bigl(14363896N^4+3739260N^3-24768426N^2
\end{aligned}
$$

$$-34435223N + 22044567\Big) \; . \qquad (428)$$

Diagram 3 displays a particularly simple structure and does only depend on the logarithms

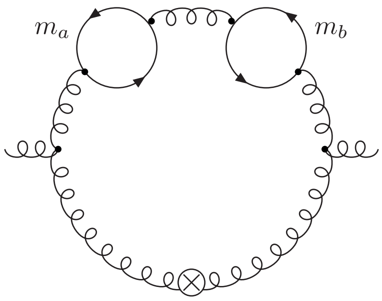


Figure 9: Diagram 3. Due to the symmetry of the diagram both mass assignments $m_a = m_1$, $m_b = m_2$ and $m_a = m_2$, $m_b = m_1$ yield identical results.

$H_0(\eta) = \ln(\eta)$ and $H_0(z) = \ln(z)$ in z-space. $D_3(z)$ reads

$$\begin{aligned} D_3(z) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{\varepsilon-3}\Bigg\{ \\ &\Bigg[-\frac{(1+\eta^3)(-1+z)}{210z\varepsilon} - \frac{(74-245\eta-245\eta^2+74\eta^3)(-1+z)}{88200z} \\ &+\frac{1}{420z}\eta^3(-1+z)H_0(\eta) - \frac{1}{420z}\left(1+\eta^3\right)(-1+z)H_0(z)\Bigg]\Bigg\} \; . \end{aligned} \qquad (429)$$

In N-space this corresponds to an expression in terms of rational functions and $\ln(\eta)$ only. It is given by

$$\begin{aligned} D_3(N) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{\varepsilon-3}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{\frac{\left(\eta^3+1\right)}{210\varepsilon N(N+1)} + \frac{P_1(\eta+1)}{88200N^2(N+1)^2} \\ &-\frac{\eta^3\ln(\eta)}{420N(N+1)}\Bigg\} \; , \end{aligned} \qquad (430)$$

with the polynomial

$$P_1 = \eta^2\left(74N^2-346N-210\right) + \eta\left(-319N^2+101N+210\right) + 74N^2 - 346N - 210 \; . \qquad (431)$$

The z-space expressions for Diagrams D_{4a} and D_{4b} are completely regular as $z \to 1$. For $D_{4a}(z)$ one obtains

$$\begin{aligned} D_{4a}(z) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\Bigg\{ \\ &\frac{1}{\varepsilon}\Bigg[-\frac{(-1+z)(-1+5z+2z^2)}{60z^2} + \frac{1}{10}G\left[\left\{\frac{1}{\tau}\right\}, z\right]\Bigg] + \frac{Q_1}{57600\eta z^2} \end{aligned}$$

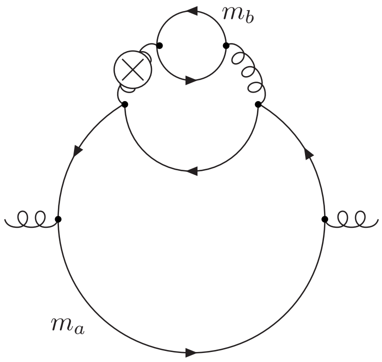


Figure 10: Topology 4. D_{4a} is given by assigning $m_a = m_2$,$m_b = m_1$ and D_{4b} by assigning $m_a = m_1$, $m_b = m_2$ respectively.

$$
\begin{aligned}
&-\frac{\eta^2\zeta_2}{60} - \frac{(-1+z)Q_2}{1920\eta z} G\left[\left\{\frac{1}{1-\tau}\right\}, z\right] + \frac{Q_5 G\left[\left\{\frac{1}{\tau}\right\}, z\right]}{28800\eta z^2} \\
&-\frac{(-1+\eta)(1-z)^{3/2} Q_3 G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right]}{240\eta z^{5/2}} + \frac{1}{60}\eta^2 G\left[\left\{\frac{1}{\tau}, \frac{1}{1-\tau}\right\}, z\right] \\
&+\frac{\eta(5+3\eta)(-7+5\eta) G\left[\left\{\frac{1}{\tau}, \frac{1}{\tau}\right\}, \eta\right]}{3840} + \frac{1}{60}\left(3+\eta^2\right) G\left[\left\{\frac{1}{\tau}, \frac{1}{\tau}\right\}, z\right] \\
&+\frac{(1+\eta)(1-z)^{3/2} Q_3}{480\eta z^{5/2}} \left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau}\right\}, z\right] \right. \\
&\left. +G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{\tau}\right\}, z\right] \right] - \frac{(-1+\eta)^2\left(-7-2\eta+\eta^2\right)}{8\eta} \times \\
&G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] + G\left[\left\{\frac{1}{\tau}\right\}, \eta\right] \left[\frac{Q_4}{1920\eta z} \right. \\
&+\frac{1}{60}\eta^2 G\left[\left\{\frac{1}{\tau}\right\}, z\right] + \frac{(1+\eta)(1-z)^{3/2} Q_3}{480\eta z^{5/2}} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] \\
&-\frac{1}{480}\eta\left(-35-3\eta+8\eta^2\right) G\left[\left\{\frac{1}{1-\tau+\eta\tau}\right\}, z\right] \\
&-\frac{(-1+\eta)^2(1-z)^{3/2} Q_3}{1920\eta z^{5/2}} G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\}, z\right] \\
&+\frac{(-1+\eta)(1+\eta)\left(-7-2\eta+\eta^2\right)}{16\eta} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] \\
&\left. -\frac{(-1+\eta)^3\left(-7-2\eta+\eta^2\right)}{64\eta} G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\}, z\right] \right] \\
&-\frac{\eta\left(-35-3\eta+8\eta^2\right)}{480} \left[G\left[\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{1-\tau}\right\}, z\right] + G\left[\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{\tau}\right\}, z\right] \right] \\
&-\frac{(-1+\eta)^2(1-z)^{3/2} Q_3}{1920\eta z^{5/2}} \left[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{1-\tau}\right\}, z\right] \right.
\end{aligned}
$$

$$
\begin{aligned}
&+G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]+\frac{(7-5\eta-3\eta^2+\eta^3)}{512\sqrt{\eta}}G\left[\left\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right]\\
&+\frac{(-1+\eta)(1+\eta)\left(-7-2\eta+\eta^2\right)}{16\eta}\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\Bigg]-\frac{(-1+\eta)^3\left(-7-2\eta+\eta^2\right)}{64\eta}\\
&\times\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]\Bigg\}, \qquad (432)
\end{aligned}
$$

with the polynomials

$$
\begin{aligned}
Q_1 &= -150\eta^3z^3\left(10+41z-62z^2+24z^3\right)+75\eta^4z^2\left(12-22z+43z^2-38z^3+12z^4\right)\\
&\quad -20\eta^2z\left(82-57z+65z^2+135z^3-240z^4+90z^5\right)\\
&\quad -15z\left(24-152z+662z^2+51z^3-830z^4+420z^5\right)\\
&\quad +2\eta\left(104+696z-2568z^2+7798z^3+3195z^4-11850z^5+5400z^6\right) \qquad (433)\\
Q_2 &= -6+67z-81z^2-85z^3+105z^4+5\eta^4(-1+z)^2z(-1+3z)\\
&\quad -10\eta^3(-1+z)^2z(1+3z)+\eta^2\left(2-50z+20z^2+160z^3-120z^4\right)\\
&\quad +2\eta\left(-6+99z+7z^2-45z^3+15z^4\right) \qquad (434)\\
Q_3 &= 3+(-13+5\eta)z-15\left(-3+2\eta+\eta^2\right)z^2+15\left(7-5\eta-3\eta^2+\eta^3\right)z^3 \qquad (435)\\
Q_4 &= -6+38z-148z^2-4z^3+190z^4-105z^5-5\eta^4z\left(3-6z+12z^2\right.\\
&\quad \left.-10z^3+3z^4\right)+10\eta^3z\left(3+6z^2-8z^3+3z^4\right)-2\eta\left(6-30z\right.\\
&\quad \left.+92z^2+52z^3-60z^4+15z^5\right)+\eta^2\left(2-17z+70z^2+140z^3\right.\\
&\quad \left.-280z^4+120z^5\right), \qquad (436)\\
Q_5 &= 150\eta^3z^4\left(6-8z+3z^2\right)\\
&\quad -75\eta^4z^3\left(-6+12z-10z^2+3z^3\right)-15z\left(6-38z+148z^2\right.\\
&\quad \left.+4z^3-190z^4+105z^5\right)+10\eta^2z\left(3-203z+105z^2\right.\\
&\quad \left.+210z^3-420z^4+180z^5\right)-6\eta\left(40-210z-86z^2+460z^3+260z^4\right.\\
&\quad \left.-300z^5+75z^6\right). \qquad (437)\\
& \qquad (438)
\end{aligned}
$$

Performing the Mellin transform yields

$$
\begin{aligned}
D_{4a}(N) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{1}{5\varepsilon(N-1)N(N+1)^2(N+2)}\\
&\quad +\Bigg[\frac{\eta\big(\big(4N^2-4N-3\big)\eta^2+(4N-6)\eta-35\big)(1-\eta)^{-N}}{1024(N+1)^2(N+2)(2N-3)}\\
&\quad -\frac{P_2 2^{-2N-9}\binom{2N-2}{N-1}}{N(N+1)^2(N+2)(2N-3)}
\end{aligned}
$$

$$
-\frac{2^{-2N-11}\binom{2N-2}{N-1}P_2}{N(N+1)^2(N+2)(2N-3)}\eta\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}}{\binom{-2+2i_1}{-1+i_1}}\Bigg]\ln^2(\eta)
$$

$$
+\Bigg[-\frac{\eta\big(3\big(4N^2-4N-3\big)\eta^2+12(2N-3)\eta-35\big)}{768(N+1)^2(N+2)(2N-3)}
$$

$$
+\frac{P_2 2^{-2N-7}\binom{2N-2}{N-1}}{N(N+1)^2(N+2)(2N-3)}
$$

$$
+\frac{2^{-2N}\binom{2N-2}{N-1}\eta P_2}{1024N(N+1)^2(N+2)(2N-3)}\Bigg[\sum_{i_1=1}^{N}\frac{2^{2i_1}}{\binom{-2+2i_1}{-1+i_1}i_1}
$$

$$
-\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}S_1(1-\eta,i_1)}{\binom{-2+2i_1}{-1+i_1}}\Bigg]
$$

$$
+\frac{\eta\big(\big(4N^2-4N-3\big)\eta^2+(4N-6)\eta-35\big)(1-\eta)^{-N}}{512(N+1)^2(N+2)(2N-3)}S_1(1-\eta,N)\Bigg]\ln(\eta)
$$

$$
+\frac{P_1}{28800(N-1)^2N^2(N+1)^3(N+2)(2N-3)}
$$

$$
+\frac{2^{-2N-7}\binom{2N-2}{N-1}P_2}{\sqrt{\eta}N(N+1)^2(N+2)(2N-3)}\Bigg[H_{-1,0,0}(\sqrt{\eta})+H_{1,0,0}(\sqrt{\eta})\Bigg]
$$

$$
-\frac{P_2 2^{-2N-6}\binom{2N-2}{N-1}}{N(N+1)^2(N+2)(2N-3)}+\frac{2^{-2N}\binom{2N-2}{N-1}\eta P_2}{1024N(N+1)^2(N+2)(2N-3)}
$$

$$
\times\Bigg[\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\Big[S_2(1-\eta,i_1)-S_{1,1}(1-\eta,1,i_1)\Big]}{\binom{-2+2i_1}{-1+i_1}}+\sum_{i_1=1}^{N}\frac{2^{2i_1}\Big(\frac{S_1(i_1)}{i_1}-\frac{1}{i_1^2}\Big)}{\binom{-2+2i_1}{-1+i_1}}\Bigg]
$$

$$
-\frac{\eta\big(3\big(4N^2-4N-3\big)\eta^2+12(2N-3)\eta-35\big)S_1(N)}{768(N+1)^2(N+2)(2N-3)}
$$

$$
+\frac{(1-\eta)^{-N}\eta\big(\big(4N^2-4N-3\big)\eta^2+(4N-6)\eta-35\big)}{512(N+1)^2(N+2)(2N-3)}\Big[S_{1,1}(1-\eta,1,N)
$$

$$
-S_2(1-\eta,N)\Big]\Bigg\}, \tag{439}
$$

with polynomials

$$
\begin{aligned}
P_1 &= 900\eta^3N^7-900\eta^2(2\eta-1)N^6-25\eta\left(27\eta^2+90\eta+163\right)N^5+\big(2475\eta^3\\
&\quad+450\eta^2+8875\eta+7296\big)N^4+\left(-225\eta^3+2250\eta^2-725\eta+6336\right)N^3\\
&\quad-\big(675\eta^3+1350\eta^2+8875\eta+33216\big)N^2+192(25\eta+27)N+8640 &(440)\\
P_2 &= \eta^3\left(8N^3-12N^2-2N+3\right)+3\eta^2\left(4N^2-8N+3\right)+\eta(45-30N)-105\,. &(441)
\end{aligned}
$$

Interchanging the masses $m_1 \leftrightarrow m_2$ yields

$$
D_{4b}(z) = \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Bigg\{\frac{1}{\varepsilon}\Bigg[-\frac{(z-1)\big(2z^2+5z-1\big)}{60z^2}+\frac{1}{10}G\big(\{\tfrac{1}{\tau}\},z\big)\Bigg]
$$

$$
+\frac{G\big(\{\frac{1}{\tau}\},z\big)Q_3}{28800\eta^3z^2}+\frac{Q_4}{57600\eta^3z^2}-\frac{1}{\eta^3z}\frac{(z-1)Q_1}{1920}G\big(\{\frac{1}{1-\tau}\},z\big)
$$

$$
\begin{aligned}
&+\frac{(1-z)^{3/2}(\eta+1)}{480\eta^3 z^{5/2}}Q_5\Biggl[G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\bigr)+G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\bigr)\Biggr]\\
&+\frac{\bigl(-35\eta^2-3\eta+8\bigr)}{480\eta^2}\Biggl[G\bigl(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\bigr)+G\bigl(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\bigr)\Biggr]\\
&+\frac{(1-z)^{3/2}}{\eta^3 z^{5/2}}\frac{(\eta-1)^2 Q_5}{1920}\Biggl[G\bigl(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\bigr)+G\bigl(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\bigr)\Biggr]\\
&+\frac{(\eta-1)(\eta+1)\bigl(7\eta^2+2\eta-1\bigr)}{16\eta^3}\Biggl[G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\bigr)\\
&+G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\bigr)\Biggr]+\frac{1}{\eta^3}\frac{1}{64}(\eta-1)^3\bigl(7\eta^2+2\eta-1\bigr)\\
&\times\Biggl[G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\bigr)\\
&+G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\bigr)\Biggr]+\Biggl[\frac{Q_2}{1920\eta^3 z}\\
&-\frac{(1-z)^{3/2}(\eta+1)Q_5}{480\eta^3 z^{5/2}}G\bigl(\{\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)+\frac{\bigl(35\eta^2+3\eta-8\bigr)}{480\eta^2}G\bigl(\{\frac{1}{-\eta-\tau+\eta\tau}\},z\bigr)\\
&-\frac{(1-z)^{3/2}}{\eta^3 z^{5/2}}\frac{(\eta-1)^2 Q_5}{1920}G\bigl(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\bigr)-\frac{1}{\eta^3}\frac{1}{16}(\eta-1)(\eta+1)\bigl(7\eta^2+2\eta-1\bigr)\\
&\times G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)-\frac{(\eta-1)^3\bigl(7\eta^2+2\eta-1\bigr)}{64\eta^3}\\
&\times G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\bigr)-\frac{G\bigl(\{\frac{1}{\tau}\},z\bigr)}{60\eta^2}\Biggr]G\bigl(\{\frac{1}{\tau}\},\eta\bigr)\\
&+\frac{(1-z)^{3/2}}{\eta^3 z^{5/2}}\frac{1}{240}(\eta-1)Q_5 G\bigl(\{\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)+\frac{1}{\eta^2}\frac{1}{60}G\bigl(\{\frac{1}{\tau},\frac{1}{1-\tau}\},z\bigr)\\
&+\frac{1}{\eta^2}\frac{\bigl(105\eta^3-180\eta^2-5\eta+64\bigr)}{3840}G\bigl(\{\frac{1}{\tau},\frac{1}{\tau}\},\eta\bigr)+\frac{1}{\eta^3}\frac{1}{8}(\eta-1)^2\bigl(7\eta^2+2\eta-1\bigr)\\
&\times G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)+\frac{1}{\eta^{5/2}}\frac{1}{512}\bigl(7\eta^3-5\eta^2-3\eta+1\bigr)\\
&\times G\bigl(\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\bigr)+\frac{1}{60}\bigl(3+\frac{1}{\eta^2}\bigr)G\bigl(\{\frac{1}{\tau},\frac{1}{\tau}\},z\bigr)-\frac{\zeta_2}{60\eta^2}\Biggr\}\,. \qquad (442)
\end{aligned}
$$

$$
\begin{aligned}
Q_1 &= \eta^4\bigl(105z^4-85z^3-81z^2+67z-6\bigr)+2\eta^3\bigl(15z^4-45z^3+7z^2+99z\\
&\quad-6\bigr)+\eta^2\bigl(-120z^4+160z^3+20z^2-50z+2\bigr)-10\eta(z-1)^2 z(3z+1)\\
&\quad+5(z-1)^2 z(3z-1)\,, \qquad (443)\\
Q_2 &= \eta^4\bigl(105z^5-190z^4+4z^3+148z^2-143z+6\bigr)+2\eta^3\bigl(15z^5-60z^4\\
&\quad+52z^3+92z^2-45z+6\bigr)-\eta^2\bigl(120z^5-280z^4+140z^3+70z^2\\
&\quad-137z+2\bigr)-10\eta z^3\bigl(3z^2-8z+6\bigr)+5z^2\bigl(3z^3-10z^2+12z-6\bigr)\,, \qquad (444)\\
Q_3 &= -15\eta^4 z\bigl(105z^5-190z^4+4z^3+148z^2-38z+6\bigr)-6\eta^3\bigl(75z^6-300z^5
\end{aligned}
$$

$$
\begin{aligned}
&+260z^4+460z^3-86z^2-210z+40\big)+10\eta^2 z\big(180z^5-420z^4+210z^3\\
&+105z^2-203z+3\big)+150\eta z^4\big(3z^2-8z+6\big)-75z^3\big(3z^3-10z^2+12z-6\big)\ , \quad (445)\\
Q_4 \ =\ &-15\eta^4 z\big(420z^5-830z^4+51z^3+662z^2-572z+24\big)+2\eta^3\big(5400z^6\\
&-11850z^5+3195z^4+7798z^3-7968z^2+696z+104\big)-20\eta^2 z\big(90z^5\\
&-240z^4+135z^3+65z^2-147z+82\big)-150\eta z^2\big(24z^4-62z^3+41z^2+10z\\
&-24\big)+75z^3\big(12z^3-38z^2+43z-22\big)\ , \quad (446)\\
Q_5 \ =\ &15z^3-15\eta z^2(3z+1)-5\eta^2 z\big(15z^2+6z-1\big)+\eta^3\big(105z^3+45z^2-13z+3\big)\ . \quad (447)
\end{aligned}
$$

In Mellin space D_{4b} takes the form

$$
\begin{aligned}
D_{4b}(N) \ =\ & \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\left[\frac{1+(-1)^N}{2}\right]\Biggl\{-\frac{1}{\varepsilon}\frac{1}{5(N-1)N(N+1)^2(N+2)}\\
&+\Biggl[-\frac{\big(35\eta^2+(6-4N)\eta-4N^2+4N+3\big)(\eta-1)^{-N}\eta^{N-3}}{1024(N+1)^2(N+2)(2N-3)}\\
&+\frac{P_1 2^{-2N-9}\binom{2N-2}{N-1}}{\eta^3 N(N+1)^2(N+2)(2N-3)}\\
&+\frac{2^{-2N-11}\binom{2N-2}{N-1}P_1}{\eta^4 N(N+1)^2(N+2)(2N-3)}\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}}{\binom{-2+2i_1}{-1+i_1}}\Biggr]\ln^2(\eta)\\
&+\Biggl[-\frac{35\eta^2+(36-24N)\eta-12N^2+12N+9}{768\eta^3(N+1)^2(N+2)(2N-3)}\\
&+\frac{P_1 2^{-2N-7}\binom{2N-2}{N-1}}{\eta^3 N(N+1)^2(N+2)(2N-3)}\\
&+\frac{2^{-2N}\binom{2N-2}{N-1}P_1}{1024\eta^4 N(N+1)^2(N+2)(2N-3)}\Biggl[\sum_{i_1=1}^N\frac{2^{2i_1}}{\binom{-2+2i_1}{-1+i_1}i_1}\\
&-\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{-2+2i_1}{-1+i_1}}\Biggr]\\
&+\frac{\big(35\eta^2+(6-4N)\eta-4N^2+4N+3\big)(\eta-1)^{-N}\eta^{N-3}}{512(N+1)^2(N+2)(2N-3)}S_1\left(\frac{\eta-1}{\eta},N\right)\Biggr]\ln(\eta)\\
&+\frac{P_2}{28800\eta^3(N-1)^2N^2(N+1)^3(N+2)(2N-3)}\\
&+\frac{P_1 2^{-2N-6}\binom{2N-2}{N-1}}{\eta^3 N(N+1)^2(N+2)(2N-3)}-\frac{2^{-2N-7}\binom{2N-2}{N-1}}{\eta^{5/2}N(N+1)^2(N+2)(2N-3)}\\
&\times\Bigl[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\Bigr]P_1+\frac{2^{-2N}\binom{2N-2}{N-1}P_1}{1024\eta^4 N(N+1)^2(N+2)(2N-3)}\\
&\times\Biggl[\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}\left[S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)-S_2\left(\frac{-1+\eta}{\eta},i_1\right)\right]}{\binom{-2+2i_1}{-1+i_1}}
\end{aligned}
$$

$$
+\sum_{i_1=1}^{N}\frac{2^{2i_1}\left[\frac{1}{i_1^2}-\frac{S_1(i_1)}{i_1}\right]}{\binom{-2+2i_1}{-1+i_1}}\Bigg]+\frac{\left(35\eta^2+(36-24N)\eta-12N^2+12N+9\right)}{768\eta^3(N+1)^2(N+2)(2N-3)}S_1(N)
$$

$$
+\frac{\left(35\eta^2+(6-4N)\eta-4N^2+4N+3\right)(\eta-1)^{-N}\eta^{N-3}}{512(N+1)^2(N+2)(2N-3)}\left[S_2\left(\frac{\eta-1}{\eta},N\right)\right.
$$

$$
\left.\left.-S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\right]\right\}, \tag{448}
$$

with polynomials

$$
P_1 = 105\eta^3+15\eta^2(2N-3)-3\eta\left(4N^2-8N+3\right)-8N^3+12N^2+2N-3 \tag{449}
$$

$$
\begin{aligned}
P_2 = {} & 192\eta^3\left(38N^4+33N^3-173N^2+27N+45\right)-25\eta^2(N-1)^2N\big(163N^2 \\
& -29N-192\big)+450\eta(N-1)^2N^2\left(2N^2-N-3\right)+225(N-1)^2N^2 \\
& \times\left(4N^3-7N-3\right) .
\end{aligned} \tag{450}
$$

In z-space diagram D_{5a} is given by

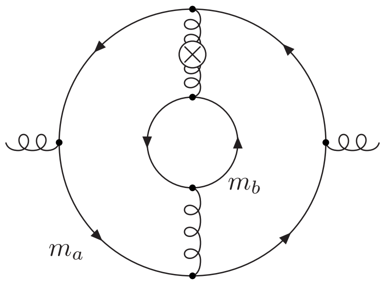


Figure 11: This diagram depicts D_{5a} with $m_a = m_2$, $m_b = m_1$ and D_{5b} with $m_a = m_1$, $m_b = m_2$ respectively.

$$
\begin{aligned}
D_{5a}(z) = {} & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left\{\frac{1}{\varepsilon}\left[-\frac{(-1+z)\left(-1+5z+2z^2\right)}{45z^2}+\frac{2}{15}G\left[\left\{\frac{1}{\tau}\right\},z\right]\right]\right. \\
& -\frac{Q_1}{201600\eta z^2}-\frac{1}{105}\eta^3 z\zeta_2+\frac{(-1+z)Q_2}{6720\eta z}G\left[\left\{\frac{1}{1-\tau}\right\},z\right] \\
& +\frac{Q_3}{100800\eta z^2}G\left[\left\{\frac{1}{\tau}\right\},z\right]+\frac{(-1+\eta)(1-z)^{3/2}Q_4}{840\eta z^{5/2}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right] \\
& +\frac{1}{105}\eta^3 zG\left[\left\{\frac{1}{\tau},\frac{1}{1-\tau}\right\},z\right]+\left(-\frac{5\eta}{384}+\eta^3\left(-\frac{1}{128}+\frac{z}{105}\right)\right)G\left[\left\{\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right] \\
& +\frac{1}{105}\left(7+\eta^3 z\right)G\left[\left\{\frac{1}{\tau},\frac{1}{\tau}\right\},z\right]+\frac{\left(5-8\eta+2\eta^2+\eta^4\right)}{4\eta}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\right\},z\right] \\
& -\frac{(1+\eta)(1-z)^{3/2}Q_4}{1680\eta z^{5/2}}\left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]\right. \\
& \left.+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\right]+G\left[\left\{\frac{1}{\tau}\right\},\eta\right]\left[\frac{Q_5}{6720\eta z}+\frac{1}{105}\eta^3 zG\left[\left\{\frac{1}{\tau}\right\},z\right]\right. \\
& -\frac{(1+\eta)(1-z)^{3/2}Q_4}{1680\eta z^{5/2}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]-\frac{\eta\left(-175-35\eta-16\eta^2 z+16\eta^3 z\right)}{1680}
\end{aligned}
$$

$$
\begin{aligned}
&\times G\left[\left\{\frac{1}{1-\tau+\eta\tau}\right\},z\right]+\frac{(-1+\eta)^2(1-z)^{3/2}Q_4}{6720\eta z^{5/2}}G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\},z\right]\\
&-\frac{(-1+\eta)(1+\eta)\left(5+2\eta+\eta^2\right)}{8\eta}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]\\
&+\frac{(-1+\eta)^3\left(5+2\eta+\eta^2\right)}{32\eta}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\},z\right]\Bigg]\\
&-\frac{\eta\left(-175-35\eta-16\eta^2z+16\eta^3z\right)}{1680}\Bigg[G\left[\left\{\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+G\left[\left\{\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]+\frac{(-1+\eta)^2(1-z)^{3/2}Q_4}{6720\eta z^{5/2}}\\
&\times\left[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\right]\\
&+\frac{\left(5-3\eta-\eta^2-\eta^3\right)}{256\sqrt{\eta}}G\left[\left\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right]-\frac{(-1+\eta)(1+\eta)\left(5+2\eta+\eta^2\right)}{8\eta}\\
&\times\left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\right]\\
&+\frac{(-1+\eta)^3\left(5+2\eta+\eta^2\right)}{32\eta}\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]\Bigg\}, \qquad (451)
\end{aligned}
$$

with polynomials

$$
\begin{aligned}
Q_1 &= 30\eta^3z^2\left(216-146z-105z^2+70z^3\right)+525\eta^4z^2\left(12-22z+43z^2\right.\\
&\left.-38z^3+12z^4\right)+75z\left(24-152z+662z^2+51z^3-830z^4+420z^5\right)\\
&+20\eta^2z\big(290-237z+535z^2+756z^3-1554z^4+630z^5\big)-2\eta\big(672\\
&+2248z-11984z^2+37234z^3+15345z^4-55590z^5+25200z^6\big) \qquad (452)\\
Q_2 &= 35\eta^4(-1+z)^2z(-1+3z)+2\eta^3z\left(3+35z-175z^2+105z^3\right)+5\big(6\\
&-67z+81z^2+85z^3-105z^4\big)-2\eta\left(-30+509z+17z^2-251z^3\right.\\
&\left.+105z^4\right)+2\eta^2\big(-5+47z-28z^2-294z^3+210z^4\big) \qquad (453)\\
Q_3 &= 525\eta^4z^3\left(-6+12z-10z^2+3z^3\right)+30\eta^3z^2\left(-108-32z+210z^2\right.\\
&\left.-280z^3+105z^4\right)-75z\left(6-38z+148z^2+4z^3-190z^4+105z^5\right)\\
&+10\eta^2z\big(15-751z+225z^2+798z^3-1512z^4+630z^5\big)-2\eta\big(560\\
&-2910z-854z^2+7380z^3+4020z^4-5340z^5+1575z^6\big) \qquad (454)\\
Q_4 &= -15+(65-21\eta)z+\left(-225+126\eta+35\eta^2\right)z^2+105\left(-5+3\eta\right.\\
&\left.+\eta^2+\eta^3\right)z^3 \qquad (455)\\
Q_5 &= 35\eta^4z\left(3-6z+12z^2-10z^3+3z^4\right)+2\eta^3z\left(-3-32z+210z^2\right.\\
&\left.-280z^3+105z^4\right)-2\eta\left(30-154z+492z^2+268z^3-356z^4+105z^5\right)\\
&-5\big(6-38z+148z^2+4z^3-190z^4+105z^5\big)+\eta^2\left(10-69z+150z^2\right.\\
&\left.+532z^3-1008z^4+420z^5\right)\ . \qquad (456)
\end{aligned}
$$

In Mellin-space one obtains

$$
\begin{aligned}
D_{5a}(N) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Biggl\{-\frac{4}{\varepsilon 15(N-1)N(N+1)^2(N+2)} \\
&+\Biggl[\frac{\eta\bigl(\bigl(4N^2-8N+3\bigr)\eta^2-4(N+1)\eta+25\bigr)(1-\eta)^{-N}}{256(\eta-1)(N+1)(N+2)(2N-3)(2N-1)} \\
&-\frac{2^{-2N}\binom{2N}{N}P_2}{512(\eta-1)(N+1)^2(N+2)(2N-3)(2N-1)}\eta\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}}{\binom{2i_1}{i_1}} \\
&-\frac{P_2 2^{-2N-9}\binom{2N}{N}}{(\eta-1)(N+1)^2(N+2)(2N-3)(2N-1)}\Biggr]\ln^2(\eta) \\
&+\Biggl[\frac{\eta\bigl(3\bigl(4N^2-4N-3\bigr)\eta^2+25\bigr)}{384(N+1)^2(N+2)(2N-3)}-\frac{2^{-2N-8}\binom{2N}{N}\eta}{(\eta-1)(N+1)^2(N+2)(2N-3)(2N-1)} \\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}S_1\left(1-\eta,i_1\right)P_2}{\binom{2i_1}{i_1}} \\
&+\frac{\eta\bigl(\bigl(4N^2-8N+3\bigr)\eta^2-4(N+1)\eta+25\bigr)(1-\eta)^{-N}}{128(\eta-1)(N+1)(N+2)(2N-3)(2N-1)}S_1\left(1-\eta,N\right) \\
&-\frac{P_2 2^{-2N-7}\binom{2N}{N}}{(N+1)^2(N+2)(2N-3)(2N-1)}\Biggr]\ln(\eta) \\
&-\frac{P_1}{14400(N-1)^2N^2(N+1)^3(N+2)(2N-3)} \\
&+\frac{2^{-2N-6}\binom{2N}{N}P_2}{(N+1)^2(N+2)(2N-3)(2N-1)} \\
&-\frac{2^{-2N-7}\binom{2N}{N}P_2}{\sqrt{\eta}(N+1)^2(N+2)(2N-3)(2N-1)}\Bigl[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\Bigr] \\
&+\frac{2^{-2N}\binom{2N}{N}\eta P_2}{256(\eta-1)(N+1)^2(N+2)(2N-3)(2N-1)} \\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\Bigl[S_2\left(1-\eta,i_1\right)-S_{1,1}\left(1-\eta,1,i_1\right)\Bigr]}{\binom{2i_1}{i_1}} \\
&+\frac{\eta\bigl(\bigl(4N^2-8N+3\bigr)\eta^2-4(N+1)\eta+25\bigr)(1-\eta)^{-N}}{128(\eta-1)(N+1)(N+2)(2N-3)(2N-1)}\Bigl[S_{1,1}\left(1-\eta,1,N\right) \\
&-S_2\left(1-\eta,N\right)\Bigr]+\frac{\eta\bigl(3\bigl(4N^2-4N-3\bigr)\eta^2+25\bigr)S_1(N)}{384(N+1)^2(N+2)(2N-3)}\Biggr\}, \qquad (457)
\end{aligned}
$$

with the polynomials

$$
\begin{aligned}
P_1 &= 900\eta^3N^7-900\eta^2(2\eta+1)N^6-25\eta\left(27\eta^2-90\eta-89\right)N^5+\bigl(2475\eta^3 \\
&-450\eta^2-4625\eta-5504\bigr)N^4-\left(225\eta^3+2250\eta^2-175\eta+3264\right)N^3 \\
&+\bigl(-675\eta^3+1350\eta^2+4625\eta+22784\bigr)N^2-96(25\eta+46)N-5760 \qquad (458)
\end{aligned}
$$

$$P_2 = \eta^3\left(8N^3-12N^2-2N+3\right)+\eta^2\left(-4N^2+8N-3\right)+9\eta(2N-3)+75\,. \qquad (459)$$

The mass reversed diagram D_{5b} obeys the z-space representation

$$\begin{aligned}
D_{5b}(z) &= \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Biggl\{\frac{1}{\varepsilon}\Biggl[-\frac{(z-1)\bigl(2z^2+5z-1\bigr)}{45z^2}+\frac{2}{15}G\bigl(\{\tfrac{1}{\tau}\},z\bigr)\Biggr] \\
&+\frac{G\bigl(\{\frac{1}{\tau}\},z\bigr)Q_3}{100800\eta^3z^2}-\frac{Q_4}{201600\eta^3z^2}+\frac{(\eta-1)(\eta+1)\bigl(5\eta^2+2\eta+1\bigr)}{8\eta^3} \\
&\Biggl[G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\bigr)+G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\bigr)\Biggr] \\
&+\frac{(\eta-1)^3\bigl(5\eta^2+2\eta+1\bigr)}{32\eta^3}\Biggl[G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\bigr) \\
&+G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\bigr)\Biggr]-\frac{1}{\eta^3z}\frac{(z-1)G\bigl(\{\frac{1}{1-\tau}\},z\bigr)Q_1}{6720} \\
&-\frac{\bigl(35\eta^2+175\eta^3-16z+16\eta z\bigr)}{1680\eta^3}\Biggl[G\bigl(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\bigr) \\
&+G\bigl(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\bigr)\Biggr]+\frac{(1-z)^{3/2}(\eta+1)Q_5}{1680\eta^3z^{5/2}}\Biggl[G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\bigr) \\
&+G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\bigr)\Biggr]+\frac{(1-z)^{3/2}(\eta-1)^2Q_5}{6720\eta^3z^{5/2}}\Biggl[G\bigl(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\bigr) \\
&+G\bigl(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\bigr)\Biggr]+\Biggl[\frac{Q_2}{6720\eta^3z}-\frac{(1-z)^{3/2}(\eta+1)Q_5}{1680\eta^3z^{5/2}} \\
&\times G\bigl(\{\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)+\frac{1}{\eta^3}\frac{\bigl(35\eta^2+175\eta^3-16z+16\eta z\bigr)}{1680}G\bigl(\{\frac{1}{-\eta-\tau+\eta\tau}\},z\bigr) \\
&-\frac{(1-z)^{3/2}}{\eta^3z^{5/2}}\frac{(\eta-1)^2Q_5}{6720}G\bigl(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\bigr)-\frac{(\eta-1)(\eta+1)\bigl(5\eta^2+2\eta+1\bigr)}{8\eta^3} \\
&\times G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)-\frac{(\eta-1)^3\bigl(5\eta^2+2\eta+1\bigr)}{32\eta^3} \\
&G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\bigr)-\frac{zG\bigl(\{\frac{1}{\tau}\},z\bigr)}{105\eta^3}\Biggr]G\bigl(\{\frac{1}{\tau}\},\eta\bigr) \\
&+\frac{(1-z)^{3/2}}{\eta^3z^{5/2}}\frac{1}{840}(\eta-1)Q_5G\bigl(\{\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)+\frac{1}{\eta^3}\frac{1}{105}zG\bigl(\{\frac{1}{\tau},\frac{1}{1-\tau}\},z\bigr) \\
&+\Bigl(-\frac{1}{16}+\frac{1}{384\eta}+\frac{5\eta}{128}+\frac{z}{105\eta^3}\Bigr)G\bigl(\{\frac{1}{\tau},\frac{1}{\tau}\},\eta\bigr)+\frac{1}{\eta^3}\frac{1}{4}(\eta-1)^2\bigl(5\eta^2+2\eta+1\bigr) \\
&G\bigl(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\bigr)+\frac{\bigl(5\eta^3-3\eta^2-\eta-1\bigr)}{256\eta^{5/2}}G\bigl(\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\bigr) \\
&+\frac{1}{105}\Bigl(7+\frac{z}{\eta^3}\Bigr)G\bigl(\{\frac{1}{\tau},\frac{1}{\tau}\},z\bigr)-\frac{z\zeta_2}{105\eta^3}\Biggr\}. \qquad (460)
\end{aligned}$$

$$
\begin{aligned}
Q_1 &= 5\eta^4\big(105z^4-85z^3-81z^2+67z-6\big)+2\eta^3\big(105z^4-251z^3+17z^2\\
&\quad +509z-30\big)+\eta^2\big(-420z^4+588z^3+56z^2-94z+10\big)-2\eta z\big(105z^3\\
&\quad -175z^2+35z+3\big)-35(z-1)^2z(3z-1)\ , \qquad (461)\\
Q_2 &= 5\eta^4\big(105z^5-190z^4+4z^3+148z^2-143z+6\big)+\eta^3\big(210z^5-712z^4\\
&\quad +536z^3+984z^2-518z+60\big)-\eta^2\big(420z^5-1008z^4+532z^3+150z^2\\
&\quad -489z+10\big)+2\eta z\big(-105z^4+280z^3-210z^2+32z+108\big)\\
&\quad -35z^2\big(3z^3-10z^2+12z-6\big)\ , \qquad (462)\\
Q_3 &= -75\eta^4z\big(105z^5-190z^4+4z^3+148z^2-38z+6\big)-2\eta^3\big(1575z^6\\
&\quad -5340z^5+4020z^4+7380z^3-854z^2-2910z+560\big)+10\eta^2z\big(630z^5\\
&\quad -1512z^4+798z^3+225z^2-751z+15\big)+30\eta z^2\big(105z^4-280z^3\\
&\quad +210z^2-32z-108\big)+525z^3\big(3z^3-10z^2+12z-6\big)\ , \qquad (463)\\
Q_4 &= 75\eta^4z\big(420z^5-830z^4+51z^3+662z^2-572z+24\big)-2\eta^3\big(25200z^6\\
&\quad -55590z^5+15345z^4+37234z^3-37184z^2+2248z+672\big)\\
&\quad +20\eta^2z\big(630z^5-1554z^4+756z^3+535z^2-867z+290\big)\\
&\quad +30\eta z^2\big(70z^3-105z^2-146z+216\big)\\
&\quad +525z^3\big(12z^3-38z^2+43z-22\big)\ , \qquad (464)\\
Q_5 &= -105z^3-35\eta z^2(3z+1)-21\eta^2z\big(15z^2+6z-1\big)\\
&\quad +5\eta^3\big(105z^3+45z^2-13z+3\big)\ . \qquad (465)
\end{aligned}
$$

In Mellin space one obtains

$$
\begin{aligned}
D_{5b}(N) &= \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{4}{\varepsilon 15(N-1)N(N+1)^2(N+2)}\\
&+\Bigg[-\frac{\big(25\eta^2-4(N+1)\eta+4N^2-8N+3\big)(\eta-1)^{-N-1}\eta^{N-2}}{256(N+1)(N+2)(2N-3)(2N-1)}\\
&+\frac{P_1 2^{-2N-9}\binom{2N}{N}}{(\eta-1)\eta^2(N+1)^2(N+2)(2N-3)(2N-1)}\\
&+\frac{2^{-2N-9}\binom{2N}{N}P_1}{(\eta-1)\eta^3(N+1)^2(N+2)(2N-3)(2N-1)}\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}}{\binom{2i_1}{i_1}}\Bigg]\ln^2(\eta)\\
&+\Bigg[-\frac{\big(25\eta^2+12N^2-12N-9\big)}{384\eta^3(N+1)^2(N+2)(2N-3)}+\frac{P_1 2^{-2N-7}\binom{2N}{N}}{\eta^3(N+1)^2(N+2)(2N-3)(2N-1)}\\
&-\frac{2^{-2N-8}\binom{2N}{N}P_1}{(\eta-1)\eta^3(N+1)^2(N+2)(2N-3)(2N-1)}\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}}\\
&+\frac{\big(25\eta^2-4(N+1)\eta+4N^2-8N+3\big)(\eta-1)^{-N-1}\eta^{N-2}}{128(N+1)(N+2)(2N-3)(2N-1)}S_1\left(\frac{\eta-1}{\eta},N\right)\Bigg]\ln(\eta)\\
&+\frac{P_2}{14400\eta^3(N-1)^2N^2(N+1)^3(N+2)(2N-3)}\\
&+\frac{P_1 2^{-2N-6}\binom{2N}{N}}{\eta^3(N+1)^2(N+2)(2N-3)(2N-1)}-\frac{2^{-2N-7}\binom{2N}{N}P_1}{\eta^{5/2}(N+1)^2(N+2)(2N-3)(2N-1)}
\end{aligned}
$$

$$\times\left[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\right]+\frac{2^{-2N}\binom{2N}{N}P_1}{256(\eta-1)\eta^3(N+1)^2(N+2)(2N-3)(2N-1)}$$
$$\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}\left[S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)-S_2\left(\frac{-1+\eta}{\eta},i_1\right)\right]}{\binom{2i_1}{i_1}}$$
$$+\frac{(\eta-1)^{-N-1}\eta^{N-2}\left(25\eta^2-4(N+1)\eta+4N^2-8N+3\right)}{128(N+1)(N+2)(2N-3)(2N-1)}\left[S_2\left(\frac{\eta-1}{\eta},N\right)\right.$$
$$\left.-S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\right]+\frac{\left(25\eta^2+12N^2-12N-9\right)}{384\eta^3(N+1)^2(N+2)(2N-3)}S_1(N)\Biggr\}, \quad (466)$$

where we abbreviated the polynomials

$$P_1 = 75\eta^3+9\eta^2(2N-3)+\eta\left(-4N^2+8N-3\right)+8N^3-12N^2-2N+3 \quad (467)$$
$$P_2 = 64\eta^3\left(86N^4+51N^3-356N^2+69N+90\right)-25\eta^2(N-1)^2N\big(89N^2 -7N-96\big)+450\eta(N-1)^2N^2\left(2N^2-N-3\right)-225(N-1)^2N^2 \times\left(4N^3-7N-3\right) . \quad (468)$$

The diagrams $D_{6a,b}$ and $D_{8a,b}$, see Figures 12 and 14 respectively, are of a topology with one

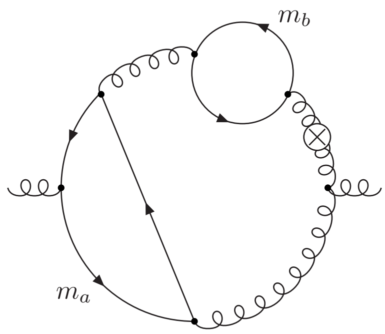


Figure 12: D_{6a} with $m_a = m_2$, $m_b = m_1$ and D_{6b} with $m_a = m_1$, $m_b = m_2$ respectively.

fermionic triangle and one fermion-bubble. For $D_{6a}(z)$ one obtains

$$D_{6a}(z) = \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Biggl\{$$
$$-\frac{1}{45\varepsilon^2}+\frac{1}{\varepsilon}\left[\frac{22+\frac{30}{\eta^2}+\frac{25}{\eta}+\frac{10}{z^2}-\frac{40}{z}}{1800}+\frac{1}{90}G\big(\{\frac{1}{1-\tau}\},z\big)-\frac{1}{90}G\big(\{\frac{1}{\tau}\},z\big)\right]$$
$$+\frac{G\big(\{\frac{1}{\tau}\},z\big)Q_3}{201600(\eta-1)\eta^3z^2}+\frac{Q_4}{18144000\eta^3z^2}+\frac{1}{(\eta-1)\eta^3z}\frac{G\big(\{\frac{1}{1-\tau}\},z\big)Q_2}{201600}$$
$$+\frac{1}{180}\left[-G\big(\{\frac{1}{1-\tau},\frac{1}{1-\tau}\},z\big)-G\big(\{\frac{1}{\tau},\frac{1}{\tau}\},z\big)\right]+\frac{\sqrt{1-z}}{\eta^3z^{5/2}}\frac{(\eta+1)Q_5}{6720}$$
$$\times\left[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\big)\right]$$

$$
\begin{aligned}
&-\frac{\big(-25\eta^3+26\eta^2+23\eta+8\big)}{3360(\eta-1)\eta^2}\Bigg[G\big(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)\\
&+G\big(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]+\frac{\sqrt{1-z}}{\eta^3 z^{5/2}}\frac{(\eta-1)^2Q_5}{26880}\Bigg[G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)\\
&+G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]-\frac{1}{\eta^3}\frac{1}{420}\Bigg[G\big(\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{1-\tau}\},z\big)\\
&+G\big(\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},z\big)\Bigg]+\frac{1}{\eta^3}\frac{1}{420}(\eta-1)\Bigg[G\big(\{\frac{1}{1-\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)\\
&+G\big(\{\frac{1}{1-\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]-\frac{1}{\eta^3}\frac{(\eta+1)\big(75\eta^3-63\eta^2-35\eta-105\big)}{1680}\\
&\times\Bigg[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&-\frac{(\eta-1)^2\big(75\eta^3-63\eta^2-35\eta-105\big)}{6720\eta^3}\Bigg[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)\\
&+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]+\Bigg[\frac{Q_1}{40320(\eta-1)\eta^3 z}-\frac{\sqrt{1-z}}{\eta^3 z^{5/2}}\frac{(\eta+1)Q_5}{6720}\\
&G\big(\{\sqrt{1-\tau}\sqrt{\tau}\},z\big)+\frac{1}{(\eta-1)\eta^2}\frac{\big(-25\eta^3+26\eta^2+23\eta+8\big)}{3360}G\big(\{\frac{1}{-\eta-\tau+\eta\tau}\},z\big)\\
&-\frac{\sqrt{1-z}}{\eta^3 z^{5/2}}\frac{(\eta-1)^2Q_5G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\big)}{26880}+\frac{1}{\eta^3}\frac{1}{420}G\big(\{\frac{1}{1-\tau},\frac{1}{\tau}\},z\big)\\
&-\frac{(\eta-1)}{\eta^3}\frac{1}{420}G\big(\{\frac{1}{1-\tau},\frac{1}{-\eta-\tau+\eta\tau}\},z\big)+\frac{1}{\eta^3}\frac{(\eta+1)\big(75\eta^3-63\eta^2-35\eta-105\big)}{1680}\\
&\times G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\big)+\frac{1}{\eta^3}\frac{(\eta-1)^2\big(75\eta^3-63\eta^2-35\eta-105\big)}{6720}\\
&G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\big)\Bigg]G\big(\{\frac{1}{\tau}\},\eta\big)\\
&+\frac{\sqrt{1-z}}{\eta^3 z^{5/2}}\frac{(\eta-1)Q_5}{3360}G\big(\{\sqrt{1-\tau}\sqrt{\tau}\},z\big)-\frac{1}{\eta^3}\frac{1}{420}G\big(\{\frac{1}{1-\tau}\},z\big)G\big(\{\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)\\
&-\frac{1}{\eta^3}\frac{1}{840}(\eta-1)\big(75\eta^3-63\eta^2-35\eta-105\big)G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\big)\\
&+\Bigg[-\frac{1}{120}+\frac{G\big(\{\frac{1}{1-\tau}\},z\big)}{420\eta^3}\Bigg]\zeta_2\Bigg\}\ . \qquad (469)
\end{aligned}
$$

$$
\begin{aligned}
Q_1 \;=\;& -15\eta^5\big(15z^5-25z^4-8z^3+34z^2-14z+3\big)+3\eta^4\big(63z^5-55z^4-96z^3\\
&-10z^2+34z-15\big)+\eta^3\big(330z^5-826z^4+294z^3+987z^2-478z+105\big)\\
&+\eta^2\big(126z^5-154z^4+294z^3-273z^2+178z-15\big)+\eta z\big(-105z^4\\
&+35z^3+210z^2-105z-12\big)-105z^2\big(3z^3-7z^2+6z-3\big)\ , \qquad (470)
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &= 75\eta^5(z-1)^2\big(15z^3+5z^2-13z+3\big)+\eta^4\big(-945z^5+825z^4+1440z^3 \\
&\quad +150z^2-1247z+225\big)-\eta^3\big(1650z^5-4130z^4+1470z^3+4935z^2 \\
&\quad -2602z+525\big)-5\eta^2\big(126z^5-154z^4+294z^3-273z^2+78z-15\big) \\
&\quad +35\eta z\big(15z^4-5z^3-30z^2+15z+53\big)+525(z-1)^2 z\big(3z^2-z+1\big)\ , \qquad (471)\\
Q_3 &= 75\eta^5 z\big(15z^5-25z^4-8z^3+34z^2-14z+3\big)+\eta^4\big(-945z^6+825z^5 \\
&\quad +1440z^4+150z^3+722z^2-2015z+560\big)+\eta^3\big(-1650z^6+4130z^5 \\
&\quad -1470z^4-4935z^3+2558z^2+1715z-560\big)-5\eta^2 z\big(126z^5-154z^4 \\
&\quad +294z^3-273z^2+122z-15\big)+5\eta z^2\big(105z^4-35z^3-210z^2+105z \\
&\quad -324\big)+525z^3\big(3z^3-7z^2+6z-3\big)\ , \qquad (472)\\
Q_4 &= 3375\eta^4 z\big(60z^5-110z^4-27z^3+146z^2-56z+12\big)-8\eta^3\big(46575z^6 \\
&\quad -93825z^5-4050z^4+114075z^3-51319z^2+7605z+3780\big) \\
&\quad +450\eta^2 z\big(168z^5-434z^4+77z^3+574z^2-232z+290\big)-360\eta z^2\big(525z^4 \\
&\quad -875z^3+700z^2-175z-853\big)+23625z^3\big(12z^3-26z^2+19z-10\big)\ , \qquad (473)\\
Q_5 &= 105(1-2z)z^3+35\eta z^2\big(-2z^2+z+1\big)-21\eta^2 z\big(6z^3-3z^2-4z+1\big) \\
&\quad +5\eta^3\big(30z^4-15z^3-23z^2+11z-3\big)\ . \qquad (474)
\end{aligned}
$$

Performing the Mellin transformation using `HarmonicSums` [175, 177–179] one obtains

$$
\begin{aligned}
D_{6a}(N) &= \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{1}{45\varepsilon^2(N+1)}+\frac{1}{\varepsilon}\Bigg[\frac{S_1(N)}{90(N+1)} \\
&\quad -\frac{P_5}{1800\eta^2(N-1)N(N+1)^2}\Bigg]+\Bigg[\frac{-75\eta^2+38\eta+41}{53760\eta^2(N+1)} \\
&\quad +\frac{\left(\frac{\eta}{\eta-1}\right)^N P_2}{53760(\eta-1)\eta^2(N+1)^2(2N-3)(2N-1)} \\
&\quad -\frac{2^{-2N-10}\binom{2N}{N}P_6}{105(\eta-1)\eta^3(N+1)^2(2N-3)(2N-1)}\Bigg[\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}}{\binom{2i_1}{i_1}}+\eta\Bigg] \\
&\quad -\frac{S_1\left(\frac{\eta}{\eta-1},N\right)}{840\eta^3(N+1)}\Bigg]\ln^2(\eta)+\Bigg[\frac{P_3}{80640\eta^3(N+1)^2(2N-3)} \\
&\quad +\frac{2^{-2N-9}\binom{2N}{N}P_6}{105(\eta-1)\eta^3(N+1)^2(2N-3)(2N-1)}\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}} \\
&\quad -\frac{2^{-2N-8}\binom{2N}{N}P_6}{105\eta^3(N+1)^2(2N-3)(2N-1)} \\
&\quad -\frac{\left(\frac{\eta}{\eta-1}\right)^N P_2}{26880(\eta-1)\eta^2(N+1)^2(2N-3)(2N-1)}S_1\left(\frac{\eta-1}{\eta},N\right) \\
&\quad +\frac{1}{\eta^3(N+1)}\Bigg[\frac{1}{420}S_{1,1}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)-\frac{S_2(N)}{420}\Bigg]\Bigg]\ln(\eta) \\
&\quad -\frac{P_7}{9072000\eta^3(N-1)^2N^2(N+1)^3(2N-3)}-\frac{2^{-2N-7}\binom{2N}{N}P_6}{105\eta^3(N+1)^2(2N-3)(2N-1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2^{-2N}\binom{2N}{N}P_6}{53760(\eta-1)\eta^3(N+1)^2(2N-3)(2N-1)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_2\left(\frac{-1+\eta}{\eta},i_1\right)-S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)\right]}{\binom{2i_1}{i_1}}\\
&+\frac{\left(\frac{\eta}{\eta-1}\right)^{N+1}P_2}{26880\eta^3(N+1)^2(2N-3)(2N-1)}\left[S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)-S_2\left(\frac{\eta-1}{\eta},N\right)\right]\\
&+\frac{P_4S_1(N)}{403200\eta^3(N+1)^2(2N-3)}-\frac{1}{360(N+1)}\left[S_1^2(N)+S_2(N)\right]+\frac{1}{420\eta^3(N+1)}\\
&\times\Biggl[-S_3(N)-S_1\left(\frac{\eta}{\eta-1},N\right)S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)-S_{1,2}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},N\right)\\
&+S_{1,2}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)+S_{1,1,1}\left(\frac{\eta-1}{\eta},1,\frac{\eta}{\eta-1},N\right)\\
&+S_{1,1,1}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},1,N\right)\Biggr]+\Biggl[\frac{\left(75\eta^3-63\eta^2-35\eta-105\right)}{13440\eta^{5/2}(N+1)}\\
&+\frac{2^{-2N-8}\binom{2N}{N}P_6}{105\eta^{5/2}(N+1)^2(2N-3)(2N-1)}\Biggr]\times\left[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\right]\\
&-\frac{\zeta_2}{120(N+1)}\Biggr\}\,.
\end{aligned}
\tag{475}
$$

Here we abbreviated the polynomials

$$
\begin{aligned}
P_1 &= 91582N^7-147453N^6-132764N^5+236946N^4+235222N^3-442293N^2\\
&\quad+32760N+75600\,, && (476)\\
P_2 &= 375\eta^2(N+1)-6\eta\left(4N^2+23N+19\right)+420N^3-676N^2-13N+123\,, && (477)\\
P_3 &= -450\eta^3\left(2N^2-N-3\right)+\eta^2\left(656N^2-328N-609\right)+18\eta\left(26N^2-11N-42\right)\\
&\quad+315\left(4N^2-4N-3\right)\,, && (478)\\
P_4 &= 448\eta^3(3-2N)^2+\eta^2\left(-5600N^2+2800N+6525\right)-30\eta\left(224N^2-106N-345\right)\\
&\quad-1575\left(4N^2-4N-3\right)\,, && (479)\\
P_5 &= 4\eta^2\left(2N^3-5N^2+3N-10\right)-25\eta N\left(N^2-1\right)-30N\left(N^2-1\right)\,, && (480)\\
P_6 &= 1125\eta^3+189\eta^2(2N-3)-35\eta\left(4N^2-8N+3\right)+105\left(8N^3-12N^2-2N+3\right) && (481)\\
P_7 &= 2\eta^3(2N-3)\bigl(49057N^6-10080N^5-62834N^4+15120N^3+215377N^2\\
&\quad-55440N-50400\bigr)-225\eta^2(N-1)^2N(N+1)\left(1376N^3-1248N^2-539N-840\right)\\
&\quad-90\eta(N-1)^2N^2(N+1)(2N-3)(2231N+581)-70875(N-1)^2N^2(N+1)\\
&\quad\times(2N-3)(2N+1)\,. && (482)
\end{aligned}
$$

Assigning the masses for this diagram the other way yields diagram D_{6b}. Its z-space representation reads

$$
D_{6b}(z) = \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\Biggl\{
$$

$$
\begin{aligned}
&-\frac{1}{45\varepsilon^2}+\frac{1}{\varepsilon}\Bigg[\frac{10-40z+(22+25\eta+30\eta^2)\,z^2}{1800z^2}+\frac{1}{90}G\left[\left\{\frac{1}{1-\tau}\right\},z\right]\\
&-\frac{1}{90}G\left[\left\{\frac{1}{\tau}\right\},z\right]\Bigg]+\frac{Q_1}{18144000\eta z^2}+\zeta_2\Bigg[-\frac{1}{120}+\frac{1}{420}\eta^3G\left[\left\{\frac{1}{1-\tau}\right\},z\right]\Bigg]\\
&+\frac{Q_2}{201600(-1+\eta)\eta z^2}G\left[\left\{\frac{1}{\tau}\right\},z\right]+\frac{(-1+\eta)\sqrt{1-z}Q_3}{3360\eta z^{5/2}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]\\
&-\frac{1}{180}G\left[\left\{\frac{1}{1-\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\frac{1}{1-\tau}\right\},z\right]\Bigg[\frac{Q_4}{201600(-1+\eta)\eta z}\\
&-\frac{1}{420}\eta^3G\left[\left\{\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right]\Bigg]-\frac{1}{180}G\left[\left\{\frac{1}{\tau},\frac{1}{\tau}\right\},z\right]-\frac{(1+\eta)\sqrt{1-z}Q_3}{6720\eta z^{5/2}}\\
&\times\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\Bigg]\\
&-\frac{(-1+\eta)\left(-75+63\eta+35\eta^2+105\eta^3\right)}{840\eta}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]\\
&+G\left[\left\{\frac{1}{\tau}\right\},\eta\right]\Bigg[-\frac{Q_6}{40320(-1+\eta)\eta z}-\frac{(1+\eta)\sqrt{1-z}Q_3}{6720\eta z^{5/2}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]\\
&-\frac{\eta\left(-25+26\eta+23\eta^2+8\eta^3\right)}{3360(-1+\eta)}G\left[\left\{\frac{1}{1-\tau+\eta\tau}\right\},z\right]+\frac{(-1+\eta)^2\sqrt{1-z}Q_3}{26880\eta z^{5/2}}\\
&\times G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\},z\right]-\frac{1}{420}\eta^3G\left[\left\{\frac{1}{1-\tau},\frac{1}{\tau}\right\},z\right]\\
&+\frac{(-1+\eta)\eta^3}{420}G\left[\left\{\frac{1}{1-\tau},\frac{1}{1-\tau+\eta\tau}\right\},z\right]+\frac{(1+\eta)\left(-75+63\eta+35\eta^2+105\eta^3\right)}{1680\eta}\\
&\times G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]-\frac{(-1+\eta)^2\left(-75+63\eta+35\eta^2+105\eta^3\right)}{6720\eta}\\
&\times G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\},z\right]\Bigg]-\frac{\eta\left(-25+26\eta+23\eta^2+8\eta^3\right)}{3360(-1+\eta)}\\
&\times\Bigg[G\left[\left\{\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]\\
&+\frac{(-1+\eta)^2\sqrt{1-z}Q_3}{26880\eta z^{5/2}}\Bigg[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]-\frac{1}{420}\eta^3G\left[\left\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{1-\tau}\right\},z\right]\\
&-\frac{\eta^3}{420}G\left[\left\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\right\},z\right]+\frac{(-1+\eta)\eta^3}{420}G\left[\left\{\frac{1}{1-\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+\frac{1}{420}(-1+\eta)\eta^3G\left[\left\{\frac{1}{1-\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]
\end{aligned}
$$

$$+\frac{(1+\eta)\left(-75+63\eta+35\eta^2+105\eta^3\right)}{1680\eta}\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]$$

$$+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\Bigg]-\frac{(-1+\eta)^2\left(-75+63\eta+35\eta^2+105\eta^3\right)}{6720\eta}$$

$$\times\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]$$

$$+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]\Bigg\}, \tag{483}$$

with

$$Q_1 = 23625\eta^4z^3\left(-10+19z-26z^2+12z^3\right)-360\eta^3z^2\big(-853-175z$$
$$+700z^2-875z^3+525z^4\big)+3375z\left(12-56z+146z^2-27z^3-110z^4+60z^5\right)$$
$$+450\eta^2z\left(290-232z+574z^2+77z^3-434z^4+168z^5\right)-8\eta\big(3780+$$
$$7605z-51319z^2+114075z^3-4050z^4-93825z^5+46575z^6\big) \tag{484}$$

$$Q_2 = -525\eta^5z^3\left(-3+6z-7z^2+3z^3\right)-5\eta^4z^2\left(-324+105z-210z^2-35z^3+105z^4\right)$$
$$-75z\left(3-14z+34z^2-8z^3-25z^4+15z^5\right)+5\eta^3z\big(-15+122z-273z^2+294z^3$$
$$-154z^4+126z^5\big)+\eta\left(-560+2015z-722z^2-150z^3-1440z^4-825z^5+945z^6\right)$$
$$+\eta^2\left(560-1715z-2558z^2+4935z^3+1470z^4-4130z^5+1650z^6\right) \tag{485}$$

$$Q_3 = 15+(-55+21\eta)z+\left(115-84\eta-35\eta^2\right)z^2-\left(-75+63\eta+35\eta^2+105\eta^3\right)z^3$$
$$+2\left(-75+63\eta+35\eta^2+105\eta^3\right)z^4 \tag{486}$$

$$Q_4 = -525\eta^5(-1+z)^2z\left(1-z+3z^2\right)-75(-1+z)^2\left(3-13z+5z^2+15z^3\right)$$
$$-35\eta^4z\left(53+15z-30z^2-5z^3+15z^4\right)+5\eta^3\big(-15+78z-273z^2+294z^3$$
$$-154z^4+126z^5\big)+\eta\left(-225+1247z-150z^2-1440z^3-825z^4+945z^5\right)$$
$$+\eta^2\left(525-2602z+4935z^2+1470z^3-4130z^4+1650z^5\right), \tag{487}$$

$$Q_6 = 105\eta^5z^2\left(3z^3-7z^2+6z-3\right)+\eta^4z\left(105z^4-35z^3-210z^2+105z+12\right)$$
$$+\eta^3\left(-126z^5+154z^4-294z^3+273z^2-178z+15\right)-\eta^2\big(330z^5-826z^4$$
$$+294z^3+987z^2-478z+105\big)-3\eta\left(63z^5-55z^4-96z^3-10z^2+34z-15\right)$$
$$+15\left(15z^5-25z^4-8z^3+34z^2-14z+3\right). \tag{488}$$

In Mellin space this corresponds to

$$D_{6b}(N) = \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{1}{45\varepsilon^2(N+1)}+\frac{1}{\varepsilon}\Bigg[\frac{P_1}{1800(N-1)N(N+1)^2}$$
$$+\frac{S_1(N)}{90(N+1)}\Bigg]+\frac{\eta^3}{420(N+1)}\Bigg[-S_3(N)-S_1\left(\frac{1}{1-\eta},N\right)S_{1,1}(1-\eta,1,N)$$
$$+S_{1,2}\left(\frac{1}{1-\eta},1-\eta,N\right)-S_{1,2}\left(1-\eta,\frac{1}{1-\eta},N\right)+S_{1,1,1}\left(1-\eta,1,\frac{1}{1-\eta},N\right)$$
$$+S_{1,1,1}\left(1-\eta,\frac{1}{1-\eta},1,N\right)\Bigg]-\frac{P_2}{9072000(N-1)^2N^2(N+1)^3(2N-3)}$$

$$
\begin{aligned}
&-\frac{2^{-2N-7}\binom{2N}{N}P_5}{105(N+1)^2(2N-3)(2N-1)}+\frac{2^{-2N}\binom{2N}{N}\eta P_5}{53760(\eta-1)(N+1)^2(2N-3)(2N-1)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\Big[S_{1,1}\left(1-\eta,1,i_1\right)-S_2\left(1-\eta,i_1\right)\Big]}{\binom{2i_1}{i_1}}-\frac{P_4S_1\left(N\right)}{403200(N+1)^2(2N-3)}\\
&+\frac{(1-\eta)^{-N}\eta P_6}{26880(\eta-1)(N+1)^2(2N-3)(2N-1)}\Bigg[S_2\left(1-\eta,N\right)-S_{1,1}\left(1-\eta,1,N\right)\Bigg]\\
&-\frac{1}{360(N+1)}\Big[S_1^2(N)+S_2\left(N\right)\Big]+\ln^2(\eta)\Bigg[\frac{41\eta^2+38\eta-75}{53760(N+1)}\\
&+\frac{2^{-2N}\binom{2N}{N}\eta P_5}{107520(\eta-1)(N+1)^2(2N-3)(2N-1)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}}{\binom{2i_1}{i_1}}\\
&+\frac{2^{-2N-10}\binom{2N}{N}P_5}{105(\eta-1)(N+1)^2(2N-3)(2N-1)}-\frac{S_1\left(\frac{1}{1-\eta},N\right)\eta^3}{840(N+1)}\\
&-\frac{(1-\eta)^{-N}\eta P_6}{53760(\eta-1)(N+1)^2(2N-3)(2N-1)}\Bigg]+\ln(\eta)\Bigg[\frac{\eta^3}{420(N+1)}\\
&\times\Bigg[S_2\left(N\right)-S_{1,1}\left(\frac{1}{1-\eta},1-\eta,N\right)\Bigg]+\frac{P_3}{80640(N+1)^2(2N-3)}\\
&+\frac{P_5 2^{-2N-8}\binom{2N}{N}}{105(N+1)^2(2N-3)(2N-1)}+\frac{2^{-2N-9}\binom{2N}{N}\eta P_5}{105(\eta-1)(N+1)^2(2N-3)(2N-1)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}S_1\left(1-\eta,i_1\right)}{\binom{2i_1}{i_1}}-\frac{(1-\eta)^{-N}\eta P_6S_1\left(1-\eta,N\right)}{26880(\eta-1)(N+1)^2(2N-3)(2N-1)}\Bigg]\\
&+\Bigg[\frac{2^{-2N-8}\binom{2N}{N}}{\sqrt{\eta}(N+1)^2(2N-3)(2N-1)}\frac{P_5}{105}-\frac{\left(105\eta^3+35\eta^2+63\eta-75\right)}{13440\sqrt{\eta}(N+1)}\Bigg]\\
&\times\Bigg[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\Bigg]-\frac{\zeta_2}{120(N+1)}\Bigg\}\,, \qquad (489)
\end{aligned}
$$

where

$$
\begin{aligned}
P_1 &= \left(30\eta^2+25\eta-8\right)N^3-\left(30\eta^2+25\eta+12\right)N+20N^2+40 \qquad (490)\\
P_2 &= -4\left(70875\eta^3+100395\eta^2+77400\eta-49057\right)N^7+6\big(94500\eta^3+149895\eta^2\\
&\quad+98400\eta-55777\big)N^6+\left(212625\eta^3+60660\eta^2+150075\eta-190856\right)N^5\\
&\quad-3\left(259875\eta^3+352080\eta^2+174225\eta-145828\right)N^4+\big(70875\eta^3+340920\eta^2\\
&\quad-29475\eta+770788\big)N^3+3\left(70875\eta^3+52290\eta^2-22575\eta-504674\right)N^2+\\
&\quad 2520(75\eta+52)N+302400 \qquad (491)\\
P_3 &= -315\eta^3\left(4N^2-4N-3\right)+\eta^2\left(-468N^2+198N+756\right)+\eta\big(-656N^2+328N\\
&\quad+609\big)+450\left(2N^2-N-3\right) \qquad (492)\\
P_4 &= 1575\eta^3\left(4N^2-4N-3\right)+30\eta^2\left(224N^2-106N-345\right)+25\eta\big(224N^2-112N\\
&\quad-261\big)-448(3-2N)^2 \qquad (493)\\
P_5 &= 105\eta^3\left(8N^3-12N^2-2N+3\right)-35\eta^2\left(4N^2-8N+3\right)+189\eta(2N-3)
\end{aligned}
$$

$$
\begin{aligned}
&+1125 && (494)\\
P_6 &= \eta^2\left(420N^3-676N^2-13N+123\right)-6\eta\left(4N^2+23N+19\right)+375(N+1)\,. && (495)
\end{aligned}
$$

The ladder-type diagram D_7 is symmetric under $m_1 \to m_2$ and only one mass assignment has

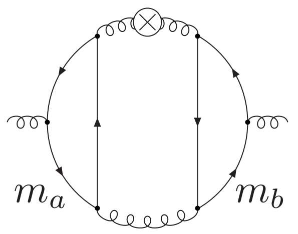


Figure 13: D_7, both mass assignments $m_a = m_2$,$m_b = m_1$ and $m_a = m_1$, $m_b = m_2$ yield the same result due to symmetry reasons.

to be considered. It evaluates to

$$
\begin{aligned}
D_7(z) &= \left(m_1^2\right)^{-3+3/2\varepsilon}\Bigg\{-\frac{1}{\varepsilon}\frac{\eta+1}{24\eta^2}+\frac{Q_2}{8640\eta^3 z}+\frac{\left(\eta^3+1\right)\left(z^2-1\right)}{180\eta^3 z}\Bigg[G\big(\{\frac{1}{1-\tau},\frac{1}{1-\tau}\},z\big)\\
&\quad+G\big(\{\frac{1}{1-\tau},\frac{1}{\tau}\},z\big)\Bigg]-\frac{1}{\eta^3}\frac{1}{180}(\eta^3+1)z\Bigg[G\big(\{\frac{1}{\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad-\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{1}{720}(\eta+1)Q_3\Bigg[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad+\frac{1}{(\eta-1)\eta^3 z}\frac{1}{720}Q_4\Bigg[G\big(\{\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad-\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta-1)^2\big(27+2(5\eta+27)z+\left(27\eta^2-10\eta-81\right)z^2\big)}{2880}\\
&\quad\times\Bigg[G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad-\frac{1}{(\eta-1)\eta^3 z}\frac{Q_6}{720}\Bigg[G\big(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad-\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta-1)^2\big(10\eta(z-1)z-27z^2+27\eta^2\left(3z^2-2z-1\right)\big)}{2880}\\
&\quad\times\Bigg[G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad+\frac{1}{\eta^3}\frac{1}{180}(\eta^3+1)\Bigg[G\big(\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&\quad+\frac{(1-\eta)}{180}\Bigg[G\big(\{\frac{1}{1-\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{1-\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(\eta-1)}{180\eta^3}\left[G\big(\{\frac{1}{1-\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{1-\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\right]\\
&-\frac{\left(\eta^3+1\right)}{90\eta^3}\left[G\big(\{\frac{1}{\tau},\frac{1}{1-\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{\tau},\frac{1}{1-\tau},\frac{1}{\tau}\},z\big)\right]\\
&-\frac{1}{\eta^3}\frac{1}{90}(\eta-1)\left[G\big(\{\frac{1}{\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\right]\\
&+\frac{1}{90}(1-\eta)\left[G\big(\{\frac{1}{\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\right]\\
&+\frac{(\eta+1)\left(27\eta^2+10\eta+27\right)}{180\eta^3}\left[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\big)\right.\\
&\left.+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\big)\right]+\frac{1}{\eta^3}\frac{(\eta-1)^2\left(27\eta^2-10\eta-81\right)}{1440}\\
&\times\left[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\right]\\
&+\frac{1}{\eta^3}\frac{(\eta-1)^2\left(81\eta^2+10\eta-27\right)}{1440}\left[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)\right.\\
&\left.+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\right]-\frac{1}{\eta^3}\frac{(\eta+1)Q_5}{1440}G\big(\{\frac{1}{1-\tau}\},z\big)\\
&+\left[\frac{Q_1}{4320(\eta-1)\eta^3}-\frac{\left(\eta^3-1\right)\left(z^2-1\right)}{180\eta^3 z}G\big(\{\frac{1}{1-\tau}\},z\big)+\frac{\left(z-\eta^3 z\right)}{180\eta^3}G\big(\{\frac{1}{\tau}\},z\big)\right.\\
&+\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{3}{80}(\eta-1)(\eta+1)^2\left(4z^2-2z-1\right)G\big(\{\sqrt{1-\tau}\sqrt{\tau}\},z\big)\\
&+\frac{1}{(\eta-1)\eta^3 z}\frac{Q_4}{720}G\big(\{\frac{1}{1-\tau+\eta\tau}\},z\big)+\frac{1}{(\eta-1)\eta^3 z}\frac{1}{720}Q_6G\big(\{\frac{1}{-\eta-\tau+\eta\tau}\},z\big)\\
&-\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta-1)^2\left(27+2(5\eta+27)z+\left(27\eta^2-10\eta-81\right)z^2\right)}{2880}G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\},z\big)\\
&+\frac{\sqrt{-(z-1)z}}{\eta^3 z}\frac{(\eta-1)^2\left(10\eta(z-1)z-27z^2+27\eta^2\left(3z^2-2z-1\right)\right)}{2880}\\
&\times G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\big)+\frac{1}{\eta^3}\frac{1}{180}\left(\eta^3-1\right)G\big(\{\frac{1}{1-\tau},\frac{1}{\tau}\},z\big)\\
&+\frac{(1-\eta)}{180}G\big(\{\frac{1}{1-\tau},\frac{1}{1-\tau+\eta\tau}\},z\big)+\frac{(\eta-1)}{180\eta^3}G\big(\{\frac{1}{1-\tau},\frac{1}{-\eta-\tau+\eta\tau}\},z\big)\\
&+\frac{1}{\eta^3}\frac{1}{90}\left(\eta^3-1\right)G\big(\{\frac{1}{\tau},\frac{1}{1-\tau}\},z\big)-\frac{1}{\eta^3}\frac{1}{90}(\eta-1)G\big(\{\frac{1}{\tau},\frac{1}{1-\tau+\eta\tau}\},z\big)\\
&-\frac{(\eta-1)(\eta+1)^2}{\eta^3}\frac{3}{10}G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\big)+\frac{1}{\eta^3}\frac{(\eta-1)^2\left(27\eta^2-10\eta-81\right)}{1440}
\end{aligned}
$$

$$\times G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\},z\big)-\frac{1}{\eta^3}\frac{(\eta-1)^2\big(81\eta^2+10\eta-27\big)}{1440}$$
$$\times G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\},z\big)+\frac{(\eta-1)}{90}G\big(\{\frac{1}{\tau},\frac{1}{-\eta-\tau+\eta\tau}\},z\big)\Bigg]G\big(\{\frac{1}{\tau}\},\eta\big)$$
$$-\frac{(\eta+1)Q_7G\big(\{\frac{1}{\tau}\},z\big)}{1440\eta^3}-\frac{3\sqrt{1-z}(\eta-1)^2(\eta+1)\big(4z^2-2z-1\big)}{40\eta^3\sqrt{z}}G\big(\{\sqrt{1-\tau}\sqrt{\tau}\},z\big)$$
$$+\Bigg[-\frac{-16\eta+\eta^2+8z+\eta^3(8z-27)}{1440\eta^3}+\frac{\big(\eta^3+1\big)}{180\eta^3}G\big(\{\frac{1}{1-\tau}\},z\big)\Bigg]G\big(\{\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)$$
$$+\frac{(\eta-1)^2(\eta+1)}{\eta^3}\frac{3}{5}G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\big)-\frac{\big(\eta^3+1\big)}{90\eta^3}G\big(\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)$$
$$+\frac{\big(27\eta^2+10\eta+27\big)}{2880\eta^{5/2}}G\big(\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)+\Bigg[\frac{\big(\eta^3+1\big)z}{180\eta^3}-\frac{\big(\eta^3+1\big)}{180\eta^3}G\big(\{\frac{1}{1-\tau}\},z\big)\Bigg]\zeta_2$$
$$-\frac{1}{90}G\big(\{\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)\Bigg\}. \tag{496}$$

$$Q_1 = 27\eta^4\big(6z^4-11z^3+3z^2+3z-6\big)-2\eta^3\big(22z^3-66z^2-15z+63\big)$$
$$+\eta^2\big(-324z^4+682z^3-426z^2-30z+84\big)+\eta\big(-44z^3+132z^2$$
$$+30z+204\big)+27z\big(6z^3-11z^2+3z+3\big), \tag{497}$$
$$Q_2 = -27\eta^3z\big(24z^4-46z^3+11z^2+16z-24\big)+\eta^2\big(648z^5-1370z^4$$
$$+873z^3-12z^2-390z-180\big)+\eta\big(648z^5-1370z^4+873z^3$$
$$-12z^2+18z-180\big)-27z^2\big(24z^3-46z^2+11z+16\big) \tag{498}$$
$$Q_3 = 20\eta(z-1)z+27\big(2z^2-2z-1\big)+27\eta^2\big(2z^2-2z-1\big), \tag{499}$$
$$Q_4 = 4\eta^4(1-2z)z+4\eta^5z^2+\eta^3z(4z+3)+4\big(z^2-1\big)-8\eta\big(z^2+z-1\big)$$
$$+\eta^2\big(4z^2-15z-4\big), \tag{500}$$
$$Q_5 = 9(z-1)^3(3z+2)+9\eta^2(z-1)^3(3z+2)+2\eta\big(5z^4-33z^3+42z^2$$
$$+14z-43\big), \tag{501}$$
$$Q_6 = -4z^2+4\eta z(2z-1)-\eta^2z(4z+3)+\eta^3\big(-4z^2+15z+4\big)$$
$$-4\eta^5\big(z^2-1\big)+8\eta^4\big(z^2+z-1\big), \tag{502}$$
$$Q_7 = 9z\big(3z^3-7z^2+3z+3\big)+9\eta^2z\big(3z^3-7z^2+3z+3\big)+2\eta\big(5z^4$$
$$-33z^3+42z^2+14z+11\big). \tag{503}$$

The Mellin space-expression for this diagram reads

$$D_7(N) = \big(m_1^2\big)^{-3+3/2\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{\eta+1}{24\varepsilon\eta^2(N+1)}$$
$$+\Bigg[-\frac{32+32\eta^3+11\eta N\big(N^2+3N+2\big)+11\eta^2N\big(N^2+3N+2\big)}{5760\eta^3N(N+1)^2(N+2)}$$
$$+\frac{1}{45}\frac{2^{-2N-9}\binom{2N}{N}P_2}{(\eta-1)\eta^3(N+1)^2(N+2)}\sum_{i_1=1}^{N}\frac{2^{2i_1}\big(\frac{\eta}{-1+\eta}\big)^{i_1}}{\binom{2i_1}{i_1}}$$

$$
\begin{aligned}
&-\frac{3}{5}\frac{2^{-2N-7}\binom{2N}{N}(\eta+1)N}{\eta^2(N+1)^2}+\frac{\left(\frac{\eta}{\eta-1}\right)^N}{11520(\eta-1)\eta^2N(N+1)^2(N+2)}P_5\\
&+\frac{(1-\eta)^{-N}P_6}{11520(\eta-1)\eta^3N(N+1)^2(N+2)}+\frac{S_1\left(\frac{1}{1-\eta},N\right)}{360(N+1)}+\frac{S_1\left(\frac{\eta}{\eta-1},N\right)}{360\eta^3(N+1)}\\
&-\frac{1}{45}P_{10}\frac{2^{-2N-9}\binom{2N}{N}}{(\eta-1)\eta(N+1)^2(N+2)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}}{\binom{2i_1}{i_1}}\Bigg]\ln^2(\eta)\\
&+\Bigg[-\frac{P_7}{5760\eta^3(N+1)^2(N+2)}-\frac{(\eta-1)P_{11}2^{-2N-7}\binom{2N}{N}}{45\eta^3(N+1)^2(N+2)}\\
&-\frac{1}{45}\frac{2^{-2N-8}\binom{2N}{N}P_2}{(\eta-1)\eta^3(N+1)^2(N+2)}\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}}\\
&+\frac{1}{90}\left(\eta^3-1\right)\frac{1}{\eta^3N(N+1)^2(N+2)}S_1\left(N\right)-\frac{\left(\eta^3-1\right)S_2\left(N\right)}{180\eta^3(N+1)}\\
&+\frac{(1-\eta)^{-N}}{5760(\eta-1)\eta^3N(N+1)^2(N+2)}P_6S_1\left(1-\eta,N\right)\\
&-\frac{\left(\frac{\eta}{\eta-1}\right)^N}{5760(\eta-1)\eta^2N(N+1)^2(N+2)}P_5S_1\left(\frac{\eta-1}{\eta},N\right)\\
&+\frac{S_{1,1}\left(\frac{1}{1-\eta},1-\eta,N\right)}{180(N+1)}-\frac{1}{180}\frac{1}{\eta^3(N+1)}S_{1,1}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)\\
&-\frac{1}{45}P_{10}\frac{2^{-2N-8}\binom{2N}{N}}{(\eta-1)\eta(N+1)^2(N+2)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}S_1\left(1-\eta,i_1\right)}{\binom{2i_1}{i_1}}\Bigg]\ln(\eta)\\
&+\frac{1}{\eta^{5/2}}\left[\frac{\left(27\eta^2+10\eta+27\right)}{1440(N+1)}-\frac{2^{-2N-6}\binom{2N}{N}P_1}{45(N+1)^2(N+2)}\right]\times\left[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\right]\\
&+\frac{P_{12}2^{-2N-6}\binom{2N}{N}}{45\eta^3(N+1)^2(N+2)}+\frac{2^{-2N}\binom{2N}{N}P_2}{11520(\eta-1)\eta^3(N+1)^2(N+2)}\\
&\times\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)-S_2\left(\frac{-1+\eta}{\eta},i_1\right)\right]}{\binom{2i_1}{i_1}}\\
&-\frac{(\eta+1)P_9}{2880\eta^3N(N+1)^2(N+2)}+\frac{(\eta+1)P_3S_1\left(N\right)}{5760\eta^3(N+1)^2(N+2)}\\
&+\frac{\left(\eta^3+1\right)}{180\eta^3N(N+1)^2(N+2)}\left[S_2\left(N\right)-S_1^2(N)\right]+\frac{\left(\eta^3+1\right)S_3(N)}{180\eta^3(N+1)}\\
&+\frac{(1-\eta)^{-N}P_6}{5760(\eta-1)\eta^3N(N+1)^2(N+2)}\left[S_{1,1}\left(1-\eta,1,N\right)-S_2\left(1-\eta,N\right)\right]\\
&+\frac{\left(\frac{\eta}{\eta-1}\right)^NP_5}{5760(\eta-1)\eta^2N(N+1)^2(N+2)}\left[S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)-S_2\left(\frac{\eta-1}{\eta},N\right)\right]\\
&+\frac{1}{180(N+1)}\Bigg[S_1\left(\frac{1}{1-\eta},N\right)S_{1,1}\left(1-\eta,1,N\right)-S_{1,2}\left(\frac{1}{1-\eta},1-\eta,N\right)
\end{aligned}
$$

$$+S_{1,2}\left(1-\eta,\frac{1}{1-\eta},N\right)-S_{1,1,1}\left(1-\eta,1,\frac{1}{1-\eta},N\right)$$
$$-S_{1,1,1}\left(1-\eta,\frac{1}{1-\eta},1,N\right)\Bigg]+\frac{1}{180\eta^3(N+1)}\Bigg[S_1\left(\frac{\eta}{\eta-1},N\right)S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)$$
$$+S_{1,2}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},N\right)-S_{1,2}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)-S_{1,1,1}\left(\frac{\eta-1}{\eta},1,\frac{\eta}{\eta-1},N\right)$$
$$-S_{1,1,1}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},1,N\right)\Bigg]+\frac{2^{-2N}\binom{2N}{N}P_{10}}{11520(\eta-1)\eta(N+1)^2(N+2)}$$
$$\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\Big[S_2\left(1-\eta,i_1\right)-S_{1,1}\left(1-\eta,1,i_1\right)\Big]}{\binom{2i_1}{i_1}}\Bigg\}\ . \qquad (504)$$

This expression contains the polynomials

$$P_1 = 27\eta^2\left(2N^2+4N+3\right)-10\eta(2N+1)+27\left(2N^2+4N+3\right)\ , \qquad (505)$$
$$P_2 = 81\eta^2-10\eta(2N+1)+27\left(4N^2+8N+3\right)\ , \qquad (506)$$
$$P_3 = 27\eta^2\left(4N^2+8N+3\right)+2\eta\left(6N^2+73N+115\right)+27\left(4N^2+8N+3\right)\ , \qquad (507)$$
$$P_4 = 27\eta^2\left(2N^2+4N+3\right)+2\eta\left(27N^2+44N-5\right)+27\left(4N^2+8N+3\right)\ , \qquad (508)$$
$$P_5 = 64\eta^3-64\eta^2(N+1)+5\eta N(N+1)-N\left(54N^2+103N+17\right)\ , \qquad (509)$$
$$P_6 = \eta^3N\left(54N^2+103N+17\right)-5\eta^2N(N+1)+64\eta(N+1)-64\ , \qquad (510)$$
$$P_7 = -27\eta^3\left(4N^2+8N+3\right)+\eta^2\left(196N^2+586N+449\right)+\eta\big(164N^2+494N+271\big)+27\left(4N^2+8N+3\right)\ , \qquad (511)$$
$$P_8 = \eta^3\left(27N^3+81N^2+54N-32\right)-\eta^2N\left(N^2+3N+2\right)+16\eta N\left(N^2+3N+2\right)-32\ , \qquad (512)$$
$$P_9 = 27\eta^2N\left(4N^2+8N+3\right)+2\eta\left(27N^3+76N^2+78N+60\right)+27N\left(4N^2+8N+3\right)\ , \qquad (513)$$
$$P_{10} = 27\big(4N^2+8N+3\big)\eta^2-10(2N+1)\eta+81\ , \qquad (514)$$
$$P_{11} = 27\eta^2\left(4N^2+8N+3\right)+2\eta\left(54N^2+98N-5\right)+27\left(4N^2+8N+3\right)\ , \qquad (515)$$
$$P_{12} = 27\eta^3\left(4N^2+8N+3\right)+\eta^2(71-20N)+\eta(71-20N)+27\left(4N^2+8N+3\right)\ . \qquad (516)$$

Finally we turn to the diagrams $D_{8a,b}$. In z-space they contain contributions which have to be

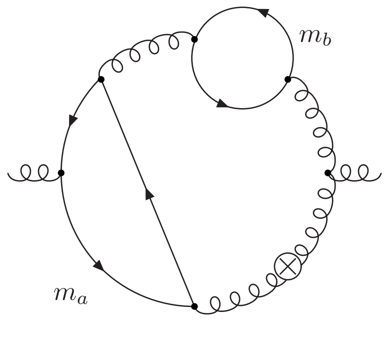


Figure 14: D_{8a} with $m_a=m_2$, $m_b=m_1$ and D_{8b} with $m_a=m_1$, $m_b=m_2$ respectively.

regularized as in Eq. 405. For D_{8a} this contribution is given by

$$
\begin{aligned}
D_{8a}^{(+)} &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\Biggl\{\frac{1}{\varepsilon}\frac{1}{90(-1+z)}+\frac{1}{450(-1+z)}+\frac{1}{180(1-z)}G\left[\left\{\frac{1}{1-\tau}\right\},z\right]\\
&+\Biggl[\frac{(-1+\eta)\left(25\sqrt{z}+(-100+63\eta)\sqrt{1-z}\sqrt{(1-z)z}\right)}{3360\eta(1-z)^{3/2}}\\
&-\frac{(1+\eta)\left(25\sqrt{z}+(-100+63\eta)\sqrt{1-z}\sqrt{-(-1+z)z}\right)G\left[\left\{\frac{1}{\tau}\right\},\eta\right]}{6720\eta(1-z)^{3/2}}\Biggr]\\
&\times G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]-\frac{(-1+\eta)^2\left(25\sqrt{z}+(-100+63\eta)(1-z)\sqrt{z}\right)}{26880\eta(1-z)^{3/2}}\\
&\times G\left[\left\{\frac{1}{\tau}\right\},\eta\right]G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\},z\right]\\
&+\frac{(1+\eta)\left(25\sqrt{z}+(-100+63\eta)\sqrt{1-z}\sqrt{-(-1+z)z}\right)}{6720\eta(1-z)^{3/2}}\Biggl[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]\\
&+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\Biggr]+\frac{(-1+\eta)^2\left(25\sqrt{z}+(-100+63\eta)(1-z)\sqrt{z}\right)}{26880\eta(1-z)^{3/2}}\\
&\times\Biggl[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Biggr]\Biggr\}. \qquad (517)
\end{aligned}
$$

The regular contribution to $D_{8a}(z)$ reads

$$
\begin{aligned}
D_{8a}^{\mathrm{Reg}} &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\Biggl\{\left(-\frac{1}{45\varepsilon^2}-\frac{1}{225\varepsilon}+\frac{7}{20250}-\frac{\zeta_2}{120}\right)\delta(1-z)\\
&-\frac{1}{45\varepsilon^2}+\frac{1}{\varepsilon}\left[\frac{\eta}{72}+\frac{1}{450}\left(8-\frac{5}{z}\right)+\frac{1}{90}\left(G\left[\left\{\frac{1}{1-\tau}\right\},z\right]-G\left[\left\{\frac{1}{\tau}\right\},z\right]\right)\right]\\
&+\frac{Q_1}{18144000\eta z}-\frac{\zeta_2}{120}+G\left[\left\{\frac{1}{1-\tau}\right\},z\right]\Biggl(\frac{Q_2}{201600(-1+\eta)\eta}\\
&+\frac{\eta^3(-1+z)G\left[\left\{\frac{1}{\tau}\right\},\eta\right]}{420z}\Biggr)+\frac{Q_3}{201600(-1+\eta)\eta z}G\left[\left\{\frac{1}{\tau}\right\},z\right]\\
&+\frac{(-1+\eta)\sqrt{1-z}Q_4}{3360\eta\sqrt{z}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]\\
&-\frac{(3\eta^3(-1+z)+7z)}{1260z}G\left[\left\{\frac{1}{1-\tau},\frac{1}{1-\tau}\right\},z\right]-\frac{\eta^3(-1+z)}{420z}G\left[\left\{\frac{1}{1-\tau},\frac{1}{\tau}\right\},z\right]\\
&-\frac{\eta\left(25-6\eta+105\eta^2\right)}{26880}G\left[\left\{\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right]-\frac{1}{180}G\left[\left\{\frac{1}{\tau},\frac{1}{\tau}\right\},z\right]\\
&+\frac{(1+\eta)\sqrt{1-z}Q_4}{6720\eta\sqrt{z}}\left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(-1+\eta)\left(-75+63\eta+35\eta^2+105\eta^3\right)}{840\eta}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\right\},z\right] \\
&+G\left[\left\{\frac{1}{\tau}\right\},\eta\right]\Bigg[\frac{Q_5}{40320(-1+\eta)\eta}-\frac{(1+\eta)\sqrt{1-z}Q_4}{6720\eta\sqrt{z}}G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right] \\
&-\frac{\eta Q_6}{3360(-1+\eta)z}G\left[\left\{\frac{1}{-\eta-\tau+\eta\tau}\right\},z\right]-\frac{(-1+\eta)^2\sqrt{1-z}Q_4}{26880\eta\sqrt{z}} \\
&\times G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\},z\right]-\frac{1}{420}\eta^3G\left[\left\{\frac{1}{\tau},\frac{1}{1-\tau}\right\},z\right] \\
&-\frac{(-1+\eta)\eta^3}{420}G\left[\left\{\frac{1}{\tau},\frac{1}{-\eta-\tau+\eta\tau}\right\},z\right]+\frac{(1+\eta)\left(-75+63\eta+35\eta^2+105\eta^3\right)}{1680\eta} \\
&\times G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\right\},z\right]+\frac{(-1+\eta)^2\left(-75+63\eta+35\eta^2+105\eta^3\right)}{6720\eta} \\
&\times G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\},z\right]\Bigg]+\frac{\eta Q_6}{3360(-1+\eta)z} \\
&\times\left[G\left[\left\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\right] \\
&+\frac{(-1+\eta)^2\sqrt{1-z}Q_4}{26880\eta\sqrt{z}}\Bigg[G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right] \\
&+G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]+\frac{1}{420}\eta^3\Bigg[G\left[\left\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right] \\
&+G\left[\left\{\frac{1}{\tau},\frac{1}{1-\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\frac{1}{\tau},\frac{1}{1-\tau},\frac{1}{\tau}\right\},z\right] \\
&+G\left[\left\{\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right]\Bigg]+\frac{1}{420}(-1+\eta)\eta^3\Bigg[G\left[\left\{\frac{1}{\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right] \\
&+G\left[\left\{\frac{1}{\tau},\frac{1}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]-\frac{\left(-75+63\eta+35\eta^2+105\eta^3\right)}{53760\sqrt{\eta}} \\
&\times G\left[\left\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\right\},\eta\right]-\frac{(1+\eta)\left(-75+63\eta+35\eta^2+105\eta^3\right)}{1680\eta} \\
&\times\left[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right]+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right]\right] \\
&-\frac{(-1+\eta)^2\left(-75+63\eta+35\eta^2+105\eta^3\right)}{6720\eta}\Bigg[G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right] \\
&+G\left[\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right]\Bigg]\Bigg\}, \qquad (518)
\end{aligned}
$$

with

$$
\begin{aligned}
Q_1 &= 16875z^2\left(2+3z-26z^2+12z^3\right)+23625\eta^4z\left(-12+14z+z^2-22z^3+12z^4\right) \\
&-12600\eta^3\left(-12-3z+25z^2+5z^3-35z^4+15z^5\right)+450\eta^2(280+30z
\end{aligned}
$$

$$
\begin{aligned}
& +238z^2-301z^3-238z^4+168z^5\big)-8\eta\big(5040-10864z-675z^2+6075z^3 \\
& -92475z^4+46575z^5\big) \qquad (519)\\
Q_2 &= 525\eta^5(-1+z)^2\left(-1+z+3z^2\right)+\eta^2\left(-1332+1845z+270z^2+2470z^3-1650z^4\right) \\
& +175\eta^4\left(-13+9z+12z^2-11z^3+3z^4\right)+375\left(1+2z^2-7z^3+3z^4\right)-5\eta^3\big(4 \\
& +315z+210z^2-350z^3+126z^4\big)-3\eta\left(-619+500z+690z^2-985z^3+315z^4\right) \qquad (520)\\
Q_3 &= 375z^3\left(2-7z+3z^2\right)+525\eta^5z^2\left(3-5z^2+3z^3\right)+5\eta^4z\big(-96+315z+420z^2 \\
& -385z^3+105z^4\big)-5\eta^3z\left(-376+315z+210z^2-350z^3+126z^4\right)+\eta^2\big(-1120 \\
& -728z+1845z^2+270z^3+2470z^4-1650z^5\big)-\eta\big(-1120+672z+1500z^2 \\
& +2070z^3-2955z^4+945z^5\big) \qquad (521)\\
Q_4 &= 75(1-2z)z+63\eta z(-1+2z)+35\eta^2z(-1+2z)+105\eta^3\left(-1-z+2z^2\right) \qquad (522)\\
Q_5 &= -75z^2\left(2-7z+3z^2\right)+3\eta z\left(100+138z-197z^2+63z^3\right)-35\eta^4\big(-3+9z \\
& +12z^2-11z^3+3z^4\big)-105\eta^5\left(-3+3z-5z^3+3z^4\right)+\eta^3\big(-395+315z \\
& +210z^2-350z^3+126z^4\big)+\eta^2\left(-25-369z-54z^2-494z^3+330z^4\right) \qquad (523)\\
Q_6 &= -8\eta^4(-1+z)-25z+26\eta z+8\eta^3(-2+3z)+\eta^2(8+15z)\;. \qquad (524)
\end{aligned}
$$

In Mellin space one obtains

$$
\begin{aligned}
D_{8a}(N) &= \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{N+2}{45\varepsilon^2(N+1)} \\
& +\frac{1}{\varepsilon}\left[\frac{(N+2)S_1(N)}{90(N+1)}-\frac{8N^3+(4-25\eta)N^2-(25\eta+24)N+20}{1800N(N+1)^2}\right] \\
& -\frac{\left(7N\left(N^2+3N+2\right)-3\eta^3\right)S_1^2(N)}{2520N(N+1)^2}-\frac{P_4S_1(N)}{403200\eta(N+1)^2} \\
& -\frac{2^{-2N-7}\binom{2N}{N}P_2}{105\eta(N+1)^2}+\frac{2^{-2N-8}\binom{2N}{N}P_2}{105\sqrt{\eta}(N+1)^2}\times\left[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\right] \\
& +\frac{2^{-2N}\binom{2N}{N}P_2}{53760(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^N\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_2\left(\frac{-1+\eta}{\eta},i_1\right)-S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)\right]}{\binom{2i_1}{i_1}} \\
& +\ln^2(\eta)\Bigg[\frac{\eta^3}{840N(N+1)^2}-\frac{\frac{(\eta-1)^{-N-1}\eta^N}{N(N+1)^2}P_1}{53760}-\frac{2^{-2N-10}\binom{2N}{N}P_2}{105(\eta-1)(N+1)^2} \\
& -\frac{2^{-2N-10}\binom{2N}{N}P_2}{105(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}}{\binom{2i_1}{i_1}}\Bigg]+\frac{P_5}{9072000\eta N^2(N+1)^3} \\
& -\frac{\left(3\eta^3+7N\left(N^2+3N+2\right)\right)S_2(N)}{2520N(N+1)^2}+\ln(\eta)\Bigg[-\frac{S_1(N)\eta^3}{420N(N+1)^2} \\
& -\frac{2^{-2N-8}\binom{2N}{N}P_2}{105\eta(N+1)^2}+\frac{2^{-2N-9}\binom{2N}{N}P_2}{105(\eta-1)\eta(N+1)^2} \\
& \times\sum_{i_1=1}^N\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}}+\frac{(\eta-1)^{-N-1}\eta^NP_1}{26880N(N+1)^2}S_1\left(\frac{\eta-1}{\eta},N\right)
\end{aligned}
$$

$$
\left. + \frac{P_3}{80640(N+1)^2\eta} \right] + \frac{(\eta-1)^{-N-1}\eta^N P_1}{26880N(N+1)^2} \Biggl[S_2\left(\frac{\eta-1}{\eta}, N\right) \\
\left. - S_{1,1}\left(\frac{\eta-1}{\eta}, 1, N\right) \right] - \frac{(N+2)\zeta_2}{120(N+1)} \Biggr\} \tag{525}
$$

with the polynomials

$$
\begin{aligned}
P_1 &= 64\eta^4 - 64\eta^3(N+1) - 3\eta^2 N(N+1) + 2\eta N\left(88N^2+245N+157\right) \\
&\quad -25N\left(4N^3+20N^2+31N+15\right) , && (526)\\
P_2 &= 105\eta^3 - 35\eta^2(2N+1) - 63\eta\left(4N^2+8N+3\right) + 25\bigl(8N^3+36N^2 \\
&\quad +46N+15\bigr) , && (527)\\
P_3 &= 192\eta^3 - \eta^2(538N+547) - 6\eta\left(76N^2+52N-93\right) + 75\bigl(8N^3+36N^2 \\
&\quad +46N+15\bigr) , && (528)\\
P_4 &= 960\eta^3 + 5\eta^2(22N+13) + \eta\left(-3176N^2-2008N+5478\right) \\
&\quad +375\left(8N^3+36N^2+46N+15\right) , && (529)\\
P_5 &= 10800\eta^3 N\left(9N^2+16N+7\right) + 225\eta^2 N\left(478N^3+945N^2+747N+280\right) \\
&\quad -4\eta\left(58616N^5+203774N^4+241285N^3+101167N^2-32760N-12600\right) \\
&\quad +16875N^2\left(8N^4+44N^3+82N^2+61N+15\right) . && (530)
\end{aligned}
$$

For diagram D_{8b} the part that requires regularization reads

$$
\begin{aligned}
D_{8b}^{(+)}(z) &= \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Biggl\{\frac{1}{\varepsilon}\frac{1}{90(-1+z)} + \frac{1}{450(-1+z)} + \frac{1}{180(1-z)} G\left[\left\{\frac{1}{1-\tau}\right\}, z\right] \\
&\quad + \Biggl(-\frac{(-1+\eta)\sqrt{z}(-63(-1+z)+25\eta(-3+4z))}{3360\eta(1-z)^{3/2}} \\
&\quad + \frac{(1+\eta)\sqrt{z}(-63(-1+z)+25\eta(-3+4z))}{6720\eta(1-z)^{3/2}} G\left[\left\{\frac{1}{\tau}\right\}, \eta\right]\Biggr) G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}\right\}, z\right] \\
&\quad - \frac{(-1+\eta)^2\sqrt{z}(-63(-1+z)+25\eta(-3+4z))}{26880\eta(1-z)^{3/2}} G\left[\left\{\frac{1}{\tau}\right\}, \eta\right] G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\right\}, z\right] \\
&\quad + \frac{(1+\eta)\sqrt{z}(-63(-1+z)+25\eta(-3+4z))}{6720\eta(1-z)^{3/2}} \Biggl(G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau}\right\}, z\right] \\
&\quad + G\left[\left\{\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{\tau}\right\}, z\right]\Biggr) - \frac{(-1+\eta)^2\sqrt{z}(-63(-1+z)+25\eta(-3+4z))}{26880\eta(1-z)^{3/2}} \\
&\quad \times \Biggl(G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{1-\tau}\right\}, z\right] + G\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, \frac{1}{\tau}\right\}, z\right]\Biggr)\Biggr\} ,
\end{aligned} \tag{531}
$$

and the regular contribution is given by

$$
D_{8b}^{\mathrm{Reg}}(z) = \left(m_1^2\right)^{-3+\varepsilon}\left(m_2^2\right)^{\varepsilon/2}\Biggl\{\left[\frac{7}{20250} - \frac{1}{45\varepsilon^2} - \frac{1}{225\varepsilon} - \frac{\zeta_2}{120}\right]\delta(1-z)
$$

$$
\begin{aligned}
&-\frac{1}{\varepsilon^2}\frac{1}{45}+\frac{1}{\varepsilon}\Bigg[\frac{1}{72\eta}+\frac{1}{450}\big(8-\frac{5}{z}\big)+\frac{1}{90}G\big(\{\frac{1}{1-\tau}\},z\big)-\frac{1}{90}G\big(\{\frac{1}{\tau}\},z\big)\Bigg]\\
&+\frac{G\big(\{\frac{1}{\tau}\},z\big)Q_3}{201600(\eta-1)\eta^3 z}+\frac{Q_4}{18144000\eta^3 z}+\frac{1}{(\eta-1)\eta^3}\frac{Q_2}{201600}G\big(\{\frac{1}{1-\tau}\},z\big)\\
&-\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta+1)Q_5}{6720}\Bigg[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&+\frac{1}{(\eta-1)\eta^3 z}\frac{\big(8-8z+26\eta^3 z-25\eta^4 z+8\eta(3z-2)+\eta^2(15z+8)\big)}{3360}\\
&\times\Bigg[G\big(\{\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&+\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta-1)^2 Q_5}{26880}\Bigg[G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&+\frac{1}{\eta^3}\frac{1}{420}\Bigg[G\big(\{\frac{1}{\tau},\frac{1}{1-\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{\tau},\frac{1}{1-\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&+\frac{1}{\eta^3}\frac{1}{420}(\eta-1)\Bigg[G\big(\{\frac{1}{\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\frac{1}{\tau},\frac{1}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&+\frac{1}{\eta^3}\frac{(\eta+1)\big(75\eta^3-63\eta^2-35\eta-105\big)}{1680}\Bigg[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\},z\big)\\
&+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\},z\big)\Bigg]-\frac{1}{\eta^3}\frac{(\eta-1)^2\big(75\eta^3-63\eta^2-35\eta-105\big)}{6720}\\
&\times\Bigg[G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{1-\tau}\},z\big)+G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\frac{1}{\tau}\},z\big)\Bigg]\\
&+\Bigg[\frac{Q_1}{40320(\eta-1)\eta^3}-\frac{1}{\eta^3 z}\frac{1}{420}(z-1)G\big(\{\frac{1}{1-\tau}\},z\big)-\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta+1)Q_5}{6720}\\
&\times G\big(\{\sqrt{1-\tau}\sqrt{\tau}\},z\big)+\frac{\big(8-8z+26\eta^3 z-25\eta^4 z+8\eta(3z-2)+\eta^2(15z+8)\big)}{3360(\eta-1)\eta^3 z}\\
&\times G\big(\{\frac{1}{1-\tau+\eta\tau}\},z\big)+\frac{\sqrt{1-z}}{\eta^3\sqrt{z}}\frac{(\eta-1)^2 Q_5}{26880}G\big(\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\},z\big)\\
&+\frac{1}{\eta^3}\frac{1}{420}G\big(\{\frac{1}{\tau},\frac{1}{1-\tau}\},z\big)+\frac{1}{\eta^3}\frac{1}{420}(\eta-1)G\big(\{\frac{1}{\tau},\frac{1}{1-\tau+\eta\tau}\},z\big)\\
&+\frac{1}{\eta^3}\frac{(\eta+1)\big(75\eta^3-63\eta^2-35\eta-105\big)}{1680}G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\big)\\
&-\frac{(\eta-1)^2\big(75\eta^3-63\eta^2-35\eta-105\big)}{6720\eta^3}G\big(\{\sqrt{1-\tau}\sqrt{\tau},\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}\},z\big)\Bigg]G\big(\{\frac{1}{\tau}\},\eta\big)\\
&+\frac{\sqrt{1-z}(\eta-1)Q_5}{3360\eta^3\sqrt{z}}G\big(\{\sqrt{1-\tau}\sqrt{\tau}\},z\big)+\frac{\big(3-3z-7\eta^3 z\big)}{1260\eta^3 z}G\big(\{\frac{1}{1-\tau},\frac{1}{1-\tau}\},z\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\eta^3 z}\frac{1}{420}(z-1)G\big(\{\frac{1}{1-\tau},\frac{1}{\tau}\},z\big)+\frac{1}{\eta^2}\frac{\big(75\eta^3-138\eta^2+3\eta-64\big)G\big(\{\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)}{26880}\\
&-\frac{1}{\eta^3}\frac{1}{840}(\eta-1)\big(75\eta^3-63\eta^2-35\eta-105\big)G\big(\{\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau}\},z\big)\\
&+\frac{1}{\eta^3}\frac{1}{420}G\big(\{\frac{1}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)+\frac{1}{\eta^{5/2}}\frac{\big(75\eta^3-63\eta^2-35\eta-105\big)G\big(\{\frac{\sqrt{\tau}}{1-\tau},\frac{1}{\tau},\frac{1}{\tau}\},\eta\big)}{53760}\\
&-\frac{1}{180}G\big(\{\frac{1}{\tau},\frac{1}{\tau}\},z\big)-\frac{\zeta_2}{120}\Bigg\}\,. \qquad (532)
\end{aligned}
$$

$$
\begin{aligned}
Q_1 &= -75\eta^5\big(3z^4-7z^3+2z^2+3\big)+3\eta^4\big(63z^4-197z^3+138z^2+100z+63\big)\\
&+\eta^3\big(330z^4-494z^3-54z^2-369z+305\big)+\eta^2\big(126z^4-350z^3+210z^2\\
&+315z-269\big)-35\eta z\big(3z^3-11z^2+12z+9\big)-105z\big(3z^3-5z^2+3\big)\,, \qquad (533)\\
Q_2 &= -375\eta^5\big(3z^4-7z^3+2z^2+1\big)+3\eta^4\big(315z^4-985z^3+690z^2+500z-619\big)\\
&+\eta^3\big(1650z^4-2470z^3-270z^2-1845z+1332\big)+5\eta^2\big(126z^4-350z^3\\
&+210z^2+315z+4\big)-175\eta\big(3z^4-11z^3+12z^2+9z-13\big)\\
&-525(z-1)^2\big(3z^2+z-1\big)\,, \qquad (534)\\
Q_3 &= -375\eta^5z^3\big(3z^2-7z+2\big)+\eta^4\big(945z^5-2955z^4+2070z^3+1500z^2+672z\\
&-1120\big)+\eta^3\big(1650z^5-2470z^4-270z^3-1845z^2+728z+1120\big)\\
&+5\eta^2z\big(126z^4-350z^3+210z^2+315z-376\big)-5\eta z\big(105z^4-385z^3+420z^2\\
&+315z-96\big)-525z^2\big(3z^3-5z^2+3\big)\,, \qquad (535)\\
Q_4 &= 16875\eta^4z\big(12z^4-26z^3+3z^2+2z+12\big)-8\eta^3\big(46575z^5-92475z^4+6075z^3\\
&-675z^2+35711z+5040\big)+450\eta^2\big(168z^5-238z^4-301z^3+238z^2\\
&+198z+280\big)-12600\eta\big(15z^5-35z^4+5z^3+25z^2+12z-12\big)\\
&+23625z^2\big(12z^3-22z^2+z+14\big)\,, \qquad (536)\\
Q_5 &= 105+\big(-75\eta^3+63\eta^2+35\eta+105\big)z+2\big(75\eta^3-63\eta^2-35\eta-105\big)z^2\,. \qquad (537)
\end{aligned}
$$

Finally one obtains the N-space representation

$$
\begin{aligned}
D_{8b}(N) &= \big(m_1^2\big)^{-3+\varepsilon}\big(m_2^2\big)^{\varepsilon/2}\left[\frac{1+(-1)^N}{2}\right]\Bigg\{-\frac{N+2}{45\varepsilon^2(N+1)}\\
&+\frac{1}{\varepsilon}\left[\frac{(N+2)S_1(N)}{90(N+1)}-\frac{\big(4\eta\big(2N^3+N^2-6N+5\big)-25N(N+1)\big)}{1800\eta N(N+1)^2}\right]\\
&-\frac{\big(7\eta^3N\big(N^2+3N+2\big)-3\big)}{2520\eta^3N(N+1)^2}S_1^2(N)-\frac{P_4S_1(N)}{403200\eta^2(N+1)^2}\\
&-\frac{2^{-2N-7}\binom{2N}{N}P_2}{105\eta^2(N+1)^2}+\frac{2^{-2N-8}\binom{2N}{N}P_2}{105\eta^{5/2}(N+1)^2}\times\Big[H_{-1,0,0}\left(\sqrt{\eta}\right)+H_{1,0,0}\left(\sqrt{\eta}\right)\Big]\\
&+\frac{2^{-2N}\binom{2N}{N}P_2}{53760(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\Big[S_{1,1}(1-\eta,1,i_1)-S_2(1-\eta,i_1)\Big]}{\binom{2i_1}{i_1}}\\
&+\ln^2(\eta)\Bigg[-\frac{(-1)^N(\eta-1)^{-N-1}P_1}{53760\eta^3N(N+1)^2}+\frac{2^{-2N}\binom{2N}{N}P_2}{107520(\eta-1)\eta(N+1)^2}
\end{aligned}
$$

$$
\times\sum_{i_1=1}^{N}\frac{(-1)^{i_1}2^{2i_1}(-1+\eta)^{-i_1}}{\binom{2i_1}{i_1}}+\frac{1}{840\eta^3N(N+1)^2}+\frac{2^{-2N-10}\binom{2N}{N}P_2}{105(\eta-1)\eta^2(N+1)^2}\Biggr]
$$

$$
+\frac{P_5}{9072000\eta^2N^2(N+1)^3}-\frac{\left(7N\left(N^2+3N+2\right)\eta^3+3\right)}{2520\eta^3N(N+1)^2}S_2\left(N\right)
$$

$$
+\ln(\eta)\Biggl[-\frac{P_3}{80640\eta^2(N+1)^2}+\frac{2^{-2N-8}\binom{2N}{N}P_2}{105\eta^2(N+1)^2}
$$

$$
+\frac{2^{-2N-9}\binom{2N}{N}P_2}{105(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}2^{2i_1}(-1+\eta)^{-i_1}S_1\left(1-\eta,i_1\right)}{\binom{2i_1}{i_1}}+\frac{S_1\left(N\right)}{420\eta^3N(N+1)^2}
$$

$$
-\frac{(-1)^N(\eta-1)^{-N-1}}{26880\eta^3N(N+1)^2}P_1S_1\left(1-\eta,N\right)\Biggr]+\frac{(-1)^N(\eta-1)^{-N-1}P_1}{26880\eta^3N(N+1)^2}\Bigl[S_2\left(1-\eta,N\right)
$$

$$
-S_{1,1}\left(1-\eta,1,N\right)\Bigr]-\frac{(N+2)}{120(N+1)}\zeta_2\Biggr\}\ . \tag{538}
$$

Here we used the polynomials

$$
\begin{aligned}
P_1 &= 25\eta^4N\left(4N^3+20N^2+31N+15\right)-2\eta^3N\left(88N^2+245N+157\right)\\
&\quad+3\eta^2N(N+1)+64\eta(N+1)-64\ , && (539)\\
P_2 &= 25\eta^3\left(8N^3+36N^2+46N+15\right)-63\eta^2\left(4N^2+8N+3\right)-35\eta(2N+1)\\
&\quad+105\ , && (540)\\
P_3 &= 75\eta^3\left(8N^3+36N^2+46N+15\right)-6\eta^2\left(76N^2+52N-93\right)-\eta(538N+547)\\
&\quad+192\ , && (541)\\
P_4 &= 375\eta^3\left(8N^3+36N^2+46N+15\right)+\eta^2\left(-3176N^2-2008N+5478\right)\\
&\quad+5\eta(22N+13)+960\ , && (542)\\
P_5 &= 16875\eta^3N^2\left(8N^4+44N^3+82N^2+61N+15\right)-4\eta^2\bigl(58616N^5+203774N^4\\
&\quad+241285N^3+101167N^2-32760N-12600\bigr)+225\eta N\bigl(478N^3+945N^2\\
&\quad+747N+280\bigr)+10800N\left(9N^2+16N+7\right)\ . && (543)
\end{aligned}
$$

With the exception of D_1 and D_3 in z-space the scalar $A_{gg,Q}$ diagrams are not expressible within the class of the usual harmonic polylogarithms [180], but generalizations thereof occur. These are given in terms of iterated integrals over the following letters

$$
\left\{\frac{d\tau}{1-\tau},\frac{d\tau}{\tau},\frac{d\tau\sqrt{\tau}}{1-\tau},d\tau\sqrt{1-\tau}\sqrt{\tau},\frac{d\tau}{\eta\tau-\tau+1},d\tau\frac{\sqrt{1-\tau}\sqrt{\tau}}{\eta\tau-\tau+1},\frac{d\tau}{\eta\tau-\eta-\tau},d\tau\frac{\sqrt{1-\tau}\sqrt{\tau}}{\eta\tau-\eta-\tau}\right\}.\tag{544}
$$

In Mellin-space all scalar $A_{gg,Q}$-diagrams are expressible in terms of $\ln(\eta)$, the harmonic polylogarithms $H_{-1,0,0}(\sqrt{\eta})$,$H_{1,0,0}(\sqrt{\eta})$, alternating harmonic sums, η-dependant generalized harmonic sums and η-dependent finite binomial sums. For fixed values of the Mellin variable N, these η-dependent sums turn into rational functions in η. Thus for fixed Mellin moments, all diagrams are given in terms of the $\ln(\eta)$ and the combination $H_{-1,0,0}(\sqrt{\eta})+H_{1,0,0}(\sqrt{\eta})$ with rational coefficients in η.

The summands of many of these sums diverge for $\eta\to 1$ due to factors as $(1-\eta)^{-j}$, where j is a summation index which assumes positive integer values. Furthermore also contributions

$\propto (1-\eta)^{-N}$ emerge. Physically the limit $\eta \to 1$ represents the equal mass case $m_1 = m_2$ and thus the diagrams are expected to be divergent in this limit. Due to the many individually divergent terms this is highly non-trivial to prove for general values of N. However, evaluating a series of Mellin moments $N = 2 \dots 30$, yields convergent results for $\eta = 1$, which agree with the results given in Ref. [289] previously. This indicates that these apparent divergences are just a relic of this specific representation which has been applied. The diagrams (D_{2a}, D_{2b}), (D_{4a}, D_{4b}), (D_{5a}, D_{5b}), (D_{6a}, D_{6b}) and (D_{8a}, D_{8b}) have all been computed independently. One notes, that as expected the respective z- and Mellin-space results can be translated into each other by interchanging the masses $m_1 \leftrightarrow m_2$, $\eta \to 1/\eta$. Furthermore the results for the mass-symmetric diagrams D_1, D_3 and D_7 turn are invariant under this interchange, which constitutes further evidence on the correctness of these results.

For all scalar $A_{gg,Q}^{(3)}$-topologies series expansions up to $O(\eta^3 \ln^3(\eta))$ for a series of fixed Mellin moments ($N = 2, 4, 6$) have been computed using the code `Q2e/Exp` [185, 186]. All the general N and general-η results agree with these expansions.

7 Integration via hyperlogarithms

In this Section we explore the possibilities to apply parametric integration via hyperlogarithmic functions to massive Feynman parameter integrals with operator insertions. This method is designed for performing convergent Feynman parameter integrals in space-time dimensions $D \in \mathbb{N}$. Most Feynman integrals carry infrared, ultraviolet or collinear divergences which require regularization. In order to compute divergent diagrams in dimensional regularization [250], the ε-expansion has to be performed before the actual integration. Special methods to achieve this have been proposed, e.g. sector decomposition [290–292], which, however, is not very well suited for this application as it usually leads to a very large number of terms and is based on variable transformations which may easily result in more complicated structures of the integrands. In Refs. [293, 294] a method to extract the ε-pole terms prior to the integration via partial integrations has been presented. This method may, however, lead to huge increase in the size of the expressions for the case of many distinct singularities in the integration domain.
Here we restrict ourselves to diagrams which are convergent in $D = 4$ dimensions and do therefore not require dimensional regularization [10]. This applies in particular to very complicated topologies for which the present method turns out to be very useful [151, 296, 297]. For some diagrams the method requires a mapping of the integration variables to obtain linearity of the denominator functions in the integration variable [270, 296].

7.1 The α-parametrization.

We consider massive Feynman diagrams at $l = 3$ loops with operator insertions. The Feynman rules of Appendix B.2 are applied, a Schwinger parameter α_i is attached to each propagator and the momentum integration is performed according to [208].
The Feynman parameter integrals of a graph G are then expressed as integrals over the Schwinger parameters [298]:

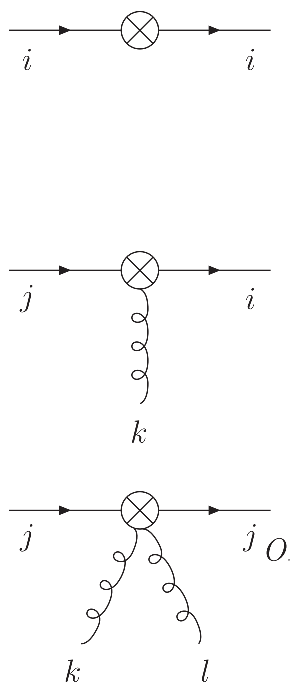
$$I_G = \frac{\Gamma\left(a - lD/2\right)}{\prod_j \Gamma\left(a_j\right)} \int_0^\infty \frac{\prod_j \alpha_j^{a_j - 1} OP_i\left(\alpha_i, N\right)}{\Psi_G^{D/2} M_G^{a - lD/2}} \delta\left(1 - \sum_{l \in v} \alpha_l\right) d\alpha_i, \tag{545}$$

where the a_i denote the powers of the different propagators, $a = \sum_{i \in edges} a_i$ and v an arbitrary subset of the edges E of G according to the Cheng-Wu-theorem [299, 300]. While M_G equals the sum of all Schwinger parameters which are attached to a massive line, Since we consider massive Feynman diagrams whose external momentum is strictly on-shell in our case the second Symanzik polynomial F just into $F = U_G M_G$. the graph polynomial Ψ_G and the operator insertion $OP_i\left(\alpha_i, N\right)$ obey nice graph theoretical descriptions.
For a graph with n_v vertices and n_e edges we define the $n_e \times n_v$ graph incidence matrix

$$(\varepsilon)_{e,v} = \begin{cases} 1, & \text{if the edge } e \text{ starts at vertex } v \\ -1, & \text{if the edge } e \text{ ends at vertex } v \\ 0, & \text{if the edge } e \text{ is not connected to vertex } v. \end{cases} \tag{546}$$

We choose ε_G as the $n_e \times (n_v - 1)$-matrix obtained from (546) by removing one arbitrary column. ε_G is thus not uniquely defined and depends on the direction of the edges and the choice of the

[10] Note that convergence is in general not a sufficient condition for not requiring any regularization [295].



$$OP_1\left(\alpha_i, N\right) = \left(\frac{\Psi^{i,L+1}_{\tilde{G}}}{\Psi_G}\right)^N$$

$$OP_2\left(\alpha_i, N\right) = \frac{1}{\left(\Psi_G\right)^N} \sum_{m=0}^{N} \left(\Psi^{j,L+1}_{\tilde{G}}\right)^m \left(\Psi^{j,L+1}_{\tilde{G}}\right)^{N-m}$$

$$OP_3\left(\alpha_i, N\right) = \frac{1}{\left(\Psi_G\right)^N} \sum_{m=0}^{N-3} \sum_{n=m+1}^{N-2} \left(\Psi^{j,L+1}_{\tilde{G}}\right)^m \left(\Psi^{i,L+1}_{\tilde{G}}\right)^{N-n-2} \times\Big[C_1\left(\Psi^{i,L+1}_{\tilde{G}} + \Psi^{l,L+1}_{\tilde{G}}\right)^{n-m-1} + C_2\left(\Psi^{i,L+1}_{\tilde{G}} + \Psi^{k,L+1}_{\tilde{G}}\right)^{n-m-1}\Big]$$

Figure 15: The different operators expressed in terms of the graph polynomial Ψ_G and different Dodgson polynomials of the graph $\tilde{G}$, where the external line of G has been closed.

removed column. The graph matrix M_G is then given by

$$M_G = \left(\begin{array}{cccc|c} \alpha_1 & & & & \\ & \ddots & & & \varepsilon_G \\ & & \ddots & & \\ & & & \alpha_{n_e} & \\ \hline & & -^T\varepsilon_G & & 0 \end{array}\right) . \tag{547}$$

The first Symanzik polynomial Ψ_G reads

$$\Psi_G = -\det(M_G) . \tag{548}$$

Although the matrix M_G is not uniquely defined Ψ_G is independent of the possible choices M_G. The first Symanzik polynomial (or first graph polynomial) is always linear in all α-parameters and of homogeneous degree d.
If I,J,K are sets of edges in the graph G with $|I| = |J|$ we define the Dodgson polynomials as

$$\Psi^{I,J}_{G,K} = \pm \det M_G\left(I, J\right)\Big|_{\alpha_e = 0 \quad \text{for all } e \in K} , \tag{549}$$

with $M_G\left(I, J\right)$ being the matrix M_G after removing all rows corresponding to the edges in I and all columns corresponding to the edges in J. If K is empty we omit it and write $\Psi^{I,J}_G$. We consider the graph $\tilde{G}$ where the external line of G has been closed. Figure 15 shows the the functions $OP_i\left(\alpha_i, N\right)$ expressed in terms of Dodgson polynomials for the different operators studied in the following. The Dodgson polynomials $\Psi^{I,J}_{G,K}$ are only defined up to a sign which generally

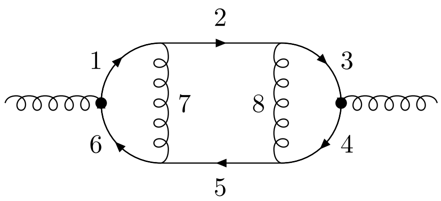


Figure 16: This ladder graph is two-edge reducible concerning the pairs of edges $1, 8$, $2, 5$ and $3, 4$

depends on the orientation of the edges in ε_G and also on the columns that were removed to define M_G. For the present paper we were able to choose $\Psi_{G,K}^{I,J} = \det M_G(I,J)\big|_{\alpha_e=0 \text{ for all } e\in K}$ when the directions of the edges were chosen according to the Feynman rules in Appendix C.
The first graph polynomial obeys the following symmetry property which we use frequently in the present computation: If a graph is two-edge-reducible, i.e. the graph can be split into two non-connected components by removing two lines, the first Symanzik polynomial is a function of the sum of the two corresponding α-parameters only [187].
Let us consider the graph in Figure 16. It is two-edge-reducible concerning the edges $1, 8$, $2, 5$ and $3, 4$ and its first Symanzik polynomial is given by

$$\begin{aligned} \Psi_G = {} & \alpha_8(\alpha_1+\alpha_6)(\alpha_2+\alpha_5) + (\alpha_1+\alpha_6)(\alpha_2+\alpha_5)(\alpha_3+\alpha_4) + (\alpha_7+\alpha_8)(\alpha_1+\alpha_6)(\alpha_3+\alpha_4) \\ & + \alpha_8\alpha_7(\alpha_1+\alpha_6) + \alpha_7(\alpha_2+\alpha_5)(\alpha_3+\alpha_4) + \alpha_8\alpha_7(\alpha_2+\alpha_5) + \alpha_8\alpha_7(\alpha_3+\alpha_4)\ . \end{aligned} \tag{550}$$

. In many cases the mass polynomial M_G shares the same symmetry. For the graph in Figure 16 we have

$$M_G = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\ . \tag{551}$$

It is thus advantageous to introduce new variables $x_1 = \alpha_1+\alpha_6$, $x_2 = \alpha_2+\alpha_5$ and $x_3 = \alpha_3+\alpha_4$ and write

$$\Psi_G = \alpha_8 x_1 x_2 + x_1 x_2 x_3 + (\alpha_7+\alpha_8)x_1 x_3 + \alpha_8\alpha_7 x_1 + \alpha_7 x_2 x_3 + \alpha_8\alpha_7 x_2 + \alpha_8\alpha_7 x_3\ , \tag{552}$$

$$M_G = x_1 + x_2 + x_3\ . \tag{553}$$

The integrals then obey the following representation

$$I_G = \frac{\Gamma(a-lD/2)}{\prod_j \Gamma(a_j)} \int_0^\infty \frac{\prod_j \alpha_j^{a_j-1} OP_i(\alpha_1,\alpha_2,\alpha_3,\alpha_7,\alpha_8,x_1,x_2,x_3,N)}{\Psi_G(x_1,x_2,x_3,\alpha_7,\alpha_8)^{D/2} M_G^{a-lD/2}} \delta\left(1-\sum_{l\in v}\alpha_l\right) d\alpha_i\ . \tag{554}$$

7.2 Linear reduction

The integration method presented in this Section only works as long as all the denominators factor into linear polynomials in the present integration variable at every integration step. In order to verify if this condition holds for a given parametric integral a linear reduction algorithm has been presented in [159, 187]. In our case we had to modify this algorithm slightly due to the Heaviside functions appearing in (554).
We start with the set of denominators of our integrand $S_0 = \{f_1, \cdots, f_k\}$. If all $s \in S_0$ are linear in x_i performing one single integration step $\int_a^b$ over the variable x_i yields the denominators

$$S_{[x]}\left\{f_1|_{x=a},\cdots,f_k|_{x=a},f_1|_{x=b},\cdots,f_k|_{x=b},\left[f_1,f_2\right]_x,\cdots,\left[f_1,f_k\right]_x,\cdots,\left[f_k,f_1\right]_x,\cdots,\left[f_k,f_{k-1}\right]\right\}\ , \tag{555}$$

where $f_i|_{x=\infty}$ is defined as the leading coefficient of f_i in x and $\left[f_1,f_j\right]_x$ denotes the resultant

$$\left[a_1+b_1x,a_2+b_2x\right]_x=b_1a_2-b_1a_1\ . \tag{556}$$

If S_0 contains one or more non-linear polynomials in x we set $S_{[x]}=\emptyset$.
Following this procedure iteratively for all integration steps one obtains a superset of the possible denominators appearing at all integration steps. The number of possible denominators is reduced by using the fact that our integrals are independent of the respective integration order. One may thus restrict the set of possible denominators at a specific integration step to the intersection of all denominator sets of all compatible integration orders up to this integration. A compatible integration order in this sense is one, where at each integration step only linear polynomials in the respective integration variable are encountered. [11]

7.3 Parametric integration

Let σ be a set of distinct points in $\mathbb{C}$ and $\vec{a}$ a word where each letter corresponds to an element in σ. The elements in σ may be constants or rational functions of further parameters. Then the hyperlogarithm [301] $L(\vec{a},z):\mathbb{C}\setminus\sigma\to\mathbb{C}$ is a function defined by

$$L(\{\underbrace{0,\cdots,0}_{\text{n times}}\},z) \;=\; =\frac{1}{n!}\ln(z)^n \tag{557}$$

and otherwise recursively by

$$L(\{a_1\},z) = \int_0^z \frac{1}{z_1-a_1}dz_1 \tag{558}$$

$$L(\{\vec{a}\},z) = L(\{a_1,\vec{a'}\},z)=\int_0^z \frac{1}{z_1-a_1}L(\{\vec{a'}\},z_1)dz_1\ . \tag{559}$$

The weight w of a hyperlogarithm is defined as the number of letters in $\vec{a}$. Being an iterated integral, the hyperlogarithm satisfies a shuffle product [166–168, 172, 174, 180]

$$L\left(\vec{a_1},z\right)L\left(\vec{a_2},z\right)=L\left(\vec{a_1}\sqcup\!\sqcup\,\vec{a_2},z\right)\ . \tag{560}$$

For example one has

$$\begin{aligned}L\left(\{a,b\},z\right)L\left(\{c,d\},z\right) &= L\left(\{a,b,c,d\},z\right)+L\left(\{a,c,b,d\},z\right)+L\left(\{a,c,d,b\},z\right)\\ &\quad +L\left(\{c,a,b,d\},z\right)+L\left(\{c,a,d,b\},z\right)\\ &\quad +L\left(\{c,d,a,b\},z\right)\ .\end{aligned} \tag{561}$$

[11] Further methods to restrict the number of spurious elements in this superset of denominators are outlined in Ref. [187].

Derivatives with respect to the argument z follow from (559)

$$\frac{d}{dz}L(\{a_1,\vec{a}\},z) \quad = \quad \frac{1}{z-a_1}L(\{\vec{a}\},z)\ . \tag{562}$$

Using these properties one may construct primitives over expressions containing rational functions and hyperlogarithms in z if all denominators factor into linear terms in z. A graph G is called linear reducible if there exists an integration order, for which this condition holds at every integration step. If and for which integration order a given graph is linear reducible can be checked a priori applying reduction algorithms presented in [159, 187]. Using the shuffle product and a partial fraction decomposition one obtains expressions of the form

$$I(b,n) \quad = \quad \int dx(x+b)^n L\left(\{a_1,\vec{a}\},x\right)\ . \tag{563}$$

If $n=-1$ the integral is identified with $L\left(\{-b,a_1,\vec{a}\},x\right)$. Otherwise we apply the integration by parts relation

$$I(b,n)=\frac{(x+b)^{n+1}}{n+1}L\left(\{a_1,\vec{a}\},x\right)-\int dx(x+b)^{n+1}\frac{1}{(n+1)(x-a_1)}L\left(\{\vec{a}\},x\right)\ , \tag{564}$$

where the last term is simpler as the weight of the hyperlogarithm is reduced by one.
In order to evaluate the primitives at the respective integration limits $x\to 0$, $x\to\infty$, we need the respective series expansions. A hyperlogarithm of weight w satisfies sum representations of the form

$$L(\{a_1,\cdots,a_n\},z) \quad = \quad \sum_{i=0}^{\infty}\sum_{j=0}^{w} c_{i,j}^{(0)}\ln^j(z)z^i\ , \tag{565}$$

$$L(\{a_1,\cdots,a_n\},z) \quad = \quad \sum_{i=0}^{\infty}\sum_{j=0}^{w} c_{i,j}^{(\infty)}\ln^j(z)z^{-i}\ . \tag{566}$$

We follow [159] and define the restricted regularization $\mathrm{RReg}_{\mathrm{z}\to\{0,\infty\}}$ as the constant parts of the generalized series expansion, which are given by

$$\mathrm{RReg}_{z\to 0}\ L(\{a_1,\cdots,a_n\},z) \quad = \quad c_{0,0}^{(0)}=0 \tag{567}$$

$$\mathrm{RReg}_{z\to\infty}\ L(\{a_1,\cdots,a_n\},z) \quad = \quad c_{0,0}^{(\infty)}\ . \tag{568}$$

The restricted regularization with respect to inner variables is defined analogously. Using the restricted regularization we define the regularized integration as

$$\int_{\mathrm{Reg}(0)}^{z} f(y)dy \quad = \quad F(z)-\mathrm{RReg}_{y\to 0}F(y)\ . \tag{569}$$

The series expansions are constructed by first differentiating w.r.t. the argument of the hyperlogarithm, performing the (simpler) series expansion of the derivative (which is of a lower weight) and then undoing the differentiation by finding a primitive and fixing the respective integration constant. With $\mathrm{Ser}_{y\to\infty}^{(k)}$ the series operator up to $O\left(y^{-k}\ln^w(y)\right)$ one obtains

$$\mathrm{Ser}_{z\to\infty}^{(k)}L\left(\{a_1,\vec{a}\},z\right) \quad = \quad \int_{\mathrm{RReg}(0)}^{z}\mathrm{Ser}_{z\to\infty}^{(k+1)}\frac{d}{dz}L\left(\{a_1,\vec{a}\},z\right)+\mathrm{RReg}_{z\to\infty}L\left(\{a_1,\vec{a}\},z\right)$$

$$= \int_{\mathrm{RReg}(0)}^{z} \mathrm{Ser}^{(k+1)}_{z\to\infty} \frac{1}{z-a_1} L\left(\{\vec{a}\}, z\right) + \mathrm{RReg}_{z\to\infty} L\left(\{a_1, \vec{a}\}, z\right) \ . \quad (570)$$

For example:

$$\mathrm{Ser}^{(4)}_{y\to\infty} \frac{d}{dz} L(\{a_1\}, z) = \frac{1}{z} - \frac{a_1}{z^2} + \frac{a_1^2}{z^3} - \frac{a_1^3}{z^4} + O\left(\frac{1}{z^5}\right) , \quad (571)$$

$$\mathrm{Ser}^{(3)}_{y\to\infty} L(\{a_1\}, z) = c^{(\infty)}_{0,0}\left(\{a_1\}\right) + L(\{0\}, z) - \frac{a_1}{z} - \frac{a_1^2}{2z^2} - \frac{a_1^3}{3z^3} + O\left(\frac{1}{z^4}\right) . \quad (572)$$

The same method is applied to construct the series representations for hyperlogarithms of higher weight. In [178, 179] closed form expressions of these expansions are given.

Rewriting the integration constants. One now rewrites the integration constants $c^{\infty}_{0,0}$. Derivatives with respect to a variable t appearing in the word $\{a_1(t), \ldots, a_n(t)\}$ of the hyperlogarithm are computed by taking the derivative under the integral

$$\frac{\partial}{\partial t} L(\{a_1(t), a_2(t), \cdots, a_n(t)\}, z) = \int\limits_{\mathrm{Reg}(0)}^{z} dz_1 \int\limits_{\mathrm{Reg}(0)}^{z_1} dz_2 \cdots \int\limits_{\mathrm{Reg}(0)}^{z_{n-1}} dz_n \prod_{i=0}^{n} \frac{\partial}{\partial t} \frac{1}{z_i - a_i(t)} \ . \quad (573)$$

Note that taking the derivative with respect to the argument or an inner variable of the hyperlogarithm always yields expressions which contain only hyperlogarithms of lower weight.
To prepare the next integration step, the constants

$$c^{(\infty)}_{0,0}\left((\{a_1, \cdots, a_n\}\right) = \mathrm{RReg}_{y\to\infty} L\left((\{a_1, \cdots, a_n\}, y\right) \quad (574)$$

have to be rewritten in terms of hyperlogarithms, such that the next integration variable does not appear in the respective alphabet. These constants are rewritten by differentiating, rewriting the (now weight reduced) expression and then undoing the differentiation again. Consider for example $c^{(\infty)}_{0,0}\left(-x, -1\right) = \mathrm{RReg}_{y\to\infty}\ L(\{-x, -1\}, y)$. From

$$\begin{aligned} \mathrm{RReg}_{y\to\infty}\ \frac{\partial}{\partial x} L\left(\{-x, -1\}, y\right) &= \mathrm{RReg}_{y\to\infty} \frac{L(\{-x\}, y)}{x-1} - \frac{(y+1)L(\{-1\}, y)}{(x-1)(x+y)} \\ &= -\frac{L(\{0\}, x)}{x-1} \end{aligned} \quad (575)$$

follows, that

$$\begin{aligned} c^{(\infty)}_{0,0}\left(-x, -1\right) &= \int_0^x \mathrm{RReg}_{y\to\infty}\ \frac{\partial}{\partial x'} L\left(\{-x', -1\}, y\right) + \mathrm{RReg}_{x\to 0} \mathrm{RReg}_{y\to\infty}\ L\left(\{-x, -1\}, y\right) \\ &= \int_0^x dx\ -\frac{L(\{0\}, x')}{x'-1} + \mathrm{RReg}_{y\to\infty}\ L\left(\{0, -1\}, y\right) \quad (576) \\ &= -L(\{1, 0\}, x) + \zeta_2 \ . \quad (577) \end{aligned}$$

Special care has to be taken when evaluating constants $c^{(\infty)}_{0,0}\left(a_1, \cdots, a_n\right)$ which contain letters of the form $x^{-i} f(x)$ with $f(x) \neq 0$ as $x \to 0$ or trailing letters of the form $x^i f(x)$ with $f(x)$

convergent as $x \to 0$. In all other cases $\mathrm{RReg}_{x\to 0} L(a_1, \cdots, a_n, y)$ is just obtained by taking the limit $x \to 0$ under the integral. In the first case the limit $x \to 0$ does not commute with the limit $y \to \infty$. If a hyperlogarithm does not have any trailing zero in its word we may substitute the integration variables $z_i \to a z_i$ in (559) to obtain

$$L(\{a_1, \cdots, a_n\}, z) \;=\; L(\{aa_1, \cdots, aa_n\}, az)\ . \tag{578}$$

In other cases trailing zeros have to be removed by means of the shuffle algebra first, e.g.

$$\begin{aligned} L(\{a_1,0,0\},z) &= L(\{a_1\},z)\,L(\{0,0\},z) - L(\{0\},z)\,L(\{0,a_1\},z) \\ &= L(\{0,0\},a)\,L(\{aa_1\},az) - L(\{0\},a)\,L(\{aa_1,0\},az) \\ &\quad + L(\{aa_1,0,0\},az)\ , \end{aligned} \tag{579}$$

after using (578), (558) and (560). Applying (578), resp. (579), we obtain

$$\begin{aligned} c^{(\infty)}_{0,0}\left(\{x^{-i}f_1(x), \cdots, f_n(x)\}\right) &= \mathrm{RReg}_{y\to\infty} L\left(\{x^{-i}f_1(x), \cdots, f_n(x)\}, y\right) \\ &= \mathrm{RReg}_{y\to\infty} L\left(\{f_1(x), \cdots, x^i f_n(x)\}, yx^i\right) \\ &= \mathrm{RReg}_{y\to\infty}\left[\mathrm{Ser}^{(0)}_{z\to\infty} L\left(\{f_1(x), \cdots, x^i f_n(x)\}, z\right)\right]\Big|_{z=yx^i} \\ &= \left[\mathrm{Ser}^{(0)}_{z\to\infty} L\left(\{f_1(x), \cdots, x^i f_n(x)\}, z\right)\right]\Big|_{z=x^i}\ . \end{aligned} \tag{580}$$

By definition $\mathrm{Ser}^{(0)}_{z\to\infty} L(\{f_1(x), \cdots, x^i f_n(x)\}, z)$ does only depend logarithmically on the variable $z = yx^i$ and the operation $\mathrm{RReg}_{y\to\infty}$ in the second last step is easily performed.
In the case of trailing letters of the shape $x^i f(x)$ with $f(x)$ convergent as $x \to 0$, the limit $x \to 0$ does not commute with the implicit limits contained in the definition of the hyperlogarithm. Here we apply the identity

$$\begin{aligned} \mathrm{RReg}_{x\to 0} L\left(\{x^{i_1}f_1(x), \cdots, x^{i_n}f_n(x)\}, y\right) &= \mathrm{RReg}_{x\to 0} L\left(\{x^{i_1-1}f_1(x), \cdots, x^{i_n-1}f_n(x)\}, \frac{y}{x}\right) \\ &= \mathrm{Ser}^{(0)}_{y\to\infty}\mathrm{RReg}_{x\to 0} L\left(\{x^{i_1-1}f_1(x), \cdots, x^{i_n-1}f_n(x)\}, y\right) \end{aligned} \tag{581}$$

on the parts containing the respective letters, if necessary repeatedly. The identity (581) is derived by considering the change of integration variables $z_i \to \frac{z_i'}{x}$ in (559).
As an example we consider

$$\begin{aligned} \mathrm{RReg}_{x\to 0} L\left(\{-2, -\frac{x}{2}\}, y\right) &= \mathrm{RReg}_{x\to 0}\int_0^y \frac{dz_1}{z_1+2}\mathrm{RReg}_{x\to 0} L\left(\{-\frac{1}{2}\}, \frac{z_1}{x}\right) \\ &= \int_0^y \frac{dz_1}{z_1+2}\left[\mathrm{Ser}^{(0)}_{z_1\to\infty} L\left(\{-\frac{1}{2}\}, z_1\right)\right] \\ &= \int_0^y \frac{dz_1}{z_1+2}\left[\ln 2 + L(\{0\}, z_1)\right] \\ &= L(\{-2\}, y)\ln 2 + L(\{-2, 0\}, y)\ . \end{aligned} \tag{582}$$

We repeat the previous steps for all further integration variables until we have rewritten all constants in a way suitable for the following integration step. These can then be performed analogously.

7.4 Treatment of local operator insertions

In all the different operators we encounter polynomials with symbolic integer exponents, which cannot be treated directly with the method of parametric integration. We thus introduce a generating function [151], by

$$OP_i\left(\alpha_i, t\right) = \sum_{n=0}^{\infty} t^n OP_i\left(\alpha_i, n\right) \ . \tag{583}$$

The function $OP_i\left(\alpha_i, N\right)$ can be re-gained by taking the Nth coefficient of the generating function. We obtain the following generating functions for the three different operators in Figure 15:

$$\begin{aligned}
OP_1\left(\alpha_i, N\right) &= \frac{\Psi_G}{\Psi_G - t\Psi_{\tilde{G}}^{i,L+1}} && (584)\\
OP_2\left(\alpha_i, N\right) &= \sum_{N=0}^{\infty} t^N \frac{1}{\Psi_G^N} \sum_{m=0}^{N} \left(\Psi_{\tilde{G}}^{i,L+1}\right)^{N-m} \left(\Psi_{\tilde{G}}^{j,L+1}\right)^{m} \\
&= \sum_{N=0}^{\infty} t^N \frac{\left(\Psi_{\tilde{G}}^{i,L+1}\right)^{N} - \left(\Psi_{\tilde{G}}^{j,L+1}\right)^{N}}{\Psi_{\tilde{G}}^{i,L+1} - \Psi_{\tilde{G}}^{j,L+1}} && (585)\\
&= \frac{\Psi_G^2}{\left(\Psi_G - t\Psi_{\tilde{G}}^{i,L+1}\right)\left(\Psi_G - t\Psi_{\tilde{G}}^{j,L+1}\right)} \\
OP_3\left(\alpha_i, N\right) &= \sum_{N=0}^{\infty} t^N \frac{1}{\left(\Psi_G\right)^N} \sum_{m=0}^{N-3} \sum_{n=m+1}^{N-2} \left(\Psi_{\tilde{G}}^{j,L+1}\right)^{m} \left(\Psi_{\tilde{G}}^{i,L+1}\right)^{N-n-2} \\
&\quad \times \left[C_1 \left(\Psi_{\tilde{G}}^{i,L+1} + \Psi_{\tilde{G}}^{l,L+1}\right)^{n-m-1} + C_2 \left(\Psi_{\tilde{G}}^{i,L+1} + \Psi_{\tilde{G}}^{k,L+1}\right)^{n-m-1}\right] \\
&= \frac{\Psi_G^3}{\left(\Psi_G - t\Psi_{\tilde{G}}^{i,L+1}\right)\left(\Psi_G - t\Psi_{\tilde{G}}^{j,L+1}\right)} \\
&\quad \times \left[C_1 \frac{1}{\Psi_G - t\left(\Psi_{\tilde{G}}^{i,L+1} + \Psi_{\tilde{G}}^{l,L+1}\right)} + C_2 \frac{1}{\Psi_G - t\left(\Psi_{\tilde{G}}^{i,L+1} + \Psi_{\tilde{G}}^{k,L+1}\right)}\right] \ . && (586)
\end{aligned}$$

For fixed values of N all massive 3-loop QCD two-point functions with operators insertions are linear reducible. If we introduce one of the generating functions (584-586) this changes drastically. Some diagrams remain linear reducible, others could be transformed into linear reducible diagrams via a variable transformation and in other cases we could not find a way to restore linear reducibility.
After evaluating all α-parameter integrals we obtain the generating expression

$$\begin{aligned}
\hat{I}_1(x) &= \frac{1}{(1+N)(2+N)x} \Big\{ \Big[2\mathrm{L}_{-1}(x) - 2(-1+2x)\mathrm{L}_1(x) - 4\mathrm{L}_{1,1}(x)\Big]\zeta_3 \\
&\quad -3\mathrm{L}_{-1,0,0,1}(x) + 2\mathrm{L}_{-1,0,1,1}(x) - 2x\mathrm{L}_{0,0,1,1}(x) + 3x\mathrm{L}_{0,1,0,1}(x)
\end{aligned}$$

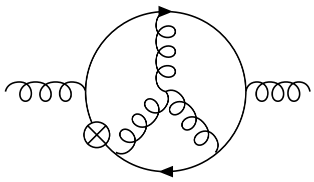


Figure 17: The 3-loop Benz diagram for $I_1(N)$, Eq. (588).

$$
\begin{aligned}
&-x\mathrm{L}_{0,1,1,1}(x) + (-3+2x)\mathrm{L}_{1,0,0,1}(x) + 2x\mathrm{L}_{1,0,1,1}(x) - \mathrm{L}_{1,0,1,1,1}(x) \\
&-(5x-1)\mathrm{L}_{1,1,0,1}(x) + x\mathrm{L}_{1,1,1,1}(x) - 2\mathrm{L}_{1,0,0,1,1}(x) + 3\mathrm{L}_{1,0,1,0,1}(x) \\
&\left.+2\mathrm{L}_{1,1,0,0,1}(x) + 2\mathrm{L}_{1,1,0,1,1}(x) - 5\mathrm{L}_{1,1,1,0,1}(x) + \mathrm{L}_{1,1,1,1,1}(x)\right\} \qquad (587)
\end{aligned}
$$

for the Benz–diagram in Figure 17. One applies the package `HarmonicSums` [175, 177–179] to obtain the Nth coefficient. It is given by

$$
\begin{aligned}
I_1(N) &= \frac{1}{(N+1)(N+2)(N+3)}\Biggl\{\frac{P_1}{(1+N)^3(2+N)^3(3+N)^3} \\
&-\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)}S_1^3 + \frac{1}{24}S_1^4 - \frac{1}{4}S_4 \\
&-\frac{(7+22N+10N^2)}{2(1+N)^2(2+N)}S_2 - \frac{19}{8}S_2^2 - \frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2 + \frac{9}{4}S_2S_1^2 - \frac{(-9+4N)}{3(1+N)}S_3 \\
&-2(-1)^N S_{-2,1} + \frac{(-1+6N)}{(1+N)}S_{2,1} + \frac{P_2}{(1+N)^3(2+N)^2(3+N)^2}S_1 \\
&+4\zeta_3S_1 - \frac{(-2+7N)}{2(1+N)}S_2S_1 + \frac{13}{3}S_3S_1 - 7S_{2,1}S_1 - 7S_{3,1} + 10S_{2,1,1}\Biggr\}, \qquad (588)
\end{aligned}
$$

with the polynomials

$$
\begin{aligned}
P_1 &= 648 + 1512N + 1458N^2 + 744N^3 + 212N^4 + 32N^5 + 2N^6 \qquad (589)\\
P_2 &= 54 + 207N + 246N^2 + 130N^3 + 32N^4 + 3N^5\ . \qquad (590)
\end{aligned}
$$

The corresponding x-space representations needed in the analysis of experimental data is obtained by the inverse Mellin transform for the corresponding structure functions represented in N-space, including their QCD-evolution in analytic form. The transform is obtained by a single numerical integral around all the singularities of the complex N-space representation [302–305]. The latter is obtained for large values of N by the asymptotic representation. For all other values, outside the singularities, one uses the algebraic shift relations of the corresponding sums from $N+1 \to N$. The latter relations are given in Appendix D. In the following we list also the asymptotic expansions for the individual diagrams. It will turn out later that their derivation needs the application of more and more sophisticated techniques corresponding to the growing complexity of the respective sums. Using the corresponding algorithms in `HarmonicSums` [175, 177–179] one obtains following asymptotic representation for $|N| \to \infty$

$$
I_1^{\rm asy}(N) \simeq \left(\frac{1}{24N^3} - \frac{1}{4N^4} + \frac{25}{24N^5} - \frac{15}{4N^6} + \frac{301}{24N^7} - \frac{161}{4N^8} + \frac{3025}{24N^9} - \frac{1555}{4N^{10}}\right)\ln^4(\bar{N})
$$

$$
\begin{aligned}
&+\Bigl(-\frac{1}{6N^3}+\frac{5}{4N^4}-\frac{421}{72N^5}+\frac{45}{2N^6}-\frac{18803}{240N^7}+\frac{10313}{40N^8}-\frac{2480627}{3024N^9}\\
&+\frac{1288247}{504N^{10}}\Bigr)\ln^3(\bar{N})+\Bigl(-\frac{3}{2N^4}+\frac{551}{48N^5}-\frac{2699}{48N^6}+\frac{652013}{2880N^7}-\frac{98339}{120N^8}+\frac{2805553}{1008N^9}\\
&-\frac{290543}{32N^{10}}\Bigr)\ln^2(\bar{N})+\Bigl(-\frac{11}{2N^4}+\frac{947}{24N^5}-\frac{8887}{48N^6}+\frac{103891}{144N^7}-\frac{36580757}{14400N^8}\\
&+\frac{2181959741}{259200N^9}-\frac{11373443593}{423360N^{10}}\Bigr)\ln(\bar{N})-\frac{16}{N^4}+\frac{2713}{24N^5}-\frac{14114}{27N^6}+\frac{773389}{384N^7}\\
&-\frac{152225303}{21600N^8}+\frac{12096164219}{518400N^9}-\frac{4428508717429}{59270400N^{10}}\\
&+\zeta_2\Biggl[\left(\frac{9}{4N^3}-\frac{27}{2N^4}+\frac{225}{4N^5}-\frac{405}{2N^6}+\frac{2709}{4N^7}-\frac{4347}{2N^8}+\frac{27225}{4N^9}-\frac{41985}{2N^{10}}\right)\ln^2(\bar{N})\\
&+\left(-\frac{7}{2N^3}+\frac{111}{4N^4}-\frac{1063}{8N^5}+\frac{1035}{2N^6}-\frac{145147}{80N^7}+\frac{239811}{40N^8}-\frac{2141827}{112N^9}\right.\\
&\left.+\frac{3342261}{56N^{10}}\right)\ln(\bar{N})-\frac{7}{N^4}+\frac{2603}{48N^5}-\frac{12755}{48N^6}+\frac{340949}{320N^7}-\frac{92045}{24N^8}+\frac{9325513}{720N^9}\Biggr]\\
&-\frac{28247675}{672N^{10}}+\zeta_3\Biggl[\left(-\frac{17}{3N^3}+\frac{34}{N^4}-\frac{425}{3N^5}+\frac{510}{N^6}-\frac{5117}{3N^7}+\frac{5474}{N^8}-\frac{51425}{3N^9}\right.\\
&\left.+\frac{52870}{N^{10}}\right)\ln(\bar{N})+\frac{26}{3N^3}-\frac{121}{2N^4}+\frac{9857}{36N^5}-\frac{1035}{N^6}+\frac{428011}{120N^7}-\frac{233281}{20N^8}\\
&+\frac{55892059}{1512N^9}-\frac{28953679}{252N^{10}}\Biggr]+\zeta_2^2\Biggl[\frac{241}{40N^3}-\frac{723}{20N^4}+\frac{1205}{8N^5}-\frac{2169}{4N^6}+\frac{72541}{40N^7}\\
&-\frac{116403}{20N^8}+\frac{145805}{8N^9}-\frac{224853}{4N^{10}}\Biggr]+O\left(\frac{\ln^4(\bar{N})}{N^{11}}\right)\,, \qquad (591)
\end{aligned}
$$

where $\overline{N} = N\exp(\gamma_E)$, with γ_E the Euler-Mascheroni constant.

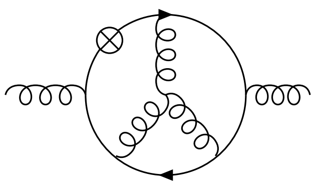


Figure 18: The 3-loop Benz diagram for $I_2(N)$, Eq. (592).

The same diagram with the with the operator on a different quark line cannot be represented in terms of harmonic sums, but requires also generalized harmonic sums, see Appendix D.3

$$
\begin{aligned}
I_2(N) \;=\; & \frac{1}{(N+1)(N+2)(N+3)}\Biggl\{\frac{2\,(N+3)}{(N+1)^3(N+2)}-\frac{4\left(-4-3N+2^{2+N}(N+1)\right)}{N+1}\zeta_3\\
&+\frac{1}{2(N+1)(N+2)}S_1^2-\frac{1}{2}S_1^3+\frac{(-1+9N+4N^2)}{2(N+1)^2(N+2)}S_2-\frac{5\,(N+2)}{2}S_2^2
\end{aligned}
$$

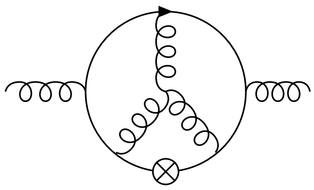


Figure 19: The 3-loop Benz diagram for $I_3(N)$, Eq. (594).

$$-3S_3 - \frac{3(N+2)}{2}S_4 - \frac{(5+3N)}{N+1}S_{2,1} - \frac{N^2-3}{(N+1)^3(N+2)}S_1 + 4(N+2)S_1\zeta_3$$
$$-\frac{7}{2}S_1S_2 - 2(N+2)S_1S_{2,1} + 2(N+2)S_{3,1} + 2^{4+N}S_{1,2}\left(\frac{1}{2},1\right)$$
$$+4(N+2)S_{2,1,1} + 2^{3+N}S_{1,1,1}\left(\frac{1}{2},1,1\right)\Bigg\}\,. \tag{592}$$

Additionally terms of $O(2^N)$ are observed, which cancel in the asymptotic expansion

$$\begin{aligned}
I_2^{\rm asy}(N) \;=\;& \left(-\frac{1}{2N^3}+\frac{3}{N^4}-\frac{25}{2N^5}+\frac{45}{N^6}-\frac{301}{2N^7}+\frac{483}{N^8}-\frac{3025}{2N^9}+\frac{4665}{N^{10}}\right)\ln^3(\bar{N})\\
&+\left(-\frac{19}{4N^4}+\frac{297}{8N^5}-\frac{196}{N^6}+\frac{72289}{80N^7}-\frac{163837}{40N^8}+\frac{6772187}{336N^9}-\frac{6652459}{56N^{10}}\right)\ln^2(\bar{N})\\
&+\Bigg(-\frac{2}{N^3}+\frac{14}{N^4}-\frac{6089}{72N^5}+\frac{33071}{72N^6}-\frac{17131999}{7200N^7}+\frac{22857919}{1800N^8}-\frac{1113784177}{14700N^9}\\
&+\frac{19063098643}{35280N^{10}}\Bigg)\ln(\bar{N})-\frac{4}{N^3}+\frac{35}{2N^4}-\frac{4181}{108N^5}-\frac{24331}{432N^6}+\frac{16232209}{12000N^7}\\
&-\frac{863086111}{72000N^8}+\frac{1575813188009}{16464000N^9}-\frac{483184825009}{592704N^{10}}\\
&+\Bigg[\left(-\frac{7}{2N^3}+\frac{21}{N^4}-\frac{175}{2N^5}+\frac{315}{N^6}-\frac{2107}{2N^7}+\frac{3381}{N^8}-\frac{21175}{2N^9}+\frac{32655}{N^{10}}\right)\ln(\bar{N})\\
&+\frac{3}{N^3}-\frac{133}{4N^4}+\frac{4819}{24N^5}-\frac{1945}{2N^6}+\frac{347613}{80N^7}-\frac{783477}{40N^8}+\frac{490035913}{5040N^9}-\frac{97672721}{168N^{10}}\Bigg]\zeta_2\\
&+\left(\frac{3}{N^3}-\frac{18}{N^4}+\frac{75}{N^5}-\frac{270}{N^6}+\frac{903}{N^7}-\frac{2898}{N^8}+\frac{9075}{N^9}-\frac{27990}{N^{10}}\right)\zeta_3+\Bigg(\frac{27}{10N^2}\\
&-\frac{54}{5N^3}+\frac{351}{10N^4}-\frac{108}{N^5}+\frac{3267}{10N^6}-\frac{4914}{5N^7}+\frac{29511}{10N^8}-\frac{8856}{N^9}+\frac{265707}{10N^{10}}\Bigg)\zeta_2^2\\
&+O\left(\frac{\ln^3(\bar{N})}{N^{11}}\right)\,,
\end{aligned}\tag{593}$$

however, and we obtain a regular representation for $|N| \to \infty$. Considering the same diagram with the operator on the lower fermionic line demonstrates how strongly the complexity of the result depends on the location of the local operator insertion. The diagram in Figure 19 yields

$$I_3(N) = \frac{1}{(N+1)(N+2)^2}\left\{\frac{4}{(N+1)^2(N+2)} - \frac{4S_1}{(N+2)} + 4S_2\right\}, \tag{594}$$

with the asymptotic representation

$$\begin{aligned} I_3^{\rm asy}(N) &= \left(-\frac{4}{N^4}+\frac{28}{N^5}-\frac{124}{N^6}+\frac{444}{N^7}-\frac{1404}{N^8}+\frac{4092}{N^9}-\frac{11260}{N^{10}}\right)\ln(\bar{N}) \\ &\quad -\frac{4}{N^4}+\frac{20}{N^5}-\frac{181}{3N^6}+\frac{133}{N^7}-\frac{2009}{10N^8}+\frac{1297}{30N^9}+\frac{728377}{630N^{10}} \\ &\quad +\left(\frac{4}{N^3}-\frac{20}{N^4}+\frac{68}{N^5}-\frac{196}{N^6}+\frac{516}{N^7}-\frac{1284}{N^8}+\frac{3076}{N^9}-\frac{7172}{N^{10}}\right)\zeta_2 + O\left(\frac{\ln(\bar{N})}{N^{11}}\right) . \end{aligned} \tag{595}$$

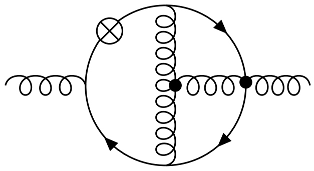


Figure 20: The 3-loop Benz diagram for $I_4(N)$, Eq. (596).

Also diagrams which do not contribute in the direct QCD computation but which are typical for master integrals after an integration by parts reduction [156,157,306–309] like the diagrams in Figures 20 and 21 are computable with the method of hyperlogarithms.

Diagram 4 can be expressed completely within the functional class of generalized harmonic sums

$$\begin{aligned} I_4(N) &= \frac{1}{(N+1)(N+2)}\left\{\frac{P_3}{(N+1)(N+2)}\zeta_3\right. \\ &\quad +\frac{1}{N+2}S_{-3}+\frac{(-1)^N}{2(N+2)}S_1^3-\frac{(-1)^N(3+2N)}{2(N+1)^2(N+2)}S_2+\frac{5(-1)^N}{2}S_2^2 \\ &\quad +\frac{(-1)^N(3+2N)}{2(N+1)^2(N+2)}S_1^2-\frac{(-1)^N}{2}S_2S_1^2+\frac{3(-1)^N(4+3N)}{(N+1)(N+2)}S_3+3(-1)^NS_4 \\ &\quad +\frac{2}{(N+2)}S_{-2,1}+2(-1)^N\zeta_3S_1(2)+\frac{2(-1)^N(3+N)}{(N+1)(N+2)}S_{2,1}-12(-1)^NS_1\zeta_3 \\ &\quad +\frac{(-1)^N(5+7N)}{2(N+1)(N+2)}S_1S_2+3(-1)^NS_1S_3+4(-1)^NS_{2,1}S_1-4(-1)^NS_{3,1} \\ &\quad -\frac{4\left((-1)^N2^{2+N}-3(-2)^NN+3(-1)^N2^{1+N}N\right)}{(N+1)(N+2)}S_{1,2}\left(\frac{1}{2},1\right) \\ &\quad +\frac{2\left(-(-1)^N2^{2+N}-13(-2)^NN+5(-1)^N2^{1+N}N\right)}{(N+1)(N+2)}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\ &\quad \left.-2(-1)^NS_{1,1,2}\left(2,\frac{1}{2},1\right)-(-1)^NS_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)-5(-1)^NS_{2,1,1}\right\}, \end{aligned} \tag{596}$$

$$P_3 = 2\left(1-13(-1)^N+(-1)^N 2^{3+N}+N-7(-1)^N N+3(-1)^N 2^{1+N}N\right). \quad (597)$$

For $|N| \to \infty$ I_4 obeys the expansion

$$\begin{aligned}
I_4^{\rm asy}(N) &= (-1)^N\Biggl\{\Biggl[-\frac{1793}{2N^{10}}+\frac{769}{2N^9}-\frac{321}{2N^8}+\frac{129}{2N^7}-\frac{49}{2N^6}+\frac{17}{2N^5}-\frac{5}{2N^4}+\frac{1}{2N^3}\Biggr]\ln^3(\bar{N})\\
&+\Biggl[-\frac{3}{2N^3}+\frac{21}{2N^4}-\frac{363}{8N^5}+\frac{1323}{8N^6}-\frac{9389}{16N^7}+\frac{183573}{80N^8}-\frac{538097}{48N^9}+\frac{123450851}{1680N^{10}}\Biggr]\\
&\times\ln^2(\bar{N})+\Biggl[-\frac{7}{N^4}+\frac{429}{8N^5}-\frac{6763}{24N^6}+\frac{662993}{480N^7}-\frac{3542309}{480N^8}+\frac{79274089}{1680N^9}\\
&-\frac{89308307}{240N^{10}}\Biggr]\ln(\bar{N})+\Biggl[\Biggl[-\frac{1}{2N^2}+\frac{3}{2N^3}-\frac{7}{2N^4}+\frac{15}{2N^5}-\frac{31}{2N^6}+\frac{63}{2N^7}-\frac{127}{2N^8}\\
&+\frac{255}{2N^9}-\frac{511}{2N^{10}}\Biggr]\ln^2(\bar{N})+\Biggl[\frac{3}{N^3}-\frac{203}{12N^4}+\frac{247}{4N^5}-\frac{7457}{40N^6}+\frac{20271}{40N^7}-\frac{3251987}{2520N^8}\\
&+\frac{528337}{168N^9}-\frac{5348629}{720N^{10}}\Biggr]\ln(\bar{N})+\Biggl[-\frac{5}{N^3}+\frac{285}{8N^4}-\frac{3887}{24N^5}+\frac{181091}{288N^6}-\frac{1151603}{480N^7}\\
&+\frac{7293811}{720N^8}-\frac{14793223}{280N^9}+\frac{217689527539}{604800N^{10}}\Biggr]\Biggr]\zeta_2+\Biggl[\Biggl[-\frac{1}{N^2}+\frac{3}{N^3}-\frac{7}{N^4}+\frac{15}{N^5}-\frac{31}{N^6}\\
&+\frac{63}{N^7}-\frac{127}{N^8}+\frac{255}{N^9}-\frac{511}{N^{10}}\Biggr]\ln(\bar{N})+\Biggl[-\frac{3}{2N^3}+\frac{67}{12N^4}-\frac{59}{4N^5}+\frac{1363}{40N^6}-\frac{2949}{40N^7}\\
&+\frac{388153}{2520N^8}-\frac{53027}{168N^9}+\frac{460691}{720N^{10}}\Biggr]\Biggr]\zeta_3+\Biggl[-\frac{12}{5N^2}+\frac{36}{5N^3}-\frac{84}{5N^4}+\frac{36}{N^5}-\frac{372}{5N^6}+\frac{756}{5N^7}\\
&-\frac{1524}{5N^8}+\frac{612}{N^9}-\frac{6132}{5N^{10}}\Biggr]\zeta_2^2+\Biggl[\frac{4}{N^3}-\frac{49}{4N^4}+\frac{181}{216N^5}+\frac{27119}{144N^6}-\frac{40222139}{27000N^7}\\
&+\frac{1251907}{125N^8}-\frac{10792338497459}{148176000N^9}+\frac{18342053050631}{29635200N^{10}}\Biggr]\Biggr\}+O\left(\frac{\ln^3(\bar{N})}{N^{11}}\right). \quad (598)
\end{aligned}$$

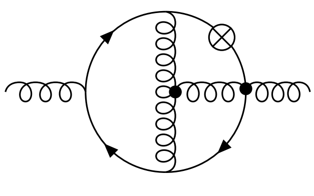


Figure 21: The 3-loop Benz diagram for $I_5(N)$, Eq. (599).

Although diagrams 4 and 5 are topologically very similar one obtains a more simple sum structure in the latter case.

$$I_5(N) = \frac{(-1)^N}{(N+1)(N+2)}\Biggl\{-\frac{2\left(2+(-1)^N(2+N)\right)}{(1+N)}\zeta_3+\frac{3}{(N+1)^2}S_2+\frac{5}{2}S_2^2+\frac{3}{2}S_4$$

$$
+\frac{2}{(N+1)}S_{2,1}-\frac{2}{(N+1)^3}S_1-4\zeta_3S_1+2S_{2,1}S_1-2S_{3,1}-4S_{2,1,1}\Biggr\}\,. \tag{599}
$$

The asymptotic representation for this diagram reads

$$
\begin{aligned}
I_5^{\rm asy}(N) &= (-1)^N\Biggl\{\Biggl(\frac{2}{N^3}-\frac{15}{2N^4}+\frac{166}{9N^5}-\frac{445}{12N^6}+\frac{59153}{900N^7}-\frac{7987}{75N^8}+\frac{1185269}{7350N^9}\\
&-\frac{227247}{980N^{10}}\Biggr)\ln(\bar{N})+\Biggl(-\frac{3}{N^3}+\frac{27}{2N^4}-\frac{41}{N^5}+\frac{105}{N^6}-\frac{2449}{10N^7}+\frac{5397}{10N^8}-\frac{40158}{35N^9}\\
&+\frac{16686}{7N^{10}}\Biggr)\zeta_2+\Biggl(\frac{4}{N^3}-\frac{25}{2N^4}+\frac{2885}{108N^5}-\frac{883}{18N^6}+\frac{381781}{4500N^7}-\frac{1312181}{9000N^8}\\
&+\frac{4756944037}{18522000N^9}-\frac{386004953}{823200N^{10}}\Biggr)+\Biggl(-\frac{27}{10N^2}+\frac{81}{10N^3}-\frac{189}{10N^4}+\frac{81}{2N^5}-\frac{837}{10N^6}\\
&+\frac{1701}{10N^7}-\frac{3429}{10N^8}+\frac{1377}{2N^9}-\frac{13797}{10N^{10}}\Biggr)\zeta_2^2\Biggr\}+\Biggl(-\frac{2}{N^2}+\frac{4}{N^3}-\frac{6}{N^4}+\frac{8}{N^5}-\frac{10}{N^6}\\
&+\frac{12}{N^7}-\frac{14}{N^8}+\frac{16}{N^9}-\frac{18}{N^{10}}\Biggr)\zeta_3+O\Biggl(\frac{\ln(\bar{N})}{N^{11}}\Biggr)\,.
\end{aligned} \tag{600}
$$

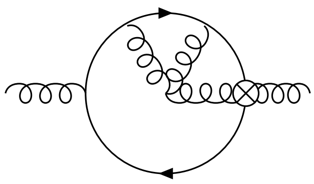


Figure 22: The 3-loop Benz diagram for $I_6(N)$, Eq. (601).

As a final Benz–diagram we consider the graph in Figure 22, which is consists of two color contributions due to the Feynman rule for the operator on the 4–vertex, see Appendix D. The complete expression is then given by

$$
\begin{aligned}
I_6(N) &= C_1\Biggl\{\frac{P_4}{(N+1)^5(N+2)^5(N+3)}-(-1)^N\frac{P_5}{(N+1)^5(N+2)^5(N+3)}+10S_{-5}\\
&+\frac{P_6}{2(N+1)^3(N+2)^3(N+3)^2}S_1^2+\frac{P_7}{2(N+1)^2(N+2)^2(N+3)^2}S_1^3+\frac{4}{N+3}S_1S_{-3}\\
&-\frac{P_8}{(N+1)^2(N+2)^2(N+3)^2}S_{-3}+3S_2S_{-3}+\frac{5}{N+3}S_4-S_5-2S_{-4,1}\\
&+\Biggl[\frac{3(-1)^NP_9}{(N+1)^2(N+2)^2(N+3)^2}+\frac{P_{10}}{(N+1)^2(N+2)^2(N+3)^2}\Biggr]S_3-\frac{2}{(N+3)^2}S_{-2,1}\\
&-8S_{-2,3}+\Biggl[(-1)^N\frac{2P_9}{(N+1)^2(N+2)^2(N+3)^2}-4S_{-2}-4S_2-\frac{2(N+2)}{N+3}S_1\,(2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2P_{11}}{(N+1)^2(N+2)^2(N+3)^2}+2^{N+2}\frac{P_{12}}{(N+1)^2(N+2)^2(N+3)^2}\Bigg]\zeta_3-5S_{2,-3}\\
&+\left[-\frac{17+23N+9N^2+N^3}{(N+1)(N+2)(N+3)^2}-(-1)^N\frac{58+84N+43N^2+10N^3+N^4}{(N+1)^2(N+2)^2(N+3)^2}\right]S_{2,1}\\
&+\frac{2\,(17+27N+15N^2+3N^3)}{(N+1)^3(N+2)^3(N+3)}S_{-2}-\frac{2}{N+3}S_{-2}S_2+2S_3S_{-2}+2S_{2,1}S_{-2}\\
&-(-1)^N\frac{P_{13}}{(N+1)^3(N+2)^3(N+3)^2}S_2+\frac{P_{14}}{2(N+1)^3(N+2)^3(N+3)^2}S_2\\
&-S_3S_2-2S_{-2,1}S_2+2S_{2,1}S_2+(-1)^N\frac{2\,(7+6N+N^2)\,(9+10N+3N^2)}{(N+1)^4(N+2)^4(N+3)^2}S_1\\
&+\left[\frac{P_{15}}{(N+1)^4(N+2)^4(N+3)^2}+(-1)^N\frac{P_9}{(N+1)^2(N+2)^2(N+3)^2}S_2\right]S_1-2S_{2,3}\\
&+\frac{7\,(61+136N+123N^2+55N^3+12N^4+N^5)}{2(N+1)^2(N+2)^2(N+3)^2}S_2S_1+\frac{5}{N+3}S_1S_3\\
&-\frac{1}{N+3}S_1S_{2,1}-\frac{(9+N)}{N+3}S_{3,1}+S_{4,1}-\frac{2^{2+N}\,(4+7N+N^2)}{(N+1)(N+3)^2}S_{1,2}\left(\frac{1}{2},1\right)-2S_{2,1,-2}\\
&-2S_{2,2,1}-\frac{2(2+N)}{3+N}S_{2,1,1}-2S_{3,1,1}-\frac{2^{1+N}\,(4+7N+N^2)}{(N+1)(N+3)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right)\\
&+\frac{2(N+2)}{N+3}S_{1,1,2}\left(2,\frac{1}{2},1\right)+3S_{2,1,1,1}+\frac{(N+2)}{N+3}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)\Bigg\}\\
&+C_2\Bigg\{-(-1)^N\frac{P_{16}}{(1+N)^5(2+N)^5(3+N)}+\frac{P_{17}}{(1+N)^5(2+N)^5(3+N)}-10S_{-5}\\
&+\frac{38+45N+16N^2+N^3}{(1+N)^2(2+N)^2(3+N)}S_{-3}-\frac{4S_1}{3+N}S_{-3}-3S_2S_{-3}+S_5+2S_{-4,1}+8S_{-2,3}\\
&+\left[-\frac{1}{2(3+N)}+\frac{(-1)^N}{(2+N)(3+N)}\right]S_2^2+S_3S_2+2S_{-2,1}S_2+5S_{2,-3}\\
&+2\Bigg[\frac{11+15N+7N^2+N^3}{(1+N)^2(2+N)^2(3+N)}-(-1)^N\frac{23+28N+10N^2+N^3}{(1+N)^2(2+N)^2(3+N)}+2S_{-2}\\
&+\left(-\frac{1}{3+N}-(-1)^N\frac{1}{(2+N)(3+N)}\right)S_1+S_2\Bigg]\zeta_3-\frac{(-1)^N}{2(2+N)(3+N)}S_1^2S_2\\
&-\frac{2(-1)^N\,(5+6N+2N^2)}{(1+N)^2(2+N)^3(3+N)}S_1^2+\frac{2}{(2+N)(3+N)}S_{-2,1}\\
&+\left(\frac{9+10N+3N^2}{(1+N)^2(2+N)^2(3+N)}-\frac{3(-1)^N\,(23+28N+10N^2+N^3)}{(1+N)^2(2+N)^2(3+N)}\right)S_3\\
&+\left(-\frac{3}{2(3+N)}+\frac{3(-1)^N}{2(2+N)(3+N)}\right)S_4\\
&-\frac{(-1)^N\,(-8-7N+N^3)}{(1+N)^3(2+N)^3(3+N)}S_2+\frac{17+27N+15N^2+3N^3}{(1+N)^3(2+N)^3(3+N)}S_2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2\left(17+27N+15N^2+3N^3\right)}{(1+N)^3(2+N)^3(3+N)}S_{-2}+\frac{2S_2}{3+N}S_{-2}-2S_3S_{-2}-2S_{2,1}S_{-2}\\
&+\frac{(-1)^N\left(23+28N+10N^2+N^3\right)}{(1+N)^2(2+N)^2(3+N)}S_{2,1}\\
&-\frac{4(-1)^N(3+2N)\left(3+3N+N^2\right)}{(1+N)^4(2+N)^3(3+N)}S_1-\frac{(-1)^N\left(23+28N+10N^2+N^3\right)}{(1+N)^2(2+N)^2(3+N)}S_1S_2\\
&+\left(-\frac{1}{3+N}-\frac{3(-1)^N}{(2+N)(3+N)}\right)S_3S_1+\frac{(-1)^N}{(2+N)(3+N)}S_1S_{2,1}\\
&+\left(\frac{1}{3+N}+\frac{5(-1)^N}{(2+N)(3+N)}\right)S_{3,1}+2S_{2,1,-2}-\frac{5(-1)^N}{(2+N)(3+N)}S_{2,1,1}\Bigg\}\,. \qquad (601)
\end{aligned}
$$

Here C_1 and C_2 represent the group-theoretic color factors. Since we consider only scalar graphs we leave them unspecified here. The polynomials in (601) read

$$
\begin{aligned}
P_4 &= -70-108N-18N^2+49N^3+30N^4+5N^5 && (602)\\
P_5 &= -70-104N-3N^2+70N^3+43N^4+8N^5 && (603)\\
P_6 &= 47+98N+81N^2+30N^3+4N^4 && (604)\\
P_7 &= 61+136N+123N^2+55N^3+12N^4+N^5 && (605)\\
P_8 &= 112+168N+89N^2+18N^3+N^4 && (606)\\
P_9 &= 58+84N+43N^2+10N^3+N^4 && (607)\\
P_{10} &= 48+213N+274N^2+150N^3+36N^4+3N^5 && (608)\\
P_{11} &= -126-284N-259N^2-116N^3-25N^4-2N^5 && (609)\\
P_{12} &= 16+60N+80N^2+47N^3+12N^4+N^5 && (610)\\
P_{13} &= 51+103N+81N^2+29N^3+4N^4 && (611)\\
P_{14} &= 325+758N+669N^2+262N^3+38N^4 && (612)\\
P_{15} &= 160+391N+396N^2+204N^3+52N^4+5N^5 && (613)\\
P_{16} &= 142+370N+388N^2+203N^3+52N^4+5N^5 && (614)\\
P_{17} &= 142+374N+403N^2+224N^3+65N^4+8N^5. && (615)
\end{aligned}
$$

The asymptotic expansion of I_6 is given by

$$
\begin{aligned}
I_6^{\rm asy}(N) \;=\; & C_1\Bigg\{\left[\frac{1}{4N^2}-\frac{19}{12N^3}+\frac{15}{2N^4}-\frac{1889}{60N^5}+\frac{247}{2N^6}-\frac{38935}{84N^7}+\frac{3371}{2N^8}-\frac{359009}{60N^9}+\frac{41679}{2N^{10}}\right]\\
&\times\ln^3(\bar{N})+\left[\frac{1}{8N^2}+\frac{23}{12N^3}-\frac{223}{12N^4}+\frac{45229}{400N^5}-\frac{280379}{480N^6}+\frac{66622583}{23520N^7}-\frac{23133233}{1680N^8}\right.\\
&\left.+\frac{724473271}{10080N^9}-\frac{2931192779}{6720N^{10}}\right]\ln^2(\bar{N})+\left[(-1)^N\left[\frac{1}{N^7}-\frac{21}{N^8}+\frac{242}{N^9}-\frac{1998}{N^{10}}\right]\right.\\
&-\frac{7}{8N^2}+\frac{95}{18N^3}-\frac{3371}{288N^4}-\frac{69017}{2000N^5}+\frac{8462677}{14400N^6}-\frac{7789424551}{1646400N^7}+\frac{323933401}{9800N^8}\\
&\left.-\frac{247879811629}{1058400N^9}+\frac{3111216830509}{1693440N^{10}}\right]\ln(\bar{N})+\left[-\frac{24}{5}+\frac{43}{10N}-\frac{129}{10N^2}+\frac{387}{10N^3}\right.
\end{aligned}
$$

$$
-\frac{1161}{10N^4}+\frac{3483}{10N^5}-\frac{10449}{10N^6}+\frac{31347}{10N^7}-\frac{94041}{10N^8}+\frac{282123}{10N^9}-\frac{846369}{10N^{10}}\Bigg]\zeta_2^2
$$

$$
+(-1)^N\Bigg[-\frac{1}{N^4}+\frac{51}{4N^5}-\frac{884}{9N^6}+\frac{14041}{24N^7}-\frac{1768501}{600N^8}+\frac{657507}{50N^9}-\frac{262301037}{4900N^{10}}\Bigg]
$$

$$
+\Bigg[(-1)^N\Bigg[-\frac{3}{N^3}+\frac{27}{N^4}-\frac{159}{N^5}+\frac{765}{N^6}-\frac{3249}{N^7}+\frac{12663}{N^8}-\frac{46443}{N^9}+\frac{163377}{N^{10}}\Bigg]-\frac{3}{2N^2}
$$

$$
+\frac{19}{2N^3}-\frac{45}{N^4}+\frac{1889}{10N^5}-\frac{741}{N^6}+\frac{38935}{14N^7}-\frac{10113}{N^8}+\frac{359009}{10N^9}-\frac{125037}{N^{10}}\Bigg]\zeta_3
$$

$$
+\zeta_2\Bigg[(-1)^N\Bigg[-\frac{5}{2N^4}+\frac{295}{12N^5}-\frac{605}{4N^6}+\frac{89029}{120N^7}-\frac{127147}{40N^8}+\frac{31520947}{2520N^9}-\frac{2616665}{56N^{10}}\Bigg]
$$

$$
+\Bigg[(-1)^N\Bigg[-\frac{1}{N^3}+\frac{9}{N^4}-\frac{53}{N^5}+\frac{255}{N^6}-\frac{1083}{N^7}+\frac{4221}{N^8}-\frac{15481}{N^9}+\frac{54459}{N^{10}}\Bigg]+\frac{7}{4N^2}
$$

$$
-\frac{133}{12N^3}+\frac{105}{2N^4}-\frac{13223}{60N^5}+\frac{1729}{2N^6}-\frac{38935}{12N^7}+\frac{23597}{2N^8}-\frac{2513063}{60N^9}+\frac{291753}{2N^{10}}\Bigg]\ln(\bar{N})
$$

$$
+3\zeta_3-\frac{11}{8N^2}+\frac{569}{36N^3}-\frac{1225}{12N^4}+\frac{216201}{400N^5}-\frac{1261231}{480N^6}+\frac{125654423}{10080N^7}-\frac{306787391}{5040N^8}
$$

$$
+\frac{9847032577}{30240N^9}-\frac{13758651023}{6720N^{10}}\Bigg]+\zeta_5-\frac{31}{16N^2}+\frac{2153}{216N^3}-\frac{5735}{128N^4}+\frac{40340069}{180000N^5}
$$

$$
-\frac{542992637}{432000N^6}+\frac{659641453013}{86436000N^7}-\frac{7397109902939}{148176000N^8}+\frac{962090042920501}{2667168000N^9}
$$

$$
-\frac{330634683598931}{111132000N^{10}}\Bigg\}
$$

$$
+C_2\Bigg\{(-1)^N\Bigg[\frac{2}{N^3}-\frac{15}{N^4}+\frac{226}{3N^5}-\frac{950}{3N^6}+\frac{18049}{15N^7}-\frac{12859}{3N^8}+\frac{511284}{35N^9}-\frac{337628}{7N^{10}}\Bigg]
$$

$$
\times\ln^2(\bar{N})+(-1)^N\Bigg[\frac{4}{N^3}-\frac{20}{N^4}+\frac{581}{9N^5}-\frac{2879}{18N^6}+\frac{132043}{450N^7}-\frac{39521}{180N^8}-\frac{1617779}{1225N^9}
$$

$$
+\frac{41782189}{4410N^{10}}\Bigg]\ln(\bar{N})+\Bigg[(-1)^N\Bigg[-\frac{19}{10N^2}+\frac{19}{2N^3}-\frac{361}{10N^4}+\frac{247}{2N^5}-\frac{4009}{10N^6}+\frac{2527}{2N^7}
$$

$$
-\frac{39121}{10N^8}+\frac{23959}{2N^9}-\frac{364249}{10N^{10}}\Bigg]-\frac{8}{5N}+\frac{24}{5N^2}-\frac{72}{5N^3}+\frac{216}{5N^4}-\frac{648}{5N^5}+\frac{1944}{5N^6}
$$

$$
-\frac{5832}{5N^7}+\frac{17496}{5N^8}-\frac{52488}{5N^9}+\frac{157464}{5N^{10}}\Bigg]\zeta_2^2+(-1)^N\Bigg[\frac{5}{N^3}-\frac{227}{8N^4}+\frac{3259}{27N^5}-\frac{395983}{864N^6}
$$

$$
+\frac{1296603}{800N^7}-\frac{488729}{90N^8}+\frac{64743036461}{3704400N^9}-\frac{40570237223}{740880N^{10}}\Bigg]+\Bigg[(-1)^N\Bigg[-\frac{1}{2N^2}+\frac{5}{2N^3}
$$

$$-\frac{19}{2N^4}+\frac{65}{2N^5}-\frac{211}{2N^6}+\frac{665}{2N^7}-\frac{2059}{2N^8}+\frac{6305}{2N^9}-\frac{19171}{2N^{10}}\Bigg]\ln^2(\bar{N})+(-1)^N\Bigg[-\frac{5}{2N^3}$$
$$+\frac{175}{12N^4}-\frac{695}{12N^5}+\frac{7763}{40N^6}-\frac{4759}{8N^7}+\frac{4409821}{2520N^8}-\frac{2570599}{504N^9}+\frac{75289517}{5040N^{10}}\Bigg]\ln(\bar{N})$$
$$+(-1)^N\Bigg[\frac{1}{2N^3}-\frac{55}{8N^4}+\frac{122}{3N^5}-\frac{51913}{288N^6}+\frac{1006891}{1440N^7}-\frac{45683}{18N^8}+\frac{14891573}{1680N^9}$$
$$-\frac{18137689541}{604800N^{10}}\Bigg]+\frac{1}{N^2}-\frac{6}{N^3}+\frac{161}{6N^4}-\frac{106}{N^5}+\frac{3901}{10N^6}-\frac{6849}{5N^7}+\frac{976349}{210N^8}-\frac{538172}{35N^9}$$
$$+\frac{2092661}{42N^{10}}\Bigg]\zeta_2+\Bigg[(-1)^N\Bigg[-\frac{15}{2N^3}+\frac{175}{4N^4}-\frac{695}{4N^5}+\frac{23289}{40N^6}-\frac{14277}{8N^7}+\frac{4409821}{840N^8}$$
$$-\frac{2570599}{168N^9}+\frac{75289517}{1680N^{10}}\Bigg]+(-1)^N\Bigg[-\frac{3}{N^2}+\frac{15}{N^3}-\frac{57}{N^4}+\frac{195}{N^5}-\frac{633}{N^6}+\frac{1995}{N^7}$$
$$-\frac{6177}{N^8}+\frac{18915}{N^9}-\frac{57513}{N^{10}}\Bigg]\ln(\bar{N})\Bigg]\zeta_3-\frac{1}{4N^3}+\frac{7}{3N^4}-\frac{691}{48N^5}+\frac{3521}{48N^6}-\frac{13331}{40N^7}$$
$$+\frac{55991}{40N^8}-\frac{6987079}{1260N^9}+\frac{26436619}{1260N^{10}}\Bigg\}+O\left(\frac{\ln^3(\bar{N})}{N^{11}}\right)\,. \qquad (616)$$

Again this series is regular as $|N|\to\infty$.

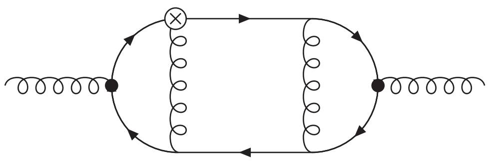


Figure 23: The ladder diagram for $I_7(N)$, Eq. (618).

One of the diagrams that could first be evaluated using this method is the ladder diagram in figure 23. Very recent implementations in `Sigma` [236, 237] do now allow to solve the physical graphs of this kind using integration by parts and differential equation techniques [310].
The generating function is given in terms of hyperlogarithms over the alphabet $\{1/2, 0, 1, -2\}$,

$$\begin{aligned}\hat{I}_7(x) \;=\; & \Bigg[-\frac{1+x}{x^3}\mathrm{L}_{-1}(x)-\frac{2x-1}{x^3}\mathrm{L}_{1/2}(x)-\frac{3(1-x)}{x^3}\mathrm{L}_1(x)-\frac{1-2x+x^2}{(1-x)x^3}\mathrm{L}_{0,-1}(x)\\
&+\frac{1-2x^2}{x^3}\mathrm{L}_{0,1/2}(x)-\frac{3-4x-3x^2+3x^3}{(1-x)x^3}\mathrm{L}_{0,1}(x)-\frac{1-2x^2}{x^3}\mathrm{L}_{1,1/2}(x)\\
&+\frac{(1-x)(2+3x)}{x^3}\mathrm{L}_{1,1}(x)\Bigg]\zeta_3\\
&+\frac{(1+x)}{2x^3}\Bigg[3\mathrm{L}_{-1,0,0,1}(x)-2\mathrm{L}_{-1,0,1,1}(x)-3\mathrm{L}_{1,0,0,1}(x)\Bigg]+\frac{1}{x^2}\Bigg[6\mathrm{L}_{0,0,1,1}(x)-4\mathrm{L}_{0,1,0,1}(x)\end{aligned}$$

$$
-\mathrm{L}_{0,1,1,1}(x)\Bigg] - \frac{(-1+2x)}{2x^3}\Bigg[3\mathrm{L}_{1/2,0,0,1}(x) - \mathrm{L}_{1/2,0,1,1}(x) - 3\mathrm{L}_{1/2,1,0,1}(x) + \mathrm{L}_{1/2,1,1,1}(x)\Bigg]
-\frac{3}{2x^2}\mathrm{L}_{1,0,1,1}(x) + \frac{2}{x^2}\mathrm{L}_{1,1,0,1}(x) - \frac{(-1+x)}{2x^3}\mathrm{L}_{1,1,1,1}(x) + \frac{2}{x^2}\Bigg[\mathrm{L}_{0,1,1}(x) - \mathrm{L}_{1,0,1}(x)\Bigg]
+\frac{(-1+2x+x^2)}{2(-1+x)x^3}\Bigg[3\mathrm{L}_{0,-1,0,0,1}(x) - 2\mathrm{L}_{0,-1,0,1,1}(x)\Bigg]
-\frac{5}{-1+x}\mathrm{L}_{0,0,0,1,1}(x) - \frac{5}{2(-1+x)}\mathrm{L}_{0,0,1,0,1}(x) + \frac{3(3+x)}{2(-1+x)x}\mathrm{L}_{0,0,1,1,1}(x)
-\frac{(-1+2x^2)}{2x^3}\Bigg[3\mathrm{L}_{0,1/2,0,0,1}(x) + \mathrm{L}_{0,1/2,0,1,1}(x) + 3\mathrm{L}_{0,1/2,1,0,1}(x) - \mathrm{L}_{0,1/2,1,1,1}(x)\Bigg]
+\frac{3\left(1-3x^2+3x^3\right)}{2(-1+x)x^3}\mathrm{L}_{0,1,0,0,1}(x) + \frac{8-14x+5x^2+3x^3}{2(-1+x)x^3}\mathrm{L}_{0,1,0,1,1}(x)
+\frac{8-15x+3x^2}{2(-1+x)x^3}\mathrm{L}_{0,1,1,0,1}(x) - \frac{3(-3+2x)}{2x^3}\mathrm{L}_{0,1,1,1,1}(x) + \frac{-6+3x+5x^2}{x^3}\mathrm{L}_{1,0,0,1,1}(x)
+\frac{2(-1+x)}{x^3}\mathrm{L}_{1,1,1,0,1}(x) + \frac{4-2x+5x^2}{2x^3}\mathrm{L}_{1,0,1,0,1}(x) - \frac{-4+6x+3x^2}{2x^3}\mathrm{L}_{1,0,1,1,1}(x)
+\frac{(-1+2x^2)}{2x^3}\Bigg[3\mathrm{L}_{1,1/2,0,0,1}(x) - \mathrm{L}_{1,1/2,0,1,1}(x)\Bigg] - \frac{3(-1+x)(4+3x)}{2x^3}\mathrm{L}_{1,1,0,0,1}(x)
-\frac{(-1+2x^2)}{2x^3}\Bigg[\mathrm{L}_{1,1/2,1,0,1}(x) + \mathrm{L}_{1,\frac{1}{2},1,1,1}(x)\Bigg] - \frac{(-1+x)(5+3x)}{2x^3}\mathrm{L}_{1,1,0,1,1}(x)\,. \qquad (617)
$$

As in generally observed expressions stemming from generating functions over this alphabet require harmonic sums, alternating harmonic sums and additional generalized harmonic sums for their representation. The N space result is given by

$$
\begin{aligned}
I_7(N) \;=\;& \frac{P_1}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{P_2}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 \\
&+\frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} + \frac{\left(-24-5N+2N^2\right)}{12(2+N)^2(3+N)^2}S_1^3 \\
&-\frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2S_2 \\
&+\frac{314+631N+578N^2+288N^3+68N^4+5N^5}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 \\
&-\frac{\left(399+2069N+2774N^2+1510N^3+349N^4+27N^5\right)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} \\
&-2\zeta_3S_{-2} - S_{-2,1}S_{-2} + \frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} \\
&+\frac{\left(59+42N+6N^2\right)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3S_1\Big(2\Big) \\
&-\frac{752+2087N+2490N^2+1580N^3+558N^4+105N^5+8N^6}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3S_2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{2}S_3S_2-2S_{2,1}S_2+\frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1}\\
&+\frac{P_3}{(1+N)^4(2+N)^4(3+N)^4}S_1-\frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3S_1\\
&-\frac{(470+1075N+996N^2+447N^3+96N^4+8N^5)}{4(1+N)^2(2+N)^2(3+N)^2}S_2S_1-S_{2,3}\\
&+\frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3S_1-\frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1S_{2,1}\\
&+\frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1}-3S_{4,1}+S_{-2,1,-2}\\
&+\frac{2^{1+N}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right)-\frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1}\\
&+5S_{2,2,1}+6S_{3,1,1}+\frac{2^N\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right)\\
&-\frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right)-\frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)\ , \qquad (618)
\end{aligned}
$$

with

$$
\begin{aligned}
P_1 &= -31104-159408N-353808N^2-446652N^3-353808N^4-182604N^5-61488N^6\\
&\quad -13044N^7-1584N^8-84N^9, \qquad (619)\\
P_2 &= -105+65(-1)^N+7\times 2^{4+N}-150N+101(-1)^NN+39\times 2^{2+N}N-73N^2\\
&\quad +56(-1)^NN^2+33\times 2^{1+N}N^2-12N^3+13(-1)^NN^3+2^{2+N}N^3+(-1)^NN^4\\
&\quad -2^{1+N}N^4, \qquad (620)\\
P_3 &= 5436+29004N+67285N^2+89175N^3+74616N^4+41120N^5+15107N^6+3659N^7\\
&\quad +562N^8+50N^9+2N^{10}\ . \qquad (621)
\end{aligned}
$$

In order to show that the 2^N-terms cancel as $N\to\infty$, we calculate the asymptotic representation of I_7 :

$$
\begin{aligned}
I_7^{\rm asy}(N) =& \left[\frac{1115231}{20N^{10}}-\frac{74121}{4N^9}+\frac{122951}{20N^8}-\frac{40677}{20N^7}+\frac{13391}{20N^6}-\frac{873}{4N^5}+\frac{1391}{20N^4}-\frac{417}{20N^3}+\frac{101}{20N^2}\right]\zeta_2^2\\
&+\Biggl[\left(-\frac{95855}{2N^{10}}+\frac{31525}{2N^9}-\frac{10295}{2N^8}+\frac{3325}{2N^7}-\frac{1055}{2N^6}+\frac{325}{2N^5}-\frac{95}{2N^4}+\frac{25}{2N^3}-\frac{5}{2N^2}\right)\ln(\bar{N})\\
&-\frac{23280115}{2016N^{10}}+\frac{2093041}{1008N^9}-\frac{177251}{1008N^8}-\frac{25843}{336N^7}+\frac{2569}{48N^6}-\frac{155}{8N^5}+\frac{91}{24N^4}+\frac{2}{3N^3}-\frac{11}{12N^2}\Biggr]\zeta_3\\
&+\Biggl[\left(\frac{19171}{N^{10}}-\frac{6305}{N^9}+\frac{2059}{N^8}-\frac{665}{N^7}+\frac{211}{N^6}-\frac{65}{N^5}+\frac{19}{N^4}-\frac{5}{N^3}+\frac{1}{N^2}\right)\ln^2(\bar{N})\\
&+\Biggl(\frac{103016863}{2520N^{10}}-\frac{3091261}{315N^9}+\frac{2571839}{1260N^8}-\frac{6215}{21N^7}-\frac{293}{20N^6}+\frac{2071}{60N^5}-\frac{103}{6N^4}+\frac{67}{12N^3}\\
&-\frac{1}{N^2}\Biggr)\ln(\bar{N})+\frac{292993001621}{302400N^{10}}-\frac{4402272031}{30240N^9}+\frac{22261739}{840N^8}-\frac{78507473}{14112N^7}+\frac{180961}{144N^6}\\
&-\frac{111807}{400N^5}+\frac{629}{12N^4}-\frac{319}{72N^3}-\frac{7}{4N^2}\Biggr]\zeta_2
\end{aligned}
$$

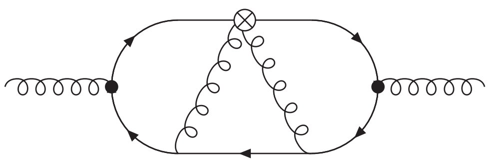


Figure 24: Diagram 8. For this diagram linear reducibility can only be gained through a suitable variable transformation.

$$
\begin{aligned}
&+\Big(\frac{249223}{6N^{10}}-\frac{145015}{12N^9}+\frac{10295}{3N^8}-\frac{11305}{12N^7}+\frac{1477}{6N^6}-\frac{715}{12N^5}+\frac{38}{3N^4}-\frac{25}{12N^3}+\frac{1}{6N^2}\Big)\ln^3(\bar{N})\\
&+\Big(\frac{193493767}{10080N^{10}}+\frac{210658237}{10080N^9}-\frac{21541697}{2520N^8}+\frac{243269}{96N^7}-\frac{30539}{48N^6}+\frac{2123}{16N^5}-\frac{59}{3N^4}\\
&+\frac{5}{8N^3}+\frac{1}{2N^2}\Big)\ln^2(\bar{N})\\
&+\Big(-\frac{2207364771673}{4233600N^{10}}+\frac{1390655509}{352800N^9}+\frac{285594061}{22050N^8}-\frac{67234111}{14400N^7}+\frac{8617073}{7200N^6}-\frac{35209}{144N^5}\\
&+\frac{116}{3N^4}-\frac{119}{24N^3}+\frac{1}{N^2}\Big)\ln(\bar{N})\\
&+\frac{1344226725047831}{889056000N^{10}}-\frac{165849841805771}{889056000N^9}+\frac{808151260279}{27783000N^8}-\frac{708430537}{120960N^7}+\frac{304474703}{216000N^6}\\
&-\frac{606811}{1728N^5}+\frac{1867}{24N^4}-\frac{1813}{144N^3}+\frac{1}{N^2}+O\left(\frac{1}{N^{11}}\right)\ ,
\end{aligned}
\tag{622}
$$

which displays a convergent behavior.
The V–type diagram in Figure 24 has the color structure

$$
I_8 = C_1 I_{8a} + C_2 I_{8b}\ . \tag{623}
$$

Here we leave the group theoretic factors C_1 and C_2 again unspecified for the scalar case. Because the two contributions I_{8a} and I_{8b} are of a very different complexity, we will treat them individually from now on.
The respective parametric integrals are given by

$$
\begin{aligned}
I_{8a} &= \int_0^\infty dx_1 dx_2 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_7 \frac{\sum_{j_1=0}^{N}\sum_{j_2=j_1+1}^{N+1}(-T_2)^{j_1}(T_1)^{N+1-j_2}(T_1+T_3)^{j_2-j_1-1}}{U^{N+2}M}\\
&\quad\times\theta(x_1-\alpha_1)\,\theta(x_2-\alpha_4) \qquad (624)\\
I_{8b} &= \int_0^\infty dx_1 dx_2 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_7 \frac{\sum_{j_1=0}^{N}\sum_{j_2=j_1+1}^{N+1}(-T_2)^{j_1}(T_1)^{N+1-j_2}(T_1-T_4)^{j_2-j_1-1}}{U^{N+2}M}\\
&\quad\theta(x_1-\alpha_1)\,\theta(x_2-\alpha_4)\ , \qquad (625)
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= \alpha_1+\alpha_6\ ,\\
x_2 &= \alpha_4+\alpha_5\ .
\end{aligned}
\tag{626}
$$

The different graph polynomials in Eqs. (624) and (625) read

$$
M = x_1 + x_2 + \alpha_7
$$

$$
\begin{aligned}
U &= -\alpha_3\alpha_2\alpha_7 - \alpha_2\alpha_7 x_2 - \alpha_2 x_2 x_1 - \alpha_3\alpha_2 x_2 - \alpha_3\alpha_2 x_1 - \alpha_7 x_2 x_1 - \alpha_3 x_2 x_1 - \alpha_3\alpha_7 x_1 \\
T_1 &= -\alpha_3\alpha_7\alpha_1 + \alpha_3\alpha_2\alpha_7 - \alpha_2\alpha_3\alpha_1 - \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_7 x_2 + \alpha_2 x_2 x_1 - \alpha_2 x_2\alpha_1 + \alpha_3\alpha_2 x_2 \\
&\quad +\alpha_3\alpha_2 x_1 + \alpha_7 x_2 x_1 - \alpha_7 x_2\alpha_1 + \alpha_3 x_2 x_1 - \alpha_3 x_2\alpha_1 + \alpha_3\alpha_7 x_1 \\
T_2 &= -(\alpha_7\alpha_4\alpha_2 - \alpha_3\alpha_2\alpha_7 + \alpha_2\alpha_3\alpha_1 + \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_7 x_2 - \alpha_2 x_2 x_1 + \alpha_2\alpha_4 x_1 - \alpha_3\alpha_2 x_2 \\
&\quad -\alpha_3\alpha_2 x_1 - \alpha_7 x_2 x_1 + \alpha_7\alpha_4 x_1 - \alpha_3 x_2 x_1 + \alpha_3\alpha_4 x_1 - \alpha_3\alpha_7 x_1) \\
T_3 &= \alpha_7 x_2\alpha_1 + \alpha_3 x_2\alpha_1 + \alpha_3\alpha_7\alpha_1 - \alpha_3\alpha_4 x_1 \\
T_4 &= -\alpha_2 x_2\alpha_1 + \alpha_7\alpha_4 x_1 + \alpha_7\alpha_4\alpha_2 + \alpha_2\alpha_4 x_1 \ . \qquad (627)
\end{aligned}
$$

The first color contribution yields an expression in hyperlogarithms over the alphabet $a_i \in \{0,-1,1\}$, which is thus expressible within the class of harmonic polylogarithms

$$
\begin{aligned}
\hat{I}_{8a}(x) &= \frac{4}{x^2(x+1)}\Biggl\{-\left[\mathrm{L}_{0,1}(x)+\mathrm{L}_{0,-1}(x)\right]\zeta_3 - 4\mathrm{L}_{0,-1,-1,0,-1}(x) \\
&\quad -2\mathrm{L}_{0,-1,0,-1,-1}(x) + 2\mathrm{L}_{0,-1,0,0,-1}(x) + 6\mathrm{L}_{0,0,-1,-1,-1}(x) \\
&\quad -4\mathrm{L}_{0,1,0,-1,-1}(x) + 2\mathrm{L}_{0,1,0,0,-1}(x)\Biggr\} \ . \qquad (628)
\end{aligned}
$$

The factors in the denominators of the rational functions correspond to the same alphabet and are all of the form $\frac{1}{x-a_i}$. As generally observed this yields a representation within the class of generalized harmonic sums

$$
\begin{aligned}
I_{8a}(N) &= (-1)^N\Biggl[-\frac{12\ (2N+3)\,(N^2+3N+3)}{(N+1)^3(N+2)^3}S_1^2 + \frac{8\ (2N^2+6N+5)}{(N+1)^2(N+2)^2}\left[2S_1S_2 - S_{2,1}\right] \\
&\quad -\frac{8\ (4N+5)}{(N+1)^2(N+2)^3}S_1 + 8S_3S_2 + 16S_{2,1}S_2 + 8S_{-2,1}S_{-2} + 8S_5 - 8S_{2,3} + 24S_{4,1} \\
&\quad -8S_{-2,1,-2} - 24S_{2,2,1} - 24S_{3,1,1} + \frac{4\ (10N^3+43N^2+65N+35)}{(N+1)^3(N+2)^3}S_2\Biggr] \\
&\quad +\frac{8(2N+3)}{(N+1)^2(N+2)^2}\left[S_{-3} - 2S_{-2,1}\right] \\
&\quad +4\left[(-1)^N\left(\frac{(2N^2+6N+5)}{(N+1)^2(N+2)^2} + S_2 + S_{-2}\right) - \frac{(2N+3)}{(N+1)^2(N+2)^2}\right]\zeta_3 \ , \qquad (629)
\end{aligned}
$$

with the asymptotic representation

$$
\begin{aligned}
&I_{8a}^{\mathrm{asy}}(N) \propto \\
&\quad \Biggl[(-1)^N\left(-\frac{16}{N} + \frac{40}{N^2} - \frac{296}{3N^3} + \frac{240}{N^4} - \frac{8632}{15N^5} + \frac{1360}{N^6} - \frac{66536}{21N^7} + \frac{7280}{N^8} - \frac{247672}{15N^9} + \frac{37008}{N^{10}}\right) \\
&\quad \times\ln(\bar{N}) + (-1)^N\Biggl[-\frac{16}{N} - \frac{2}{N^2} + \frac{538}{9N^3} - \frac{721}{3N^4} + \frac{18996}{25N^5} - \frac{6514}{3N^6} + \frac{12902497}{2205N^7} - \frac{954313}{63N^8} \\
&\quad +\frac{7190138}{189N^9} - \frac{19586179}{210N^{10}} + 12\zeta_3\Biggr]\Biggr]\zeta_2 + (-1)^N\Biggl[\Biggl(\frac{6}{N^2} - \frac{30}{N^3} + \frac{111}{N^4} - \frac{360}{N^5} + \frac{1079}{N^6}
\end{aligned}
$$

$$
-\frac{3060}{N^7}+\frac{8317}{N^8}-\frac{21840}{N^9}+\frac{278631}{5N^{10}}\Biggr)\ln^2(\bar{N})+\Biggl(\frac{10}{N^2}-\frac{20}{N^3}+\frac{11}{6N^4}+\frac{485}{3N^5}-\frac{15469}{18N^6}
+\frac{19465}{6N^7}-\frac{13226411}{1260N^8}+\frac{216849}{7N^9}-\frac{9020336}{105N^{10}}\Biggr)\ln(\bar{N})+\frac{1}{N^2}+\frac{62}{3N^3}-\frac{7457}{72N^4}+\frac{31339}{90N^5}
-\frac{5369077}{5400N^6}+\frac{6553031}{2520N^7}-\frac{3416761097}{529200N^8}+\frac{820719223}{52920N^9}-\frac{192478383749}{5292000N^{10}}-5\zeta_5\Biggr]
+(-1)^N\Biggl(\frac{12}{N}-\frac{30}{N^2}+\frac{74}{N^3}-\frac{180}{N^4}+\frac{2158}{5N^5}-\frac{1020}{N^6}+\frac{16634}{7N^7}-\frac{5460}{N^8}+\frac{61918}{5N^9}
-\frac{27756}{N^{10}}\Biggr)\zeta_3+O\left(\frac{\ln^2(\bar{N})}{N^{11}}\right)\,. \qquad (630)
$$

7.5 Obtaining linear reducibility via transformations of the integration variables

If a diagram is not linearly reducible this does generally mean that it is not calculable using hyperlogarithms. In some cases linear reducibility may be obtained by a suitable transformation of integration variables. For the integral (625) no completely linear reducible integration order exists a priori. In the last integration step quadratic denominators appear. Since these non-linear denominators are not of a higher polynomial degree than 2 and appear in the last integration step only it is, however, possible to remap the tracing variable to gain linear reducibility, e.g:

$$
\int_0^\infty dy \frac{L(\{\cdots\},y)}{y^2+y(2+t)+1} \;=\; \int_0^\infty dy \frac{L(\{\cdots\},y)}{\left(y+1+t/2+\sqrt{t^2+4t}/2\right)\left(y+1+t/2-\sqrt{t^2+4t}/2\right)}\,. \qquad (631)
$$

Applying the transformation

$$
t=\frac{4x^2}{1-x^2} \qquad (632)
$$

yields

$$
\int_0^\infty dy \left(x^2-1\right)^2 \frac{L(\{\cdots\},y)}{(y(x^2-1)-1-3x^2+2x)\,(y(x^2-1)-1-3x^2-2x)}\,. \qquad (633)
$$

As a result, the final expression will consist of hyperlogarithms of argument

$$
x=\sqrt{\frac{t}{t+4}}\,. \qquad (634)
$$

In cases that no variable transformation is needed to obtain linear reducibility the generating function is given in terms of a linear combination of hyperlogarithms with argument t. In this case obtaining the N-space representation is achieved by evaluating the Cauchy-products between the corresponding sum representations of the rational prefactors and the hyperlogarithms [179]. If transformations like (631) have been applied the $N$the coefficient is extracted by generating difference equations which are then solved by using the package `Sigma` [236, 237] . The second method is more time consuming but has the advantage of working in much more general setups.

The generating expression in terms of hyperlogarithms is rather lengthy and will not be presented here. It is however given completely in terms of hyperlogarithms with argument x over the alphabet

$$\left\{1, 0, -1, -4, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{i}{\sqrt{3}}, \tfrac{i}{\sqrt{3}} - \tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}\right\}. \tag{635}$$

Due to the more complex dependence on the generating parameter t, the N-space results also contain classes of sums beyond harmonic and alternating sums. This class is given by finite binomial and inverse binomial sums over harmonic or alternating harmonic sums [311].
The complete Mellin space representation of I_{8b} is then given by

$$\begin{aligned}
I_{8b} \;=\;& -\frac{2(3N+2)}{(N+1)^5(N+2)^2} + \frac{2\big(4N^3+35N^2+82N+58\big)}{(N+1)^3(N+2)^3}\left[S_2+3S_{-2}\right] \\
& -\frac{4\big(N^3+8N^2+23N+20\big)}{(N+1)^2(N+2)^2}S_3 - \frac{4\big(N^3+8N^2+27N+26\big)}{(N+1)^2(N+2)^2}S_{-3} \\
& -\frac{8\big(N^2+6N+7\big)}{(N+1)^2(N+2)}S_{-2,1} + 2^{N+2}\frac{\big(2N^3+12N^2+31N+26\big)}{(N+1)^2(N+2)^2}\left[S_{1,2}\left(\frac{1}{2},1,N\right)\right. \\
& \left.+3S_{1,2}\left(\frac{1}{2},-1,N\right)\right] + \frac{(-1)^N}{\binom{2N}{N}}\left\{-\frac{3\big(4N^2+6N-3\big)}{(N+1)(N+2)(2N+1)}\sum_{i=1}^N(-2)^i\binom{2i}{i}S_{1,2}\Big(\frac{1}{2},1,i\Big)\right. \\
& -\frac{9\big(4N^2+6N-3\big)}{(N+1)(N+2)(2N+1)}\sum_{i=1}^N(-2)^i\binom{2i}{i}S_{1,2}\Big(\frac{1}{2},-1,i\Big) + \frac{(N+1)}{(N+2)(2N+1)}\Bigg[\\
& -\sum_{i=1}^N\frac{(-1)^i\binom{2i}{i}}{i^3} - 2\sum_{i=1}^N\frac{\binom{2i}{i}S_1(i)}{i^2} + \frac{3}{2}\sum_{i=1}^N\frac{\binom{2i}{i}S_1^2(i)}{i} + \frac{9}{2}\sum_{i=1}^N\frac{\binom{2i}{i}S_2(i)}{i} \\
& \left.+2\sum_{i=1}^N\frac{(-1)^i\binom{2i}{i}S_2(i)}{i} + 3\sum_{i=1}^N\frac{\binom{2i}{i}S_{-2}(i)}{i} + 6\sum_{i=1}^N\frac{(-1)^i\binom{2i}{i}S_{-2}(i)}{i}\Bigg]\right\} \\
& +(-1)^N\left\{-\frac{8\big(N^3+6N^2+11N+7\big)}{3(N+1)^2(N+2)^2}S_1^3 + \frac{\big(-4N^3-7N^2+6N+10\big)}{(N+1)^3(N+2)^3}S_1^2\right. \\
& +\left[\frac{2\big(16N^3+107N^2+222N+146\big)}{(N+1)^4(N+2)^3} - \frac{12\big(N^3+6N^2+11N+7\big)}{(N+1)^2(N+2)^2}S_2\right]S_1 \\
& +4S_2^2+6S_{-2}^2+10S_4+\frac{2(3N+2)}{(N+1)^5(N+2)^2} - \frac{8\big(5N^3+24N^2+37N+20\big)}{3(N+1)^2(N+2)^2}S_3 \\
& -8(5N+12)S_5+8S_{-4}-10(N+2)S_{-5} \\
& +\left[-\frac{8\big(2N^3+10N^2+16N+9\big)}{(N+1)^2(N+2)^2} - 2(5N+12)S_2 - 6(5N+12)S_{-2}\right]S_{-3} \\
& +\left[\frac{-36N^3-165N^2-270N-158}{(N+1)^3(N+2)^3} - 2(5N+12)S_3 - 4S_{2,1}\right]S_2 \\
& +\frac{4\big(N^3+6N^2+11N+7\big)}{(N+1)^2(N+2)^2}S_{2,1} + 2(5N+12)S_{2,3} + 2(5N+16)S_{2,-3} - 12S_{3,1}
\end{aligned}$$

$$
\begin{aligned}
&+16(N+2)S_{4,1}+\frac{16\big(N^3+6N^2+11N+7\big)}{(N+1)^2(N+2)^2}S_{-2,1}+\Bigg[-\frac{2\big(4N^3+7N^2-6N-10\big)}{(N+1)^3(N+2)^3}\\
&-\frac{16\big(N^3+6N^2+11N+7\big)}{(N+1)^2(N+2)^2}S_1+4S_2+6(N+4)S_3+8(N+2)S_{-2,1}\Bigg]S_{-2}\\
&+2NS_{-2,3}+2(23N+60)S_{-2,-3}+4S_{2,1,1}-16S_{2,1,-2}+8S_{2,2,1}\\
&+6(N+4)S_{3,1,1}-8(N+2)S_{-2,1,-2}-16S_{2,1,1,1}\\
&-2(3N+8)\left[S_{1,2}\left(\frac{1}{2},1,N\right)+3S_{1,2}\left(\frac{1}{2},-1,N\right)\right]S_2(-2)\\
&+2(3N+8)\Bigg[-3S_{1,4}\left(\frac{1}{2},2,N\right)-S_{1,4}\left(\frac{1}{2},-2,N\right)+S_{1,2,2}\left(\frac{1}{2},1,-2,N\right)\\
&+3S_{1,2,2}\left(\frac{1}{2},-1,-2,N\right)+S_{1,2,2}\left(\frac{1}{2},-2,1,N\right)+3S_{1,2,2}\left(\frac{1}{2},-2,-1,N\right)\Bigg]\\
&-6(3N+8)\sum_{i=1}^{N}(-2)^i\binom{2i}{i}\left[S_{1,2}\left(\frac{1}{2},1,i\right)+3S_{1,2}\left(\frac{1}{2},-1,i\right)\right]\sum_{j=1}^{i}\frac{1}{\binom{2j}{j}j^2}\\
&+36\sum_{i=1}^{N}(-2)^i\binom{2i}{i}S_{1,2}\left(\frac{1}{2},1,i\right)\sum_{j=1}^{i}\frac{1}{\binom{2j}{j}j}\\
&+108\sum_{i=1}^{N}(-2)^i\binom{2i}{i}S_{1,2}\left(\frac{1}{2},-1,i\right)\sum_{j=1}^{i}\frac{1}{\binom{2j}{j}j}\\
&+6(3N+8)\Bigg[\sum_{i=1}^{N}(-2)^i\binom{2i}{i}S_{1,2}\left(\frac{1}{2},1,i\right)\\
&+3\sum_{i=1}^{N}(-2)^i\binom{2i}{i}S_{1,2}\left(\frac{1}{2},-1,i\right)\Bigg]\sum_{i=1}^{N}\frac{1}{\binom{2i}{i}i^2}\\
&-36\left[\sum_{i=1}^{N}(-2)^i\binom{2i}{i}S_{1,2}\left(\frac{1}{2},1,i\right)+3\sum_{i=1}^{N}(-2)^i\binom{2i}{i}S_{1,2}\left(\frac{1}{2},-1,i\right)\right]\sum_{i=1}^{N}\frac{1}{\binom{2i}{i}i}\\
&+\frac{3}{2}(3N+8)\left[\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_1^2(j)}{j}}{\binom{2i}{i}(1+i)}+3\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_2(j)}{j}}{\binom{2i}{i}(1+i)}\right]
\end{aligned}
$$

$$+2(3N+8)\left[\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}S_2(j)}{j}}{\binom{2i}{i}(1+i)}+\frac{3}{2}\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_{-2}(j)}{j}}{\binom{2i}{i}(1+i)}\right]$$

$$+6(3N+8)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}S_{-2}(j)}{j}}{\binom{2i}{i}(1+i)}+2(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}}{j^3}}{\binom{2i}{i}(1+2i)}$$

$$+4(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_1(j)}{j^2}}{\binom{2i}{i}(1+2i)}-3(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_1^2(j)}{j}}{\binom{2i}{i}(1+2i)}$$

$$-9(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_2(j)}{j}}{\binom{2i}{i}(1+2i)}-4(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}S_2(j)}{j}}{\binom{2i}{i}(1+2i)}$$

$$-6(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_{-2}(j)}{j}}{\binom{2i}{i}(1+2i)}-12(3N+5)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}S_{-2}(j)}{j}}{\binom{2i}{i}(1+2i)}$$

$$+(-3N-8)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{(-1)^j\binom{2j}{j}}{j^3}}{\binom{2i}{i}(1+i)}-2(3N+8)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\binom{2j}{j}S_1(j)}{j^2}}{\binom{2i}{i}(1+i)}\quad\Bigg\}$$

$$+\Bigg\{(-1)^N\Bigg[6(N^2+1)\frac{1}{(N-1)N^2\binom{2N}{N}}\sum_{i=1}^{N}(-2)^i\binom{2i}{i}$$

$$-6(3N-1)\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}(-2)^j\binom{2j}{j}}{i^2\binom{2i}{i}}+36\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}(-2)^j\binom{2j}{j}}{i\binom{2i}{i}}$$

$$-36S_1(-2,N)+8(3N-1)S_2(-2,N)+\frac{4(N^2-N+1)}{(N-1)N^2}+4S_2$$

$$
-4(2N-1)S_{-2}\Biggr] + \frac{4\left(N^2-3N+1\right)}{(N-1)N^2} - \frac{2^{N+3}\left(N^2-N+1\right)}{(N-1)N^2}\Biggr\}\zeta_3 \; . \tag{636}
$$

Here as in all other N-space representations the code `Sigma` has been used to solve the difference equations and we obtain a representation in which all occurring sums are transcendental to each other. Some of the individual sums contributing to (636) diverge $\propto 8^N, 4^N$ and 2^N for large values of N. Performing the asymptotic expansion yields that the divergences $\propto 8^N$ and $\propto 4^N$ cancel while this is not the case for the divergences 2^N.

The asymptotic expansion in this case takes the following form

$$
I_{8b}(N) \propto 2^N \hat{I}_{8b,1}(N) + \hat{I}_{8b,2}(N) \; , \tag{637}
$$

with

$$
\begin{aligned}
I_{8b,1}(N) &\simeq \left[-\frac{112}{9N^3} + \frac{7568}{81N^4} - \frac{27280}{81N^5} + \frac{2256112}{2187N^6} - \frac{52719920}{19683N^7} + \frac{373195088}{59049N^8}\right]\zeta_3 \qquad (638) \\
I_{8b,2}(N) &\simeq \left[\frac{1}{N^4} - \frac{12}{N^5} + \frac{91}{N^6} - \frac{574}{N^7} + \frac{3451}{N^8}\right]\zeta_2 \\
&+2^{-N}\Biggl[\Biggl[-\frac{3}{2N^2} + \frac{1}{2N^3} + \frac{6}{N^4} - \frac{35}{2N^5} + \frac{17}{N^6} + \frac{79}{2N^7} - \frac{152}{N^8}\Biggr]\ln^2(2) \\
&+\left[-\frac{3}{N^2} + \frac{1}{N^3} + \frac{12}{N^4} - \frac{35}{N^5} + \frac{34}{N^6} + \frac{79}{N^7} - \frac{304}{N^8}\right]\left(\mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{2}\zeta_2\right) \\
&+\left[-\frac{3}{2N^2} + \frac{1}{2N^3} + \frac{6}{N^4} - \frac{35}{2N^5} + \frac{17}{N^6} + \frac{79}{2N^7} - \frac{152}{N^8}\right]\zeta_2\Biggr] + \Biggl[\frac{2}{N^2} - \frac{6}{N^3} \\
&+\frac{8}{N^4} + \frac{14}{N^5} - \frac{128}{N^6} + \frac{478}{N^7} - \frac{1272}{N^8}\Biggr]\zeta_3 + (-1)^N\Biggl[\Biggl[-\frac{4}{3N^2} + \frac{52}{9N^3} - \frac{56}{3N^4} \\
&+\frac{2396}{45N^5} - \frac{424}{3N^6} + \frac{22516}{63N^7} - \frac{872}{N^8}\Biggr]\ln^3(\bar{N}) + \Biggl[-\frac{74}{9N^3} + \frac{133}{3N^4} - \frac{4103}{25N^5} + \frac{15439}{30N^6} \\
&-\frac{6456953}{4410N^7} + \frac{1230668}{315N^8}\Biggr]\ln^2(\bar{N}) + \Biggl[\Biggl[-\frac{2}{N^2} + \frac{26}{3N^3} - \frac{28}{N^4} + \frac{1198}{15N^5} - \frac{212}{N^6} + \frac{11258}{21N^7} \\
&-\frac{1308}{N^8}\Biggr]\zeta_2 + \frac{4}{N^2} - \frac{436}{27N^3} + \frac{29}{N^4} + \frac{32}{375N^5} - \frac{8489}{36N^6} + \frac{8193131}{6860N^7} - \frac{778753}{180N^8}\Biggr]\ln(\bar{N}) \\
&+A_1 + A_2 N + \Biggl[-\frac{8}{N} + \frac{21}{N^2} - \frac{520}{9N^3} + \frac{476}{3N^4} - \frac{21473}{50N^5} + \frac{68569}{60N^6} - \frac{26328833}{8820N^7} \\
&+\frac{4823873}{630N^8}\Biggr]\zeta_2 + 2^{-N}\Biggl[\Biggl[\Biggl[-\frac{3}{2} - \frac{1}{N} - \frac{1}{N^2} + \frac{15}{N^3} - \frac{121}{N^4} + \frac{1023}{N^5} - \frac{9721}{N^6} \\
&+\frac{104415}{N^7} - \frac{1259161}{N^8}\Biggr]\zeta_2 - \frac{3}{N} + \frac{11}{2N^2} - \frac{55}{2N^3} + \frac{602}{3N^4} - \frac{50497}{30N^5} + \frac{239851}{15N^6} \\
&-\frac{36068621}{210N^7} + \frac{43495976}{21N^8}\Biggr]\ln^2(2) + \Biggl[-\frac{3}{2} - \frac{1}{N} - \frac{1}{N^2} + \frac{15}{N^3} - \frac{121}{N^4}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1023}{N^5}-\frac{9721}{N^6}+\frac{104415}{N^7}-\frac{1259161}{N^8}\Biggr]\zeta_2^2+\Biggl[-\frac{3}{N}+\frac{11}{2N^2}-\frac{55}{2N^3}+\frac{602}{3N^4}\\
&-\frac{50497}{30N^5}+\frac{239851}{15N^6}-\frac{36068621}{210N^7}+\frac{43495976}{21N^8}\Biggr]\zeta_2+\left(\mathrm{Li}_2\left(-\frac{1}{2}\right)+\frac{1}{2}\zeta_2\right)\\
&\times\Biggl[\Biggl[-3-\frac{2}{N}-\frac{2}{N^2}+\frac{30}{N^3}-\frac{242}{N^4}+\frac{2046}{N^5}-\frac{19442}{N^6}+\frac{208830}{N^7}-\frac{2518322}{N^8}\Biggr]\zeta_2\\
&-\frac{6}{N}+\frac{11}{N^2}-\frac{55}{N^3}+\frac{1204}{3N^4}-\frac{50497}{15N^5}+\frac{479702}{15N^6}-\frac{36068621}{105N^7}+\frac{86991952}{21N^8}\Biggr]\Biggr]\\
&+\Biggl[-\frac{2}{N}+\frac{10}{3N^2}-\frac{46}{9N^3}+\frac{20}{3N^4}-\frac{242}{45N^5}-\frac{20}{3N^6}+\frac{3194}{63N^7}-\frac{180}{N^8}\Biggr]\zeta_3\\
&+\frac{6}{N^2}-\frac{1732}{81N^3}+\frac{793}{12N^4}-\frac{1217029}{5625N^5}+\frac{130343}{180N^6}-\frac{10153834441}{4321800N^7}+\frac{1632850801}{226800N^8}\Biggr]\\
&+\frac{4}{N^5}-\frac{62}{N^6}+\frac{1759}{3N^7}-\frac{4530}{N^8}\\
&+\Biggl\{-\frac{2}{N^2}-\frac{6}{N^3}-\frac{8}{N^4}-\frac{2}{N^5}+\frac{8}{N^6}-\frac{10}{N^7}-\frac{72}{N^8}\\
&+(-1)^N\Biggl[\frac{10}{3}\zeta_2-\frac{4\pi}{\sqrt{3}}+\frac{2}{N^2}+\frac{10}{3N^3}+\frac{4}{N^4}+\frac{62}{15N^5}+\frac{4}{N^6}+\frac{82}{21N^7}+\frac{4}{N^8}\Biggr]\\
&+\left(-\frac{1}{4}\right)^N\sqrt{\pi}\Biggl[-\frac{64}{3}\left(\frac{1}{N}\right)^{5/2}+\frac{232}{9}\left(\frac{1}{N}\right)^{7/2}-\frac{6697}{54}\left(\frac{1}{N}\right)^{9/2}\\
&+\frac{65167}{144}\left(\frac{1}{N}\right)^{11/2}-\frac{30311555}{13824}\left(\frac{1}{N}\right)^{13/2}+\frac{3942221963}{331776}\left(\frac{1}{N}\right)^{15/2}\Biggr]\Biggr]\Biggr\}\zeta_3\,. \qquad (639)
\end{aligned}
$$

Here the constants A_1 and A_2 are

$$
A_1 \approx 6.233478110414834, \quad A_2 < 10^{-15}\,. \qquad (640)
$$

It has, however, been observed, that in the physical case including the numerator structures the 2^N-divergences cancel as well, cf. [310].

7.6 Fixed Mellin moments

The strategy to obtain general N representations via a generating function breaks down in some cases, e.g. for the diagrams in figure 25, because the respective integrals are not linearly reducible. If one is not interested in the general N representation but values for fixed integer values of N the integrals for all diagrams at 3–loop level are linearly reducible, however. In theory it would thus be possible to compute a large amount of Mellin moments, for all practical purposes one is, however, constrained by the available computing power and memory sizes. Here the resource demands rise exponentially in N due to the larger rational expressions and the intensive use of partial fraction decompositions in the algorithm. As an example we computed a series of Mellin moments for the most complicated class of diagrams contributing to the 3–loop OMEs. For the diagrams in figure 25 the computation of the 9th Mellin moment took about 8 hours of CPU time and about 35 Gigabyte of memory. In comparison the `FORM` based code `MATAD` [272] has similar resource demands on CPU time and memory storage but allows the computation of a few higher Mellin moments.

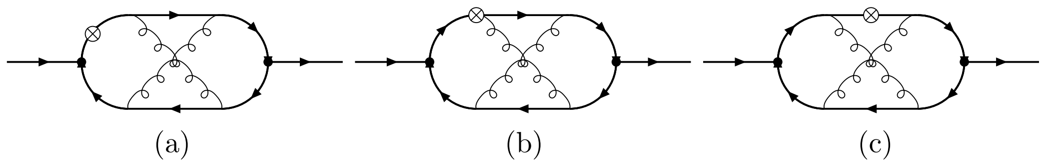


Figure 25: Crossed-box topologies with local operator insertions.

N	(a)	(b)	(c)
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1	$-\frac{1}{8}$	$\frac{1}{16} - \frac{7}{32}\zeta_3$	$-\frac{3}{16} + \frac{7}{32}\zeta_3$
2	$\frac{145}{1536} - \frac{145}{9216}\zeta_3$	$-\frac{89}{4608} + \frac{445}{3072}\zeta_3$	$-\frac{935}{13824} + \frac{87}{1024}\zeta_3$
3	$-\frac{81}{1024} + \frac{145}{6144}\zeta_3$	$\frac{8519}{55296} - \frac{2813}{12288}\zeta_3$	$-\frac{4993}{36864} + \frac{3145}{24576}\zeta_3$
4	$\frac{5582479}{82944000} - \frac{10489}{409600}\zeta_3$	$\frac{369197}{5529600} + \frac{18623}{737280}\zeta_3$	$\frac{2379019}{82944000} - \frac{719}{49152}\zeta_3$
5	$-\frac{1899679}{33177600} + \frac{36401}{1474560}\zeta_3$	$\frac{18015269}{99532800} - \frac{4794311}{22118400}\zeta_3$	$-\frac{39045971}{298598400} + \frac{507679}{4423680}\zeta_3$
6	$\frac{141912342181}{2913258700800} - \frac{695736571}{30828134400}\zeta_3$	$\frac{278864978351}{1248539443200} - \frac{5175109523}{39636172800}\zeta_3$	$\frac{1058933976943}{8739776102400} - \frac{255461723}{2642411520}\zeta_3$
7	$-\frac{11526313783}{277453209600} + \frac{59076777}{2936012800}\zeta_3$	$\frac{25191975655421}{74912366592000} - \frac{120819716411}{369937612800}\zeta_3$	$-\frac{7247023939349}{33294385152000} + \frac{370501349}{2013265920}\zeta_3$
8	$\frac{266608033463}{7491236659200} - \frac{29536680029}{1664719257600}\zeta_3$	$\frac{47884345670443}{89894839910400} - \frac{1916259725321}{4756340736000}\zeta_3$	$\frac{13258221091439}{44947419955200} - \frac{115670928497}{475634073600}\zeta_3$
9	$-\frac{255303766759}{8323596288000} + \frac{5768976713}{369937612800}\zeta_3$	$\frac{499790324842646\,47}{62926387937280000} - \frac{75636078173}{108716359680}\zeta_3$	$-\frac{3310967262876383}{6991820881920000} + \frac{4778989541}{12079595520}\zeta_3$

Table 2: Moments of the finite crossed-box graphs (a–c) shown in Figure 25.

8 Conclusions

For a precise analysis of the deeply inelastic scattering data recorded at HERA within one percent accuracy, a precise understanding of the heavy quark flavor contributions to the structure function F_2 at three loop level is essential. At $O(\alpha_s^3)$ the basic description of a single heavy quark to the unpolarized Wilson coefficients $C_{(q,g),2}^{\mathrm{NS,PS,S}}$ in the asymptotic region $Q^2 \gg m^2$ via massive operator matrix elements is known. It relies on a factorization theorem [128, 139] which allows to represent the heavy flavor Wilson coefficients at $Q^2 \gtrsim 10m^2$ [128] as a convolution of the light flavor Wilson coefficients which are known up to NNLO [104, 161] and the process independent massive operator matrix elements. This kinematic domain thereby covers the most interesting kinematic region a very important kinematic region of the deep-inelastic scattering experiments at HERA [129–132]. At $O(\alpha_s^3)$ a series of fixed Mellin moments $N = 2, \ldots, 10(12, 14)$ for all relevant OMEs are known [139].The main challenge for a complete understanding of the single heavy quark contributions is thus to obtain these operator matrix elements for general values of the Mellin variable N. Generally this thesis aims at extending the present understanding of the $O(\alpha_s^3)$ heavy flavor contributions to DIS at two fronts:

Firstly, there is no strong hierarchy between the charm- and bottom- quark masses and thus there exists a kinematic region where a description based on one single heavy and N_F light fermion flavors does not suffice. We aim to extend the present description to a scenario with two heavy quark flavors, to work out the respective renormalization prescriptions and to lay out the groundwork for a complete calculation of the respective contributions.

Secondly, despite recent success [156–158], a general algorithm to evaluate the respective Feynman diagrams remains unknown and one is thus forced to restrict oneself to subclasses and subtopologies of the complete set of Feynman diagrams. Therefore new gauge-invariant color contributions to different OMEs are calculated and new algorithms are explored in order to break the ground for a systematic evaluation of further topologies.

Section 3 presents the computation of the OME $A_{qg,Q}$. This completes the $O(C_{A,F}T_F^2 N_F)$ contributions to operator matrix elements contributing to the structure function F_2 [140], which constitutes the first complete color factors of the general N computation. The computation relied on representations within the functional class of generalized hypergeomtric functions and their respective sum representations. These allow for a Laurent series expansion in the dimensional regularization parameter ε and were afterwards evaluated using advanced summation algorithms encoded in the packages `Sigma` [236, 237] , `SumProduction` and `EvaluateMultiSums` [238, 239] . Furthermore, the method by representations in terms of special functions encouraged an automatized approach, with which the subtopologies which contain two completely massless bubble diagrams contributing to all relevant OMEs for the structure function F_2 and the variable flavor number scheme (VFNS) where solved.

In the case of a single heavy quark flavor all logarithmic contributions $\propto \ln(Q^2/m^2)$ to the heavy flavor Wilson coefficients are a direct consequence of the factorization relation and the renormalization prescription. They have been determined for the massive operator matrix elements $A_{qq,Q}^{\mathsf{NS}}$, A_{Qq}^{PS},$A_{qq,Q}^{\mathsf{PS}}$, A_{Qg}, $A_{qg,Q}$, $A_{gq,Q}$ and $A_{gg,Q}$ and the associated heavy flavor Wilson coefficients [220] in Section 4. The computation of the OME $A_{qg,Q}$ completed the Wilson coefficient $L_{g,2}^{\mathrm{S}}$ which has been presented together with the completed Wilson coefficient $L_{q,2}^{\mathrm{PS}}$ in this Section. Quantitatively the contributions of these two Wilson coefficients, however, turn out to be exceeded by other contributions to the structure function F_2.

The renormalization prescription for massive operator matrix elements which receive contributions from two different masses has been worked out in the Section 5. Here we apply an on-shell-

scheme for the mass renormalization. In order to renormalize the coupling constant we use the background field mechanism to obtain an intermediary MOM-scheme. Finally, we perform the transformation into the $\overline{\mathsf{MS}}$-scheme, where we assume the decoupling of the heavy quark contribution. We present all renormalization formulas for the respective two flavor contributions $\tilde{A}_{ij}$ in the on shell scheme for the masses and the $\overline{\mathsf{MS}}$-scheme for the coupling constant. Furthermore we provide the respective two-mass transformations to the $\overline{\mathsf{MS}}$, respectively MOM-scheme.
In Section 6 we extended the present description of the massive contributions to the VFNS and the structure function F_2 to the case of two heavy quark flavors up to $O(\alpha_s^3)$. This yields five OMEs $A_{qq,Q}^{\mathsf{NS}}(m_1^2, m_2^2, \mu^2)$, $A_{Qq}^{\mathsf{PS}}(m_1^2, m_2^2, \mu^2)$, $A_{Qg}(m_1^2, m_2^2, \mu^2)$, $A_{gq,Q}(m_1^2, m_2^2, \mu^2)$ and $A_{gg,Q}(m_1^2, m_2^2, \mu^2)$ which receive genuine contributions from Feynman diagrams with two heavy fermionic lines. We present the computation of a series of fixed Mellin moments $N = 2, 4, 6$ up to $O(\eta^3 \ln^3(\eta))$, with $\eta = m_2^2/m_1^2$ for all these OMEs. The respective Feynman diagrams were generated using `QGRAF`. Projection operators, cf. Ref. [139], were applied to map the diagrams with operator insertions onto pure tadpole diagrams. These were expanded in the mass ratio η and evaluated using the codes `Q2e`, `EXP` and `MATAD`. We computed the complete two-flavor contributions to the OME $A_{qq,Q}^{\mathrm{NS}}$ and the transversity OME $A_{qq,Q}^{\mathrm{NS,trans}}$. The computation was performed using Mellin-Barnes representations and yielded a structure in which the η dependence completely factors of the N-dependent parts. This property allows to expand the respective generalized hypergeometric functions using `HypExp` and to obtain the general N solution. The same approach is used to evaluate the respective contributions to the single-mass case with $m_1 = m_2$. For this case also the $O(C_F T_F^2)$-contribution to the OME A_{Qq}^{PS} could be evaluated. For other OMEs a larger entanglement between the Mellin variable N and the mass-ratio occurs. This demands more sophisticated approaches which have been described in Section 6.5. Here all scalar topologies contributing to the OME $A_{gg,Q}$ have been computed. The method is based on various mappings of the integration variables in order to obtain representations in which only one single Feynman parameter integral depends on the Mellin variable N and the Mellin Barnes variable ξ. These integral representations allow for a splitting of the integration domain which ensures the convergence of the associated associated contour integrals. Collecting the residues yielded sums which were evaluated using the packages `Sigma` [236,237] and `EvaluateMultiSums`. The remaining integral was remapped to obtain the x–space representations for the respective scalar diagrams. The Mellin space representations were obtained in a final step using difference equation methods encoded in `HarmonicSums` [175,177–179] and `Sigma` [236,237] to solve these equations. The method required new iterated integrals and sums in intermediary and final steps. On the integral side advanced methods to rewrite these integrals in suitable representations were developed.
In Section 7 the application of parametric integration methods via hyperlogarithms [159,187] to massive Feynman diagrams with operator insertions, which are convergent in $D = 4$ space-time dimensions has been explored. This approach turns out to be relatively efficient to compute fixed Mellin moments. In order to obtain general N representations for scalar diagrams of more evolved topologies, generating functions were constructed. These allowed for an evaluation using the methods of hyperlogarithms and in a final step the general N-results were reconstructed from these generating functions using methods encoded in `Sigma` [236,237] and `HarmonicSums` [175,177–179] . The method allowed to evaluate various new scalar topologies for the first time. While some diagrams could be made accessible to this method by a suitable transformation of the integration variables, it turns out that not all scalar topologies are solvable by this approach. This is due to the emergence of non-linearity of the denominator polynomials, which occur for some topologies.

Acknowledgement

First and foremost I would like to thank Prof. Johannes Blümlein for supervising this Thesis and for being a constant source of support and guidance. I would like to thank Carsten Schneider and Jakob Ablinger for many helpful discussions and their support with the use of `Sigma` and `HarmonicSums`. Furthermore, I would like to express my gratitude to Francis Brown for his hospitality and very enriching discussions. I thank Prof. Matthias Steinhauser for providing me with the codes `Q2e` and `EXP`. I am indebted to Arnd Behring for providing scripts and support to check the mathematical formulas within this Thesis for typographical errors. Furthermore, I would like to thank Alexander Hasselhuhn, Abilio de Freitas, Sebastian Klein, Clemens Raab and Mark Round for many interesting exchanges of ideas. Last but not least I would like to express my gratitude towards my family, my friends and especially Zinovia for providing me with moral support during this time.
This work has been supported in part by DESY, Studienstiftung des Deutschen Volkes, the EU Networks LHCPHENOnet PITN-GA-2010-264564 and by the FP7 ERC Starting Grant 257638 PAGAP.

A Conventions

We use natural units

$$\hbar = 1 \;, \quad c = 1 \;, \quad \varepsilon_0 = 1 \;, \tag{A.1}$$

where $\hbar$ denotes Planck's constant, c the vacuum speed of light and ε_0 the permittivity of vacuum. The electromagnetic fine–structure constant α is given by [312]

$$\alpha = \alpha'(\mu^2 = 0) = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{e^2}{4\pi} \approx \frac{1}{137.03599911(46)} \;. \tag{A.2}$$

In this convention, energies and momenta are given in the same units, electron volt (eV).
The space–time dimension is taken to be $D = 4+\varepsilon$ and the metric tensor $g_{\mu\nu}$ in Minkowski–space is defined as

$$g_{00} = 1 \;, \quad g_{ii} = -1 \;, i = 1 \ldots D-1 \;, \quad g_{ij} = 0 \;, i \neq j \;. \tag{A.3}$$

Einstein's summation convention is used in form of

$$x_\mu y^\mu := \sum_{\mu=0}^{D-1} x_\mu y^\mu \;. \tag{A.4}$$

Minkowski-space vectors are represented by

$$x = (x_0, \vec{x}) \;. \tag{A.5}$$

If not stated otherwise, Greek indices refer to the D–component space–time vector and Latin ones to the $D-1$ spatial components only. The Minkowski product of two vectors is defined by

$$p.q = p_0 q_0 - \sum_{i=1}^{D-1} p_i q_i \;. \tag{A.6}$$

The Dirac–matrices γ_μ are taken to be of dimension D and obey the anti–commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \;. \tag{A.7}$$

It follows that

$$\gamma_\mu\gamma^\mu = D \tag{A.8}$$
$$Tr\,(\gamma_\mu\gamma_\nu) = 4g_{\mu\nu} \tag{A.9}$$
$$Tr\,(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) = 4[g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}] \;. \tag{A.10}$$

The Feynman–dagger for a D-momentum p is defined by

$$\not{p} := \gamma_\mu p^\mu \;. \tag{A.11}$$

The conjugate of a bi–spinor u of a particle is given by

$$\overline{u} = u^\dagger\gamma_0 \;, \tag{A.12}$$

where † denote Hermitian and $*$ complex conjugation, respectively. The bi–spinors u and v fulfill the free Dirac–equation,

$$
\begin{aligned}
(\not{p}-m)u(p) &= 0 \;, \quad \overline{u}(p)(\not{p}-m)=0 && \text{(A.13)}\\
(\not{p}+m)v(p) &= 0 \;, \quad \overline{v}(p)(\not{p}+m)=0 \;. && \text{(A.14)}
\end{aligned}
$$

Bi–spinors and polarization vectors are normalized to

$$
\begin{aligned}
\sum_{\sigma} u(p,\sigma)\overline{u}(p,\sigma) &= \not{p}+m && \text{(A.15)}\\
\sum_{\sigma} v(p,\sigma)\overline{v}(p,\sigma) &= \not{p}-m && \text{(A.16)}\\
\sum_{\lambda} \epsilon^{\mu}(k,\lambda)\epsilon^{\nu}(k,\lambda) &= -g^{\mu\nu} \;, && \text{(A.17)}
\end{aligned}
$$

where λ and σ represent the spin.
The commonly used caret "$\hat{}$" to signify an operator, e.g. $\hat{O}$, is omitted if confusion is not to be expected.
The gauge symmetry group of QCD is the Lie–Group $SU(3)_c$. We consider the general case of $SU(N_c)$. The non–commutative generators are denoted by t^a, where a runs from 1 to N_c^2-1. The generators can be represented by Hermitian, traceless matrices, [201]. The structure constants f^{abc} and d^{abc} of $SU(N_c)$ are defined via the commutation and anti–commutation relations of its generators, [204, 313],

$$
\begin{aligned}
[t^a, t^b] &= i f^{abc} t^c && \text{(A.18)}\\
\{t^a, t^b\} &= d^{abc} t^c + \frac{1}{N_c}\delta_{ab} \;. && \text{(A.19)}
\end{aligned}
$$

The indices of the color matrices, in a given representation, are denoted by $i, j, k, l, ...$ The color invariants most commonly encountered are

$$
\begin{aligned}
\delta_{ab} C_A &= f^{acd} f^{bcd} && \text{(A.20)}\\
\delta_{ij} C_F &= t^a_{il} t^a_{lj} && \text{(A.21)}\\
\delta_{ab} T_F &= t^a_{ik} t^b_{ki} \;. && \text{(A.22)}
\end{aligned}
$$

These constants evaluate to

$$
C_A = N_c \;, \quad C_F = \frac{N_c^2-1}{2N_c} \;, \quad T_F = \frac{1}{2} \;, \qquad \text{(A.23)}
$$

in $SU(N_c)$. At higher loops, more irreducible color–invariants emerge. At 3–loop order, one additionally obtains

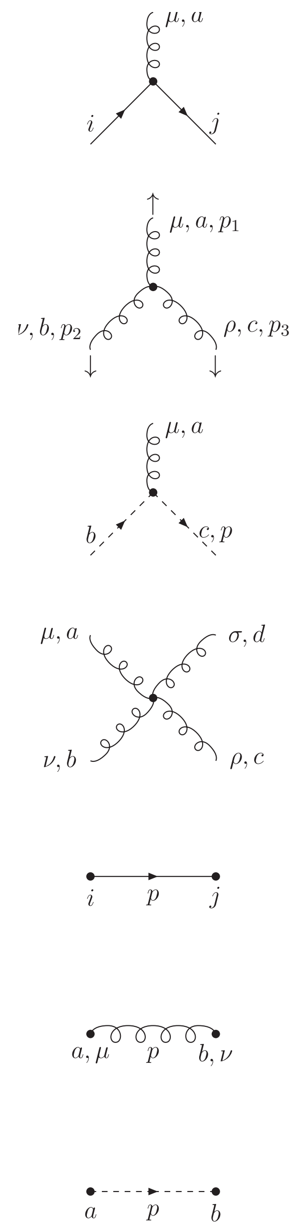
$$
d^{abd} d_{abc} = (N_c^2-1)(N_c^2-4)/N_c \;. \qquad \text{(A.24)}
$$

In case of $SU(3)_c$, $C_A = 3\;,\; C_F = 4/3\;,\; d^{abc} d_{abc} = 40/3$ holds.

B Feynman Rules

B.1 QCD-Feynman rules

For the QCD Feynman rules, Figure 26, we follow Ref. [139, 204], cf. also Refs. [314, 315]. D–dimensional momenta are denoted by p_i and Lorentz-indices by Greek letters. Color indices are $a, b, ...$ and i, j are indices of the color matrices. Solid directed lines represent fermions, curvy lines gluons and dashed lines ghosts. Arrows denote the direction of the momenta. A factor (-1) has to be included for each closed fermion– or ghost loop.



$$i g_s \gamma_\mu t^a_{ji}$$

$$-g_s f^{abc}[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho}]$$

$$-g_s f^{abc} p_\mu$$

$$-i g_s^2 \sum_e \Big\{ f^{abe} f^{cde} [g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}] + f^{ace} f^{bde} [g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}] + f^{ade} f^{cbe} [g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma}] \Big\}$$

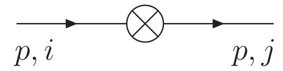
$$\frac{i}{\not{p} - m + i0} \delta_{ij}$$

$$\frac{i}{p^2 + i0} (-g_{\mu\nu} + \xi p_\mu p_\nu / (p^2 + i0)) \delta_{ab}$$

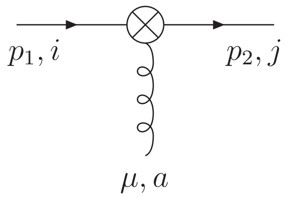
$$\frac{i}{p^2 + i0} \delta_{ab}$$

Figure 26: Feynman rules of QCD.

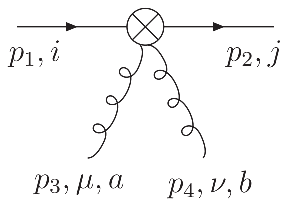
The Feynman rules for the quarkonic composite operators are given in Figure 27. Up to $O(g^2)$ they can be found in Ref. [316] and also in [317]. Note that the $O(g)$ term in the former reference contains a typographical error. In Ref. [139] these rules were checked and agree up to normalization factors, which may be due to different conventions. There also the new rule with three external gluons was given. The terms $\gamma_\pm$ refer to the unpolarized $(+)$ and polarized $(-)$ case, respectively. Gluon momenta are taken to be incoming.



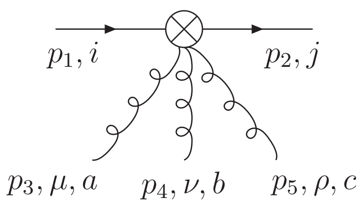
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} \;, \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} \;, \quad N \geq 2$$



$$\begin{gathered} g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \\ \left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] \;, \\ N \geq 3 \end{gathered}$$



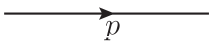
$$\begin{aligned} & g^3 \Delta^\mu \Delta^\nu \Delta^\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2} \\ & \Big[\quad (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\ & \quad + (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\ & \quad + (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\ & \quad + (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \\ & \quad + (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\ & \quad + (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big] \;, \\ & \qquad N \geq 4 \end{aligned}$$

$\gamma_+ = 1 \;, \quad \gamma_- = \gamma_5 \;.$ For transversity, one has to replace: $\not{\Delta} \gamma_\pm \to \; \sigma^{\mu\nu} \Delta_\nu$.

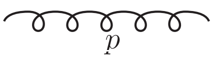
Figure 27: Feynman rules for quarkonic composite operators. Δ denotes a light-like 4-vector, $\Delta^2 = 0$; N is a suitably large positive integer, Ref. [139]

B.2 Scalar Feynman Rules

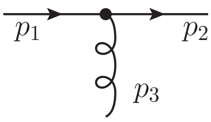
Additionally to complete QCD diagrams, in Sections 6.5 and 7 we calculated massive scalar Feynman integrals with a local operator insertion. The corresponding scalar Feynman rules read :



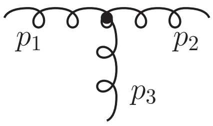
$$\frac{1}{(p^2 - m^2)} \tag{B.1}$$



$$\frac{1}{p^2} \tag{B.2}$$

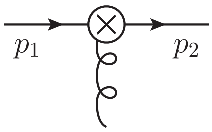


$$g \tag{B.3}$$

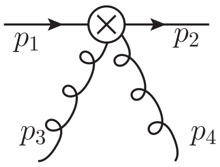


$$g \tag{B.4}$$

$$(\Delta.p)^N \tag{B.5}$$



$$g \sum_{j=0}^{N} (\Delta.p_1)^j (\Delta.p_2)^{N-j} \tag{B.6}$$



$$\begin{aligned} &\text{a)} \quad g^2 \sum_{j=0}^{N} \sum_{l=0}^{N-j} (\Delta.p_2)^j (\Delta.p_1)^{N-l-j} (\Delta.p_1 + \Delta.p_4)^l \\ &\text{b)} \quad g^2 \sum_{j=0}^{N} \sum_{l=0}^{N-j} (\Delta.p_2)^j (\Delta.p_1)^{N-l-j} (\Delta.p_1 + \Delta.p_3)^l \end{aligned} \tag{B.7}$$

C D-dimensional integrals

In the calculation of the D-dimensional loop integrals [250, 318, 319] with $D = 4+\varepsilon$, we perform first a Wick-rotation to Euclidean momenta

$$\prod_{i=1}^{M}\int\frac{d^Dk_i}{(2\pi)^D}f\left(k_i|_{i=1}^{M}\right)\prod_{i=1}^{M}\frac{1}{(k_i^2-m_i^2)^{a_i}} = (-1)^{-\sum_{j=1}^{M}a_j}\prod_{i=1}^{M}\int\frac{d^Dk_i}{(2\pi)^D}f\left(-k_i|_{i=1}^{M}\right) \times\prod_{i=1}^{M}\frac{1}{(-k_i^2-m_i^2)^{a_i}},$$

with $\forall a_i \in \mathbb{N}$. One obtains the following Euclidean integrals, where, cf. [204],

$$k_E^2 = k_0^2 + \vec{k}^2,$$

$$\begin{aligned}\int\frac{d^Dk}{(2\pi)^D}\frac{(k^2)^r}{(k^2-\varphi)^m} &= i(-1)^{(r-m)}\int\frac{d^Dk_E}{(2\pi)^D}\frac{(k_E^2)^r}{(k_E^2+\varphi)^m}\\ &= i\frac{(-1)^{(r-m)}}{(4\pi)^{D/2}}\frac{\Gamma(r+D/2)}{\Gamma(D/2)}\frac{\Gamma(m-r-D/2)}{\Gamma(m)}\varphi^{r-m+D/2},\\ &\quad m,r\in\mathbb{N}\end{aligned} \tag{C.8}$$

$$\int d^Dk\ \ k_1^\alpha k_2^\alpha\ \ f(k^2) = \frac{g^{\alpha_1\alpha_2}}{D}\int d^Dk\ \ k^2\ \ f(k^2) \tag{C.9}$$

$$\int d^Dk\ \ k_1^\alpha k_2^\alpha k_3^\alpha k_4^\alpha\ \ f(k^2) = \frac{g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_4}+g^{\alpha_1\alpha_3}g^{\alpha_2\alpha_4}+g^{\alpha_1\alpha_4}g^{\alpha_2\alpha_3}}{D^2+2D}\times\int d^Dk\ \ k^4 f(k^2) \tag{C.10}$$

$$\int d^Dk\ \ k_1^\alpha k_2^\alpha k_3^\alpha k_4^\alpha k_5^\alpha k_6^\alpha\ \ f(k^2) = \frac{Q(g^{\alpha_i\alpha_j})}{D^3+6D^2+8D}\int d^Dk\ \ k^6 f(k^2), \tag{C.11}$$

with

$$\begin{aligned}Q(g^{\alpha_i\alpha_j}) = \; & g^{\alpha_1\alpha_2}\left[g^{\alpha_3\alpha_4}g^{\alpha_5\alpha_6}+g^{\alpha_3\alpha_5}g^{\alpha_4\alpha_6}+g^{\alpha_3\alpha_6}g^{\alpha_4\alpha_5}\right]\\ &+g^{\alpha_1\alpha_3}\left[g^{\alpha_2\alpha_4}g^{\alpha_5\alpha_6}+g^{\alpha_2\alpha_5}g^{\alpha_4\alpha_6}+g^{\alpha_2\alpha_6}g^{\alpha_4\alpha_5}\right]\\ &+g^{\alpha_1\alpha_4}\left[g^{\alpha_2\alpha_3}g^{\alpha_5\alpha_6}+g^{\alpha_3\alpha_5}g^{\alpha_2\alpha_6}+g^{\alpha_3\alpha_6}g^{\alpha_2\alpha_5}\right]\\ &+g^{\alpha_1\alpha_5}\left[g^{\alpha_3\alpha_4}g^{\alpha_2\alpha_6}+g^{\alpha_2\alpha_3}g^{\alpha_4\alpha_6}+g^{\alpha_3\alpha_6}g^{\alpha_2\alpha_4}\right]\\ &+g^{\alpha_1\alpha_6}\left[g^{\alpha_3\alpha_4}g^{\alpha_2\alpha_5}+g^{\alpha_3\alpha_5}g^{\alpha_2\alpha_4}+g^{\alpha_2\alpha_3}g^{\alpha_4\alpha_5}\right]\end{aligned} \tag{C.12}$$

and

$$\int d^Dk\ \ \prod_{i=1}^{2M+1}k_i^\alpha\ \ f(k^2) = 0\ . \tag{C.13}$$

For each loop integral a universal factor

$$S_\varepsilon = \exp\left[(\gamma_E - \ln(4\pi))\ \frac{\varepsilon}{2}\right] \tag{C.14}$$

emerges, where γ_E denotes the Euler–Mascheroni constant

$$\gamma_E = \lim_{k\to\infty}\left[\sum_{l=1}^{k}\frac{1}{l} - \ln(l)\right] . \tag{C.15}$$

The factors S_ε are kept separately and are not expanded in ε. In the $\overline{\mathrm{MS}}$–scheme [88] they are set to $S_\varepsilon = 1$ at the end of the calculation.
The Γ-function obeys the relation

$$\Gamma(1+\varepsilon) = \exp\left[-\gamma_E\varepsilon + \sum_{n=2}^{\infty}\frac{(-\varepsilon)^n}{n}\zeta_n\right] , \tag{C.16}$$

with

$$\zeta_n = \sum_{k=1}^{\infty}\frac{1}{k^n}, \quad n\in\mathbb{N}, \quad n\geq 2 , \tag{C.17}$$

the Riemann ζ-function at integer arguments, $n \geq 2$.
We apply the following Feynman parametrization to combine denominators

$$\frac{1}{A_1\ldots A_n} = \Gamma(n)\int_0^1 dx_1\ldots\int_0^1 dx_n\delta\left(\sum_{k=1}^{n}x_k - 1\right)\frac{1}{(x_1A_1+\ldots+x_nA_n)^n} \tag{C.18}$$

resp.

$$\begin{aligned}\frac{1}{A_1^{a_1}\ldots A_n^{a_n}} &= \frac{\Gamma\left(\sum_{k=1}^{n}a_k\right)}{\prod_{k=1}^{N}\Gamma(a_k)}\int_0^1 dx_1\ldots\int_0^1 dx_n\delta\left(\sum_{k=1}^{n}x_k - 1\right)\\ &\quad\times\frac{\prod_{k=1}^{n}x_k^{a_k-1}}{(x_1A_1+\ldots+x_nA_n)^{\left(\sum_{k=1}^{n}a_k\right)}} ,\end{aligned} \tag{C.19}$$

with $\forall a_i \in \mathbb{N}$.
The integral over the δ–distribution yields

$$\begin{aligned}\int_0^1 dx_l\;\delta\left(\sum_{k=1}^{n}x_k - 1\right) &= \int_{-\infty}^{+\infty} dx_l\;\delta\left(\sum_{k=1}^{n}x_k - 1\right)\theta\left(1-\sum_{k=1}^{n}x_k\right)\prod_{m=1}^{n}\theta(x_m)\\ &= \theta\left(1-\sum_{k=1,k\neq l}^{n}x_k\right)\prod_{m=1,m\neq l}^{n}\theta(x_m) ,\end{aligned} \tag{C.20}$$

where $\theta(z)$ denotes the Heaviside function

$$\theta(z) = \begin{cases}1, & z\geq 0\\ 0, & z<0\end{cases} . \tag{C.21}$$

D Special functions

D.1 The Euler Integrals

The Γ-function, cf. [320, 321], is analytic in the whole complex plane except at the non-positive integers, where it possesses single poles. Euler's infinite product defines

$$\frac{1}{\Gamma(z)} = z\exp(\gamma_E z)\prod_{i=1}^{\infty}\left[\left(1+\frac{z}{i}\right)\exp\left(-\frac{z}{i}\right)\right] . \tag{D.22}$$

The residues of the Γ-function at its poles are given by

$$\mathsf{Res}[\Gamma(z)]_{z=-N} = \frac{(-1)^N}{N!} , \quad N \in \mathbb{N}\cup 0 . \tag{D.23}$$

In case of $\mathsf{Re}(z) > 0$, the Γ-function can be expressed by Euler's integral

$$\Gamma(z) = \int_0^\infty dt \exp(-t)\, t^{z-1} , \tag{D.24}$$

from which one infers the well known functional equation of the Γ-function

$$\Gamma(z+1) = z\Gamma(z) \ , z \neq -n, n \in \mathbb{N}\cup 0. \tag{D.25}$$

Eq. (D.25) may be used to continue the Γ-function analytically. Around $z = 1$, the following series expansion is obtained

$$\Gamma(1-\varepsilon) \ = \ \exp(\varepsilon\gamma_E)\exp\left\{\sum_{i=2}^{\infty}\zeta_i\frac{\varepsilon^i}{i}\right\} , |\varepsilon| < 1 . \tag{D.26}$$

Here and in (D.22), γ_E denotes the Euler-Mascheroni constant, cf. (C.15), and ζ_k Riemann's ζ–function for integer arguments k, cf. (C.17). A shorthand notation for rational functions of Γ–functions is

$$\Gamma\begin{bmatrix} a_1, ..., a_i \\ b_1, ..., b_j \end{bmatrix} := \frac{\Gamma(a_1)...\Gamma(a_i)}{\Gamma(b_1)...\Gamma(b_j)} . \tag{D.27}$$

A further useful identity involving the Γ-function is given by

$$\Gamma(\varepsilon - N) = (-1)^N\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(N+1-\varepsilon)} . \tag{D.28}$$

Functions closely related to the Γ-function are the Euler Beta-function $B(A,C)$ function, the $\psi(x)-$, and the $\beta(x)$–functions.
The Beta-function can be defined by Eq. (D.27)

$$B(A,C) = \Gamma\begin{bmatrix} A, C \\ A+C \end{bmatrix} . \tag{D.29}$$

If $\mathsf{Re}(A), \mathsf{Re}(C) > 0$, the following integral representation is valid

$$B(A,C) = \int_0^1 dx\ x^{A-1}(1-x)^{C-1} . \tag{D.30}$$

For arbitrary values of A and C, (D.30) can be continued analytically outside of the respective singularities using Eqs. (D.22, D.29). Its expansion around singularities can be performed via Eqs. (D.23, D.26). The ψ-function and $\beta(x)$ are defined as logarithmic derivatives of the Γ-function via

$$\psi(x) = \frac{1}{\Gamma(x)}\frac{d}{dx}\Gamma(x) , \tag{D.31}$$

$$\beta(x) = \frac{1}{2}\left[\psi\Big(\frac{x+1}{2}\Big) - \psi\Big(\frac{x}{2}\Big)\right] . \tag{D.32}$$

D.2 Polylogarithms

The polylogarithms [283–285] are defined on the unit disk in the complex plane by

$$\mathrm{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k} . \tag{D.33}$$

This definition can be uniquely extended to the whole complex plane by a analytic continuation. In the special case $z = 1$ (D.33) becomes the sum representation of the Riemann ζ–function

$$\mathrm{Li}_k(1) = \zeta_k , k \geq 2 . \tag{D.34}$$

The polylogarithm (D.33) furthermore has the representation

$$\mathrm{Li}_0(z) = \frac{z}{1-z} \tag{D.35}$$

and

$$\mathrm{Li}_{k+1}(z) = \int_0^z \mathrm{d}t \frac{\mathrm{Li}_k(t)}{t} , k \geq 0 . \tag{D.36}$$

When considering the combination $\mathrm{Li}_k(-z) + \mathrm{Li}_k(z)$ all odd powers in definition (D.33) cancel and one obtains the identity

$$\mathrm{Li}_k(-z) + \mathrm{Li}_k(z) = 2^{1-k}\mathrm{Li}_k(z^2) . \tag{D.37}$$

D.2.1 Harmonic polylogarithms

A more general form of the iterated integrals are the harmonic polylogarithms [180]. A harmonic polylogarithm of weight w is uniquely defined by

$$H_{\{\}}(x) = 1 , \tag{D.38}$$

$$H_{\{\underbrace{0,\cdots,0}_{\text{n times}}\}}(x) = \frac{1}{n!}\ln^n(x) , \tag{D.39}$$

and otherwise

$$H_{a,\vec{w}}(x) = \int_0^1 dx\ f_a(x) H_{\vec{w}}(x) . \tag{D.40}$$

Here $a \in \{-1,0,1\}$ and f_a is defined by

$$f_1 = \frac{1}{1-x}, \quad f_0 = \tfrac{1}{x}, \quad f_{-1} = \frac{1}{1+x} \ . \tag{D.41}$$

All these iterated integral structures are associated to specific sum structures in two different ways:
They emerge in the coefficients c_n, d_n of the series expansions around 0 and ∞, e.g.

$$L_{\vec{v}}(z) = \sum_{k=0}^{w(v)} \sum_{n=0}^{\infty} c_n z^n \ , \tag{D.42}$$

$$L_{\vec{v}}(z) = \sum_{k=0}^{w(v)} \sum_{n=0}^{\infty} d_n z^{-n} \ . \tag{D.43}$$

Furthermore the same sum structures are observed in the Mellin transforms of the same iterated integrals.

D.3 Sum structures

We distinguish the following classes of sums [286, 311]:
Harmonic sums have been observed in physical results for some time by now. Many of their algebraic [166–168], differential and structural relations [169–171] have been studied. They are defined by

$$\begin{aligned} S_{a_1,\ldots,a_m}(N) &= \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \cdots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}} \ , \\ & N \in \mathbb{N}, \ \forall \ l \ a_l \in \mathbb{Z} \setminus 0 \ , \qquad \text{(D.44)} \\ S_{\emptyset} &= 1 \ . \qquad \text{(D.45)} \end{aligned}$$

A first generalization, the *generalized harmonic sums* are obtained by allowing for additional weights in the numerators [174, 175]:

$$\begin{aligned} S_{a_1,\ldots,a_m}(x_1,\ldots,x_m,N) &= \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(x_1)^{n_1}}{n_1^{a_1}} \frac{(x_2)^{n_2}}{n_2^{a_2}} \cdots \frac{(x_m)^{n_m}}{n_m^{a_m}} \ , \\ & N \in \mathbb{N}, \ \forall \ l \ a_l \in \mathbb{N} \ , \qquad \text{(D.46)} \\ S_{\emptyset}(\emptyset) &= 1 \ . \qquad \text{(D.47)} \end{aligned}$$

Furthermore, in recent calculations finite sums weighted by binomials $\binom{2i}{i}$ in both, numerators and denominators have been frequently observed [287, 296]. They are of a similar form as

$$\begin{aligned} & \sum_{i=1}^{N} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{j\binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2}, -1; j\right) \qquad \text{(D.48)} \\ &= \int_0^1 dx \frac{x^N - 1}{x-1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{w_{17},-1,0}(x) - 2\mathrm{H}^*_{w_{18},-1,0}(x)\right] \end{aligned}$$

$$
+\frac{\zeta_2}{2}\int_0^1 dx \frac{(-x)^N-1}{x+1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}_{12}^*(x)-2\mathrm{H}_{13}^*(x)\right]+c_3\int_0^1 dx \frac{(-8x)^N-1}{x+\frac{1}{8}}\sqrt{\frac{x}{1-x}}\ ,
$$

where in this example

$$
\begin{aligned}
w_{12} &= \frac{1}{\sqrt{x(8-x)}}, \qquad & w_{13} &= \frac{1}{(2-x)\sqrt{x(8-x)}}, \\
w_{17} &= \frac{1}{\sqrt{x(8+x)}}, \qquad & w_{18} &= \frac{1}{(2+x)\sqrt{x(8+x)}}\ .
\end{aligned} \tag{D.49}
$$

Here the $\mathrm{H}_{\mathrm{xy}}^*$ represent integrals over the word w_{xy} on the interval $[x,1]$.

E The logarithmic contributions to the heavy flavor Wilson coefficients

In this Appendix we present all logarithmic contributions to asymptotic heavy flavor Wilson coefficients $H_{q,2}^{\rm PS}$ and $H_{g,2}^{S}$ with respect to the single-mass case, see also Ref. [220].

$$
\begin{aligned}
H_{q,2}^{\mathsf{PS}} &= \tfrac{1}{2}[1+(-1)^N] \\
&\times\Biggl\{a_s^2\Biggl\{C_FT_F\Biggl[-\frac{4L_M^2\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{\bigl(4S_1^2-12S_2\bigr)\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)} \\
&+\frac{4L_Q^2\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{32(-1)^NP_{45}}{3(N-1)N^2(N+1)^3(N+2)^3} \\
&+\frac{8P_{75}}{3(N-1)N^4(N+1)^4(N+2)^3}+\frac{8P_{57}S_1}{(N-1)N^3(N+1)^3(N+2)^2} \\
&+L_Q\Biggl[-\frac{8S_1\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{8P_{57}}{(N-1)N^3(N+1)^3(N+2)^2}\Biggr] \\
&+\frac{64S_{-2}}{(N-1)N(N+1)(N+2)}-\frac{8\bigl(N^2+5N+2\bigr)\bigl(5N^3+7N^2+4N+4\bigr)L_M}{(N-1)N^3(N+1)^3(N+2)^2}\Biggr]\Biggr\} \\
&+a_s^3\Biggl\{C_F^2T_F\Biggl[L_Q^3\Biggl[\frac{8\bigl(N^2+N+2\bigr)^2\bigl(3N^2+3N+2\bigr)}{3(N-1)N^3(N+1)^3(N+2)}-\frac{32\bigl(N^2+N+2\bigr)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Biggr] \\
&+L_Q^2\Biggl[\frac{\bigl(24S_1^2-24S_2\bigr)\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{4P_{73}}{(N-1)N^4(N+1)^4(N+2)^2} \\
&+\frac{8P_{62}S_1}{(N-1)N^3(N+1)^3(N+2)^2}\Biggr]+L_Q\Biggl[\frac{\bigl(104S_1S_2-\frac{56}{3}S_1^3\bigr)\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)} \\
&-\frac{16\bigl(N^2+N-22\bigr)S_3\bigl(N^2+N+2\bigr)}{3(N-1)N^2(N+1)^2(N+2)}+\frac{\bigl(128S_{-3}-256S_{-2,1}-384\zeta_3\bigr)\bigl(N^2+N+2\bigr)}{(N-1)N^2(N+1)^2(N+2)} \\
&-\frac{4P_{67}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2}-\frac{64(-1)^NP_{98}}{15(N-2)(N-1)^3N^3(N+1)^5(N+2)^4(N+3)^3} \\
&+\frac{4P_{100}}{15(N-1)^3N^5(N+1)^5(N+2)^4(N+3)^3}+\frac{128(-1)^NP_{45}S_1}{3(N-1)N^2(N+1)^3(N+2)^3} \\
&-\frac{8P_{79}S_1}{3(N-1)N^4(N+1)^4(N+2)^3}+\frac{512S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)}+\frac{4P_{65}S_2}{(N-1)N^3(N+1)^3(N+2)^2} \\
&+L_M^2\Biggl[\frac{16\bigl(N^2+N+2\bigr)^2S_1}{(N-1)N^2(N+1)^2(N+2)}-\frac{4\bigl(N^2+N+2\bigr)^2\bigl(3N^2+3N+2\bigr)}{(N-1)N^3(N+1)^3(N+2)}\Biggr] \\
&+L_M\Biggl[\frac{32\bigl(N^2+5N+2\bigr)\bigl(5N^3+7N^2+4N+4\bigr)S_1}{(N-1)N^3(N+1)^3(N+2)^2} \\
&-\frac{8\bigl(N^2+5N+2\bigr)\bigl(3N^2+3N+2\bigr)\bigl(5N^3+7N^2+4N+4\bigr)}{(N-1)N^4(N+1)^4(N+2)^2}\Biggr] \\
&+\frac{32P_{69}S_{-2}}{(N-2)(N-1)N^3(N+1)^3(N+2)(N+3)}\Biggr]-\frac{8(3N+2)\bigl(N^2+N+2\bigr)S_1^3}{3(N-1)N^3(N+1)(N+2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4P_{74}S_1^2}{(N-1)N^4(N+1)^4(N+2)^3}-2\frac{\big(N^2+N+2\big)\zeta_2}{(N-1)N^4(N+1)^4(N+2)}P_{55}-\frac{4P_{94}}{(N-1)N^5(N+1)^6(N+2)^3}\\
&-\frac{4}{3}\frac{\big(N^2+N+2\big)^2\big(3N^2+3N+2\big)\zeta_3}{(N-1)N^3(N+1)^3(N+2)}+4\frac{\big(N^2+N+2\big)\big(5N^4+4N^3+N^2-10N-8\big)\zeta_2}{(N-1)N^3(N+1)^3(N+2)}S_1\\
&+\frac{4P_{91}S_1}{(N-1)N^5(N+1)^5(N+2)^3}+L_M^3\Biggl[\frac{4\big(N^2+N+2\big)^2\big(3N^2+3N+2\big)}{3(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{16\big(N^2+N+2\big)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Biggr]-\frac{8\big(N^2+N+2\big)\big(3N^4+9N^3+15N^2+11N-2\big)S_1S_2}{(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{4P_{76}S_2}{(N-1)N^4(N+1)^4(N+2)^3}+L_M^2\Biggl[-\frac{4\big(13N^2+5N-6\big)S_1\big(N^2+N+2\big)^2}{(N-1)N^3(N+1)^3(N+2)}\\
&+\frac{\big(24S_2-8S_1^2\big)\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{4P_{51}\big(N^2+N+2\big)}{(N-1)N^4(N+1)^4(N+2)}\Biggr]\\
&-\frac{8\big(N^2+N+2\big)\big(3N^4+48N^3+43N^2-22N-8\big)S_3}{3(N-1)N^3(N+1)^3(N+2)}+\frac{32\big(N^2-3N-2\big)\big(N^2+N+2\big)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)}\\
&+\frac{\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}\Biggl[\frac{2}{3}S_1^4-12S_2S_1^2+\Biggl[\frac{16S_3}{3}+32S_{2,1}\Biggr]S_1+\frac{16}{3}\zeta_3S_1+18S_2^2-12S_4\\
&+32S_{3,1}-64S_{2,1,1}+\big(4S_1^2-12S_2\big)\zeta_2\Biggr]+L_M\Biggl[\frac{\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}\Biggl[\frac{8}{3}S_1^3-24S_2S_1\\
&-\frac{80S_3}{3}+32S_{2,1}+96\zeta_3\Biggr]-\frac{4P_{60}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2}-\frac{4P_{92}}{(N-1)N^5(N+1)^5(N+2)^3}\\
&+\frac{8P_{77}S_1}{(N-1)N^4(N+1)^4(N+2)^3}-\frac{4P_{59}S_2}{(N-1)N^3(N+1)^3(N+2)^2}\Biggr]\Biggr]\\
&+C_FT_F^2\Biggl[\Biggl[\frac{32\big(N^2+N+2\big)^2L_Q^3}{9(N-1)N^2(N+1)^2(N+2)}-\frac{32P_{66}L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+L_Q\Biggl[\frac{32\big(N^2+N+2\big)^2L_M^2}{3(N-1)N^2(N+1)^2(N+2)}+\Biggl[\frac{64\big(N^2+N+2\big)\big(8N^3+13N^2+27N+16\big)}{9(N-1)N^2(N+1)^3(N+2)}\\
&-\frac{64\big(N^2+N+2\big)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Biggr]L_M-\frac{256(-1)^NP_{45}}{9(N-1)N^2(N+1)^3(N+2)^3}\\
&+\frac{64P_{82}}{27(N-1)N^4(N+1)^4(N+2)^3}+\frac{512S_{-2}}{3(N-1)N(N+1)(N+2)}\Biggr]\\
&-\frac{128\big(N^2+N+2\big)^2L_M^3}{9(N-1)N^2(N+1)^2(N+2)}+\frac{16\big(N^2+N+2\big)\big(7N^4+16N^3+32N^2+19N+2\big)S_1^2}{3(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{32\big(11N^5+26N^4+57N^3+142N^2+84N+88\big)L_M^2}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{32\zeta_2P_{48}}{9(N-1)N^3(N+1)^2(N+2)^2}\\
&+\frac{32P_{95}}{81(N-1)N^5(N+1)^5(N+2)^4}-\frac{32P_{68}S_1}{27(N-1)N^3(N+1)^4(N+2)}\\
&+L_M\Biggl[\frac{\Bigl[\frac{32}{3}S_1^2-32S_2\Bigr]\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{64P_{80}}{27(N-1)N^4(N+1)^4(N+2)^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{64P_{57}S_1}{3(N-1)N^3(N+1)^3(N+2)^2}\Biggr]+\frac{16P_{61}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\frac{\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}\Bigl[-\frac{64}{9}S_1^3-\frac{32}{3}S_2S_1-\frac{32}{3}\zeta_2S_1+\frac{160S_3}{9}+\frac{128\zeta_3}{9}\Bigr]\Biggr]\Biggr]\\
&+N_FT_F^2C_F\Biggl[\frac{32\bigl(N^2+N+2\bigr)^2L_Q^3}{9(N-1)N^2(N+1)^2(N+2)}-\frac{32P_{66}L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\Biggl[\frac{\bigl[-\frac{16}{3}S_1^2-\frac{16S_2}{3}\bigr]\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{32\bigl(8N^3+13N^2+27N+16\bigr)S_1\bigl(N^2+N+2\bigr)}{9(N-1)N^2(N+1)^3(N+2)}\\
&-\frac{256(-1)^NP_{45}}{9(N-1)N^2(N+1)^3(N+2)^3}+\frac{32P_{84}}{27(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{512S_{-2}}{3(N-1)N(N+1)(N+2)}\Biggr]L_Q-\frac{32\bigl(N^2+N+2\bigr)^2L_M^3}{9(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{16}{9}\frac{\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}P_{63}-\frac{32P_{87}}{3(N-1)N^5(N+1)^5(N+2)^4}\\
&+L_M^2\Biggl[\frac{32P_{64}}{9(N-1)N^3(N+1)^3(N+2)^2}-\frac{32\bigl(N^2+N+2\bigr)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Biggr]\\
&+L_M\Biggl[\frac{\bigl[-\frac{16}{3}S_1^2-\frac{80S_2}{3}\bigr]\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{32P_{78}}{27(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{32P_{63}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\Biggr]+\frac{64\bigl(N^2+5N+2\bigr)\bigl(5N^3+7N^2+4N+4\bigr)S_2}{3(N-1)N^3(N+1)^3(N+2)^2}\\
&+\frac{\bigl(N^2+N+2\bigr)^2\bigl[\frac{64S_3}{3}+\frac{16}{3}S_1\zeta_2+\frac{32\zeta_3}{9}\bigr]}{(N-1)N^2(N+1)^2(N+2)}\Biggr]\\
&+C_AC_FT_F\Biggl[L_Q^3\Biggl[-\frac{16S_1\bigl(N^2+N+2\bigr)^2}{3(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{8\bigl(11N^4+22N^3-23N^2-34N-12\bigr)\bigl(N^2+N+2\bigr)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2}\Biggr]\\
&+L_Q^2\Biggl[-\frac{16\bigl(5N^2-1\bigr)S_1\bigl(N^2+N+2\bigr)^2}{(N-1)^2N^3(N+1)^3(N+2)}+\frac{\bigl(16S_1^2-16S_2-32S_{-2}\bigr)\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{16(-1)^N\bigl(N^5+9N^4+24N^3+36N^2+32N+8\bigr)\bigl(N^2+N+2\bigr)}{(N-1)N^3(N+1)^4(N+2)^3}\\
&+\frac{8P_{85}}{9(N-1)^2N^4(N+1)^3(N+2)^3}\Biggr]+L_Q\Biggl[\frac{\bigl(N^2+N+2\bigr)^2}{(N-1)N^2(N+1)^2(N+2)}\Bigl[-\frac{40}{3}S_1^3+40S_2S_1\\
&-144S_{-3}+96S_{-2,1}\Bigr]+\frac{4P_{47}S_1^2\bigl(N^2+N+2\bigr)}{3(N-1)^2N^3(N+1)^3(N+2)^2}+48\frac{\bigl(N^2+N+6\bigr)\bigl(N^2+N+2\bigr)\zeta_3}{(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{32S_{-2}S_1\bigl(N^2+N+2\bigr)}{N^2(N+1)^2}+\frac{4P_{52}S_2\bigl(N^2+N+2\bigr)}{3(N-1)^2N^3(N+1)^3(N+2)^2}+\frac{32(-1)^NP_{93}}{9(N-1)^2N^4(N+1)^5(N+2)^4}\\
&-\frac{8\bigl(13N^2+13N+62\bigr)S_3\bigl(N^2+N+2\bigr)}{3(N-1)N^2(N+1)^2(N+2)}-\frac{8P_{97}}{27(N-1)^2N^5(N+1)^5(N+2)^4}
\end{aligned}
$$

$$
-\frac{32(-1)^N P_{72}S_1}{3(N-1)N^3(N+1)^4(N+2)^3}-\frac{8P_{88}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^3}
$$
$$
+\frac{16P_{71}S_{-2}}{3(N-1)^2N^3(N+1)^3(N+2)^2}\Bigg]+\frac{8\big(N^2+N+2\big)\big(N^3+8N^2+11N+2\big)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2}
$$
$$
+\frac{4\big(N^2+N+2\big)P_{43}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3}+8\big(N^2+N+2\big)\frac{(-1)^N\zeta_2}{(N-1)N^3(N+1)^4(N+2)^3}P_{44}
$$
$$
-\frac{8}{9}\frac{\big(N^2+N+2\big)\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2}P_{49}+\frac{4}{9}\frac{\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3}P_{89}
$$
$$
+\frac{8P_{99}}{3(N-1)^2N^6(N+1)^6(N+2)^5}-\frac{4}{3}\frac{\big(N^2+N+2\big)\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2}P_{53}S_1
$$
$$
-\frac{8\big(N^2+N+2\big)P_{70}S_1}{(N-1)N^2(N+1)^5(N+2)^4}+L_M^3\Bigg[\frac{16S_1\big(N^2+N+2\big)^2}{3(N-1)N^2(N+1)^2(N+2)}
$$
$$
+\frac{8\big(11N^4+22N^3-23N^2-34N-12\big)\big(N^2+N+2\big)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2}\Bigg]+\frac{4P_{86}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^3}
$$
$$
-\frac{8\big(N^2+N+2\big)\big(3N^3-12N^2-27N-2\big)S_1S_2}{(N-1)N^2(N+1)^3(N+2)^2}-\frac{16\big(N^2+N+2\big)P_{50}S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2}
$$
$$
+\frac{32(-1)^N\big(N^2+N+2\big)\big(N^4+2N^3+7N^2+22N+20\big)S_{-2}}{(N-1)N(N+1)^4(N+2)^3}
$$
$$
+L_M^2\Bigg[\frac{\big(16S_2+32S_{-2}\big)\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{8P_{54}S_1\big(N^2+N+2\big)}{3(N-1)^2N^3(N+1)^3(N+2)^2}
$$
$$
-\frac{16(-1)^N\big(N^5+9N^4+24N^3+36N^2+32N+8\big)\big(N^2+N+2\big)}{(N-1)N^3(N+1)^4(N+2)^3}
$$
$$
-\frac{8P_{83}}{9(N-1)^2N^4(N+1)^3(N+2)^3}\Bigg]+\frac{\big(N^2-N-4\big)\big(N^2+N+2\big)}{(N-1)N(N+1)^3(N+2)^2}\Big[-64(-1)^NS_1S_{-2}
$$
$$
-32(-1)^NS_{-3}+64S_{-2,1}-32(-1)^NS_1\zeta_2-24(-1)^N\zeta_3\Big]
$$
$$
+\frac{\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}\Big[-\frac{2}{3}S_1^4-20S_2S_1^2-32(-1)^NS_{-3}S_1+\Big(64S_{-2,1}-\frac{160S_3}{3}\Big)S_1
$$
$$
-\frac{8}{3}\big(-7+9(-1)^N\big)\zeta_3S_1-2S_2^2+S_{-2}\big(-32(-1)^NS_1^2-32(-1)^NS_2\big)-36S_4-16(-1)^NS_{-4}
$$
$$
+16S_{3,1}+32S_{-2,2}+32S_{-3,1}+16S_{2,1,1}-64S_{-2,1,1}+\big(-4\big(-3+4(-1)^N\big)S_1^2
$$
$$
-4\big(-1+4(-1)^N\big)S_2-8\big(1+2(-1)^N\big)S_{-2}\big)\zeta_2\Big]+L_M\Bigg[\frac{\big(N^2+N+2\big)^2}{(N-1)N^2(N+1)^2(N+2)}\Big[-\frac{8}{3}S_1^3
$$
$$
+40S_2S_1+32\big(1+(-1)^N\big)S_{-2}S_1+16(-1)^NS_{-3}-32S_{2,1}+12\big(-9+(-1)^N\big)\zeta_3\Big]
$$
$$
+\frac{4\big(17N^4-6N^3+41N^2-16N-12\big)S_1^2\big(N^2+N+2\big)}{3(N-1)^2N^3(N+1)^2(N+2)}+\frac{4P_{56}S_2\big(N^2+N+2\big)}{3(N-1)^2N^3(N+1)^3(N+2)^2}
$$
$$
+\frac{8\big(31N^2+31N+74\big)S_3\big(N^2+N+2\big)}{3(N-1)N^2(N+1)^2(N+2)}+\frac{16\big(7N^2+7N+10\big)S_{-3}\big(N^2+N+2\big)}{(N-1)N^2(N+1)^2(N+2)}
$$
$$
-\frac{128\big(N^2+N+1\big)S_{-2,1}\big(N^2+N+2\big)}{(N-1)N^2(N+1)^2(N+2)}+\frac{\big(N^2-N-4\big)\big(N^2+N+2\big)32(-1)^NS_{-2}}{(N-1)N(N+1)^3(N+2)^2}
$$

$$
-\frac{64(-1)^N P_{81}}{9(N-1)N^3(N+1)^5(N+2)^4}+\frac{8P_{96}}{27(N-1)^2N^5(N+1)^5(N+2)^4}
$$
$$
+\frac{64(-1)^N P_{46}S_1}{3(N-1)N^2(N+1)^3(N+2)^3}-\frac{8P_{90}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^3}
$$
$$
+\frac{16P_{58}S_{-2}}{(N-1)N^3(N+1)^3(N+2)^2}\Bigg]\Bigg]+a_{Qq}^{\mathsf{PS},(3)}+\tilde{C}_{2,q}^{\mathsf{PS},(3)}(N_F+1)\Bigg\}\Bigg\}, \tag{E.50}
$$

with the polynomials

$$P_{43} = N^6+6N^5+7N^4+4N^3+18N^2+16N-8 \tag{E.51}$$
$$P_{44} = 2N^6+7N^5+31N^4+82N^3+86N^2+32N+8 \tag{E.52}$$
$$P_{45} = 4N^6+22N^5+48N^4+53N^3+45N^2+36N+8 \tag{E.53}$$
$$P_{46} = 5N^6+29N^5+78N^4+118N^3+114N^2+72N+16 \tag{E.54}$$
$$P_{47} = 5N^6+135N^5+327N^4+329N^3+220N^2-176N-120 \tag{E.55}$$
$$P_{48} = 8N^6+29N^5+84N^4+193N^3+162N^2+124N+24 \tag{E.56}$$
$$P_{49} = 11N^6+6N^5+75N^4+68N^3-200N^2-80N-24 \tag{E.57}$$
$$P_{50} = 11N^6+29N^5-7N^4-25N^3-56N^2-72N-24 \tag{E.58}$$
$$P_{51} = 16N^6+35N^5+33N^4-11N^3-41N^2-36N-12 \tag{E.59}$$
$$P_{52} = 17N^6-57N^5-213N^4-175N^3-140N^2+64N+72 \tag{E.60}$$
$$P_{53} = 17N^6+27N^5+75N^4+149N^3-20N^2-80N-24 \tag{E.61}$$
$$P_{54} = 17N^6+51N^5+51N^4+89N^3+40N^2-80N-24 \tag{E.62}$$
$$P_{55} = 38N^6+108N^5+151N^4+106N^3+21N^2-28N-12 \tag{E.63}$$
$$P_{56} = 73N^6+189N^5+45N^4+31N^3-238N^2-412N-120 \tag{E.64}$$
$$P_{57} = N^7-15N^5-58N^4-92N^3-76N^2-48N-16 \tag{E.65}$$
$$P_{58} = 2N^7+14N^6+37N^5+102N^4+155N^3+158N^2+132N+40 \tag{E.66}$$
$$P_{59} = 3N^7-15N^6-153N^5-577N^4-854N^3-652N^2-408N-128 \tag{E.67}$$
$$P_{60} = 5N^7+19N^6+61N^5+197N^4+266N^3+212N^2+136N+32 \tag{E.68}$$
$$P_{61} = 5N^7+37N^6+188N^5+643N^4+925N^3+742N^2+460N+120 \tag{E.69}$$
$$P_{62} = 7N^7+21N^6+5N^5-117N^4-244N^3-232N^2-192N-80 \tag{E.70}$$
$$P_{63} = 8N^7+37N^6+68N^5-11N^4-86N^3-56N^2-104N-48 \tag{E.71}$$
$$P_{64} = 8N^7+37N^6+83N^5+85N^4+61N^3+58N^2-20N-24 \tag{E.72}$$
$$P_{65} = 9N^7+15N^6-103N^5-575N^4-998N^3-948N^2-696N-256 \tag{E.73}$$
$$P_{66} = 11N^7+37N^6+53N^5+7N^4-68N^3-56N^2-80N-48 \tag{E.74}$$
$$P_{67} = 25N^7+91N^6+101N^5-195N^4-546N^3-556N^2-520N-224 \tag{E.75}$$
$$P_{68} = 62N^7+329N^6+986N^5+1790N^4+2242N^3+1653N^2+650N+96 \tag{E.76}$$
$$P_{69} = N^8+8N^7+8N^6-14N^5-53N^4-82N^3+60N^2+104N+96 \tag{E.77}$$
$$P_{70} = 2N^8+22N^7+117N^6+386N^5+759N^4+810N^3+396N^2+72N+32 \tag{E.78}$$
$$P_{71} = 6N^8-42N^7-241N^6-579N^5-307N^4+477N^3+602N^2+492N+168 \tag{E.79}$$
$$P_{72} = 10N^8+71N^7+244N^6+497N^5+698N^4+720N^3+512N^2+248N+48 \tag{E.80}$$
$$
\begin{aligned}
P_{73} = {} & 19N^9+86N^8+144N^7-38N^6-535N^5-1016N^4-1180N^3-872N^2\\
& -416N-96
\end{aligned} \tag{E.81}
$$
$$
\begin{aligned}
P_{74} = {} & N^{10}+15N^9+105N^8+361N^7+660N^6+828N^5+814N^4+384N^3\\
& -112N^2-128N-32
\end{aligned} \tag{E.82}
$$

$$P_{75} = 6N^{10} + 49N^9 + 197N^8 + 472N^7 + 833N^6 + 1469N^5 + 2142N^4 + 1904N^3 + 1040N^2 + 432N + 96 \quad \text{(E.83)}$$

$$P_{76} = 11N^{10} + 123N^9 + 541N^8 + 1273N^7 + 1806N^6 + 1672N^5 + 1006N^4 + 320N^3 - 16N^2 - 64N - 32 \quad \text{(E.84)}$$

$$P_{77} = 19N^{10} + 143N^9 + 412N^8 + 426N^7 - N^6 + 159N^5 + 1066N^4 + 1552N^3 + 1456N^2 + 848N + 224 \quad \text{(E.85)}$$

$$P_{78} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288 \quad \text{(E.86)}$$

$$P_{79} = 60N^{10} + 397N^9 + 1073N^8 + 1111N^7 + 623N^6 + 4328N^5 + 12432N^4 + 15944N^3 + 12704N^2 + 6816N + 1728 \quad \text{(E.87)}$$

$$P_{80} = 67N^{10} + 383N^9 + 867N^8 + 696N^7 - 755N^6 - 2391N^5 - 3027N^4 - 2744N^3 - 1256N^2 - 48N + 144 \quad \text{(E.88)}$$

$$P_{81} = 77N^{10} + 646N^9 + 2553N^8 + 6903N^7 + 14498N^6 + 22898N^5 + 24861N^4 + 17068N^3 + 7040N^2 + 1760N + 192 \quad \text{(E.89)}$$

$$P_{82} = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 + 1864N^2 + 1248N + 432 \quad \text{(E.90)}$$

$$P_{83} = 118N^{10} + 675N^9 + 1588N^8 + 1652N^7 + 326N^6 + 357N^5 + 876N^4 + 1672N^3 + 3440N^2 + 2544N + 576 \quad \text{(E.91)}$$

$$P_{84} = 127N^{10} + 740N^9 + 1737N^8 + 1308N^7 - 1592N^6 - 2226N^5 + 1386N^4 + 3064N^3 + 3040N^2 + 2496N + 864 \quad \text{(E.92)}$$

$$P_{85} = 151N^{10} + 708N^9 + 1156N^8 + 464N^7 - 967N^6 + 372N^5 + 3672N^4 + 5236N^3 + 6152N^2 + 3792N + 864 \quad \text{(E.93)}$$

$$P_{86} = 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3 + 6512N^2 + 2432N + 384 \quad \text{(E.94)}$$

$$P_{87} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32 \quad \text{(E.95)}$$

$$P_{88} = 118N^{11} + 529N^{10} + 1264N^9 + 3846N^8 + 11353N^7 + 23684N^6 + 32793N^5 + 31801N^4 + 22836N^3 + 10448N^2 + 2592N + 432 \quad \text{(E.96)}$$

$$P_{89} = 127N^{11} + 820N^{10} + 2197N^9 + 1890N^8 - 1847N^7 - 1960N^6 + 3843N^5 + 9730N^4 + 13632N^3 + 10688N^2 + 4944N + 864 \quad \text{(E.97)}$$

$$P_{90} = 136N^{11} + 1039N^{10} + 3100N^9 + 3534N^8 - 1295N^7 - 6352N^6 - 8421N^5 - 11729N^4 - 7644N^3 + 1376N^2 + 1920N + 144 \quad \text{(E.98)}$$

$$P_{91} = 7N^{12} + 47N^{11} + 123N^{10} + 76N^9 - 598N^8 - 2178N^7 - 3626N^6 - 3933N^5 - 3254N^4 - 1608N^3 - 144N^2 + 112N + 32 \quad \text{(E.99)}$$

$$P_{92} = 37N^{12} + 305N^{11} + 1017N^{10} + 1462N^9 + 592N^8 + 408N^7 + 4064N^6 + 9645N^5 + 12222N^4 + 10280N^3 + 6064N^2 + 2192N + 352 \quad \text{(E.100)}$$

$$P_{93} = 242N^{12} + 1853N^{11} + 6173N^{10} + 12711N^9 + 18608N^8 + 17040N^7 - 302N^6 - 24986N^5 - 32225N^4 - 20010N^3 - 7904N^2 - 2016N - 288 \quad \text{(E.101)}$$

$$P_{94} = 5N^{13} + 27N^{12} - 97N^{11} - 1410N^{10} - 5754N^9 - 12428N^8 - 16530N^7 - 14531N^6 - 7956N^5 - 1038N^4 + 2176N^3 + 1632N^2 + 448N + 32 \quad \text{(E.102)}$$

$$P_{95} = 119N^{13} + 1897N^{12} + 12595N^{11} + 48221N^{10} + 124877N^9 + 239946N^8$$

$$
\begin{aligned}
& +345670N^7+356234N^6+253043N^5+129982N^4+55768N^3+20112N^2 \\
& +5616N+864 \qquad \text{(E.103)}\\
P_{96} &= 686N^{14}+8408N^{13}+39228N^{12}+89257N^{11}+113445N^{10}+109336N^9 \\
& +76360N^8-109649N^7-393915N^6-482272N^5-376932N^4-263440N^3 \\
& -155472N^2 \\
& -56448N-8640 \qquad \text{(E.104)}\\
P_{97} &= 1790N^{14}+15938N^{13}+56250N^{12}+90805N^{11}+43917N^{10}-38450N^9-42314N^8 \\
& -169217N^7-616623N^6-992860N^5-964980N^4-697072N^3-376464N^2 \\
& -127872N-19008 \qquad \text{(E.105)}\\
P_{98} &= 30N^{16}+397N^{15}+1996N^{14}+3786N^{13}-3905N^{12}-30084N^{11}-44372N^{10} \\
& +5100N^9+71344N^8+27709N^7-104744N^6-146534N^5-30293N^4+77346N^3 \\
& +33768N^2-23544N-3888 \qquad \text{(E.106)}\\
P_{99} &= 12N^{17}+162N^{16}+1030N^{15}+4188N^{14}+11527N^{13}+19051N^{12}+11176N^{11} \\
& -17182N^{10}-36527N^9-27469N^8-11770N^7+5554N^6+32640N^5+46456N^4 \\
& +34528N^3+14816N^2+3584N+384 \qquad \text{(E.107)}\\
P_{100} &= 1245N^{18}+19980N^{17}+133282N^{16}+461805N^{15}+787161N^{14}+185392N^{13} \\
& -1368400N^{12}-225082N^{11}+6978631N^{10}+13143336N^9+5808466N^8 \\
& -11433627N^7-19928573N^6-12013164N^5+1462668N^4+8209584N^3 \\
& +6906384N^2+2980800N+544320\ . \qquad \text{(E.108)}
\end{aligned}
$$

The Wilson coefficient $H_{g,2}^{\mathsf{S}}$, except for the constant contribution $a_{Qg}^{(3)}$, has a similar structure. It is given by :

$$
\begin{aligned}
& H_{g,2}^{\mathsf{S}} = \tfrac{1}{2}[1+(-1)^N] \\
& \times\Biggl\{a_sT_F\Biggl\{-\tilde{\gamma}_{qg}^0L_Q-\frac{4\bigl(N^3-4N^2-N-2\bigr)}{N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0S_1+\tilde{\gamma}_{qg}^0L_M\Biggr\} \\
& +a_s^2\Biggl\{T_F^2\Biggl[\frac{4}{3}\tilde{\gamma}_{qg}^0L_M^2-\frac{4}{3}\tilde{\gamma}_{qg}^0L_QL_M+\Biggl[\frac{4}{3}\tilde{\gamma}_{qg}^0S_1-\frac{16\bigl(N^3-4N^2-N-2\bigr)}{3N^2(N+1)(N+2)}\Biggr]L_M\Biggr] \\
& +C_FT_F\Biggl[\Biggl[\Biggl[\frac{2\bigl(N^2+N+2\bigr)\bigl(3N^2+3N+2\bigr)}{N^2(N+1)^2(N+2)}+2\tilde{\gamma}_{qg}^0S_1\Biggr]L_Q^2+\Biggl[-\frac{4P_{111}}{N^3(N+1)^3(N+2)} \\
& +\frac{4\bigl(3N^4+2N^3-9N^2-16N-12\bigr)S_1}{N^2(N+1)^2(N+2)}+L_M\Biggl[-\frac{4\bigl(N^2+N+2\bigr)\bigl(3N^2+3N+2\bigr)}{N^2(N+1)^2(N+2)} \\
& -4\tilde{\gamma}_{qg}^0S_1\Biggr]+\tilde{\gamma}_{qg}^0\bigl(4S_2-4S_1^2\bigr)\Biggr]L_Q-\frac{2\bigl(9N^4+6N^3-15N^2-28N-20\bigr)S_1^2}{N^2(N+1)^2(N+2)} \\
& +\frac{16(-1)^NP_{252}}{15(N-2)(N-1)^2N^2(N+1)^4(N+2)^4(N+3)^3}+L_M^2\Biggl[\frac{2\bigl(N^2+N+2\bigr)\bigl(3N^2+3N+2\bigr)}{N^2(N+1)^2(N+2)} \\
& +2\tilde{\gamma}_{qg}^0S_1\Biggr]+\frac{4P_{265}}{15(N-1)^2N^4(N+1)^4(N+2)^4(N+3)^3}+\frac{4P_{106}S_1}{N^3(N+1)^3(N+2)} \\
& +L_M\Biggl[\frac{4P_{111}}{N^3(N+1)^3(N+2)}-\frac{4\bigl(3N^4+2N^3-9N^2-16N-12\bigr)S_1}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Bigl[4S_1^2-4S_2\Bigr]\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\big(11N^4+42N^3+43N^2-4N-12\big)S_2}{N^2(N+1)^2(N+2)}+\frac{16P_{103}S_{-2}}{(N-2)N^2(N+1)^2(N+2)(N+3)}\\
&+\tilde{\gamma}_{qg}^0\Big[2S_1^3-2S_2S_1-4S_{2,1}\Big]+\frac{128S_1S_{-2}+64S_{-3}-128S_{-2,1}}{N(N+1)(N+2)}+\frac{(N-1)\Big[48\zeta_3-16S_3\Big]}{N(N+1)}\Bigg]\\
&+C_AT_F\Bigg[\Bigg[\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)}{(N-1)N^2(N+1)^2(N+2)^2}+2\tilde{\gamma}_{qg}^0S_1\Bigg]L_Q^2+\Bigg[-\frac{32(-1)^N\big(N^3+4N^2+7N+5\big)}{(N+1)^3(N+2)^3}\\
&+\frac{8P_{160}}{(N-1)N^3(N+1)^2(N+2)^3}+\frac{16\big(N^2+1\big)\big(N^2-4N-1\big)S_1}{(N-1)N(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[-2S_1^2+2S_2+4S_{-2}\Big]\Bigg]L_Q\\
&-\frac{16P_{165}S_1}{(N-1)N^3(N+1)^2(N+2)^3}-\frac{4\big(2N^5-3N^4-5N^3-3N^2-33N-6\big)S_1^2}{(N-1)N(N+1)^2(N+2)^2}\\
&-\frac{16(-1)^NP_{167}}{3(N-1)N^2(N+1)^4(N+2)^4}-\frac{8P_{239}}{3(N-1)N^4(N+1)^4(N+2)^4}\\
&+\frac{32(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{(N+1)^3(N+2)^3}+L_M^2\Bigg[-\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)}{(N-1)N^2(N+1)^2(N+2)^2}-2\tilde{\gamma}_{qg}^0S_1\Bigg]\\
&+\frac{4P_{105}S_2}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{8\big(3N^2+3N+2\big)S_3}{N(N+1)(N+2)}+L_M\Bigg[\frac{32(-1)^N\big(N^3+4N^2+7N+5\big)}{(N+1)^3(N+2)^3}\\
&-\frac{8P_{161}}{(N-1)N^3(N+1)^2(N+2)^3}-\frac{32(2N+3)S_1}{(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[-2S_1^2-2S_2-4S_{-2}\Big]\Bigg]\\
&+\frac{16\big(N^5-N^4-5N^3+3N^2+14N+12\big)S_{-2}}{(N-1)N(N+1)^2(N+2)^2}-\frac{16\big(N^2+N+4\big)S_{-3}}{N(N+1)(N+2)}\\
&+\frac{(N-1)\Big[16S_{-2}S_1-16S_{-2,1}\Big]}{N(N+1)}+\frac{\big(N^2-N-4\big)16(-1)^NS_{-2}}{(N+1)^2(N+2)^2}-\frac{6\big(5N^2+5N-6\big)\zeta_3}{N(N+1)(N+2)}\\
&+\tilde{\gamma}_{qg}^0\Big[-8S_1S_2-4(-1)^NS_{-2}S_1-2(-1)^NS_{-3}+4S_{2,1}-\frac{3}{2}(-1)^N\zeta_3\Big]\Bigg]\Bigg\}\\
&+a_s^3\Bigg\{T_F^3\Bigg[\frac{16}{9}\tilde{\gamma}_{qg}^0L_M^3-\frac{16}{9}\tilde{\gamma}_{qg}^0L_QL_M^2+\Bigg[\frac{16}{9}\tilde{\gamma}_{qg}^0S_1-\frac{64\big(N^3-4N^2-N-2\big)}{9N^2(N+1)(N+2)}\Bigg]L_M^2-\frac{16\tilde{\gamma}_{qg}^0\zeta_3}{9}\Bigg]\\
&+C_AT_F^2\Bigg[\Bigg[\frac{64\big(N^2+N+1\big)\big(N^2+N+2\big)}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\Bigg]L_Q^3+\Bigg[-\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)}{3(N+1)^3(N+2)^3}\\
&+\frac{8P_{194}}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32\big(8N^4-7N^3+5N^2-17N-13\big)S_1}{9(N-1)N(N+1)^2(N+2)}\\
&+L_M\Bigg[\frac{64\big(N^2+N+1\big)\big(N^2+N+2\big)}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{8}{3}\tilde{\gamma}_{qg}^0S_1\Bigg]+\tilde{\gamma}_{qg}^0\Big[-\frac{4}{3}S_1^2+\frac{4S_2}{3}+\frac{8}{3}S_{-2}\Big]\Bigg]L_Q^2\\
&+\Bigg[-\frac{32\big(8N^4-7N^3+5N^2-17N-13\big)S_1^2}{9(N-1)N(N+1)^2(N+2)}+\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{3(N+1)^3(N+2)^3}\\
&-\frac{32P_{186}S_1}{27(N-1)N^2(N+1)^3(N+2)^3}+\frac{64(-1)^NP_{170}}{9(N-1)N^2(N+1)^4(N+2)^4}\\
&-\frac{16P_{233}}{27(N-1)N^3(N+1)^4(N+2)^4}+L_M^2\Bigg[\frac{64\big(N^2+N+1\big)\big(N^2+N+2\big)}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{8}{3}\tilde{\gamma}_{qg}^0S_1\Bigg]
\end{aligned}
$$

$$
+\frac{32\big(8N^4+13N^3-22N^2-9N-26\big)S_2}{9(N-1)N(N+1)(N+2)^2}+\frac{64\big(8N^5+15N^4+6N^3+11N^2+16N+16\big)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}
$$
$$
+\frac{128\big(N^2+N-1\big)S_3}{9N(N+1)(N+2)}+L_M\Bigg[-\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)}{3(N+1)^3(N+2)^3}+\frac{32P_{195}}{9(N-1)N^3(N+1)^3(N+2)^3}
$$
$$
-\frac{64(2N-1)\big(N^3+9N^2+7N+7\big)S_1}{9(N-1)N(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[-\frac{8}{3}S_1^2+\frac{8S_2}{3}+\frac{16}{3}S_{-2}\Big]\Bigg]-\frac{128\big(N^2+N+3\big)S_{-3}}{3N(N+1)(N+2)}
$$
$$
+\tilde{\gamma}_{qg}^0\Big[\frac{8}{9}S_1^3-8S_2S_1+\frac{32}{3}S_{2,1}\Big]+\frac{256S_{-2,1}}{3N(N+1)(N+2)}+\frac{(N-1)\Big[\frac{64}{3}S_{-2}S_1-32\zeta_3\Big]}{N(N+1)}\Bigg]L_Q
$$
$$
+\frac{32\big(N^3+8N^2+11N+2\big)S_1^3}{9N(N+1)^2(N+2)^2}+\frac{8P_{152}S_1^2}{27N(N+1)^3(N+2)^3}+\frac{4}{9}\frac{P_{201}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^3}
$$
$$
+\frac{8P_{264}}{81(N-1)N^5(N+1)^5(N+2)^5}+\frac{32}{3}\big(2N^4+3N^3+10N^2+37N+35\big)\frac{(-1)^N\zeta_2}{(N+1)^3(N+2)^3}
$$
$$
+\frac{32}{9}\frac{\big(9N^5-4N^4+N^3+92N^2+42N+28\big)\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}-\frac{8P_{249}S_1}{81(N-1)N^4(N+1)^4(N+2)^4}
$$
$$
-\frac{32}{9}\frac{\big(5N^4+8N^3+17N^2+43N+20\big)\zeta_2}{N(N+1)^2(N+2)^2}S_1+L_M^3\Bigg[-\frac{448\big(N^2+N+1\big)\big(N^2+N+2\big)}{9(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
-\frac{56}{9}\tilde{\gamma}_{qg}^0S_1\Bigg]+\frac{8P_{192}S_2}{3(N-1)N^3(N+1)^3(N+2)^3}-\frac{32\big(3N^3-12N^2-27N-2\big)S_1S_2}{3N(N+1)^2(N+2)^2}
$$
$$
+\frac{256\big(N^5+10N^4+9N^3+3N^2+7N+6\big)S_3}{9(N-1)N^2(N+1)^2(N+2)^2}+L_M^2\Bigg[\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)}{(N+1)^3(N+2)^3}
$$
$$
-\frac{8P_{199}}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32\big(8N^5+9N^4-57N^3-31N^2+25N-26\big)S_1}{9(N-1)N(N+1)^2(N+2)^2}
$$
$$
+\tilde{\gamma}_{qg}^0\Big[-\frac{20}{3}S_1^2-4S_2-8S_{-2}\Big]\Bigg]+\frac{128(-1)^N\big(N^4+2N^3+7N^2+22N+20\big)S_{-2}}{3(N+1)^3(N+2)^3}
$$
$$
+\frac{\big(N^2-N-4\big)}{(N+1)^2(N+2)^2}\Big[-\frac{256}{3}(-1)^NS_1S_{-2}-\frac{128}{3}(-1)^NS_{-3}+\frac{256}{3}S_{-2,1}-\frac{128}{3}(-1)^NS_1\zeta_2
$$
$$
-32(-1)^N\zeta_3\Big]+\tilde{\gamma}_{qg}^0\Big[\frac{2}{9}S_1^4+\frac{20}{3}S_2S_1^2+\frac{32}{3}(-1)^NS_{-3}S_1+\big(\frac{160S_3}{9}-\frac{64}{3}S_{-2,1}\big)S_1+\frac{2}{3}S_2^2
$$
$$
+\frac{8}{9}\big(-2+9(-1)^N\big)\zeta_3S_1+S_{-2}\big(\frac{32}{3}(-1)^NS_1^2+\frac{32}{3}(-1)^NS_2\big)+12S_4
$$
$$
+\frac{16}{3}(-1)^NS_{-4}-\frac{16}{3}S_{3,1}-\frac{32}{3}S_{-2,2}-\frac{32}{3}S_{-3,1}-\frac{16}{3}S_{2,1,1}+\frac{64}{3}S_{-2,1,1}+\Big[\frac{2}{3}\big(-3+8(-1)^N\big)S_1^2
$$
$$
+\frac{2}{3}\big(-3+8(-1)^N\big)S_2+\frac{4}{3}\big(1+4(-1)^N\big)S_{-2}\Big]\zeta_2\Big]+L_M\Bigg[\frac{64\big(N^5+9N^4+3N^3+N^2+26N-4\big)S_1^2}{9(N-1)N(N+1)^2(N+2)^2}
$$
$$
+\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{3(N+1)^3(N+2)^3}+\frac{16P_{198}S_1}{27(N-1)N^3(N+1)^3(N+2)^3}+\frac{64(N-1)S_{-2}S_1}{3N(N+1)}
$$
$$
-\frac{64(-1)^NP_{157}}{9(N-1)N^2(N+1)^3(N+2)^4}+\frac{16P_{242}}{27(N-1)N^4(N+1)^4(N+2)^4}+\frac{64\big(7N^2+7N+8\big)S_3}{9N(N+1)(N+2)}
$$
$$
-\frac{64P_{104}S_2}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{64\big(2N^5+21N^4+36N^3-7N^2-68N-56\big)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}
$$

$$
-\frac{128S_{-3}}{3N(N+1)(N+2)}-\frac{128S_{-2,1}}{3(N+2)}+\frac{\big(N^2-N-4\big)\frac{128}{3}(-1)^NS_{-2}}{(N+1)^2(N+2)^2}-\frac{16\big(3N^2+3N-2\big)\zeta_3}{N(N+1)(N+2)}
$$
$$
+\tilde{\gamma}_{qg}^0\Big[-\frac{8}{9}S_1^3-\frac{40}{3}S_2S_1-\frac{32}{3}(-1)^NS_{-2}S_1-\frac{16}{3}(-1)^NS_{-3}-4(-1)^N\zeta_3\Big]\Bigg]\Bigg]
$$
$$
+C_AT_F^2N_F\Bigg[\Bigg[\Bigg[\frac{64\big(N^2+N+1\big)\big(N^2+N+2\big)}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\Bigg]L_Q^3+\Bigg[-\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)}{3(N+1)^3(N+2)^3}
$$
$$
+\frac{8P_{194}}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32\big(8N^4-7N^3+5N^2-17N-13\big)S_1}{9(N-1)N(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[-\frac{4}{3}S_1^2+\frac{4S_2}{3}
$$
$$
+\frac{8}{3}S_{-2}\Big]\Bigg]L_Q^2+\Bigg[-\frac{32\big(8N^4-7N^3+5N^2-17N-13\big)S_1^2}{9(N-1)N(N+1)^2(N+2)}+\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{3(N+1)^3(N+2)^3}
$$
$$
-\frac{16P_{187}S_1}{27(N-1)N^2(N+1)^3(N+2)^3}+\frac{64(-1)^NP_{170}}{9(N-1)N^2(N+1)^4(N+2)^4}+\frac{128\big(N^2+N-1\big)S_3}{9N(N+1)(N+2)}
$$
$$
-\frac{8P_{248}}{27(N-1)N^4(N+1)^4(N+2)^4}+\frac{32\big(8N^4+13N^3-22N^2-9N-26\big)S_2}{9(N-1)N(N+1)(N+2)^2}
$$
$$
+\frac{64\big(8N^5+15N^4+6N^3+11N^2+16N+16\big)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}-\frac{128\big(N^2+N+3\big)S_{-3}}{3N(N+1)(N+2)}
$$
$$
+\tilde{\gamma}_{qg}^0\Big[\frac{8}{9}S_1^3-8S_2S_1+\frac{32}{3}S_{2,1}\Big]+\frac{256S_{-2,1}}{3N(N+1)(N+2)}+\frac{(N-1)\Big[\frac{64}{3}S_{-2}S_1-32\zeta_3\Big]}{N(N+1)}\Bigg]L_Q
$$
$$
+\frac{16\big(N^3+8N^2+11N+2\big)S_1^3}{9N(N+1)^2(N+2)^2}+\frac{8P_{102}S_1^2}{3N(N+1)^3(N+2)^3}+\frac{4}{9}\frac{P_{175}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}
$$
$$
+\frac{16P_{261}}{3(N-1)N^5(N+1)^5(N+2)^5}+\frac{16}{9}\frac{\big(9N^5-14N^4-19N^3+52N^2+12N+8\big)\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
-\frac{16P_{166}S_1}{3N(N+1)^4(N+2)^4}+\frac{16}{9}\frac{\big(5N^4+32N^3+47N^2+28N+20\big)\zeta_2}{N(N+1)^2(N+2)^2}S_1
$$
$$
+L_M^3\Bigg[-\frac{64\big(N^2+N+1\big)\big(N^2+N+2\big)}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\Bigg]+\frac{8P_{191}S_2}{3(N-1)N^3(N+1)^3(N+2)^3}
$$
$$
-\frac{16\big(3N^3-12N^2-27N-2\big)S_1S_2}{3N(N+1)^2(N+2)^2}+\frac{128\big(N^5+10N^4+9N^3+3N^2+7N+6\big)S_3}{9(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
+L_M^2\Bigg[\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)}{3(N+1)^3(N+2)^3}-\frac{8P_{172}}{9(N-1)N^2(N+1)^3(N+2)^3}
$$
$$
-\frac{32\big(5N^4+20N^3+47N^2+58N+20\big)S_1}{9N(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[-\frac{4}{3}S_1^2-\frac{4S_2}{3}-\frac{8}{3}S_{-2}\Big]\Bigg]
$$
$$
+\frac{\big(N^4+2N^3+7N^2+22N+20\big)\Big[\frac{64}{3}(-1)^NS_{-2}+\frac{32}{3}(-1)^N\zeta_2\Big]}{(N+1)^3(N+2)^3}
$$
$$
+\frac{\big(N^2-N-4\big)\Big[-\frac{128}{3}(-1)^NS_1S_{-2}-\frac{64}{3}(-1)^NS_{-3}+\frac{128}{3}S_{-2,1}-\frac{64}{3}(-1)^NS_1\zeta_2-16(-1)^N\zeta_3\Big]}{(N+1)^2(N+2)^2}
$$
$$
+\tilde{\gamma}_{qg}^0\Big[\frac{1}{9}S_1^4+\frac{10}{3}S_2S_1^2+\frac{16}{3}(-1)^NS_{-3}S_1+\Big[\frac{80S_3}{9}-\frac{32}{3}S_{-2,1}\Big]S_1+\frac{4}{9}\big(-7+9(-1)^N\big)\zeta_3S_1+\frac{1}{3}S_2^2
$$
$$
+S_{-2}\Big[\frac{16}{3}(-1)^NS_1^2+\frac{16}{3}(-1)^NS_2\Big]+6S_4+\frac{8}{3}(-1)^NS_{-4}-\frac{8}{3}S_{3,1}-\frac{16}{3}S_{-2,2}-\frac{16}{3}S_{-3,1}-\frac{8}{3}S_{2,1,1}
$$

$$
\begin{aligned}
&+\frac{32}{3}S_{-2,1,1}+\left[\frac{4}{3}\big(-1+2(-1)^N\big)S_1^2+\frac{4}{3}\big(-1+2(-1)^N\big)S_2+\frac{8}{3}(-1)^NS_{-2}\right]\zeta_2\Bigg]\\
&+L_M\Bigg[-\frac{16\big(10N^4+43N^3+106N^2+131N+46\big)S_1^2}{9N(N+1)^2(N+2)^2}+\frac{16P_{140}S_1}{27N(N+1)^3(N+2)^3}\\
&-\frac{64(-1)^N\big(7N^5+43N^4+117N^3+166N^2+107N+16\big)}{9(N+1)^4(N+2)^4}+\frac{8P_{246}}{27(N-1)N^4(N+1)^4(N+2)^4}\\
&-\frac{16P_{122}S_2}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{64\big(5N^2+8N+10\big)S_{-2}}{9N(N+1)(N+2)}+\frac{\big(N^2-N-4\big)\frac{64}{3}(-1)^NS_{-2}}{(N+1)^2(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[-\frac{8}{9}S_1^3-\frac{8}{3}S_2S_1-\frac{16}{3}(-1)^NS_{-2}S_1-\frac{40S_3}{9}-\frac{8}{3}\big(2+(-1)^N\big)S_{-3}-\frac{16}{3}S_{2,1}+\frac{16}{3}S_{-2,1}\Big]\Bigg]\Bigg]\\
&+C_F^2T_F\Bigg[-\frac{\big(15N^4+6N^3-25N^2-32N-28\big)S_1^4}{3N^2(N+1)^2(N+2)}+\frac{2P_{159}S_1^2}{N^3(N+1)^4(N+2)}\\
&-\frac{4\big(3N^5-47N^4-147N^3-93N^2+8N+12\big)S_1^3}{3N^3(N+1)^2(N+2)}-4\frac{\big(3N^4+14N^3+43N^2+48N+20\big)\zeta_2}{N^2(N+1)^2(N+2)}S_1^2\\
&-\frac{2\big(5N^4-14N^3+53N^2+120N+28\big)S_2S_1^2}{N^2(N+1)^2(N+2)}-\frac{2P_{211}S_1}{N^5(N+1)^5(N+2)}-\frac{4P_{118}S_2S_1}{N^3(N+1)^3(N+2)}\\
&-\frac{8\big(3N^4+90N^3+83N^2-44N-4\big)S_3S_1}{3N^2(N+1)^2(N+2)}-\frac{16\big(3N^4+2N^3+19N^2+28N+12\big)S_{2,1}S_1}{N^2(N+1)^2(N+2)}\\
&-\frac{8P_{132}\zeta_2S_1}{N^3(N+1)^3(N+2)}-\frac{\big(11N^4+142N^3+147N^2-32N-12\big)S_2^2}{N^2(N+1)^2(N+2)}-\frac{P_{247}}{N^6(N+1)^6(N+2)}\\
&+16\frac{(-1)^N\big(N^2+N+2\big)\zeta_2}{N(N+1)^4(N+2)}+\frac{2}{3}\big(N^2+N+2\big)\frac{\big(153N^4+306N^3+165N^2+12N+4\big)\zeta_3}{N^3(N+1)^3(N+2)}\\
&+L_Q^3\Bigg[\frac{2\big(N^2+N+2\big)\big(3N^2+3N+2\big)^2}{3N^3(N+1)^3(N+2)}-\frac{16\big(N^2+N+2\big)S_1\big(3N^2+3N+2\big)}{3N^2(N+1)^2(N+2)}\\
&-\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2\Bigg]+L_M^3\Bigg[-\frac{2\big(N^2+N+2\big)\big(3N^2+3N+2\big)^2}{3N^3(N+1)^3(N+2)}+\frac{16\big(N^2+N+2\big)S_1\big(3N^2+3N+2\big)}{3N^2(N+1)^2(N+2)}\\
&+\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2\Bigg]-\frac{2P_{182}S_2}{N^4(N+1)^4(N+2)}+\frac{4\big(21N^5+217N^4+415N^3+351N^2+152N-4\big)S_3}{3N^2(N+1)^3(N+2)}\\
&-\frac{16\big(N^2-3N-2\big)\big(3N^2+3N+2\big)S_{2,1}}{N^3(N+1)^2(N+2)}+L_M^2\Bigg[\frac{32\big(N^3+5N^2+6N+4\big)S_1^2}{N^2(N+1)^2(N+2)}\\
&+\frac{2P_{138}S_1}{N^3(N+1)^3(N+2)}-\frac{32(-1)^N\big(N^2+N+2\big)}{N(N+1)^4(N+2)}-\frac{P_{178}}{N^4(N+1)^4(N+2)}\\
&-\frac{16\big(N^2+N+2\big)\big(3N^2+3N+2\big)S_2}{N^2(N+1)^2(N+2)}-\frac{32\big(N^2+N+2\big)S_{-2}}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[8S_1^3-16S_2S_1\\
&-16S_{-2}S_1-8S_3-8S_{-3}+16S_{-2,1}\Big]\Bigg]+L_Q^2\Bigg[\frac{32\big(N^3+5N^2+6N+4\big)S_1^2}{N^2(N+1)^2(N+2)}+\frac{2P_{138}S_1}{N^3(N+1)^3(N+2)}\\
&-\frac{32(-1)^N\big(N^2+N+2\big)}{N(N+1)^4(N+2)}-\frac{P_{178}}{N^4(N+1)^4(N+2)}+L_M\Bigg[-\frac{2\big(N^2+N+2\big)\big(3N^2+3N+2\big)^2}{N^3(N+1)^3(N+2)}\\
&+\frac{16\big(N^2+N+2\big)S_1\big(3N^2+3N+2\big)}{N^2(N+1)^2(N+2)}+8\tilde{\gamma}_{qg}^0S_1^2\Bigg]-\frac{16\big(N^2+N+2\big)\big(3N^2+3N+2\big)S_2}{N^2(N+1)^2(N+2)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{32\big(N^2+N+2\big)S_{-2}}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[8S_1^3-16S_2S_1-16S_{-2}S_1-8S_3-8S_{-3}+16S_{-2,1}\Big]\Big]\\
&+\frac{P_{190}\zeta_2}{2N^4(N+1)^4(N+2)}+\frac{16\big(N^2+N+2\big)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)}+96\tilde{\gamma}_{qg}^0\log(2)\zeta_2\\
&+\frac{\big(N^2+N+2\big)\big(3N^2+3N+2\big)\Big[6S_4-16S_{3,1}+32S_{2,1,1}+8S_2\zeta_2-\frac{16}{3}S_1\zeta_3\Big]}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[\frac{1}{3}S_1^5\\
&-\frac{2}{3}S_2S_1^3+\big(-\frac{16}{3}S_3-16S_{2,1}\big)S_1^2-\frac{8}{3}\zeta_3S_1^2+\Big[-3S_2^2+6S_4-16S_{3,1}+32S_{2,1,1}\Big]S_1\\
&+\frac{8}{3}S_2S_3+\Big[-4S_1^3+8S_2S_1+8S_{-2}S_1+4S_3+4S_{-3}-8S_{-2,1}\Big]\zeta_2\Big]\\
&+L_Q\Bigg[\frac{16\big(3N^4-13N^2-18N-12\big)S_1^3}{N^2(N+1)^2(N+2)}-\frac{2P_{136}S_1^2}{N^3(N+1)^3(N+2)}+\frac{64(-1)^N\big(N^2+N+2\big)S_1}{N(N+1)^4(N+2)}\\
&-\frac{2P_{139}S_1}{N^4(N+1)^3}-\frac{16\big(7N^4+20N^3+7N^2-22N-20\big)S_2S_1}{N^2(N+1)^2(N+2)}-\frac{64\big(2N^3+N^2+3N-10\big)S_{-2}S_1}{N^2(N+1)^2(N+2)}\\
&-\frac{32(-1)^NP_{266}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^5(N+3)^3}+\frac{2P_{277}}{5(N-1)^2N^5(N+1)^5(N+2)^5(N+3)^3}\\
&-48\frac{\big(9N^4+10N^3+9N^2-8N+12\big)\zeta_3}{N^2(N+1)^2(N+2)}+L_M^2\Bigg[\frac{2\big(N^2+N+2\big)\big(3N^2+3N+2\big)^2}{N^3(N+1)^3(N+2)}\\
&-\frac{16\big(N^2+N+2\big)S_1\big(3N^2+3N+2\big)}{N^2(N+1)^2(N+2)}-8\tilde{\gamma}_{qg}^0S_1^2\Bigg]+\frac{2P_{147}S_2}{N^3(N+1)^3(N+2)}\\
&-\frac{16\big(3N^4+10N^3+15N^2+16N-12\big)S_3}{N^2(N+1)^2(N+2)}+\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)S_{-2}}{(N+1)^3(N+2)^3}\\
&-\frac{64P_{212}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)}-\frac{64\big(N^2+3N+4\big)S_{-3}}{N(N+1)^2(N+2)}\\
&+\frac{16\big(N^2+N+2\big)\big(3N^2+3N+2\big)S_{2,1}}{N^2(N+1)^2(N+2)}+L_M\Bigg[-\frac{64\big(N^3+5N^2+6N+4\big)S_1^2}{N^2(N+1)^2(N+2)}\\
&-\frac{4P_{138}S_1}{N^3(N+1)^3(N+2)}+\frac{64(-1)^N\big(N^2+N+2\big)}{N(N+1)^4(N+2)}+\frac{2P_{178}}{N^4(N+1)^4(N+2)}\\
&+\frac{32\big(N^2+N+2\big)\big(3N^2+3N+2\big)S_2}{N^2(N+1)^2(N+2)}+\frac{64\big(N^2+N+2\big)S_{-2}}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[-16S_1^3+32S_2S_1+32S_{-2}S_1\\
&+16S_3+16S_{-3}-32S_{-2,1}\Big]\Bigg]+\frac{128(N-1)\big(N^2+2N+4\big)S_{-2,1}}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[-8S_1^4+24S_2S_1^2-48S_{-3}S_1\\
&+\Big[16S_{2,1}-32S_{-2,1}\Big]S_1-48\zeta_3S_1-16S_2^2-32S_{-2}^2+S_{-2}\Big[32S_1^2-32S_2\Big]-32S_4-80S_{-4}-32S_{3,1}\\
&+32S_{-2,2}+64S_{-3,1}\Big]\Bigg]+L_M\Bigg[-\frac{16\big(3N^4-13N^2-18N-12\big)S_1^3}{N^2(N+1)^2(N+2)}+\frac{2P_{136}S_1^2}{N^3(N+1)^3(N+2)}\\
&-\frac{64(-1)^N\big(N^2+N+2\big)S_1}{N(N+1)^4(N+2)}+\frac{2P_{139}S_1}{N^4(N+1)^3}+\frac{16\big(7N^4+20N^3+7N^2-22N-20\big)S_2S_1}{N^2(N+1)^2(N+2)}\\
&+\frac{64\big(2N^3+N^2+3N-10\big)S_{-2}S_1}{N^2(N+1)^2(N+2)}+\frac{32(-1)^NP_{266}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^5(N+3)^3}\\
&-\frac{2P_{277}}{5(N-1)^2N^5(N+1)^5(N+2)^5(N+3)^3}+48\frac{\big(9N^4+10N^3+9N^2-8N+12\big)\zeta_3}{N^2(N+1)^2(N+2)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2P_{147}S_2}{N^3(N+1)^3(N+2)}+\frac{16\big(3N^4+10N^3+15N^2+16N-12\big)S_3}{N^2(N+1)^2(N+2)}\\
&-\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)S_{-2}}{(N+1)^3(N+2)^3}+\frac{64P_{212}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)}\\
&+\frac{64\big(N^2+3N+4\big)S_{-3}}{N(N+1)^2(N+2)}-\frac{16\big(N^2+N+2\big)\big(3N^2+3N+2\big)S_{2,1}}{N^2(N+1)^2(N+2)}-\frac{128(N-1)\big(N^2+2N+4\big)S_{-2,1}}{N^2(N+1)^2(N+2)}\\
&+\tilde{\gamma}_{qg}^0\Big[8S_1^4-24S_2S_1^2+48S_{-3}S_1+\Big[32S_{-2,1}-16S_{2,1}\Big]S_1+48\zeta_3S_1+16S_2^2+32S_{-2}^2\\
&+S_{-2}\Big[32S_2-32S_1^2\Big]+32S_4+80S_{-4}+32S_{3,1}-32S_{-2,2}-64S_{-3,1}\Big]\Big]\Big]\\
&+C_A^2T_F\Bigg[\frac{P_{123}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{4P_{176}S_1^3}{9(N-1)N^2(N+1)^3(N+2)^3}\\
&-\frac{4}{3}\frac{P_{131}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{2P_{215}S_1^2}{3(N-1)N^2(N+1)^4(N+2)^4}\\
&+\frac{2(N-2)\big(55N^5+347N^4+379N^3+213N^2+326N+120\big)S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{4}{9}\frac{P_{217}S_1\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^3}-\frac{4P_{262}S_1}{3(N-1)N^5(N+1)^5(N+2)^5}\\
&-\frac{4P_{193}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3}+\frac{16P_{144}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{8}{3}\frac{(-1)^NP_{213}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^4}-\frac{2}{9}\frac{P_{251}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^4}\\
&-\frac{4\big(11N^4+22N^3-35N^2-46N-24\big)P_{261}}{3(N-1)^2N^6(N+1)^6(N+2)^6}-\frac{4}{9}\frac{\zeta_3P_{148}S_1}{(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{4}{9}\frac{\big(11N^4+22N^3-35N^2-46N-24\big)\big(9N^5-14N^4-19N^3+52N^2+12N+8\big)\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^3}\\
&+L_Q^3\Bigg[-\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2+\frac{8\big(N^2+N+2\big)\big(11N^4+22N^3-59N^2-70N-48\big)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)\big(11N^4+22N^3-35N^2-46N-24\big)}{9(N-1)^2N^3(N+1)^3(N+2)^3}\Bigg]+L_M^3\Bigg[\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2\\
&-\frac{8\big(N^2+N+2\big)\big(11N^4+22N^3-59N^2-70N-48\big)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\\
&+\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)\big(11N^4+22N^3-35N^2-46N-24\big)}{9(N-1)^2N^3(N+1)^3(N+2)^3}\Bigg]\\
&-\frac{2\big(11N^4+22N^3-35N^2-46N-24\big)P_{191}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^4}\\
&-\frac{32\big(11N^4+22N^3-35N^2-46N-24\big)\big(N^5+10N^4+9N^3+3N^2+7N+6\big)S_3}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&-\frac{16(-1)^N\big(N^4+2N^3+7N^2+22N+20\big)\big(11N^4+22N^3-35N^2-46N-24\big)S_{-2}}{3(N-1)N(N+1)^4(N+2)^4}\\
&-32\frac{\big(N^2+N+1\big)\big(N^2+N+2\big)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2}S_{-2}+L_M^2\Bigg[-\frac{4P_{127}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{(N+1)^3(N+2)^3}+\frac{8P_{216}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&-\frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4}+\frac{16P_{241}}{9(N-1)^2N^4(N+1)^3(N+2)^4}\\
&-\frac{4\big(N^2+N+2\big)\big(11N^4+22N^3-83N^2-94N-72\big)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{8\big(N^2+N+2\big)\big(11N^4+22N^3-59N^2-70N-48\big)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[4S_1^3+12S_2S_1+16S_{-2}S_1\\
&+4S_3+4S_{-3}-8S_{-2,1}\Big]\Bigg]+L_Q^2\Bigg[-\frac{4P_{141}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&+\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{(N+1)^3(N+2)^3}-\frac{8P_{218}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&+\frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4}-\frac{16P_{240}}{9(N-1)^2N^4(N+1)^3(N+2)^4}\\
&+\frac{4\big(N^2+N+2\big)\big(11N^4+22N^3-83N^2-94N-72\big)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&+\frac{8\big(N^2+N+2\big)\big(11N^4+22N^3-59N^2-70N-48\big)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[4S_1^3-12S_2S_1-16S_{-2}S_1-4S_3-4S_{-3}+8S_{-2,1}\Big]\Bigg]\\
&+\frac{\big(5N^5-131N^3-58N^2+232N+96\big)}{(N-1)N(N+1)^2(N+2)^3}\Big[\frac{32}{3}(-1)^N S_1S_{-2}+\frac{16}{3}(-1)^N S_1\zeta_2\Big]\\
&+\frac{\big(N^2+N+2\big)\big(11N^4+22N^3-35N^2-46N-24\big)}{(N-1)N^2(N+1)^2(N+2)^2}\Big[\frac{1}{3}S_2^2+\frac{16}{3}(-1)^N S_{-2}S_2\\
&+6S_4+\frac{8}{3}(-1)^N S_{-4}-\frac{8}{3}S_{3,1}-\frac{16}{3}S_{-2,2}-\frac{16}{3}S_{-3,1}-\frac{8}{3}S_{2,1,1}+\frac{32}{3}S_{-2,1,1}\\
&+\big(\frac{8}{3}(-1)^N S_2+\frac{8}{3}(-1)^N S_{-2}\big)\zeta_2\Big]+\frac{\big(N^2-N-4\big)\big(11N^4+22N^3-35N^2-46N-24\big)}{(N-1)N(N+1)^3(N+2)^3}\\
&\times\Big[4(-1)^N\zeta_3+\frac{16}{3}(-1)^N S_{-3}-\frac{32}{3}S_{-2,1}\Big]+\frac{P_{130}\Big[\frac{16}{3}(-1)^N S_{-2}S_1^2+\frac{8}{3}(-1)^N\zeta_2S_1^2\Big]}{(N-1)N^2(N+1)^2(N+2)^2}\\
&+\frac{\big(11N^5+34N^4-49N^3-24N^2-68N-48\big)\Big[\frac{16}{3}(-1)^N S_1S_{-3}-\frac{32}{3}S_1S_{-2,1}+4(-1)^N S_1\zeta_3\Big]}{(N-1)N^2(N+1)(N+2)^2}\\
&+L_Q\Bigg[\frac{8\big(47N^4-133N^3+11N^2-155N-58\big)S_1^3}{9(N-1)N(N+1)^2(N+2)}+\frac{8P_{223}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&-\frac{32(N-2)(N+3)S_{-2}S_1^2}{N(N+1)(N+2)}-\frac{32(-1)^N P_{197}S_1}{3(N-1)N^3(N+1)^4(N+2)^4}+\frac{8P_{257}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4}\\
&-\frac{8P_{137}S_2S_1}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{32\big(N^2+N+6\big)S_3S_1}{N(N+1)(N+2)}-\frac{16P_{142}S_{-2}S_1}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&+\frac{32\big(9N^2+9N+22\big)S_{-3}S_1}{N(N+1)(N+2)}-\frac{64\big(3N^2+3N+10\big)S_{-2,1}S_1}{N(N+1)(N+2)}-\frac{384\zeta_3S_1}{N(N+1)(N+2)}\\
&-8\frac{P_{117}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^N P_{256}}{9(N-1)^2N^4(N+1)^5(N+2)^5}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4P_{269}}{27(N-1)^2N^5(N+1)^5(N+2)^5}-\frac{8P_{219}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&-\frac{8P_{149}S_3}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{16P_{220}S_{-2}}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&+\frac{\big(N^3+4N^2+7N+5\big)\big(-64(-1)^NS_1^2+64(-1)^NS_2+192(-1)^NS_{-2}\big)}{(N+1)^3(N+2)^3}\\
&+\frac{16P_{133}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{32P_{112}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[-2S_1^4+16S_2S_1^2-2S_2^2-12S_{-2}^2-16S_{-2}S_2-4S_4-44S_{-4}-\frac{88}{3}S_{2,1}-16S_{3,1}+56S_{-2,2}\\
&+64S_{-3,1}-96S_{-2,1,1}\Big]\Bigg]+\tilde{\gamma}_{qg}^0\Big[-\frac{1}{3}S_1^5-10S_2S_1^3-16(-1)^NS_{-3}S_1^2+\Big[32S_{-2,1}-\frac{80S_3}{3}\Big]S_1^2\\
&-\frac{4}{3}\big(-7+9(-1)^N\big)\zeta_3S_1^2-8(-1)^NS_{-4}S_1+\Big[-S_2^2-18S_4+8S_{3,1}+16S_{-2,2}+16S_{-3,1}+8S_{2,1,1}\\
&-32S_{-2,1,1}\Big]S_1+S_{-2}\Big[-16(-1)^NS_1^3-16(-1)^NS_2S_1\Big]+\Big[-4\big(-1+2(-1)^N\big)S_1^3-8(-1)^NS_2S_1\\
&-4\big(1+2(-1)^N\big)S_{-2}S_1+\frac{11S_2}{3}-2S_3-2S_{-3}+4S_{-2,1}\Big]\zeta_2\Big]+L_M\Bigg[\frac{128(-1)^NP_{171}S_1}{3(N-1)N^2(N+1)^4(N+2)^4}\\
&-\frac{8\big(11N^4+26N^3-139N^2-218N+8\big)S_1^3}{9N(N+1)^2(N+2)^2}+\frac{4P_{221}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3}\\
&-\frac{8P_{254}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4}-\frac{8P_{124}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{32\big(2N^5-23N^4-32N^3+13N^2+4N-12\big)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)^2}+2\frac{P_{151}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{16(-1)^NP_{237}}{9(N-1)N^3(N+1)^5(N+2)^5}-\frac{4P_{268}}{27(N-1)^2N^5(N+1)^5(N+2)^5}\\
&+\frac{4P_{222}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3}+\frac{\big(N^3+4N^2+7N+5\big)\big(-64(-1)^NS_1^2-64(-1)^NS_2\big)}{(N+1)^3(N+2)^3}\\
&-\frac{8P_{145}S_3}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{16(-1)^NP_{129}S_{-2}}{3(N-1)N(N+1)^3(N+2)^3}+\frac{16P_{202}S_{-2}}{9(N-1)N^3(N+1)^3(N+2)^3}\\
&-\frac{16P_{126}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}-\frac{16\big(N^2+N+2\big)\big(11N^4+22N^3+13N^2+2N+24\big)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&+\frac{\big(N^2+N+2\big)\big(11N^4+22N^3-35N^2-46N-24\big)}{(N-1)N^2(N+1)^2(N+2)^2}\Big[-\frac{8}{3}(-1)^NS_{-3}-2(-1)^N\zeta_3\Big]\\
&+\frac{\big(11N^5+34N^4-49N^3-24N^2-68N-48\big)\Big[-\frac{16}{3}(-1)^NS_1S_{-2}\Big]}{(N-1)N^2(N+1)(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[2S_1^4+32S_2S_1^2+8\big(8+(-1)^N\big)S_{-3}S_1+\Big[40S_3-16S_{2,1}-80S_{-2,1}\Big]S_1\\
&+6\big(-5+(-1)^N\big)\zeta_3S_1+2S_2^2+12S_{-2}^2+S_{-2}\Big[8\big(3+2(-1)^N\big)S_1^2+16S_2\Big]+4S_4+44S_{-4}+16S_{3,1}\\
&-56S_{-2,2}-64S_{-3,1}+96S_{-2,1,1}\Big]+\frac{16P_{125}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}\Bigg]\Bigg]\\
&+C_FT_F^2\Bigg[\Bigg[\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)\big(3N^4+6N^3-N^2-4N+12\big)}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{16}{9}\tilde{\gamma}_{qg}^0S_1\Bigg]L_Q^3
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[-\frac{16L_M\big(N^2+N+2\big)^3}{(N-1)N^3(N+1)^3(N+2)^2}-\frac{4P_{232}}{9(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{16P_{177}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[\frac{20S_2}{3}-4S_1^2\Big]\Bigg]L_Q^2+\Bigg[-\frac{16P_{181}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&-\frac{8P_{229}S_1}{9(N-1)N^4(N+1)^4(N+2)^3}+\frac{64(-1)^N P_{259}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}\\
&+\frac{4P_{275}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}+L_M^2\Bigg[-\frac{16\big(N^2+N+2\big)P_{109}}{3(N-1)N^3(N+1)^3(N+2)^2}\\
&-\frac{16}{3}\tilde{\gamma}_{qg}^0S_1\Bigg]+L_M\Bigg[\frac{8P_{231}}{9(N-1)N^4(N+1)^4(N+2)^3}-\frac{32P_{173}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[-\frac{8}{3}S_1^2-\frac{8S_2}{3}\Big]\Bigg]+\frac{16P_{183}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}-\frac{256\big(N^2+N+1\big)S_3}{3N(N+1)(N+2)}\\
&+\frac{64P_{163}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}+\tilde{\gamma}_{qg}^0\Big[\frac{8}{3}S_1^3-8S_2S_1-\frac{32}{3}S_{2,1}\Big]\\
&+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}+\frac{64(N-1)\zeta_3}{N(N+1)}\Bigg]L_Q-\frac{16\big(N^4-5N^3-32N^2-18N-4\big)S_1^2}{3N^2(N+1)^2(N+2)}\\
&-\frac{16}{9}\big(N^2+N+2\big)\frac{\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}P_{113}+\frac{2}{9}\frac{P_{234}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3}\\
&-\frac{2P_{253}}{3(N-1)N^5(N+1)^6(N+2)^2}+\frac{4P_{244}S_1}{3(N-1)N^5(N+1)^5(N+2)^2}\\
&-\frac{80}{9}\frac{\big(N^3+4N^2+11N+6\big)\zeta_2}{N^2(N+1)(N+2)}S_1+L_M^3\Bigg[\frac{16\big(N^2+N+2\big)P_{113}}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\frac{32}{9}\tilde{\gamma}_{qg}^0S_1\Bigg]+L_M^2\Bigg[-\frac{4P_{227}}{9(N-1)N^4(N+1)^4(N+2)^3}+\frac{16P_{164}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[\frac{20}{3}S_1^2-4S_2\Big]\Bigg]-\frac{16P_{110}S_2}{3N^3(N+1)^3(N+2)}+\frac{(3N+2)\Big[-\frac{32}{9}S_1^3-\frac{32}{3}S_2S_1\Big]}{N^2(N+2)}\\
&-\frac{32\big(3N^4+48N^3+43N^2-22N-8\big)S_3}{9N^2(N+1)^2(N+2)}+\frac{128\big(N^2-3N-2\big)S_{2,1}}{3N^2(N+1)(N+2)}\\
&+L_M\Bigg[\frac{16P_{174}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{8P_{230}S_1}{9(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{64(-1)^N P_{260}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}+\frac{4P_{274}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}\\
&-\frac{16P_{179}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[\frac{8}{3}S_1^3+\frac{8}{3}S_2S_1\Big]+\frac{128\big(N^2+N+4\big)S_3}{3N(N+1)(N+2)}\\
&+\frac{64P_{162}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}\\
&+\frac{64(N-1)\zeta_3}{N(N+1)}\Bigg]+\tilde{\gamma}_{qg}^0\Big[-\frac{2}{9}S_1^4-\frac{4}{3}S_2S_1^2+\Big[-\frac{16}{9}S_3-\frac{32}{3}S_{2,1}\Big]S_1-\frac{32}{9}\zeta_3S_1-\frac{2}{3}S_2^2+4S_4\\
&-\frac{32}{3}S_{3,1}+\frac{64}{3}S_{2,1,1}+\Big[2S_2-\frac{10}{3}S_1^2\Big]\zeta_2\Big]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+C_F N_F T_F^2\Bigg[\Bigg[\Bigg[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1+\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)\big(3N^4+6N^3-N^2-4N+12\big)}{9(N-1)N^3(N+1)^3(N+2)^2}\Bigg]L_Q^3\\
&+\Bigg[-\frac{4P_{232}}{9(N-1)N^4(N+1)^4(N+2)^3}+\frac{16P_{177}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+L_M\Bigg[-\frac{8\big(N^2+N+2\big)\big(3N^2+3N+2\big)}{3N^2(N+1)^2(N+2)}-\frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\Bigg]+\tilde{\gamma}_{qg}^0\Big[\frac{20S_2}{3}-4S_1^2\Big]\Bigg]L_Q^2\\
&+\Bigg[-\frac{16P_{181}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}-\frac{8P_{229}S_1}{9(N-1)N^4(N+1)^4(N+2)^3}\\
&+\frac{64(-1)^N P_{259}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}+\frac{8P_{276}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}\\
&+L_M\Bigg[\frac{8\big(N^2+N+2\big)\big(57N^4+72N^3+29N^2-22N-24\big)}{9N^3(N+1)^3(N+2)}+\tilde{\gamma}_{qg}^0\Big[\frac{8}{3}S_1^2-8S_2\Big]\\
&-\frac{16\big(N^2+N+2\big)\big(29N^2+29N-6\big)S_1}{9N^2(N+1)^2(N+2)}\Bigg]-\frac{256\big(N^2+N+1\big)S_3}{3N(N+1)(N+2)}+\frac{16P_{183}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}\\
&+\frac{64P_{163}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}+\tilde{\gamma}_{qg}^0\Big[\frac{8}{3}S_1^3-8S_2S_1-\frac{32}{3}S_{2,1}\Big]\\
&+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}+\frac{64(N-1)\zeta_3}{N(N+1)}\Bigg]L_Q-\frac{2}{9}\frac{P_{235}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3}\\
&-\frac{8\big(N^4-5N^3-32N^2-18N-4\big)S_1^2}{3N^2(N+1)^2(N+2)}-\frac{8}{9}\frac{\big(N^2+N+2\big)\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}P_{108}\\
&-\frac{4P_{271}}{3(N-1)N^6(N+1)^6(N+2)^5}+\frac{16\big(2N^5-2N^4-11N^3-19N^2-44N-12\big)S_1}{3N^2(N+1)^3(N+2)}\\
&-\frac{16}{9}\frac{\big(5N^3+11N^2+28N+12\big)\zeta_2}{N^2(N+1)(N+2)}S_1+L_M^3\Bigg[\frac{8\big(N^2+N+2\big)P_{108}}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\Bigg]\\
&-\frac{8P_{225}S_2}{3(N-1)N^4(N+1)^4(N+2)^3}+\frac{(3N+2)\Big[-\frac{16}{9}S_1^3-\frac{16}{3}S_2S_1\Big]}{N^2(N+2)}\\
&+L_M^2\Bigg[-\frac{4P_{228}}{9(N-1)N^4(N+1)^4(N+2)^3}+\frac{32\big(5N^3+8N^2+19N+6\big)S_1}{9N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\Big[\frac{4}{3}S_1^2+\frac{4S_2}{3}\Big]\Bigg]\\
&-\frac{16P_{169}S_3}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{64\big(N^2-3N-2\big)S_{2,1}}{3N^2(N+1)(N+2)}\\
&+L_M\Bigg[\frac{8\big(49N^4+122N^3+213N^2+164N+12\big)S_1^2}{9N^2(N+1)^2(N+2)}-\frac{8P_{258}}{9(N-1)N^5(N+1)^5(N+2)^4}\\
&+\frac{8P_{146}S_1}{9N^3(N+1)^3(N+2)}-\frac{8P_{184}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[\frac{16}{3}S_1S_2-16S_3+\frac{16}{3}S_{2,1}\Big]\Bigg]\\
&+\tilde{\gamma}_{qg}^0\Big[-\frac{1}{9}S_1^4-\frac{2}{3}S_2S_1^2-\frac{4}{3}\zeta_2S_1^2+\Big[-\frac{8}{9}S_3-\frac{16}{3}S_{2,1}\Big]S_1-\frac{8}{9}\zeta_3S_1-\frac{1}{3}S_2^2+2S_4-\frac{16}{3}S_{3,1}+\frac{32}{3}S_{2,1,1}\Big]\Bigg]\\
&+C_AC_FT_F\Bigg[\frac{2P_{134}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{4P_{180}S_1^3}{9(N-1)N^3(N+1)^3(N+2)^2}
\end{aligned}
$$

$$
-\frac{4P_{245}S_1^2}{3(N-1)N^4(N+1)^4(N+2)^4}-\frac{8}{3}\frac{\big(19N^5-11N^4-8N^3-49N^2-17N+18\big)\zeta_2}{(N-1)N^2(N+1)^2(N+2)}S_1^2
+\frac{4P_{143}S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}-\frac{2}{9}\frac{P_{155}S_1\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}
+\frac{4}{9}\frac{P_{189}S_1\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}+\frac{4P_{263}S_1}{3(N-1)N^5(N+1)^5(N+2)^5}
+\frac{4P_{200}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3}+\frac{8P_{153}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2}
-\frac{16\big(11N^5+45N^4-3N^3-145N^2-176N-20\big)S_{2,1}S_1}{3(N-1)N(N+1)^2(N+2)^2}-\frac{2P_{101}S_2^2}{3(N-1)N^2(N+1)^2(N+2)^2}
-\frac{256(-1)^N\big(N^3+4N^2+7N+5\big)}{(N+1)^3(N+2)^3}+8\frac{(-1)^N\zeta_2}{N^2(N+1)^4(N+2)^3}P_{115}-\frac{2}{9}\frac{P_{188}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}
-\frac{1}{18}\frac{P_{208}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^2}+\frac{P_{273}}{3(N-1)N^6(N+1)^6(N+2)^5}
+L_M^3\Bigg[-\frac{16}{3}\tilde{\gamma}_{qg}^0S_1^2-\frac{8\big(N^2+N+2\big)\big(N^2+N+6\big)\big(7N^2+7N+4\big)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}
-\frac{2\big(N^2+N+2\big)\big(3N^2+3N+2\big)\big(11N^4+22N^3-59N^2-70N-48\big)}{9(N-1)N^3(N+1)^3(N+2)^2}\Bigg]+L_Q^3\Bigg[-\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2
+\frac{8\big(N^2+N+2\big)\big(13N^4+26N^3-43N^2-56N-12\big)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}
-\frac{4\big(N^2+N+2\big)\big(3N^2+3N+2\big)\big(11N^4+22N^3-23N^2-34N-12\big)}{9(N-1)N^3(N+1)^3(N+2)^2}\Bigg]
-4\frac{\zeta_2P_{107}S_2}{(N-1)N^2(N+1)^2(N+2)^2}-\frac{4P_{243}S_2}{3(N-1)N^4(N+1)^4(N+2)^4}+\frac{4P_{185}S_3}{9(N-1)N^3(N+1)^3(N+2)^2}
+\frac{4\big(N^2+N+2\big)\big(19N^4+38N^3-22N^2-41N-30\big)S_4}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^NP_{116}S_{-2}}{N^2(N+1)^4(N+2)^3}
-\frac{16\big(N^2-3N-2\big)\big(11N^4+22N^3-35N^2-46N-24\big)S_{2,1}}{3(N-1)N^3(N+1)^2(N+2)^2}
-\frac{8\big(N^2+N+2\big)\big(31N^4+62N^3-73N^2-104N-60\big)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)^2}
+L_Q^2\Bigg[-\frac{8\big(10N^5+6N^4+5N^3-38N^2-17N+2\big)S_1^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{(N+1)^3(N+2)^3}
-\frac{4P_{204}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}-\frac{16(-1)^N\big(3N^4+11N^3+19N^2+15N+2\big)}{N(N+1)^3(N+2)^3}
+\frac{P_{238}}{9(N-1)N^4(N+1)^4(N+2)^3}+L_M\Bigg[\frac{22\big(N^2+N+2\big)\big(3N^2+3N+2\big)}{3N^2(N+1)^2(N+2)}
+\frac{22}{3}\tilde{\gamma}_{qg}^0S_1\Bigg]+\frac{8\big(N^2+N+2\big)\big(23N^4+46N^3-50N^2-73N-18\big)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[4S_1^3-12S_2S_1
+4S_3+6S_{-2}+4S_{-3}-8S_{-2,1}\Big]\Bigg]+L_M^2\Bigg[\frac{8P_{121}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}
+\frac{64(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{(N+1)^3(N+2)^3}-\frac{8P_{203}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}
$$

$$
\begin{aligned}
&-\frac{16(-1)^N\big(3N^4+11N^3+19N^2+15N+2\big)}{N(N+1)^3(N+2)^3}+\frac{P_{224}}{9(N-1)N^4(N+1)^3(N+2)^3}\\
&+\frac{8\big(N^2+N+2\big)\big(N^4+2N^3+20N^2+19N+30\big)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[-12S_1^3+4S_2S_1+4S_3+6S_{-2}\\
&+4S_{-3}-8S_{-2,1}\Big]\Bigg]+\frac{8\big(N^2+N+2\big)\big(35N^4+70N^3-137N^2-172N-84\big)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)^2}\\
&-\frac{8\big(N^2+N+2\big)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)}-48\tilde{\gamma}_{qg}^0\log(2)\zeta_2+\frac{P_{168}\Big[16(-1)^NS_1S_{-2}+8(-1)^NS_1\zeta_2\Big]}{N^3(N+1)^3(N+2)^3}\\
&+\frac{\big(N^4+3N^3+8N^2+12N+4\big)\Big[96(-1)^NS_{-2}S_1^2+48(-1)^N\zeta_2S_1^2\Big]}{N(N+1)^2(N+2)^2}\\
&+\frac{\big(3N^5+10N^4+25N^3+38N^2+20N+8\big)\Big[16(-1)^NS_2S_{-2}+8(-1)^NS_2\zeta_2\Big]}{N^2(N+1)^2(N+2)^2}\\
&+\frac{\big(N^2+N+2\big)\big(3N^2+3N+2\big)}{N^2(N+1)^2(N+2)}\Big[8(-1)^NS_{-2}\zeta_2+8(-1)^NS_{-4}-16S_{-2,2}-16S_{-3,1}+32S_{-2,1,1}\Big]\\
&+\frac{P_{156}\Big[8(-1)^NS_{-3}-16S_{-2,1}+6(-1)^N\zeta_3\Big]}{N^3(N+1)^3(N+2)^2}\\
&+\frac{\big(9N^5+28N^4+73N^3+110N^2+44N+8\big)\Big[8(-1)^NS_1S_{-3}-16S_1S_{-2,1}+6(-1)^NS_1\zeta_3\Big]}{N^2(N+1)^2(N+2)^2}\\
&+\tilde{\gamma}_{qg}^0\Big[\frac{20}{3}S_2S_1^3+\Big[24S_3+16S_{2,1}-24S_{-2,1}\Big]S_1^2+8(-1)^NS_{-4}S_1+\Big[8S_2^2+12S_4+8S_{3,1}-16S_{-2,2}\\
&-16S_{-3,1}-40S_{2,1,1}+32S_{-2,1,1}\Big]S_1+S_{-3}\Big[12(-1)^NS_1^2+4(-1)^NS_2\Big]+S_2\Big[\frac{16S_3}{3}-8S_{-2,1}\Big]\\
&+S_{-2}\Big[8(-1)^NS_1^3+24(-1)^NS_2S_1+32\Big]+\Big[4\big(1+(-1)^N\big)S_1^3+4\big(-1+2(-1)^N\big)S_{-2}S_1-2S_3\\
&-2S_{-3}+4S_{-2,1}\Big]\zeta_2+\Big[\frac{1}{3}\big(-11+27(-1)^N\big)S_1^2\Big]\zeta_3\Big]\\
&+L_Q\Bigg[\frac{8\big(29N^5+81N^4+117N^3-49N^2-362N-104\big)S_1^3}{3(N-1)N(N+1)^2(N+2)^2}+\frac{4P_{206}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3}\\
&-\frac{32(-1)^NP_{255}S_1}{15(N-2)(N-1)^2N^2(N+1)^4(N+2)^4(N+3)^3}+\frac{2P_{270}S_1}{45(N-1)^2N^4(N+1)^4(N+2)^4(N+3)^3}\\
&-\frac{8\big(17N^5+33N^4-43N^3-153N^2-254N-80\big)S_2S_1}{(N-1)N(N+1)^2(N+2)^2}+\frac{32(N-2)(N+3)S_3S_1}{N(N+1)(N+2)}\\
&-\frac{16P_{158}S_{-2}S_1}{3(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)}+\frac{512S_{-2,1}S_1}{N(N+1)(N+2)}-\frac{24\big(11N^2+11N-10\big)\zeta_3S_1}{N(N+1)(N+2)}\\
&+2\frac{\zeta_3P_{154}}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^NP_{272}}{45(N-2)(N-1)^3N^3(N+1)^5(N+2)^5(N+3)^3}\\
&-\frac{2P_{279}}{45(N-1)^3N^5(N+1)^5(N+2)^5(N+3)^3}\\
&+L_M^2\Bigg[8\tilde{\gamma}_{qg}^0S_1^2+\frac{8\big(N^2+N+2\big)\big(3N^4+6N^3+7N^2+4N+4\big)S_1}{(N-1)N^2(N+1)^2(N+2)^2}
\end{aligned}
$$

$$
-\frac{16\big(N^2+N+1\big)\big(N^2+N+2\big)\big(3N^2+3N+2\big)}{(N-1)N^3(N+1)^3(N+2)^2}\Bigg]+\frac{\big(N^2+N+10\big)\Big[-64S_{-2}S_1^2-32S_{-3}S_1\Big]}{N(N+1)(N+2)}
$$
$$
-\frac{4P_{207}S_2}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{\big(N^3+4N^2+7N+5\big)\big(128(-1)^NS_2-128(-1)^NS_1^2\big)}{(N+1)^3(N+2)^3}
$$
$$
+\frac{32P_{135}S_3}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^N\big(3N^5-6N^4-61N^3-124N^2-96N-16\big)S_{-2}}{N(N+1)^3(N+2)^3}
$$
$$
-\frac{16P_{119}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{32P_{226}S_{-2}}{3(N-2)(N-1)N^3(N+1)^3(N+2)^3(N+3)}
$$
$$
-\frac{16\big(N^2+N+2\big)\big(31N^2+31N+6\big)S_{2,1}}{3N^2(N+1)^2(N+2)}-\frac{16P_{120}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
+L_M\Bigg[\frac{16\big(10N^5+40N^4+121N^3+161N^2+52N+12\big)S_1^2}{3N^2(N+1)^2(N+2)^2}+\frac{4P_{205}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}
$$
$$
-\frac{128(-1)^N\big(N^3+4N^2+7N+5\big)S_1}{(N+1)^3(N+2)^3}+\frac{32(-1)^N\big(3N^4+11N^3+19N^2+15N+2\big)}{N(N+1)^3(N+2)^3}
$$
$$
-\frac{2P_{236}}{9(N-1)N^4(N+1)^4(N+2)^3}-\frac{16\big(N^2+N+2\big)\big(4N^2+4N-1\big)S_2}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[8S_1^3+8S_2S_1-8S_3
$$
$$
-12S_{-2}-8S_{-3}+16S_{-2,1}\Big]\Bigg]+\frac{\big(3N^5+8N^4+27N^3+46N^2+20N+8\big)16(-1)^NS_1S_{-2}}{N^2(N+1)^2(N+2)^2}
$$
$$
+\frac{\big(N^2+N+2\big)\big(3N^2+3N+2\big)\Big[8(-1)^NS_{-3}+6(-1)^N\zeta_3\Big]}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[40S_2S_1^2+16(-1)^NS_{-2}S_1^2
$$
$$
+8(-1)^NS_{-3}S_1-16S_{2,1}S_1+6(-1)^N\zeta_3S_1-8S_2^2+24S_{-2}^2+8S_4+40S_{-4}
$$
$$
+32S_{3,1}-16S_{-3,1}-32S_{-2,1,1}\Big]\Bigg]+L_M\Bigg[\frac{8\big(3N^5+2N^4-61N^3-112N^2-56N-24\big)S_1^3}{N^2(N+1)^2(N+2)^2}
$$
$$
-\frac{2P_{209}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32(-1)^N\big(15N^5+97N^4+260N^3+328N^2+158N-4\big)S_1}{N(N+1)^3(N+2)^4}
$$
$$
-\frac{2P_{250}S_1}{9(N-1)N^4(N+1)^4(N+2)^4}+\frac{8P_{114}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
+\frac{16\big(3N^3+2N^2-47N-62\big)S_{-2}S_1}{N(N+1)^2(N+2)}-6\frac{\zeta_3P_{150}}{(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
-\frac{16(-1)^NP_{267}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^5(N+3)^3}+\frac{2P_{278}}{45(N-1)^2N^5(N+1)^5(N+2)^5(N+3)^3}
$$
$$
+\frac{2P_{210}S_2}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{\big(N^3+4N^2+7N+5\big)\big(128(-1)^NS_1^2-128(-1)^NS_2\big)}{(N+1)^3(N+2)^3}
$$
$$
-\frac{16P_{128}S_3}{(N-1)N^2(N+1)^2(N+2)^2}-\frac{16(-1)^N\big(3N^5-6N^4-61N^3-124N^2-96N-16\big)S_{-2}}{N(N+1)^3(N+2)^3}
$$
$$
-\frac{16P_{214}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)}+\frac{16\big(3N^4+4N^3-9N^2-14N+8\big)S_{-3}}{N^2(N+1)^2(N+2)}
$$
$$
+\frac{32\big(N^2+N+2\big)\big(10N^4+20N^3+5N^2-5N+6\big)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}
$$
$$
+\frac{16\big(3N^4+10N^3+43N^2+44N-20\big)S_{-2,1}}{N^2(N+1)^2(N+2)}
$$

$$
+\frac{\left(3N^5+8N^4+27N^3+46N^2+20N+8\right)\Big[-16(-1)^N S_1 S_{-2}\Big]}{N^2(N+1)^2(N+2)^2}
$$
$$
+\frac{\left(N^2+N+2\right)\left(3N^2+3N+2\right)\Big[-8(-1)^N S_{-3}-6(-1)^N\zeta_3\Big]}{N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\Big[-8S_1^4-32S_2S_1^2
$$
$$
-16\left(1+(-1)^N\right)S_{-2}S_1^2-8\left(1+(-1)^N\right)S_{-3}S_1+\Big[32S_{2,1}-8S_3\Big]S_1
$$
$$
-6\left(3+(-1)^N\right)\zeta_3S_1+8S_2^2-24S_{-2}^2-8S_4-40S_{-4}-32S_{3,1}+16S_{-3,1}+32S_{-2,1,1}\Big]\Big]\Big]
$$
$$
+a_{Qg}^{(3)}+\tilde{C}_{2,g}^{\mathsf{S},(3)}(N_F+1)\Bigg\}\Bigg\}\ , \tag{E.109}
$$

with the polynomials

$$P_{101} = N^6-81N^5-264N^4-185N^3-307N^2-256N-204 \tag{E.110}$$
$$P_{102} = N^6+6N^5+7N^4+4N^3+18N^2+16N-8 \tag{E.111}$$
$$P_{103} = N^6+7N^5-7N^4-39N^3+14N^2+40N+48 \tag{E.112}$$
$$P_{104} = N^6+21N^5+57N^4+31N^3+26N^2+20N+24 \tag{E.113}$$
$$P_{105} = 2N^6-7N^5-41N^4-31N^3-29N^2-22N-16 \tag{E.114}$$
$$P_{106} = 2N^6-7N^5-24N^4-35N^3-44N^2-44N-16 \tag{E.115}$$
$$P_{107} = 3N^6+5N^5+27N^4+35N^3+6N^2+12N+8 \tag{E.116}$$
$$P_{108} = 3N^6+9N^5-N^4-17N^3-38N^2-28N-24 \tag{E.117}$$
$$P_{109} = 3N^6+9N^5+2N^4-11N^3-23N^2-16N-12 \tag{E.118}$$
$$P_{110} = 3N^6+30N^5+15N^4-64N^3-56N^2-20N-8 \tag{E.119}$$
$$P_{111} = 4N^6+5N^5-10N^4-39N^3-40N^2-24N-8 \tag{E.120}$$
$$P_{112} = 6N^6-12N^5+17N^4+106N^3+127N^2+104N+84 \tag{E.121}$$
$$P_{113} = 6N^6+18N^5+7N^4-16N^3-31N^2-20N-12 \tag{E.122}$$
$$P_{114} = 7N^6-93N^5-327N^4-287N^3-316N^2-112N-24 \tag{E.123}$$
$$P_{115} = 7N^6-20N^5-176N^4-335N^3-276N^2-116N-16 \tag{E.124}$$
$$P_{116} = 7N^6-19N^5-171N^4-325N^3-264N^2-108N-16 \tag{E.125}$$
$$P_{117} = 7N^6+21N^5+5N^4-25N^3-204N^2-188N-192 \tag{E.126}$$
$$P_{118} = 8N^6+13N^5-111N^4-193N^3-89N^2-56N-20 \tag{E.127}$$
$$P_{119} = 9N^6+21N^5+11N^4-5N^3-104N^2-76N-144 \tag{E.128}$$
$$P_{120} = 9N^6+39N^5+53N^4+25N^3+94N^2+44N+312 \tag{E.129}$$
$$P_{121} = 10N^6+18N^5-111N^4-164N^3-61N^2-16N+36 \tag{E.130}$$
$$P_{122} = 10N^6+63N^5+105N^4+31N^3+17N^2+14N+48 \tag{E.131}$$
$$P_{123} = 11N^6-15N^5-327N^4-181N^3+292N^2-20N-48 \tag{E.132}$$
$$P_{124} = 11N^6+15N^5-285N^4-319N^3-254N^2-368N-240 \tag{E.133}$$
$$P_{125} = 11N^6+33N^5-189N^4-361N^3-194N^2-92N-72 \tag{E.134}$$
$$P_{126} = 11N^6+33N^5-114N^4-247N^3-263N^2-176N-108 \tag{E.135}$$
$$P_{127} = 11N^6+33N^5-87N^4-85N^3+4N^2-116N-48 \tag{E.136}$$
$$P_{128} = 11N^6+35N^5+59N^4+57N^3-38N^2-68N+40 \tag{E.137}$$
$$P_{129} = 11N^6+47N^5+7N^4+9N^3+90N^2+28N+96 \tag{E.138}$$

$$P_{130} = 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 \tag{E.139}$$
$$P_{131} = 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 \tag{E.140}$$
$$P_{132} = 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 \tag{E.141}$$
$$P_{133} = 16N^6 + 78N^5 - 23N^4 - 228N^3 - 503N^2 - 408N - 228 \tag{E.142}$$
$$P_{134} = 17N^6 + 111N^5 + 234N^4 + 203N^3 - 89N^2 - 296N - 36 \tag{E.143}$$
$$P_{135} = 22N^6 + 69N^5 + 71N^4 + 23N^3 - 57N^2 - 68N + 84 \tag{E.144}$$
$$P_{136} = 23N^6 - 7N^5 - 237N^4 - 593N^3 - 678N^2 - 548N - 200 \tag{E.145}$$
$$P_{137} = 23N^6 + 9N^5 - 71N^4 - 53N^3 - 184N^2 - 92N - 16 \tag{E.146}$$
$$P_{138} = 25N^6 + 35N^5 - 55N^4 - 243N^3 - 286N^2 - 204N - 72 \tag{E.147}$$
$$P_{139} = 29N^6 + 91N^5 + 235N^4 + 405N^3 + 272N^2 + 288N + 120 \tag{E.148}$$
$$P_{140} = 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 \tag{E.149}$$
$$P_{141} = 35N^6 - 15N^5 - 183N^4 - 133N^3 - 356N^2 - 164N - 48 \tag{E.150}$$
$$P_{142} = 35N^6 - 15N^5 - 101N^4 + 31N^3 + 54N^2 + 164N + 120 \tag{E.151}$$
$$P_{143} = 44N^6 + 96N^5 + 369N^4 + 290N^3 - 695N^2 - 428N - 108 \tag{E.152}$$
$$P_{144} = 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 \tag{E.153}$$
$$P_{145} = 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 \tag{E.154}$$
$$P_{146} = 57N^6 + 161N^5 - 25N^4 - 193N^3 - 172N^2 - 36N + 48 \tag{E.155}$$
$$P_{147} = 65N^6 + 199N^5 + 197N^4 - 143N^3 - 330N^2 - 316N - 120 \tag{E.156}$$
$$P_{148} = 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 \tag{E.157}$$
$$P_{149} = 80N^6 + 60N^5 + 9N^4 + 230N^3 + 901N^2 + 988N + 1188 \tag{E.158}$$
$$P_{150} = 81N^6 + 211N^5 - 23N^4 - 355N^3 - 334N^2 - 4N - 344 \tag{E.159}$$
$$P_{151} = 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 \tag{E.160}$$
$$P_{152} = 130N^6 + 865N^5 + 2316N^4 + 3811N^3 + 4434N^2 + 2884N + 536 \tag{E.161}$$
$$P_{153} = 133N^6 + 699N^5 + 1395N^4 + 217N^3 - 880N^2 + 164N + 288 \tag{E.162}$$
$$P_{154} = 155N^6 + 369N^5 + 211N^4 - 65N^3 - 1002N^2 - 556N - 1416 \tag{E.163}$$
$$P_{155} = 215N^6 + 429N^5 + 891N^4 + 491N^3 - 2486N^2 - 1436N - 408 \tag{E.164}$$
$$P_{156} = 3N^7 + 28N^6 + 66N^5 + 90N^4 + 107N^3 + 78N^2 + 36N + 8 \tag{E.165}$$
$$P_{157} = 9N^7 + 71N^6 + 214N^5 + 320N^4 + 275N^3 + 215N^2 + 160N + 32 \tag{E.166}$$
$$P_{158} = 21N^7 + 120N^6 - 128N^5 - 1038N^4 - 89N^3 + 2382N^2 + 1636N - 600 \tag{E.167}$$
$$P_{159} = 81N^7 + 247N^6 + 291N^5 + 277N^4 + 108N^3 - 56N^2 + 20N + 24 \tag{E.168}$$
$$P_{160} = N^8 + 5N^7 + 10N^6 + 27N^5 + 65N^4 + 112N^3 + 124N^2 + 80N + 32 \tag{E.169}$$
$$P_{161} = N^8 + 5N^7 + 14N^6 + 23N^5 + 25N^4 + 52N^3 + 56N^2 + 48N + 16 \tag{E.170}$$
$$P_{162} = N^8 + 8N^7 - 2N^6 - 60N^5 - 23N^4 + 108N^3 + 96N^2 + 16N + 48 \tag{E.171}$$
$$P_{163} = N^8 + 8N^7 - 2N^6 - 60N^5 + N^4 + 156N^3 + 24N^2 - 80N - 240 \tag{E.172}$$
$$P_{164} = N^8 + 22N^7 + 111N^6 + 211N^5 + 42N^4 - 281N^3 - 406N^2 - 204N - 72 \tag{E.173}$$
$$P_{165} = 2N^8 + N^7 - 6N^6 + 26N^5 + 64N^4 + 51N^3 + 54N^2 + 28N + 8 \tag{E.174}$$
$$P_{166} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \tag{E.175}$$
$$P_{167} = 2N^8 + 44N^7 + 211N^6 + 485N^5 + 654N^4 + 581N^3 + 391N^2 + 192N + 32 \tag{E.176}$$
$$P_{168} = 3N^8 + 41N^7 + 136N^6 + 233N^5 + 331N^4 + 360N^3 + 208N^2 + 80N + 16 \tag{E.177}$$
$$P_{169} = 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 \tag{E.178}$$
$$P_{170} = 5N^8 - 8N^7 - 137N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 \tag{E.179}$$

$$P_{171} = 7N^8+40N^7+110N^6+193N^5+261N^4+313N^3+260N^2+96N+16 \quad (E.180)$$
$$P_{172} = 9N^8+54N^7+80N^6-110N^5-645N^4-1168N^3-1132N^2-672N-160 \quad (E.181)$$
$$P_{173} = 10N^8+46N^7+87N^6+85N^5-75N^4-251N^3-274N^2-132N-72 \quad (E.182)$$
$$P_{174} = 11N^8+74N^7+213N^6+281N^5-30N^4-427N^3-446N^2-180N-72 \quad (E.183)$$
$$P_{175} = 15N^8+36N^7+50N^6-252N^5-357N^4+152N^3-68N^2+88N+48 \quad (E.184)$$
$$P_{176} = 18N^8+101N^7+128N^6+208N^5+190N^4-769N^3-1200N^2-212N-48 \quad (E.185)$$
$$P_{177} = 19N^8+70N^7+63N^6-41N^5-192N^4-221N^3-142N^2-60N-72 \quad (E.186)$$
$$P_{178} = 21N^8+42N^7-38N^6-360N^5-631N^4-730N^3-472N^2-216N-48 \quad (E.187)$$
$$P_{179} = 23N^8+2N^7-135N^6+29N^5+210N^4-151N^3-350N^2-132N-72 \quad (E.188)$$
$$P_{180} = 27N^8-36N^7-956N^6-1724N^5+187N^4+1288N^3+70N^2-224N-72 \quad (E.189)$$
$$P_{181} = 38N^8+146N^7+177N^6+35N^5-249N^4-373N^3-218N^2-60N-72 \quad (E.190)$$
$$P_{182} = 41N^8+5N^7-195N^6-97N^5+326N^4+424N^3+208N^2+72N+16 \quad (E.191)$$
$$P_{183} = 56N^8+194N^7+213N^6+83N^5-231N^4-469N^3-290N^2-60N-72 \quad (E.192)$$
$$P_{184} = 79N^8+196N^7+132N^6+274N^5+465N^4+82N^3+332N^2+456N+288 \quad (E.193)$$
$$P_{185} = 105N^8+978N^7+1688N^6-1330N^5-5245N^4-4672N^3-2212N^2-544N-288 \quad (E.194)$$
$$P_{186} = 113N^8+348N^7+109N^6-289N^5-272N^4-859N^3-778N^2-172N+72 \quad (E.195)$$
$$P_{187} = 170N^8+369N^7-521N^6-1393N^5-761N^4-952N^3-544N^2+32N+144 \quad (E.196)$$
$$P_{188} = 264N^8+1407N^7+2246N^6+1746N^5+804N^4-1069N^3-674N^2-92N-24 \quad (E.197)$$
$$P_{189} = 283N^8+838N^7+1482N^6+628N^5-1497N^4-1130N^3-772N^2+456N+288 \quad (E.198)$$
$$P_{190} = 633N^8+2532N^7+5036N^6+6142N^5+4275N^4+1118N^3-176N^2-184N-48 \quad (E.199)$$
$$P_{191} = N^9+21N^8+85N^7+105N^6+42N^5+290N^4+600N^3+456N^2+256N+64 \quad (E.200)$$
$$P_{192} = 4N^9+53N^8+193N^7+233N^6+87N^5+554N^4+1172N^3+904N^2+512N+128 \quad (E.201)$$
$$P_{193} = 6N^9+93N^8+576N^7+1296N^6+586N^5+359N^4+2000N^3+1996N^2 +1488N+384 \quad (E.202)$$
$$P_{194} = 9N^9+54N^8+56N^7-110N^6-381N^5-568N^4-364N^3-72N^2+128N+96 \quad (E.203)$$
$$P_{195} = 9N^9+54N^8+167N^7+397N^6+780N^5+1241N^4+1448N^3+1200N^2+608N +144 \quad (E.204)$$
$$P_{196} = 11N^9+78N^8+214N^7+335N^6+383N^5+571N^4+916N^3+876N^2+480N+96 \quad (E.205)$$
$$P_{197} = 35N^9+150N^8+232N^7+137N^6+119N^5+661N^4+1174N^3+876N^2 +480N+96 \quad (E.206)$$
$$P_{198} = 37N^9+210N^8-52N^7-2738N^6-7249N^5-9368N^4-8216N^3-5888N^2 -2448N-576 \quad (E.207)$$
$$P_{199} = 45N^9+270N^8+820N^7+1478N^6+1683N^5+1996N^4+2356N^3+2328N^2 +1408N+288 \quad (E.208)$$
$$P_{200} = 57N^9+624N^8+1756N^7+1092N^6-1803N^5-1512N^4+966N^3+1116N^2 +920N+528 \quad (E.209)$$
$$P_{201} = 69N^9+366N^8+1124N^7+1966N^6+2523N^5+5228N^4+7340N^3+5352N^2 +3008N+672 \quad (E.210)$$
$$P_{202} = 94N^9+597N^8+1616N^7+2410N^6+1841N^5+1165N^4+2191N^3+3802N^2 +2916N+648 \quad (E.211)$$
$$P_{203} = 121N^9+696N^8+1535N^7+1585N^6+416N^5-749N^4-836N^3+16N^2$$

$$+528N+144 \tag{E.212}$$

$$P_{204} = 197N^9+1242N^8+2938N^7+3524N^6+2713N^5+2234N^4+3680N^3+6176N^2 +4080N+864 \tag{E.213}$$

$$P_{205} = 439N^9+2634N^8+6008N^7+6694N^6+3545N^5+736N^4+2008N^3+6208N^2 +5136N+1152 \tag{E.214}$$

$$P_{206} = 538N^9+3333N^8+7802N^7+7630N^6+458N^5-1415N^4+7786N^3+12340N^2 +5592N+864 \tag{E.215}$$

$$P_{207} = 664N^9+3861N^8+9038N^7+11830N^6+9344N^5+3793N^4+3874N^3+11044N^2 +9624N+2592 \tag{E.216}$$

$$P_{208} = 891N^9+4455N^8+16078N^7+28774N^6+37047N^5+45835N^4+42192N^3+28888N^2 +10640N+1776 \tag{E.217}$$

$$P_{209} = 923N^9+5208N^8+11824N^7+12854N^6+2185N^5-7030N^4+1436N^3+15032N^2 +12864N+3456 \tag{E.218}$$

$$P_{210} = 965N^9+4884N^8+10816N^7+20810N^6+36895N^5+40442N^4+27692N^3+22712N^2 +14496N+3456 \tag{E.219}$$

$$P_{211} = 2N^{10}-46N^9-98N^8+282N^7+1063N^6+1569N^5+1275N^4+403N^3-94N^2 -108N-24 \tag{E.220}$$

$$P_{212} = 2N^{10}+12N^9+24N^8+11N^7-48N^6-151N^5-282N^4-480N^3-664N^2 -576N-288 \tag{E.221}$$

$$P_{213} = 11N^{10}+44N^9+74N^8+196N^7+31N^6-1426N^5-3044N^4-2762N^3-1476N^2 -480N-96 \tag{E.222}$$

$$P_{214} = 11N^{10}+76N^9+138N^8-204N^7-1041N^6-988N^5+752N^4+1896N^3+944N^2 -384N-576 \tag{E.223}$$

$$P_{215} = 37N^{10}+392N^9+2106N^8+6514N^7+9211N^6+1258N^5-9218N^4-6116N^3-72N^2 -752N-192 \tag{E.224}$$

$$P_{216} = 85N^{10}+425N^9+902N^8+932N^7-521N^6-685N^5+2022N^4+2928N^3+968N^2 -1296N-576 \tag{E.225}$$

$$P_{217} = 103N^{10}+575N^9+1124N^8-334N^7-1505N^6+3755N^5+4926N^4+36N^3-472N^2 -2160N-864 \tag{E.226}$$

$$P_{218} = 118N^{10}+425N^9+197N^8+86N^7+1240N^6+2489N^5+4401N^4+3480N^3+524N^2 -1728N-864 \tag{E.227}$$

$$P_{219} = 118N^{10}+557N^9+461N^8-94N^7+1300N^6+3521N^5+4509N^4+1920N^3 -1132N^2-2376N-1008 \tag{E.228}$$

$$P_{220} = 127N^{10}+536N^9+611N^8+602N^7+1474N^6+2099N^5+798N^4-2301N^3 -4486N^2-3708N-936 \tag{E.229}$$

$$P_{221} = 170N^{10}+883N^9+2041N^8+2998N^7-448N^6-5465N^5+129N^4+6624N^3 +1132N^2-2016N-864 \tag{E.230}$$

$$P_{222} = 170N^{10}+1213N^9+3235N^8+2794N^7-2692N^6-3767N^5-1293N^4 -1632N^3-5324N^2-6240N-2016 \tag{E.231}$$

$$P_{223} = 226N^{10}+317N^9-811N^8+662N^7+4552N^6+3857N^5+3933N^4+2364N^3 +236N^2-1656N-720 \tag{E.232}$$

$$\begin{aligned}
P_{224} &= 489N^{10}+2934N^9+9364N^8+18830N^7+18627N^6+124N^5-19856N^4-19296N^3 \\
&\quad -10640N^2-2880N-1152 &\text{(E.233)}\\
P_{225} &= 3N^{11}+42N^{10}+144N^9+74N^8-459N^7-1060N^6-1152N^5-1424N^4-1688N^3 \\
&\quad -1232N^2-736N-192 &\text{(E.234)}\\
P_{226} &= 11N^{11}+37N^{10}-27N^9-118N^8+21N^7-249N^6-1097N^5-1138N^4+552N^3 \\
&\quad +3448N^2+3456N+2016 &\text{(E.235)}\\
P_{227} &= 21N^{11}+231N^{10}+1334N^9+4086N^8+6277N^7+1775N^6-9488N^5-18076N^4 \\
&\quad -18208N^3-11344N^2-5568N-1728 &\text{(E.236)}\\
P_{228} &= 33N^{11}+231N^{10}+698N^9+1290N^8+1513N^7+1463N^6+2236N^5+5096N^4 \\
&\quad +7328N^3+5456N^2+3456N+1152 &\text{(E.237)}\\
P_{229} &= 45N^{11}+383N^{10}+958N^9+526N^8-763N^7+1375N^6+7808N^5+13028N^4 \\
&\quad +12976N^3+8016N^2+4608N+1728 &\text{(E.238)}\\
P_{230} &= 51N^{11}+269N^{10}+46N^9-1934N^8-3973N^7-875N^6+7364N^5+14972N^4 \\
&\quad +16768N^3+10896N^2+5376N+1728 &\text{(E.239)}\\
P_{231} &= 51N^{11}+357N^{10}+1238N^9+2586N^8+2755N^7-1435N^6-9212N^5-15028N^4 \\
&\quad -15280N^3-9808N^2-5184N-1728 &\text{(E.240)}\\
P_{232} &= 81N^{11}+483N^{10}+1142N^9+1086N^8-767N^7-4645N^6-8936N^5-11980N^4 \\
&\quad -12352N^3-8272N^2-4800N-1728 &\text{(E.241)}\\
P_{233} &= 120N^{11}+1017N^{10}+2737N^9+1292N^8-8086N^7-20743N^6-24563N^5-16702N^4 \\
&\quad -6840N^3+120N^2+2432N+960 &\text{(E.242)}\\
P_{234} &= 243N^{11}+1701N^{10}+5378N^9+10350N^8+11479N^7+1193N^6-14684N^5-20572N^4 \\
&\quad -16288N^3-8944N^2-4992N-1728 &\text{(E.243)}\\
P_{235} &= 333N^{11}+2331N^{10}+6556N^9+9270N^8+5081N^7-6701N^6-17554N^5-20036N^4 \\
&\quad -15680N^3-9200N^2-5664N-1728 &\text{(E.244)}\\
P_{236} &= 753N^{11}+4809N^{10}+13174N^9+20466N^8+17717N^7+6829N^6+3908N^5 \\
&\quad +15304N^4+25408N^3+20272N^2+8448N+1152 &\text{(E.245)}\\
P_{237} &= 837N^{11}+7757N^{10}+30120N^9+68575N^8+119176N^7+191350N^6+262979N^5 \\
&\quad +258308N^4+163106N^3+63360N^2+14848N+1536 &\text{(E.246)}\\
P_{238} &= 1017N^{11}+6195N^{10}+14050N^9+12738N^8-2023N^7-5093N^6+27548N^5 \\
&\quad +69760N^4+80752N^3+54064N^2+20928N+3456 &\text{(E.247)}\\
P_{239} &= 3N^{12}+21N^{11}+17N^{10}-202N^9-842N^8-1924N^7-3378N^6-5059N^5 \\
&\quad -6008N^4-4860N^3-2536N^2-960N-192 &\text{(E.248)}\\
P_{240} &= 9N^{12}+63N^{11}+38N^{10}-414N^9-1035N^8-1341N^7-1511N^6-2972N^5 \\
&\quad -6011N^4-8038N^3-6892N^2-3432N-864 &\text{(E.249)}\\
P_{241} &= 9N^{12}+63N^{11}+71N^{10}-381N^9-1536N^8-2529N^7-1946N^6-1331N^5 \\
&\quad -2096N^4-4036N^3-4144N^2-2304N-576 &\text{(E.250)}\\
P_{242} &= 39N^{12}+585N^{11}+2938N^{10}+7136N^9+9083N^8+7745N^7+14668N^6+38246N^5 \\
&\quad +59856N^4+55560N^3+32144N^2+12480N+2304 &\text{(E.251)}\\
P_{243} &= 48N^{12}+459N^{11}+2322N^{10}+8290N^9+20159N^8+30862N^7+28247N^6+16109N^5 \\
&\quad +9312N^4+7488N^3+4064N^2+1328N+192 &\text{(E.252)}\\
P_{244} &= 61N^{12}+302N^{11}+531N^{10}+348N^9-349N^8-786N^7+457N^6+2524N^5
\end{aligned}$$

$$
\begin{aligned}
& +2012N^4+204N^3-360N^2-240N-96 && \text{(E.253)}\\
P_{245} &= 92N^{12}+796N^{11}+3089N^{10}+7550N^9+10547N^8+1029N^7-19496N^6\\
& -24199N^5-8960N^4+736N^3+1744N^2+816N+192 && \text{(E.254)}\\
P_{246} &= 201N^{12}+1845N^{11}+6910N^{10}+12854N^9+8915N^8-7741N^7-17126N^6\\
& -4294N^5+16260N^4+22080N^3+12416N^2+4128N+576 && \text{(E.255)}\\
P_{247} &= 239N^{12}+1338N^{11}+3137N^{10}+3164N^9-983N^8-6640N^7-8123N^6\\
& -4526N^5-342N^4+1232N^3+848N^2+256N+32 && \text{(E.256)}\\
P_{248} &= 255N^{12}+2169N^{11}+6496N^{10}+7694N^9-127N^8-6973N^7+4132N^6\\
& +25502N^5+31956N^4+22656N^3+9632N^2+864N-576 && \text{(E.257)}\\
P_{249} &= 581N^{12}+7035N^{11}+37826N^{10}+112904N^9+190293N^8+174327N^7+92032N^6\\
& +69438N^5+78364N^4+44464N^3+11520N^2-3168N-1728 && \text{(E.258)}\\
P_{250} &= 825N^{12}+7363N^{11}+25396N^{10}+40686N^9+26213N^8-12749N^7-55498N^6\\
& -89796N^5-110552N^4-134960N^3-127584N^2-64704N-12672 && \text{(E.259)}\\
P_{251} &= 69N^{13}+420N^{12}+794N^{11}-1357N^{10}-10401N^9-15678N^8+532N^7+239N^6\\
& -40018N^5-69432N^4-69152N^3-43792N^2-18336N-3456 && \text{(E.260)}\\
P_{252} &= 76N^{13}+922N^{12}+4479N^{11}+9107N^{10}-3747N^9-52973N^8-76133N^7+42261N^6\\
& +199307N^5+123839N^4-77470N^3-84132N^2-2160N-432 && \text{(E.261)}\\
P_{253} &= 295N^{13}+2387N^{12}+8005N^{11}+13687N^{10}+10883N^9+389N^8-2641N^7+6029N^6\\
& +11034N^5+6644N^4+1384N^3+80N^2+128N+64 && \text{(E.262)}\\
P_{254} &= 296N^{13}+2368N^{12}+10916N^{11}+27006N^{10}+23644N^9-19764N^8-61931N^7\\
& -63733N^6-52001N^5-56865N^4-38104N^3+2664N^2+7344N+432 && \text{(E.263)}\\
P_{255} &= 377N^{13}+4649N^{12}+21813N^{11}+38539N^{10}-39339N^9-272611N^8-332971N^7\\
& +220377N^6+801934N^5+384958N^4-362030N^3-297864N^2-1080N-864 && \text{(E.264)}\\
P_{256} &= 859N^{13}+7376N^{12}+25294N^{11}+47088N^{10}+63868N^9+80876N^8+63648N^7\\
& -35856N^6-146697N^5-157168N^4-91320N^3-34800N^2-8640N-1152 && \text{(E.265)}\\
P_{257} &= 1211N^{13}+5680N^{12}+3338N^{11}-17355N^{10}-31517N^9-48486N^8-139667N^7\\
& -278026N^6-340745N^5-269457N^4-138568N^3-34632N^2+9072N+3888 && \text{(E.266)}\\
P_{258} &= 70N^{14}+555N^{13}+1599N^{12}+1192N^{11}-4430N^{10}-13305N^9-11835N^8+8440N^7\\
& +35816N^6+57126N^5+60340N^4+44464N^3+27808N^2+12768N+2880 && \text{(E.267)}\\
P_{259} &= 76N^{14}+802N^{13}+2979N^{12}+1847N^{11}-19377N^{10}-58253N^9-26543N^8+170601N^7\\
& +362177N^6+225119N^5-103240N^4-193092N^3-137160N^2-117072N-25920 && \text{(E.268)}\\
P_{260} &= 76N^{14}+1042N^{13}+5979N^{12}+16367N^{11}+11883N^{10}-47693N^9-125723N^8-86079N^7\\
& +36437N^6+22559N^5-51700N^4+24828N^3+132840N^2+116208N+25920 && \text{(E.269)}\\
P_{261} &= 4N^{15}+50N^{14}+267N^{13}+765N^{12}+1183N^{11}+682N^{10}-826N^9-1858N^8-1116N^7\\
& +457N^6+1500N^5+2268N^4+2400N^3+1392N^2+448N+64 && \text{(E.270)}\\
P_{262} &= 26N^{15}+314N^{14}+1503N^{13}+3222N^{12}+2510N^{11}+1996N^{10}+15041N^9+40728N^8\\
& +54008N^7+44956N^6+31936N^5+30416N^4+29568N^3+16704N^2+5376N+768 && \text{(E.271)}\\
P_{263} &= 101N^{15}+1234N^{14}+6867N^{13}+21904N^{12}+40098N^{11}+32226N^{10}-22057N^9\\
& -86972N^8-114557N^7-111416N^6-89204N^5-37312N^4+13392N^3+23040N^2\\
& +9792N+1536 && \text{(E.272)}\\
P_{264} &= 390N^{15}+5121N^{14}+30556N^{13}+114173N^{12}+321958N^{11}+771597N^{10}
\end{aligned}
$$

$$+1583594N^9+2637549N^8+3381542N^7+3199120N^6+2183360N^5+1123200N^4$$
$$+489952N^3+178176N^2+48384N+6912 \quad (E.273)$$
$$P_{265} = 75N^{16}+1245N^{15}+8291N^{14}+27609N^{13}+43437N^{12}+14221N^{11}-5995N^{10}$$
$$+182937N^9+488696N^8+296818N^7-452292N^6-730430N^5-186180N^4$$
$$+259728N^3+241056N^2+116640N+25920 \quad (E.274)$$
$$P_{266} = 115N^{16}+1838N^{15}+11829N^{14}+36114N^{13}+30900N^{12}-133946N^{11}-454068N^{10}$$
$$-457420N^9+249211N^8+864716N^7+312979N^6-634466N^5-587862N^4$$
$$-19556N^3+104832N^2+9504N+1728 \quad (E.275)$$
$$P_{267} = 185N^{16}+2988N^{15}+19694N^{14}+62954N^{13}+64470N^{12}-207876N^{11}-792388N^{10}$$
$$-861230N^9+437231N^8+1750616N^7+869954N^6-1016136N^5-1130122N^4$$
$$-96596N^3+199872N^2+31104N+1728 \quad (E.276)$$
$$P_{268} = 939N^{16}+10527N^{15}+37207N^{14}+18679N^{13}-202006N^{12}-617170N^{11}-930025N^{10}$$
$$-882917N^9-157123N^8+1388549N^7+2739376N^6+2837500N^5+2088640N^4$$
$$+1259696N^3+622464N^2+211392N+34560 \quad (E.277)$$
$$P_{269} = 1155N^{16}+12417N^{15}+37693N^{14}-12293N^{13}-285754N^{12}-613900N^{11}-571735N^{10}$$
$$-134309N^9+778901N^8+2698745N^7+4995724N^6+5915740N^5+4978144N^4$$
$$+3161840N^3+1498752N^2+479808N+76032 \quad (E.278)$$
$$P_{270} = 1665N^{16}+33005N^{15}+287646N^{14}+1402624N^{13}+4031902N^{12}+6199846N^{11}$$
$$+1054640N^{10}-16668628N^9-37272559N^8-38892027N^7-17387942N^6+3962700N^5$$
$$+15625800N^4+26960688N^3+27379296N^2+12985920N+2332800 \quad (E.279)$$
$$P_{271} = 87N^{17}+1099N^{16}+6055N^{15}+19019N^{14}+37119N^{13}+45159N^{12}+29583N^{11}-2639N^{10}$$
$$-30218N^9-40778N^8-39994N^7-35844N^6-30808N^5-30384N^4-28256N^3$$
$$-16064N^2-5248N-768 \quad (E.280)$$
$$P_{272} = 829N^{17}+13413N^{16}+83461N^{15}+226391N^{14}+55508N^{13}-1239070N^{12}-2862466N^{11}$$
$$-1217372N^{10}+3372689N^9+2779147N^8-2705687N^7+171733N^6$$
$$+8617302N^5+5817902N^4-3127236N^3-3652560N^2-336096N-25920 \quad (E.281)$$
$$P_{273} = 1407N^{17}+18107N^{16}+103463N^{15}+347083N^{14}+760095N^{13}+1142715N^{12}+1220067N^{11}$$
$$+983393N^{10}+702746N^9+533822N^8+337702N^7-3552N^6$$
$$-300296N^5-332160N^4-188128N^3-63232N^2-13184N-1536 \quad (E.282)$$
$$P_{274} = 95N^{18}+3940N^{17}+48989N^{16}+308380N^{15}+1166094N^{14}+2843192N^{13}+4428234N^{12}$$
$$+3171928N^{11}-4692053N^{10}-19875244N^9-34305831N^8-34774388N^7-16392680N^6$$
$$+11584912N^5+30493776N^4+29700864N^3+18783360N^2+8294400N+1866240 \quad (E.283)$$
$$P_{275} = 325N^{18}+4280N^{17}+17759N^{16}-14880N^{15}-412326N^{14}-1696848N^{13}-3216546N^{12}$$
$$-1169232N^{11}+8956857N^{10}+23914216N^9+31536899N^8+25361392N^7+9982840N^6$$
$$-10154128N^5-26098704N^4-26761536N^3-17642880N^2-8087040N-1866240 \quad (E.284)$$
$$P_{276} = 500N^{18}+8215N^{17}+56287N^{16}+201810N^{15}+361782N^{14}+98826N^{13}-759348N^{12}$$
$$-495786N^{11}+3942186N^{10}+11896133N^9+16709737N^8+13315736N^7+3779660N^6$$
$$-7306454N^5-14232852N^4-13254768N^3-8367840N^2-3771360N-855360 \quad (E.285)$$
$$P_{277} = 150N^{19}+2815N^{18}+24285N^{17}+131358N^{16}+511310N^{15}+1515954N^{14}$$
$$+3372978N^{13}+5213980N^{12}+4715522N^{11}+980739N^{10}-2709391N^9-3741506N^8$$
$$-4630558N^7-5623132N^6-2333736N^5+3419632N^4+5238496N^3+3231936N^2$$

$$
\begin{aligned}
& +1123200N + 172800 && \text{(E.286)}\\
P_{278} &= 5410N^{19} + 98215N^{18} + 764965N^{17} + 3280996N^{16} + 8031920N^{15} + 8939378N^{14}\\
& -7608074N^{13} - 44964380N^{12} - 74768226N^{11} - 57879177N^{10} - 5243187N^{9} + 13745888N^{8}\\
& -28158216N^{7} - 49672024N^{6} + 14757808N^{5} + 94650144N^{4} + 100507392N^{3}\\
& +53764992N^{2} + 15655680N + 1866240 && \text{(E.287)}\\
P_{279} &= 7060N^{20} + 123495N^{19} + 898682N^{18} + 3394183N^{17} + 6222824N^{16} + 376386N^{15}\\
& -22032204N^{14} - 39912378N^{13} - 13976964N^{12} + 31985011N^{11} + 4994394N^{10}\\
& -91499501N^{9} - 97243208N^{8} + 54501988N^{7} + 183103272N^{6} + 127073120N^{5}\\
& -20272608N^{4} - 88410816N^{3} - 62225280N^{2} - 21772800N - 3110400\,. && \text{(E.288)}
\end{aligned}
$$

The logarithmic contributions to all heavy flavor Wilson coefficients at 3–loop level are expressible within the class of alternating harmonic sums only.

F Fixed Mellin moments for $m_1 \neq m_2$

In this Section we present the Mellin moments $N = 4 \mathrm{and} 6$ to the constant parts of the contributions with two massive lines of unequal mass to the OMEs $A_{qq,Q}^{\mathsf{NS}}$, A_{Qq}^{PS}, A_{Qg}, $A_{gq,Q}$ and $A_{gg,Q}$. We present expansions in the mass ratio $\eta = \frac{m_2^2}{m_1^2}$ up to $O(x^3 \ln^3(x))$. In order to obtain a compact notation we furthermore use the abbreviations

$$L_1 = \ln\left(\frac{m_1^2}{\mu^2}\right), \quad L_2 = \ln\left(\frac{m_2^2}{\mu^2}\right), \quad L_\eta = \ln(\eta)\,. \tag{F.289}$$

$\underline{\tilde{a}_{qq,Q}^{\mathsf{NS},(3)}:}$

$$\begin{aligned}
\tilde{a}_{qq,Q}^{\mathsf{NS},(3)}(N=4) &= C_F T_F^2 \Biggl\{ \left(-\frac{10048}{42525} L_\eta^2 - \frac{1869287216}{4219543125} - \frac{7330016}{13395375} L_\eta \right) \eta^3 \\
&+ \left(-\frac{70417954}{5788125} - \frac{628}{525} L_\eta^2 - \frac{332212}{55125} L_\eta \right) \eta^2 + \left(-\frac{118064}{3375} - \frac{5024}{225} L_\eta \right) \eta \\
&+ \left(-\frac{53084}{2025} - \frac{1256}{45} (L_2 + L_1) \right) \zeta_2 - \frac{388370299}{2733750} + \frac{5024}{405} \zeta_3 - \frac{3509323}{30375} L_1 \\
&- \frac{520841}{10125} L_2 - \frac{53084}{2025} L_2 L_1 - \frac{2512}{135} L_1 L_2^2 - \frac{5024}{405} L_2^3 - \frac{53084}{2025} \left(L_2^2 + L_1^2\right) \\
&- \frac{1256}{81} L_1^3 - \frac{1256}{135} L_1^2 L_2 \Biggr\} + O\left(\eta^4 L_\eta^3\right)
\end{aligned} \tag{F.290}$$

$$\begin{aligned}
\tilde{a}_{qq,Q}^{\mathsf{NS},(3)}(N=6) &= C_F T_F^2 \Biggl\{ \left(-\frac{90752}{297675} L_\eta^2 - \frac{16883116384}{29536801875} - \frac{66203584}{93767625} L_\eta \right) \eta^3 \\
&+ \left(-\frac{636004196}{40516875} - \frac{5672}{3675} L_\eta^2 - \frac{3000488}{385875} L_\eta \right) \eta^2 + \left(-\frac{1066336}{23625} - \frac{45376}{1575} L_\eta \right) \eta \\
&+ \left(-\frac{3424952}{99225} - \frac{11344}{315} (L_2 + L_1) \right) \zeta_2 - \frac{202733427313}{1093955625} + \frac{45376}{2835} \zeta_3 \\
&- \frac{520819486}{3472875} L_1 - \frac{700881658}{10418625} L_2 - \frac{3424952}{99225} L_2 L_1 - \frac{22688}{945} L_1 L_2^2 \\
&- \frac{45376}{2835} L_2^3 - \frac{3424952}{99225} \left(L_2^2 + L_1^2\right) - \frac{11344}{567} L_1^3 - \frac{11344}{945} L_1^2 L_2 \Biggr\} \\
&+ O\left(\eta^4 L_\eta^3\right)
\end{aligned} \tag{F.291}$$

$\underline{\tilde{a}_{Qq}^{\mathsf{PS},(3)}:}$

$$\begin{aligned}
\tilde{a}_{Qq}^{\mathsf{PS},(3)}(N=4) &= C_F T_F^2 \Biggl\{ \left(\frac{190292193776}{1123242379875} + \frac{8509216}{324168075} L_\eta - \frac{1472}{93555} L_\eta^2 \right) \eta^3 + \Biggl(-\frac{71844302}{31255875} \\
&- \frac{76}{315} L_\eta^2 - \frac{89252}{99225} L_\eta \Biggr) \eta^2 + \left(-\frac{5008}{945} - \frac{32}{9} L_\eta \right) \eta + \Biggl(-\frac{2236}{2025} - \frac{968}{225} \Bigl(L_2
\end{aligned}$$

$$
\begin{aligned}
&\quad +L_1\Big)\Big)\zeta_2 - \frac{968}{405}L_1^3 - \frac{968}{675}L_1^2L_2 + \frac{3872}{2025}\zeta_3 - \frac{2406319}{151875}L_1 - \frac{297941}{50625}L_2 \\
&\quad -\frac{2236}{2025}L_2L_1 - \frac{1936}{675}L_1L_2^2 - \frac{195482623}{13668750} - \frac{3872}{2025}L_2^3 - \frac{2236}{2025}\left(L_2^2+L_1^2\right)\Bigg\} \\
&\quad +O\left(\eta^4L_\eta^3\right) \qquad \text{(F.292)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_{Qq}^{\mathsf{PS},(3)}(N=6) &= C_FT_F^2\Bigg\{\left(\frac{19353315711436064}{86371722800488125} + \frac{112677158848}{1917454163625}L_\eta + \frac{385408}{42567525}L_\eta^2\right)\eta^3 \\
&\quad +\left(-\frac{5015464079432}{4368164810625} - \frac{45616}{363825}L_\eta^2 - \frac{432844912}{1260653625}L_\eta\right)\eta^2 + \Bigg(-\frac{2455328}{1157625} \\
&\quad -\frac{1984}{1323}L_\eta\Bigg)\eta + (-\frac{15184}{99225} - \frac{3872}{2205}\left(L_2+L_1\right))\zeta_2 - \frac{3872}{3969}L_1^3 - \frac{3872}{6615}L_1^2L_2 \\
&\quad +\frac{15488}{19845}\zeta_3 - \frac{52387796}{8103375}L_1 - \frac{172633556}{72930375}L_2 - \frac{15184}{99225}L_2L_1 - \frac{7744}{6615}L_1L_2^2 \\
&\quad -\frac{7819198418}{1531537875} - \frac{15488}{19845}L_2^3 - \frac{15184}{99225}\left(L_2^2+L_1^2\right)\Bigg\} + O\left(\eta^4L_\eta^3\right) \qquad \text{(F.293)}
\end{aligned}
$$

$\underline{\tilde{a}_{Qg}^{(3)}:}$

$$
\begin{aligned}
\tilde{a}_{Qg}^{(3)}(N=4) &= C_AT_F^2\Bigg\{\left(\frac{250077164867}{11232423798750} - \frac{156082853}{3241680750}L_\eta - \frac{744283}{1871100}L_\eta^2\right)\eta^3 \\
&\quad +\left(\frac{1634774}{1488375}L_\eta + \frac{1255194149}{468838125} - \frac{142}{525}L_\eta^2\right)\eta^2 + \Bigg(\frac{496855133}{14883750} + \frac{1877399}{141750}L_\eta \\
&\quad +\frac{707}{2700}L_\eta^2\Bigg)\eta + \left(\frac{5807}{360} + \frac{17963}{900}\left(L_2+L_1\right)\right)\zeta_2 + \frac{47956573}{1620000}L_2 + \frac{3817}{324}L_1^3 \\
&\quad -\frac{23573}{2025}\zeta_3 + \frac{384762007}{4860000}L_1 + \frac{17963}{1350}L_1L_2^2 + \frac{532373}{32400}\left(L_2^2+L_1^2\right) + \frac{7579}{1350}L_1^2L_2 \\
&\quad +\frac{62893}{4050}L_2L_1 + \frac{74657}{8100}L_2^3 + \frac{4887988511}{48600000}\Bigg\} \\
&\quad +C_FT_F^2\Bigg\{\left(\frac{23024568781}{44929695195} + \frac{285046646}{324168075}L_\eta + \frac{879808}{467775}L_\eta^2\right)\eta^3 + \Bigg(\frac{2876423}{595350}L_\eta \\
&\quad +\frac{582667691}{75014100} + \frac{27101}{9450}L_\eta^2\Bigg)\eta^2 + \left(-\frac{59657237}{4134375} + \frac{184214}{39375}L_\eta + \frac{2228}{1125}L_\eta^2\right)\eta \\
&\quad +\left(\frac{1473641}{405000} - \frac{18601}{4500}\left(L_2+L_1\right)\right)\zeta_2 + \frac{76621423}{8100000}L_2 - \frac{204611}{40500}L_1^3 + \frac{130207}{10125}\zeta_3 \\
&\quad -\frac{37307959}{4860000}L_1 - \frac{18601}{6750}L_1L_2^2 + \frac{530371}{162000}\left(L_2^2+L_1^2\right) + \frac{18601}{6750}L_1^2L_2 + \frac{442267}{101250}L_2L_1 \\
&\quad -\frac{130207}{40500}L_2^3 - \frac{33406758667}{2187000000}\Bigg\} + O\left(\eta^4L_\eta^3\right) \qquad \text{(F.294)}
\end{aligned}
$$

$$\tilde{a}_{Qg}^{(3)}(N=6) = C_A T_F^2 \Biggl\{ \Biggl(-\frac{84840004938801319}{1381947564807810000} - \frac{2287164970759}{15339633309000} L_\eta - \frac{31340489}{68108040} L_\eta^2 \Biggr) \eta^3$$
$$+ \Biggl(\frac{105157957}{360186750} L_\eta + \frac{755537213056}{624023544375} - \frac{49373}{103950} L_\eta^2 \Biggr) \eta^2 + \Biggl(\frac{832369820129}{29172150000}$$
$$+ \frac{1406143531}{138915000} L_\eta + \frac{112669}{1323000} L_\eta^2 \Biggr) \eta + \Biggl(\frac{1316809}{79380} + \frac{39248}{2205} (L_2 + L_1) \Biggr) \zeta_2$$
$$+ \frac{11771644229}{388962000} L_2 + \frac{206404}{19845} L_1^3 - \frac{197648}{19845} \zeta_3 + \frac{83755534727}{1166886000} L_1 + \frac{78496}{6615} L_1 L_2^2$$
$$+ \frac{2668087}{158760} \left(L_2^2 + L_1^2 \right) + \frac{34166}{6615} L_1^2 L_2 + \frac{64117}{3969} L_2 L_1 + \frac{162074}{19845} L_2^3$$
$$+ \frac{69882273800453}{735138180000} \Biggr\}$$
$$+ C_F T_F^2 \Biggl\{ \Biggl(\frac{990283034941336}{2467763508585375} + \frac{1255768040}{2191376187} L_\eta + \frac{63929464}{42567525} L_\eta^2 \Biggr) \eta^3$$
$$+ \Biggl(\frac{11478584}{3361743} L_\eta + \frac{524351089261}{97070329125} + \frac{88972}{40425} L_\eta^2 \Biggr) \eta^2 + \Biggl(-\frac{32427817736}{2552563125}$$
$$+ \frac{64271512}{24310125} L_\eta + \frac{376216}{231525} L_\eta^2 \Biggr) \eta + \Biggl(\frac{4784009}{4862025} - \frac{55924}{15435} (L_2 + L_1) \Biggr) \zeta_2$$
$$+ \frac{1786067629}{408410100} L_2 - \frac{615164}{138915} L_1^3 + \frac{223696}{19845} \zeta_3 - \frac{24797875607}{2042050500} L_1 - \frac{111848}{46305} L_1 L_2^2$$
$$+ \frac{3232799}{9724050} \left(L_2^2 + L_1^2 \right) + \frac{111848}{46305} L_1^2 L_2 + \frac{11119228}{4862025} L_2 L_1 - \frac{55924}{19845} L_2^3$$
$$- \frac{31618111182177}{142943535000} \Biggr\} + O\left(\eta^4 L_\eta^3\right) \tag{F.295}$$

$\tilde{a}_{gq,Q}^{(3)}$:

$$\tilde{a}_{gq,Q}^{(3)}(N=4) = C_F T_F^2 \Biggl\{ \Biggl(\frac{261938336}{1406514375} + \frac{1027136}{4465125} L_\eta + \frac{1408}{14175} L_\eta^2 \Biggr) \eta^3 + \Biggl(\frac{9867484}{1929375}$$
$$+ \frac{88}{175} L_\eta^2 + \frac{46552}{18375} L_\eta \Biggr) \eta^2 + \Biggl(\frac{16544}{1125} + \frac{704}{75} L_\eta \Biggr) \eta + \Biggl(\frac{2504}{675}$$
$$+ \frac{176}{15} (L_2 + L_1) \Biggr) \zeta_2 + \frac{176}{45} L_1^2 L_2 - \frac{704}{135} \zeta_3 + \frac{436138}{10125} L_1 + \frac{54446}{3375} L_2$$
$$+ \frac{2504}{675} L_2 L_1 + \frac{352}{45} L_1 L_2^2 + \frac{704}{135} L_2^3 + \frac{2504}{675} \left(L_2^2 + L_1^2 \right) + \frac{176}{27} L_1^3 + \frac{18480197}{455625} \Biggr\}$$
$$+ O\left(\eta^4 L_\eta^3\right) \tag{F.296}$$
$$\tilde{a}_{gq,Q}^{(3)}(N=6) = C_F T_F^2 \Biggl\{ \Biggl(\frac{1047753344}{9845600625} + \frac{4108544}{31255875} L_\eta + \frac{5632}{99225} L_\eta^2 \Biggr) \eta^3 + \Biggl(\frac{39469936}{13505625} + \frac{352}{1225} L_\eta^2$$

$$+\frac{186208}{128625}L_\eta\Bigg)\eta^2+\Bigg(\frac{66176}{7875}+\frac{2816}{525}L_\eta\Bigg)\eta+\Bigg(\frac{17632}{33075}+\frac{704}{105}\left(L_2+L_1\right)\Bigg)\zeta_2$$
$$+\frac{704}{315}L_1^2L_2-\frac{2816}{945}\zeta_3+\frac{28089976}{1157625}L_1+\frac{30801128}{3472875}L_2+\frac{17632}{33075}L_2L_1$$
$$+\frac{1408}{315}L_1L_2^2+\frac{2816}{945}L_2^3+\frac{17632}{33075}\left(L_2^2+L_1^2\right)+\frac{704}{189}L_1^3+\frac{779635012}{40516875}\Bigg\}$$
$$+O\left(\eta^4L_\eta^3\right) \qquad \text{(F.297)}$$

$\underline{\tilde{a}_{gg,Q}^{(3)}:}$

$$\tilde{a}_{gg,Q}^{(3)}(N=4) = C_AT_F^2\Bigg\{\Bigg(-\frac{311441927}{1687817250}-\frac{1293167}{5358150}L_\eta+\frac{1681}{34020}L_\eta^2\Bigg)\eta^3+\Bigg(-\frac{205123}{99225}L_\eta$$
$$-\frac{36414571}{10418625}-\frac{61}{189}L_\eta^2\Bigg)\eta^2+\Bigg(-\frac{19131223}{303750}-\frac{524233}{20250}L_\eta-\frac{6943}{2700}L_\eta^2\Bigg)\eta$$
$$+\Bigg(-\frac{28979}{675}-\frac{1558}{45}\left(L_2+L_1\right)\Bigg)\zeta_2-\frac{1057309}{13500}L_2-\frac{1546}{81}L_1^3+\frac{5992}{405}\zeta_3$$
$$-\frac{19874881}{121500}L_1-\frac{3116}{135}L_1L_2^2-\frac{173999}{4050}\left(L_2^2+L_1^2\right)-\frac{1588}{135}L_1^2L_2-\frac{86812}{2025}L_2L_1$$
$$-\frac{6202}{405}L_2^3-\frac{781640551}{3645000}\Bigg\}$$
$$+C_FT_F^2\Bigg\{\Bigg(-\frac{22991704}{602791875}+\frac{64016}{1913625}L_\eta-\frac{2624}{6075}L_\eta^2\Bigg)\eta^3+\Bigg(-\frac{988}{7875}L_\eta$$
$$-\frac{507478}{2480625}-\frac{4}{9}L_\eta^2\Bigg)\eta^2+\Bigg(\frac{1088008}{253125}+\frac{5456}{5625}L_\eta-\frac{416}{1125}L_\eta^2\Bigg)\eta$$
$$+\Bigg(\frac{14263}{10125}+\frac{242}{225}\left(L_2+L_1\right)\Bigg)\zeta_2-\frac{173327}{202500}L_2+\frac{2662}{2025}L_1^3-\frac{6776}{2025}\zeta_3+\frac{3676019}{607500}L_1$$
$$+\frac{484}{675}L_1L_2^2+\frac{31969}{20250}\left(L_2^2+L_1^2\right)-\frac{484}{675}L_1^2L_2+\frac{2164}{2025}L_2L_1+\frac{1694}{2025}L_2^3$$
$$+\frac{314275147}{54675000}\Bigg\}+O\left(\eta^4L_\eta^3\right) \qquad \text{(F.298)}$$
$$\tilde{a}_{gg,Q}^{(3)}(N=6) = C_AT_F^2\Bigg\{\Bigg(-\frac{843352247}{4219543125}-\frac{4137452}{13395375}L_\eta+\frac{6431}{42525}L_\eta^2\Bigg)\eta^3+\Bigg(-\frac{178615462}{52093125}$$
$$-\frac{1086976}{496125}L_\eta-\frac{424}{1575}L_\eta^2\Bigg)\eta^2+\Bigg(-\frac{435055073}{5315625}-\frac{11624708}{354375}L_\eta-\frac{11717}{3375}L_\eta^2\Bigg)\eta$$
$$+\Bigg(-\frac{5429062}{99225}-\frac{13684}{315}\left(L_2+L_1\right)\Bigg)\zeta_2-\frac{2106313681}{20837250}L_2-\frac{68252}{2835}L_1^3+\frac{54064}{2835}\zeta_3$$
$$-\frac{1459416547}{6945750}L_1-\frac{27368}{945}L_1L_2^2-\frac{54652}{2835}L_2^3-\frac{13768}{945}L_1^2L_2-\frac{1808152}{33075}L_2L_1$$

$$
\begin{aligned}
&-\frac{3604631677201}{13127467500}-\frac{120697}{2205}\left(L_2^2+L_1^2\right)\Bigg\}\\
&+C_FT_F^2\Bigg\{\left(-\frac{91864096}{5907360375}+\frac{979936}{3750705}L_\eta-\frac{130048}{297675}L_\eta^2\right)\eta^3+\Bigg(-\frac{41008}{2701125}\\
&+\frac{10816}{46305}L_\eta-\frac{384}{1225}L_\eta^2\Bigg)\eta^2+\left(\frac{2133088}{826875}+\frac{3616}{3375}L_\eta-\frac{1216}{11025}L_\eta^2\right)\eta\\
&+\left(\frac{197492}{231525}+\frac{968}{2205}\left(L_2+L_1\right)\right)\zeta_2+\frac{413083}{24310125}L_2+\frac{10648}{19845}L_1^3-\frac{3872}{2835}\zeta_3\\
&+\frac{281801489}{72930375}L_1+\frac{1936}{6615}L_1L_2^2+\frac{968}{2835}L_2^3-\frac{1936}{6615}L_1^2L_2+\frac{81176}{99225}L_2L_1\\
&+\frac{14596284331}{5105126250}+\frac{604598}{694575}\left(L_2^2+L_1^2\right)\Bigg\}+O\left(\eta^4L_\eta^3\right) \qquad \text{(F.299)}
\end{aligned}
$$

References

[1] R. Frisch and O. Stern. Über die magnetische Ablenkung von Wasserstoffmolekülen und das magnetische Moment des Protons. *Z. Phys.*, **85** (1933) 4.16.

[2] L. W. Alvarez and F. Bloch. A Quantitative Determination of the Neutron Moment in Absolute Nuclear Magnetons. *Phys. Rev.*, **57** (1940) 111–122.

[3] R. Hofstadter. Electron scattering and nuclear structure. *Rev.Mod.Phys.*, **28** (1956) 214–254.

[4] R.W. Mcallister and R. Hofstadter. Elastic Scattering of 188-MeV Electrons From the Proton and the α Particle. *Phys.Rev.*, **102** (1956) 851–856.

[5] R. Hofstadter and R.W. McAllister. Electron Scattering From the Proton. *Phys.Rev.*, **98** (1955) 217–218.

[6] R. Hofstadter. Nuclear and nucleon scattering of high-energy electrons. *Ann. Rev. Nucl. Part. Sci.*, **7** (1957) 231–316.

[7] Y. Ne'eman. Derivation of strong interactions from a gauge invariance. *Nucl. Phys.*, **26** (1961) 222–229.

[8] M. Gell-Mann. The Eightfold Way: A Theory of strong interaction symmetry. CTSL-20, (1961).

[9] M. Gell-Mann and Y. Neemam. The eightfold way: a review with a collection of reprints. (Benjamin Press, New York, 1964), 317 p.

[10] M. Gell-Mann. A Schematic Model of Baryons and Mesons. *Phys. Lett.*, **8** (1964) 214–215.

[11] G. Zweig. An $SU(3)$ model for the strong interaction symmetry and its breaking. CERN-TH-401, 412 (1964).

[12] W. Pauli. The Connection Between Spin and Statistics. *Phys. Rev.*, **58** (1940) 716–722.

[13] G. Lüders and B. Zumino. Connection between Spin and Statistics. *Phys. Rev.*, **110** (1958) 1450–1453.

[14] I. Duck and G. Sudarshan. Toward an understanding of the spin-statistics theorem. *Am. J. Phys.*, **66** (1998) 284–303.

[15] O. W. Greenberg. Spin and Unitary Spin Independence in a Paraquark Model of Baryons and Mesons. *Phys. Rev. Lett.*, **13** (1964) 598–602.

[16] M. Y. Han and Y. Nambu. Three-triplet model with double SU(3) symmetry. *Phys. Rev.*, **139** (1965) B1006–B1010.

[17] Y. Nambu. A systematics of hadrons in subnuclear physics. (1966) 133–142. in: Preludes in Theoretical Physics, eds. A. De-Shalit, H. Fehsbach and L. van Hove (North-Holland, Amsterdam).

[18] W.A. Bardeen, H. Fritzsch, and M. Gell-Mann. Light cone current algebra, π^0 decay, and e^+e^- annihilation. in: Scale and Conformal Symmetry in Hadron Physics, ed. R. Gatto (John Wiley & Sons, 1973) pp. 139, hep-ph/0211388.

[19] H. Fritzsch and M. Gell-Mann. Current algebra: Quarks and what else? in: Proceedings of 16th International Conference on High-Energy Physics, Batavia, Illinois, 6-13 Sep Vol. **2**, J.D. Jackson, A. Roberts, R. Donaldson, eds. (1972) 135-165.

[20] L. W. Mo and C. Peck. 8 GeV/c Spectrometer. SLAC-TN-65-029, (1965).

[21] R. E. Taylor. Nucleon form–factors above 6-GeV. in: Proc. of the Int. Symp. on Electron and Photon Interactions at High Energies, SLAC, September 5–9, 1967, (SLAC, Stanford CA, 1967), SLAC-PUB-0372, pp. 78.

[22] S. D. Drell and J. D. Walecka. Electrodynamic Processes with Nuclear Targets. *Ann. Phys.*, **28** (1964) 18–33.

[23] E. Derman. Parity violation in polarized electron-deuteron scattering without the parton model. *Phys. Rev.*, **D19** (1979) 133–143.

[24] J. D. Bjorken. Asymptotic Sum Rules at Infinite Momentum. *Phys. Rev.*, **179** (1969) 1547–1553.

[25] D. H. Coward et al. Electron-proton elastic scattering at high momentum transfers. *Phys. Rev. Lett.*, **20** (1968) 292–295.

[26] W. K. H. Panofsky. Low q^2 electrodynamics, elastic and inelastic electron (and muon) scattering. Proc. 14th International Conference on High-Energy Physics, Vienna, 1968, J. Prentki and J. Steinberger, eds., (CERN, Geneva, 1968), p. 23-29.

[27] E. D. Bloom et al. High-Energy Inelastic *ep* Scattering at 6 Degrees and 10 Degrees. *Phys. Rev. Lett.*, **23** (1969) 930–934.

[28] M. Breidenbach et al. Observed Behavior of Highly Inelastic Electron-Proton Scattering. *Phys. Rev. Lett.*, **23** (1969) 935–939.

[29] H. W. Kendall. Deep inelastic scattering: Experiments on the proton and the observation. *Rev. Mod. Phys.*, **63** (1991) 597–614.

[30] R. E. Taylor. Deep inelastic scattering: The Early Years. *Rev. Mod. Phys.*, **63** (1991) 573–595.

[31] J. I. Friedman. Deep inelastic scattering: Comparisons with the quark model. *Rev. Mod. Phys.*, **63** (1991) 615–629.

[32] R. P. Feynman. The behavior of hadron collisions at extreme energies. Proc. of 3rd International Conference on High Energy Collisions, Stony Brook, 1969, C.N. Yang, J.A. Cole, M. Good, R. Hwa, and J. Lee-Franzini, eds., (Gordon and Breach, New York, 1970), pp. 237.

[33] R. P. Feynman. Very high-energy collisions of hadrons. *Phys. Rev. Lett.*, **23** (1969) 1415–1417.

[34] R. P. Feynman. Photon-hadron interactions. (Benjamin Press, Reading, 1972), 282 p.

[35] C. G. Callan and D. J. Gross. High-energy electroproduction and the constitution of the electric current. *Phys. Rev. Lett.*, **22** (1969) 156–159.

[36] J. D. Bjorken and E. A. Paschos. Inelastic Electron Proton and Gamma-Proton Scattering and the Structure of the Nucleon. *Phys. Rev.*, **185** (1969) 1975–1982.

[37] C.-N. Yang and R. L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.*, **96** (1954) 191–195.

[38] G. 't Hooft. Renormalization of Massless Yang-Mills Fields. *Nucl. Phys.*, **B33** (1971) 173–199.

[39] H. Fritzsch, M. Gell-Mann, and H. Leutwyler. Advantages of the Color Octet Gluon Picture. *Phys. Lett.*, **B47** (1973) 365–368.

[40] D. J. Gross and F. Wilczek. Ultraviolet behavior of non-abelian gauge theories. *Phys. Rev. Lett.*, **30** (1973) 1343–1346.

[41] H. D. Politzer. Reliable perturbative results for strong interactions? *Phys. Rev. Lett.*, **30** (1973) 1346–1349.

[42] E598 collaboration, J. J. Aubert et al. Experimental Observation of a Heavy Particle J. *Phys. Rev. Lett.*, **33** (1974) 1404–1406.

[43] SLAC-SP-017 collaboration, J. E. Augustin et al. Discovery of a Narrow Resonance in e^+e^- Annihilation. *Phys. Rev. Lett.*, **33** (1974) 1406–1408.

[44] Z. Maki and M. Nakagawa. An approach to the non-leptonic decays of hyperons. *Prog. Theor. Phys.*, **31** (1964) 115–125.

[45] Y. Hara. Unitary triplets and the eightfold way. *Phys. Rev.*, **134** (1964) B701–B704.

[46] J. D. Bjorken and S. L. Glashow. Elementary Particles and SU(4). *Phys. Lett.*, **11** (1964) 255–257.

[47] S. L. Glashow, J. Iliopoulos, and L. Maiani. Weak Interactions with Lepton-Hadron Symmetry. *Phys. Rev.*, **D2** (1970) 1285–1292.

[48] S. W. Herb et al. Observation of a dimuon resonance at 9.5 GeV in 400 GeV proton-nucleus collisions. *Phys. Rev. Lett.*, **39** (1977) 252–255.

[49] CDF Collaboration, F. Abe et al. Evidence for top quark production in $\bar{p}p$ collisions at $\sqrt{s} = 1.8$ TeV. *Phys.Rev.Lett.*, **73** (1994) 225–231. (arXiv:hep-ex/9405005 [hep-ex]).

[50] CDF Collaboration, F. Abe et al. Evidence for top quark production in $\bar{p}p$ collisions at $\sqrt{s} = 1.8$ TeV. *Phys.Rev.*, **D50** (1994) 2966–3026.

[51] CDF Collaboration, F. Abe et al. Observation of top quark production in $\bar{p}p$ collisions. *Phys.Rev.Lett.*, **74** (1995) 2626–2631. (arXiv:hep-ex/9503002 [hep-ex]).

[52] D0 collaboration, S. Abachi et al. Observation of the top quark. *Phys. Rev. Lett.*, **74** (1995) 2632–2637. (arXiv:hep-ex/9503003).

[53] S. L. Glashow. Partial Symmetries of Weak Interactions. *Nucl. Phys.*, **22** (1961) 579–588.

[54] S. Weinberg. A Model of Leptons. *Phys. Rev. Lett.*, **19** (1967) 1264–1266.

[55] A. Salam and J. C. Ward. Electromagnetic and weak interactions. *Phys. Lett.*, **13** (1964) 168–171.

[56] A. Salam. Weak and Electromagnetic Interactions. Proc. of the 8th Nobel Symposium, Göteborg, Sweden, 19–25 May 1968, ed. N. Svartholm, (Almqvist and Wiskell, Stockholm, 1968), pp. 367.

[57] G. 't Hooft and M. J. G. Veltman. Combinatorics of gauge fields. *Nucl. Phys.*, **B50** (1972) 318–353.

[58] J. C. Taylor. Ward Identities and Charge Renormalization of the Yang-Mills Field. *Nucl. Phys.*, **B33** (1971) 436–444.

[59] A. A. Slavnov. Ward Identities in Gauge Theories. *Theor. Math. Phys.*, **10** (1972) 99–107.

[60] B.W. Lee and J. Zinn-Justin. Spontaneously Broken Gauge Symmetries. 1. Preliminaries. *Phys.Rev.*, **D5** (1972) 3121–3137.

[61] B.W. Lee and J. Zinn-Justin. Spontaneously Broken Gauge Symmetries. 4. General Gauge Formulation. *Phys.Rev.*, **D7** (1973) 1049–1056.

[62] K. G. Wilson. Nonlagrangian models of current algebra. *Phys. Rev.*, **179** (1969) 1499–1512.

[63] W. Zimmermann. Local operator products and renormalization. Lect. on Elementary Particle Physics and Quantum Field Theory, Brandeis Summer Inst., Vol. **1**, (MIT Press, Cambridge, 1970), p. 395-582.

[64] Y. Frishman. Operator products at almost light like distances. *Annals Phys.*, **66** (1971) 373–389.

[65] R. A. Brandt and G. Preparata. Operator product expansions near the light cone. *Nucl. Phys.*, **B27** (1972) 541–567.

[66] D. J. Gross and S. B. Treiman. Light cone structure of current commutators in the gluon quark model. *Phys. Rev.*, **D4** (1971) 1059–1072.

[67] J. C. Collins, D. E. Soper, and G. Sterman. Factorization for Short Distance Hadron-Hadron Scattering. *Nucl. Phys.*, **B261** (1985) 104–168.

[68] J. C. Collins, D. E. Soper, and G. Sterman. Factorization of Hard Processes in QCD. *Adv. Ser. Direct. High Energy Phys.* (Perturbative QCD), ed. A.H. Mueller, **5** (1988) 1–91. (arXiv:hep-ph/0409313).

[69] D. J. Gross and F. Wilczek. Asymptotically Free Gauge Theories. 1; Asymptotically Free Gauge Theories. 2. Phys. Rev., **D8** (1973) 3633-3652; **D9** (1974) 980-993.

[70] H. Georgi and H. D. Politzer. Electroproduction scaling in an asymptotically free theory of strong interactions. *Phys. Rev.*, **D9** (1974) 416–420.

[71] S. Ferrara, R. Gatto, and A. F. Grillo. Conformal algebra in space-time and operator product expansion. *Springer Tracts Mod. Phys.*, **67** (1973) 1–64 and references therein.

[72] C. Chang et al. Observed Deviations from Scale Invariance in High-Energy Muon Scattering. *Phys. Rev. Lett.*, **35** (1975) 901–910.

[73] Y. Watanabe et al. Test of Scale Invariance in Ratios of Muon Scattering Cross-Sections at 150-GeV and 56-GeV. *Phys. Rev. Lett.*, **35** (1975) 898–908.

[74] M. Diemoz, F. Ferroni, and E. Longo. Nucleon structure functions from neutrino scattering. *Phys. Rept.*, **130** (1986) 293–380.

[75] F. Eisele. High-energy neutrino interactions. *Rept. Prog. Phys.*, **49** (1986) 233–339.

[76] T. Sloan, R. Voss, and G. Smadja. The Quark Structure of the Nucleon from the CERN Muon Experiments. *Phys. Rept.*, **162** (1988) 45–167.

[77] S. R. Mishra and F. Sciulli. Deep inelastic lepton-nucleon scattering. *Ann. Rev. Nucl. Part. Sci.*, **39** (1989) 259–310.

[78] K. Winter. Neutrino physics. Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. Vol. **1**, (Cambridge University Press, Cambridge, 1991), 670 p.

[79] N. Schmitz. Neutrinophysik. (Teubner, Stuttgart, 1997), 478 p.

[80] HERA - a proposal for a large electron proton colliding beam facility at DESY. (Hamburg, DESY, 1981), DESY HERA 81-10, 292 p.

[81] H1 collaboration, I. Abt et al. The H1 detector at HERA. DESY-93-103 (1993), 194 p.

[82] ZEUS collaboration, M. Derrick et al. Initial study of deep inelastic scattering with ZEUS at HERA. *Phys. Lett.*, **B303** (1993) 183–197.

[83] HERMES collaboration, K. Ackerstaff et al. HERMES spectrometer. *Nucl. Instrum. Meth.*, **A417** (1998) 230–265. (arXiv:hep-ex/9806008).

[84] H1 and ZEUS Collaboration, F.D. Aaron et al. Combined Measurement and QCD Analysis of the Inclusive e+- p Scattering Cross Sections at HERA. *JHEP*, **1001** (2010) 109–170. (arXiv:arXiv:0911.0884 [hep-ex]).

[85] H1, ZEUS, I. Abt. Proton structure functions at HERA. *Int.J.Mod.Phys.Conf.Ser.*, **31** (2014) 1460280.

[86] H1 Collaboration, V. Andreev et al. Measurement of inclusive ep cross sections at high Q^2 at $\sqrt{s} = 225$ and 252 GeV and of the longitudinal proton structure function F_L at HERA. *Eur.Phys.J.*, **C74** (2014) 2814. (arXiv:1312.4821 [hep-ex]).

[87] A. Zee, F. Wilczek, and S. B. Treiman. Scaling Deviations for Neutrino Reactions in Asymptotically Free Field Theories. *Phys. Rev.*, **D10** (1974) 2881–2891.

[88] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta. Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories. *Phys. Rev.*, **D18** (1978) 3998–4017.

[89] W. Furmanski and R. Petronzio. Lepton - Hadron Processes Beyond Leading Order in Quantum Chromodynamics. *Z. Phys.*, **C11** (1982) 293–352. and references therein.

[90] D. W. Duke, J. D. Kimel, and G. A. Sowell. Fourth order QCD corrections to the longitudinal coefficient function in deep inelastic scattering. *Phys. Rev.*, **D25** (1982) 71–116.

[91] A. Devoto, D. W. Duke, J. D. Kimel, and G. A. Sowell. Analytic calculation of the fourth order quantum chromodynamic contribution to the nonsinglet quark longitudinal structure function. *Phys. Rev.*, **D30** (1984) 541.

[92] D. I. Kazakov and A. V. Kotikov. Total α_s correction to deep inelastic scattering cross-section ratio $R = \sigma_L/\sigma_T$ in QCD. Calculation of the longitudinal structure function. *Nucl. Phys.*, **B307** (1988) 721–762. Erratum ibid. **B345** (1990) 200-300.

[93] D. I. Kazakov, A. V. Kotikov, G. Parente, O. A. Sampayo, and J. Sanchez Guillen. Complete quartic α_s^2 correction to the deep inelastic longitudinal structure function F_L in QCD. *Phys. Rev. Lett.*, **65** (1990) 1535–1538.

[94] J. Sanchez Guillen, J. Miramontes, M. Miramontes, G. Parente, and O. A. Sampayo. Next-to-leading order analysis of the deep inelastic $R = \sigma_L/\sigma_T$. *Nucl. Phys.*, **B353** (1991) 337–345.

[95] W.L. van Neerven and E.B. Zijlstra. $O(\alpha_s^2)$ contributions to the deep inelastic Wilson coefficient. *Phys.Lett.*, **B272** (1991) 127–133.

[96] E.B. Zijlstra and W.L. van Neerven. Contribution of the second order gluonic Wilson coefficient to the deep inelastic structure function. *Phys.Lett.*, **B273** (1991) 476–482.

[97] E.B. Zijlstra and W.L. van Neerven. $O(\alpha_s^2)$ QCD corrections to the deep inelastic proton structure functions F_2 and F_L. *Nucl.Phys.*, **B383** (1992) 525–574.

[98] D. I. Kazakov and A. V. Kotikov. On the value of the $O(\alpha_s)$ correction to the Callan-Gross relation. *Phys. Lett.*, **B291** (1992) 171–176.

[99] S. A. Larin and J. A. M. Vermaseren. Two loop QCD corrections to the coefficient functions of the deep inelastic structure functions F_2 and F_L. *Z. Phys.*, **C57** (1993) 93–98.

[100] S. Moch and J. A. M. Vermaseren. Deep inelastic structure functions at two loops. *Nucl. Phys.*, **B573** (2000) 853–907. (arXiv:hep-ph/9912355).

[101] S. A. Larin, T. van Ritbergen, and J. A. M. Vermaseren. The Next next-to-leading QCD approximation for nonsinglet moments of deep inelastic structure functions. *Nucl. Phys.*, **B427** (1994) 41–52.

[102] S. A. Larin, P. Nogueira, T. van Ritbergen, and J. A. M. Vermaseren. The 3-loop QCD calculation of the moments of deep inelastic structure functions. *Nucl. Phys.*, **B492** (1997) 338–378. (arXiv:hep-ph/9605317).

[103] A. Retey and J. A. M. Vermaseren. Some higher moments of deep inelastic structure functions at next-to-next-to leading order of perturbative QCD. *Nucl. Phys.*, **B604** (2001) 281–311. (arXiv:hep-ph/0007294).

[104] S. Moch, J. A. M. Vermaseren, and A. Vogt. The longitudinal structure function at the third order. *Phys. Lett.*, **B606** (2005) 123–129. (arXiv:hep-ph/0411112).

[105] J. Blümlein and J. A. M. Vermaseren. The 16th moment of the non-singlet structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ to $O(\alpha_s^3)$. *Phys. Lett.*, **B606** (2005) 130–138. (arXiv:hep-ph/0411111).

[106] J. A. M. Vermaseren, A. Vogt, and S. Moch. The third-order QCD corrections to deep-inelastic scattering by photon exchange. *Nucl. Phys.*, **B724** (2005) 3–182. (arXiv:hep-ph/0504242).

[107] P. D. Thompson. Comparison of inclusive charm and beauty cross sections in deep-inelastic scattering at HERA with theoretical predictions. *J. Phys.*, **G34** (2007) N177–N192. (arXiv:hep-ph/0703103).

[108] H1 and ZEUS collaboration, K. Lipka. Heavy flavour production at HERA. *Nucl. Phys. (Proc. Suppl.)*, **152** (2006) 128–135.

[109] ZEUS Collaboration, S. Chekanov et al. Measurement of $D^{\pm}$ and D^0 production in deep inelastic scattering using a lifetime tag at HERA. *Eur.Phys.J.*, **C63** (2009) 171–188. (arXiv:0812.3775 [hep-ex]).

[110] J. Blümlein and S. Riemersma. QCD corrections to $F_L(x, Q^2)$. (1996). (arXiv:hep-ph/9609394 [hep-ph]).

[111] S. Alekhin, J. Blümlein, K. Daum, K. Lipka, and S. Moch. Precise charm-quark mass from deep-inelastic scattering. *Phys.Lett.*, **B720** (2013) 172–176. (arXiv:1212.2355 [hep-ph]).

[112] S. Moch S. Alekhin, J. Blümlein. In preparation. (2015).

[113] S. Alekhin, J. Blümlein, and S. Moch. Parton Distribution Functions and Benchmark Cross Sections at NNLO. *Phys. Rev.*, **D86** (2012) 054009. (arXiv:arXiv:1202.2281 [hep-ph]).

[114] J. Gao, M. Guzzi, J. Huston, H.-L. Lai, Z. Li, et al. CT10 next-to-next-to-leading order global analysis of QCD. *Phys. Rev.*, **D89**(3) (2014) 033009. (arXiv:arXiv:1302.6246 [hep-ph]).

[115] P. Jimenez-Delgado and E. Reya. Dynamical NNLO parton distributions. *Phys.Rev.*, **D79** (2009) 074023. (arXiv:0810.4274 [hep-ph]).

[116] A.D. Martin, W.J. Stirling, R.S. Thorne, and G. Watt. Parton distributions for the LHC. *Eur.Phys.J.*, **C63** (2009) 189–285. (arXiv:0901.0002 [hep-ph]).

[117] R. D. Ball, V. Bertone, F. Cerutti, L. Del Debbio, S. Forte, et al. Impact of Heavy Quark Masses on Parton Distributions and LHC Phenomenology. *Nucl.Phys.*, **B849** (2011) 296–363. (arXiv:arXiv:1101.1300 [hep-ph]).

[118] HERAPDF collab. `https://www.desy.de/h1zeus/combined_results/herapdftable/`.

[119] E. Witten. Heavy Quark Contributions to Deep Inelastic Scattering. *Nucl. Phys.*, **B104** (1976) 445–476.

[120] J. Babcock, D. W. Sivers, and S. Wolfram. QCD Estimates for Heavy Particle Production. *Phys. Rev.*, **D18** (1978) 162–227.

[121] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov. Remarks on Charm Electroproduction in QCD. *Nucl. Phys.*, **B136** (1978) 157–189.

[122] J. P. Leveille and T. J. Weiler. Characteristics of Heavy Quark Leptoproduction in QCD. *Nucl. Phys.*, **B147** (1979) 147–200.

[123] M. Glück, E. Hoffmann, and E. Reya. Scaling Violations and the Gluon Distribution of the Nucleon. *Z. Phys.*, **C13** (1982) 119–162.

[124] E. Laenen, S. Riemersma, J. Smith, and W.L. van Neerven. Complete $O(\alpha_s)$ corrections to heavy flavor structure functions in electroproduction. *Nucl.Phys.*, **B392** (1993) 162–228.

[125] E. Laenen, S. Riemersma, J. Smith, and W.L. van Neerven. $O(\alpha_s)$ corrections to heavy flavor inclusive distributions in electroproduction. *Nucl.Phys.*, **B392** (1993) 229–250.

[126] S. Riemersma, J. Smith, and W. L. van Neerven. Rates for inclusive deep inelastic electroproduction of charm quarks at HERA. *Phys. Lett.*, **B347** (1995) 143–151. (arXiv:hep-ph/9411431).

[127] S. I. Alekhin and J. Blümlein. Mellin representation for the heavy flavor contributions to deep inelastic structure functions. *Phys. Lett.*, **B594** (2004) 299–307. (arXiv:hep-ph/0404034).

[128] M. Buza, Y. Matiounine, J. Smith, R. Migneron, and W. L. van Neerven. Heavy quark coefficient functions at asymptotic values $Q^2 \gg m^2$. *Nucl. Phys.*, **B472** (1996) 611–658. (arXiv:hep-ph/9601302).

[129] S. Alekhin. Parton distributions from deep-inelastic scattering data. *Phys. Rev.*, **D68** (2003) 014002. (arXiv:hep-ph/0211096).

[130] J. Blümlein, H. Böttcher, and A. Guffanti. Non-singlet QCD analysis of deep inelastic world data at $O(\alpha_s^3)$. *Nucl.Phys.*, **B774** (2007) 182–207. (arXiv:hep-ph/0607200 [hep-ph]).

[131] J. Blümlein and H. Böttcher. Higher Twist Contributions to the Structure Functions $F_2^p(x,Q^2)$ and $F_2^d(x,Q^2)$ at Large x and Higher Orders. *Phys.Lett.*, **B662** (2008) 336–340. (arXiv:0802.0408 [hep-ph]).

[132] J. Blümlein and H. Böttcher. Higher Twist contributions to the Structure Functions $F_2(x,Q^2)$ and $g_2(x,Q^2)$. Proceedings of DIS 2012, (2012) 237–241. (arXiv:1207.3170 [hep-ph]).

[133] J. Blümlein, A. De Freitas, W. L. van Neerven, and S. Klein. The longitudinal heavy quark structure function $F_L(Q^2,x)$ in the region $Q^2 \gg m^2$ at $O(\alpha_s^3)$. *Nucl. Phys.*, **B755** (2006) 272–285. (arXiv:hep-ph/0608024).

[134] M. Buza, Y. Matiounine, J. Smith, and W. L. van Neerven. Charm electroproduction viewed in the variable-flavour number scheme versus fixed-order perturbation theory. *Eur. Phys. J.*, **C1** (1998) 301–320. (arXiv:hep-ph/9612398).

[135] I. Bierenbaum, J. Blümlein, and S. Klein. Two-loop massive operator matrix elements and unpolarized heavy flavor production at asymptotic values $Q^2 \gg m^2$. *Nucl. Phys.*, **B780** (2007) 40–75. (arXiv:hep-ph/0703285).

[136] I. Bierenbaum, J. Blümlein, and S. Klein. Calculation of massive 2-loop operator matrix elements with outer gluon lines. *Phys.Lett.*, **B648** (2007) 195–200. (arXiv:hep-ph/0702265 [hep-ph]).

[137] I. Bierenbaum, J. Blümlein, S. Klein, and C. Schneider. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl. Phys.*, **B803** (2008) 1–41. (arXiv:hep-ph/0803.0273 [hep-ph]).

[138] I. Bierenbaum, J. Blümlein, and S. Klein. The Gluonic Operator Matrix Elements at $O(\alpha_s^2)$ for DIS Heavy Flavor Production. *Phys. Lett.*, **B672** (2009) 401–406. (arXiv:hep-ph/0901.0669 [hep-ph]).

[139] I. Bierenbaum, J. Blümlein, and S. Klein. Mellin Moments of the $O(\alpha_s^3)$ Heavy Flavor Contributions to unpolarized Deep-Inelastic Scattering at $Q^2 \gg m^2$ and Anomalous Dimensions. *Nucl.Phys.*, **B820** (2009) 417–482. (arXiv:0904.3563 [hep-ph]).

[140] J. Ablinger, J. Blümlein, S. Klein, C. Schneider, and F. Wißbrock. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(N_F)$ for the Structure Function $F_2(x, Q^2)$ and Transversity. *Nucl.Phys.*, **B844** (2011) 26–54. (arXiv:1008.3347 [hep-ph]).

[141] F. P. Wißbrock. $O(\alpha_s^3 T_F^2 N_F)$ Contributions to the Heavy Flavor Wilson Coefficients of the Structure Function $F_2(x, Q^2)$ at $Q^2 \gg m^2$. (2010). Diploma thesis, (arXiv:arXiv:1010.4251 [hep-ph]).

[142] L.J. Slater. Generalized Hypergeometric Functions. Generalized Hypergeometric Functions, (Cambridge University Press, Cambridge, 1966), 273 p.

[143] W.N. Bailey. Generalized Hypergeometric Series. Generalized Hypergeometric Series, (Cambridge University Press, Cambridge, 1935), 108 p.

[144] P. Appell and J. Kampé de Fériet. Fonctions Hypergéométriques et Hyperspériques, Polynomes D' Hermite. (Gauthier-Villars, Paris, 1926).

[145] P. Appell. Les Fonctions Hypergéométriques de Plusieur Variables. (Gauthier-Villars, Paris, 1925).

[146] J. Kampé de Fériet. La fonction hypergéométrique. (Gauthier-Villars, Paris, 1937).

[147] H. Exton. Multiple Hypergeometric Functions and Applications. (Ellis Horwood, Chichester, 1976).

[148] H. Exton. Handbook of Hypergeometric Integrals. (Ellis Horwood, Chichester, 1978).

[149] H.M. Srivastava and P.W. Karlsson. Multiple Gaussian Hypergeometric Series. (Ellis Horwood, Chicester, 1985).

[150] J. Blümlein, A. Hasselhuhn, S. Klein, and C. Schneider. The $O(\alpha_s^3 N_F T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements. *Nucl.Phys.*, **B866** (2013) 196–211. (arXiv:1205.4184 [hep-ph]).

[151] J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, et al. Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements. *Nucl.Phys.*, **B864** (2012) 52–84. (arXiv:1206.2252 [hep-ph]).

[152] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, et al. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. *Nucl.Phys.*, **B885** (2014) 280–317. (arXiv:arXiv:1405.4259 [hep-ph]).

[153] K.G. Chetyrkin, A.L. Kataev, and F.V. Tkachov. Higher Order Corrections to $\sigma_t(e^+e^- \to Hadrons)$ in Quantum Chromodynamics. *Phys.Lett.*, **B85** (1979) 277–283.

[154] A.V. Kotikov. Differential equations method: New technique for massive Feynman diagrams calculation. *Phys.Lett.*, **B254** (1991) 158–164.

[155] E. Remiddi. Differential equations for Feynman graph amplitudes. *Nuovo Cim.*, **A110** (1997) 1435–1452. (arXiv:hep-th/9711188 [hep-th]).

[156] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, et al. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. *Nucl.Phys.*, **B882** (2014) 263–288. (arXiv:1402.0359 [hep-ph]).

[157] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, et al. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. *Nucl.Phys.*, **B886** (2014) 733–823. (arXiv:1406.4654 [hep-ph]).

[158] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, et al. The 3-loop pure singlet heavy flavor contributions to the structure function $F_2(x, Q^2)$ and the anomalous dimension. *Nucl.Phys.*, **B890** (2015) 48–151. (arXiv:1409.1135 [hep-ph]).

[159] F.C.S. Brown. The massless higher-loop two-point function. *Commun. Math. Phys.*, **287** (2009) 925–958. (arXiv:arxiv: 0804.1660 [math.AG]).

[160] J. Blümlein. Mathematical structure of anomalous dimensions and QCD Wilson coefficients in higher order. *Nucl. Phys. Proc. Suppl.*, **135** (2004) 225–231. (arXiv:hep-ph/0407044).

[161] S. Moch, J. A. M. Vermaseren, and A. Vogt. The three-loop splitting functions in QCD: The non-singlet case. *Nucl. Phys.*, **B688** (2004) 101–134. (arXiv:hep-ph/0403192).

[162] A. Vogt, S. Moch, and J. A. M. Vermaseren. The three-loop splitting functions in QCD: The singlet case. *Nucl. Phys.*, **B691** (2004) 129–181. (arXiv:hep-ph/0404111).

[163] M. Dittmar et al. Parton distributions: Summary report for the HERA - LHC workshop, (2005), hep-ph/0511119.

[164] J. Blümlein and V. Ravindran. Mellin moments of the next-to-next-to leading order coefficient functions for the Drell-Yan process and hadronic Higgs-boson production. *Nucl. Phys.*, **B716** (2005) 128–172. (arXiv:hep-ph/0501178).

[165] J. Blümlein and V. Ravindran. $O(\alpha_s^2)$ timelike Wilson coefficients for parton- fragmentation functions in Mellin space. *Nucl. Phys.*, **B749** (2006) 1–24. (arXiv:hep-ph/0604019).

[166] J. Blümlein and S. Kurth. Harmonic sums and Mellin transforms up to two-loop order. *Phys. Rev.*, **D60** (1999) 014018. (arXiv:hep-ph/9810241).

[167] J. A. M. Vermaseren. Harmonic sums, Mellin transforms and integrals. *Int. J. Mod. Phys.*, **A14** (1999) 2037–2076. (arXiv:hep-ph/9806280).

[168] J. Blümlein. Algebraic relations between harmonic sums and associated quantities. *Comput. Phys. Commun.*, **159** (2004) 19–54. (arXiv:hep-ph/0311046).

[169] J. Blümlein and S. Klein. Structural Relations between Harmonic Sums up to w=6. *PoS*, ACAT 2007 (2007) 084. (arXiv:hep-ph/0706.2426 [hep-ph]).

[170] J. Blümlein. Structural Relations of Harmonic Sums and Mellin Transforms up to Weight $w = 5$. *Comput.Phys.Commun.*, **180** (2009) 2218–2249. (arXiv:0901.3106 [hep-ph]).

[171] J. Blümlein. Structural Relations of Harmonic Sums and Mellin Transforms at Weight $w = 6$, in Proc. of the Workshop "Motives, Quantum Field Theory, and Pseudodifferential Operators, June (2008) Eds. A. Carey D. Ellwood, S. Paycha, S. Rosenberg." (Clay Institute, Boston University, 2009), in Clay Mathematics Proc. **12** (2010) 167-186, math-ph/0901.0837.

[172] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek. Special Values of Multiple Polylogarithms. *Trans. Am. Math. Soc.*, **353** (2001) 907–941. (arXiv:math/9910045).

[173] J. Blümlein, D.J. Broadhurst, and J.A.M. Vermaseren. The Multiple Zeta Value Data Mine. *Comput.Phys.Commun.*, **181** (2010) 582–625. (arXiv:arXiv:0907.2557 [math-ph]).

[174] S. Moch, P. Uwer, and S. Weinzierl. Nested sums, expansion of transcendental functions and multi-scale multi-loop integrals. *J. Math. Phys.*, **43** (2002) 3363–3386. (arXiv:hep-ph/0110083).

[175] J. Ablinger, J. Blümlein, and C. Schneider. Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms. *J.Math.Phys.*, **54** (2013) 082301. (arXiv:1302.0378 [math-ph]).

[176] J. Ablinger, J. Blümlein, and C. Schneider. Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials. *J.Math.Phys.*, **52** (2011) 102301. (arXiv:1105.6063 [math-ph]).

[177] J. Ablinger, J. Blümlein, C.G. Raab, and C. Schneider. Iterated Binomial Sums and their Associated Iterated Integrals. *J.Math.Phys.*, **55** (2014) 112301. (arXiv:1407.1822 [hep-th]).

[178] J. Ablinger. A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics. Diploma Thesis, J. Kepler University, Linz, 2009.

[179] J. Ablinger. Computer Algebra Algorithms for Special Functions in Particle Physics. (2013). PHD Thesis Johannes Kepler Universität Linz, (arXiv:1305.0687 [math-ph]).

[180] E. Remiddi and J. A. M. Vermaseren. Harmonic polylogarithms. *Int. J. Mod. Phys.*, **A15** (2000) 725–754. (arXiv:hep-ph/9905237).

[181] E. G. Floratos, D. A. Ross, and C. T. Sachrajda. Higher Order Effects in Asymptotically Free Gauge Theories. 2. Flavor Singlet Wilson Operators and Coefficient Functions. *Nucl. Phys.*, **B152** (1979) 493.

[182] L. F. Abbott. The Background Field Method Beyond One Loop. *Nucl. Phys.*, **B185** (1981) 189–207.

[183] A. Rebhan. Momentum Subtraction Scheme and the Background Field Method in QCD. *Z. Phys.*, **C30** (1986) 309–335.

[184] F. Jegerlehner and O. V. Tarasov. Exact mass dependent two-loop $\overline{\alpha}_s(Q^2)$ in the background MOM renormalization scheme. *Nucl. Phys.*, **B549** (1999) 481–498. (arXiv:hep-ph/9809485).

[185] R. Harlander, T. Seidensticker, and M. Steinhauser. Complete corrections of $O(\alpha\ \alpha_s)$ to the decay of the Z boson into bottom quarks. *Phys.Lett.*, **B426** (1998) 125–132. (arXiv:9712228 [hep-ph]).

[186] T. Seidensticker. Automatic application of successive asymptotic expansions of Feynman diagrams. (1999). (arXiv:hep-ph/9905298 [hep-ph]).

[187] F.C.S. Brown. On the periods of some Feynman integrals. (2009). (arXiv:0910.0114 [math.AG]).

[188] MapleSoft, Mathematics, Modeling Simulation, http://www.maplesoft.com.

[189] A. Arbuzov, D. Yu. Bardin, J. Blümlein, L. Kalinovskaya, and T. Riemann. HECTOR 1.00 - A program for the calculation of QED, QCD and electroweak corrections to *ep* and $l^{\pm}N$ deep inelastic neutral and charged current scattering. *Comput. Phys. Commun.*, **94** (1996) 128–184. (arXiv:hep-ph/9511434).

[190] J. Blümlein. $O(\alpha^2 L^2)$ radiative corrections to deep inelastic *ep* scattering for different kinematical variables. *Z.Phys.*, **C65** (1995) 293–298. (arXiv:hep-ph/9403342 [hep-ph]).

[191] J. Blümlein, S. Riemersma, and A. Vogt. On the resummation of the $\alpha \ln^2(z)$ terms for QED corrections to deep inelastic *ep* scattering and e^+e^- annihilation. *Eur.Phys.J.*, **C1** (1998) 255–259. (arXiv:hep-ph/9611214 [hep-ph]).

[192] J. Blümlein and H. Kawamura. $O(\alpha^2 L)$ radiative corrections to deep inelastic *ep* scattering. *Phys.Lett.*, **B553** (2003) 242–250. (arXiv:hep-ph/0211191 [hep-ph]).

[193] J. Blümlein and H. Kawamura. Universal higher order QED corrections to polarized lepton scattering. *Nucl.Phys.*, **B708** (2005) 467–510. (arXiv:hep-ph/0409289 [hep-ph]).

[194] J. Blümlein and H. Kawamura. Universal higher order singlet QED corrections to unpolarized lepton scattering. *Eur.Phys.J.*, **C51** (2007) 317–333. (arXiv:hep-ph/0701019 [hep-ph]).

[195] J. D. Bjorken. Inelastic Scattering of Polarized Leptons from Polarized Nucleons. *Phys. Rev.*, **D1** (1970) 1376–1379.

[196] J. Blümlein, M. Klein, T. Naumann, and T. Riemann. Structure functions, quark distributions and $\Lambda_{\rm QCD}$ at HERA. in Proc. of DESY Theory Workshop on Physics at HERA (ed. R.D. Peccei), Hamburg, F.R. Germany, Oct 12-14, 1987, Vol **1**, 67-106.

[197] B. Povh, C. Scholz, K. Rith, and F. Zetsche. Particles and nuclei: An introduction to the physical conceptions. (In German). Particles and nuclei: An introduction to the physical conceptions, (Springer, Berlin(1993), 316 p.

[198] E. Reya. Perturbative Quantum Chromodynamics. *Phys. Rept.*, **69** (1981) 195.

[199] R. Field. Applications of perturbative QCD. Applications of perturbative QCD, (Addison-Wesley, Redwood City, 1989), 366 p.

[200] R. G. Roberts. The Structure of the proton: Deep inelastic scattering. (Cambridge University Press, Cambridge, 1990), 182 p.

[201] T. Muta. Foundations of Quantum Chromodynamics. Second edition. World Sci. Lect. Notes Phys. **57**, (World Scientific, Singapore, 1998), 2nd edition.

[202] J. Blümlein. The Theory of Deeply Inelastic Scattering. *Prog.Part.Nucl.Phys.*, **69** (2013) 28–84. (arXiv:1208.6087 [hep-ph]).

[203] J. Blümlein, B. Geyer, and D. Robaschik. The virtual Compton amplitude in the generalized Bjorken region: Twist-2 contributions. *Nucl. Phys.*, **B560** (1999) 283–344. (arXiv:hep-ph/9903520).

[204] F. J. Yndurain. The theory of quark and gluon interactions. The theory of quark and gluon interactions, (Springer, Berlin, 2006), 474 p, 4th edition.

[205] J. Blümlein and A. Tkabladze. Target mass corrections for polarized structure functions and new sum rules. *Nucl. Phys.*, **B553** (1999) 427–464. (arXiv:hep-ph/9812478).

[206] R.D. Tangerman and P.J. Mulders. Intrinsic transverse momentum and the polarized Drell-Yan process. *Phys.Rev.*, **D51** (1995) 3357–3372. (arXiv:hep-ph/9403227 [hep-ph]).

[207] J. Blümlein and N. Kochelev. On the twist 2 and twist 3 contributions to the spin-dependent electroweak structure functions. *Nucl. Phys.*, **B498** (1997) 285–309. (arXiv:hep-ph/9612318).

[208] C. Itzykson and J. Zuber. Quantum Field Theory. (McGraw-Hill, New York, 1980), 705 p.

[209] A. J. Buras. Asymptotic Freedom in Deep Inelastic Processes in the Leading Order and Beyond. *Rev. Mod. Phys.*, **52** (1980) 199–276.

[210] E. C. G. Stückelberg and A. Petermann. The normalization group in quantum theory. *Helv. Phys. Acta*, **24** (1951) 317–319.

[211] M. Gell-Mann and F. E. Low. Quantum electrodynamics at small distances. *Phys. Rev.*, **95** (1954) 1300–1312.

[212] N. N. Bogolyubov and D. V. Shirkov. Introduction to the theory of quantized fields. (New York, Interscience, 1959), 720 p.

[213] C. G. Callan. Broken scale invariance in scalar field theory. *Phys. Rev.*, **D2** (1970) 1541–1547.

[214] K. Symanzik. Small distance behavior in field theory and power counting. *Commun. Math. Phys.*, **18** (1970) 227–246.

[215] M. Gluck, S. Kretzer, and E. Reya. The Strange sea density and charm production in deep inelastic charged current processes. *Phys.Lett.*, **B380** (1996) 171–176. (arXiv:hep-ph/9603304 [hep-ph]).

[216] J. Blümlein, A. Hasselhuhn, P. Kovacikova, and S. Moch. $O(\alpha_s)$ Heavy Flavor Corrections to Charged Current Deep-Inelastic Scattering in Mellin Space. *Phys.Lett.*, **B700** (2011) 294–304. (arXiv:1104.3449 [hep-ph]).

[217] M. Buza and W. L. van Neerven. $O(\alpha_s^2)$ contributions to charm production in charged-current deep-inelastic lepton hadron scattering. *Nucl. Phys.*, **B500** (1997) 301–324. (arXiv:hep-ph/9702242).

[218] J. Blümlein, A. Hasselhuhn, and T. Pfoh. The $O(\alpha_s^2)$ heavy quark corrections to charged current deep-inelastic scattering at large virtualities. *Nucl.Phys.*, **B881** (2014) 1–41. (arXiv:1401.4352 [hep-ph]).

[219] M. Buza, Y. Matiounine, J. Smith, and W. L. van Neerven. $O(\alpha_s^2)$ corrections to polarized heavy flavour production at $Q^2 \gg m^2$. *Nucl. Phys.*, **B485** (1997) 420–456. (arXiv:hep-ph/9608342).

[220] A. Behring, I. Bierenbaum, J. Blümlein, A. De Freitas, S. Klein, et al. The logarithmic contributions to the $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering. *Eur.Phys.J.*, **C74**(9) (2014) 3033. (arXiv:1403.6356 [hep-ph]).

[221] S. D. Drell and T.-M. Yan. Partons and their applications at high energies. *Ann. Phys.*, **66** (1971) 578–623.

[222] J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, et al. Heavy Flavor DIS Wilson coefficients in the asymptotic regime. *Nucl.Phys.Proc.Suppl.*, **205-206** (2010) 242–249. (arXiv:1007.0375 [hep-ph]).

[223] A. Behring, J. Blümlein, A. De Freitas, T. Pfoh, C. Raab, et al. New Results on the 3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering. *PoS*, **RADCOR2013** (2013) 058. (arXiv:1312.0124 [hep-ph]).

[224] J. A. M. Vermaseren. New features of `FORM`. (2000), math-ph/0010025.

[225] G.E. Andrews, R. Askey, and R. Roy. Special Functions. Special Functions, Encyclopedia of Mathematics and its Applications **71**, (Cambridge University Press, Cambridge, 2001), 663 p.

[226] P. Paule. Contiguous relations and creative telescoping, Technical Report, RISC, Austria, 2001.

[227] R. Vidūnas. Contiguous Relations of Hypergeometric Series. *Journal of Computational and Applied Mathematics*, **153** (2003) 507–519.

[228] M. Yu. Kalmykov, V. Bytev, B. A. Kniehl, B.F.L. Ward, and S. A. Yost. Feynman Diagrams, Differential Reduction, and Hypergeometric Functions. *PoS*, ACAT08 (2009) 125. (arXiv:arXiv:0901.4716 [hep-th]).

[229] M. Karr. Summation in finite terms. *J. ACM,* **28**, (1981) 305–350.

[230] C. Schneider. Symbolic Summation in Difference Fields. PhD Thesis, RISC–Linz, J. Kepler University, Linz, 2001.

[231] C.Schneider. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, **43(9)** (208) 611–644. (arXiv:0808.2543 [cs.SC]).

[232] C. Schneider. A difference ring theory for symbolic summation. *J. Symbol. Comput.* in press, (arXiv:1408.2776 [cs.SC]).

[233] C. Schneider. Parameterized Telescoping Proves Algebraic Independence of Sums. (2008). (arXiv:0808.2596 [cs.SC]).

[234] C. Schneider. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Difference Equ. Appl.*, **11 (9)** (2005) 799–821.

[235] C. Schneider. Simplifying sums in $\Pi\Sigma$-extensions. *J. Algebra Appl.*, **6(3)** (2007) 415–441.

[236] C. Schneider. Symbolic Summation Assists Combinatorics. *Sém. Lothar. Combin.*, **56** (2006) Article B56b.

[237] C. Schneider. in: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions Texts and Monographs in Symbolic Computation eds. C. Schneider and J. Blümlein (Springer, Wien, 2013) 325, (arXiv:1304.4134 [cs.SC]).

[238] J. Ablinger, J. Blümlein, S. Klein, and C. Schneider. Modern Summation Methods and the Computation of 2- and 3-loop Feynman Diagrams. *Nucl. Phys. (Proc. Suppl.)*, **205–206** (2010) 110–115. (arXiv:arXiv:1006.4797 [math-ph]).

[239] J. Blümlein, A. Hasselhuhn, and C. Schneider. Evaluation of Multi-Sums for Large Scale Problems. *PoS*, RADCOR2011 (2011) 032. (arXiv:1202.4303 [math-ph]).

[240] J. Ablinger, J. Blümlein, M. Round, and C. Schneider. Advanced Computer Algebra Algorithms for the Expansion of Feynman Integrals. *PoS*, LL2012 (2012) 050. (arXiv:1210.1685 [cs.SC]).

[241] C. Schneider. A new sigma approach to multi-summation. *Advances in Applied Mathematics*, **34(4)** (2005) 740–767.

[242] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, et al. 3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering with Two Heavy Quark Lines. (2014). (arXiv:1407.2821 [hep-ph]).

[243] J. Ablinger, J. Blümlein, S. Klein, C. Schneider, and F. Wißbrock. 3-Loop Heavy Flavor Corrections to DIS with two Massive Fermion Lines. (2011). (arXiv:1106.5937 [hep-ph]).

[244] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, et al. 3-loop heavy flavor Wilson coefficients in deep-inelastic scattering. (2014). (arXiv:1409.1804 [hep-ph]).

[245] A. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, et al. Recent progress on the calculation of three-loop heavy flavor Wilson coefficients in deep-inelastic scattering. *PoS*, **LL2014** (2014) 041. (arXiv:1407.3638 [hep-ph]).

[246] S. Klein. Mellin Moments of Heavy Flavor Contributions to $F_2(x,Q^2)$ at NNLO. PhD thesis, TU Dortmund University, 2009. (arXiv:0910.3101 [hep-ph]).

[247] W.L. van Neerven and A. Vogt. NNLO evolution of deep inelastic structure functions: The Singlet case. *Nucl.Phys.*, **B588** (2000) 345–373. (arXiv:hep-ph/0006154 [hep-ph]).

[248] J. Blumlein, M. Kauers, S. Klein, and C. Schneider. Determining the closed forms of the $O(\alpha_s^3)$ anomalous dimensions and Wilson coefficients from Mellin moments by means of computer algebra. *Comput.Phys.Commun.*, **180** (2009) 2143–2165. (arXiv:0902.4091 [hep-ph]).

[249] S. Alekhin, J. Blümlein, and S. Moch. The ABM parton distributions tuned to LHC data. *Phys.Rev.*, **D89**(5) (2014) 054028. (arXiv:1310.3059 [hep-ph]).

[250] G. 't Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. *Nucl. Phys.*, **B44** (1972) 189–213.

[251] R. Tarrach. The Pole Mass in Perturbative QCD. *Nucl. Phys.*, **B183** (1981) 384.

[252] O. Nachtmann and W. Wetzel. The Beta Function for Effective Quark Masses to Two Loops in QCD. *Nucl. Phys.*, **B187** (1981) 333.

[253] N. Gray, D. J. Broadhurst, W. Grafe, and K. Schilcher. Three loop relation of quark (modified) MS and pole masses. *Z. Phys.*, **C48** (1990) 673–680.

[254] J. Fleischer, F. Jegerlehner, O. V. Tarasov, and O. L. Veretin. Two-loop QCD corrections of the massive fermion propagator. *Nucl. Phys.*, **B539** (1999) 671–690. (arXiv:hep-ph/9803493).

[255] G. t'Hooft. (1972), unpublished.

[256] I. B. Khriplovich. Green's functions in theories with non-abelian gauge group. *Yad. Fiz.*, **10** (1969) 409–424.

[257] W. E. Caswell. Asymptotic Behavior of Nonabelian Gauge Theories to Two Loop Order. *Phys. Rev. Lett.*, **33** (1974) 244–254.

[258] D. R. T. Jones. Two Loop Diagrams in Yang-Mills Theory. *Nucl. Phys.*, **B75** (1974) 531–543.

[259] B. S. DeWitt. Quantum Theory of Gravity. 2. The Manifestly Covariant Theory. *Phys.Rev.*, **162** (1967) 1195–1239.

[260] Y. Matiounine, J. Smith, and W. L. van Neerven. Two-loop operator matrix elements calculated up to finite terms. *Phys. Rev.*, **D57** (1998) 6701–6722. (arXiv:hep-ph/9801224).

[261] K.G. Chetyrkin and M. Steinhauser. Short distance mass of a heavy quark at $O(\alpha_s^3)$. *Phys.Rev.Lett.*, **83** (1999) 4001–4004. (arXiv:9907509 [hep-ph]).

[262] K.G. Chetyrkin and M. Steinhauser. The Relation between the MS-bar and the on-shell quark mass at $O(\alpha_s^3)$. *Nucl.Phys.*, **B573** (2000) 617–651. (arXiv:hep-ph/9911434 [hep-ph]).

[263] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser. Strong-Coupling Constant at Three Loops in Momentum Subtraction Scheme. *Nucl. Phys.*, **B814** (2009) 231–245. (arXiv:hep-ph/0812.1337 [hep-ph]).

[264] D. J. Broadhurst. Three loop on-shell charge renormalization without integration: $\Lambda^{QED}_{\overline{MS}}$ to four loops. *Z. Phys.*, **C54** (1992) 599–606.

[265] L. Avdeev, J. Fleischer, S. Mikhailov, and O. Tarasov. $O(\alpha\alpha_s^2)$ correction to the electroweak rho parameter. *Phys. Lett.*, **B336** (1994) 560–566. (arXiv:hep-ph/9406363).

[266] S. Laporta and E. Remiddi. The analytical value of the electron $(g-2)$ at $O(\alpha^3)$ in QED. *Phys. Lett.*, **B379** (1996) 283–291. (arXiv:hep-ph/9602417).

[267] D. J. Broadhurst. Massive 3-loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity. *Eur. Phys. J.*, **C8** (1999) 311–333. (arXiv:hep-th/9803091).

[268] R. Boughezal, J. B. Tausk, and J. J. van der Bij. Three-loop electroweak correction to the rho parameter in the large Higgs mass limit. *Nucl. Phys.*, **B713** (2005) 278–290. (arXiv:hep-ph/0410216).

[269] J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, et al. New Heavy Flavor Contributions to the DIS Structure Function $F_2(x, Q^2)$ at $O(\alpha_s^3)$. *PoS*, **RADCOR2011** (2011) 031. (arXiv:1202.2700 [hep-ph]).

[270] J. Blümlein, A. De Freitas, C. Raab, F. Wißbrock, J. Ablinger, et al. Recent Results on the 3-Loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering. *PoS*, **DIS2013** (2013) 301. (arXiv:1307.7548 [hep-ph]).

[271] M. Kauers. Fast solvers for dense linear systems. *Nucl. Phys. Proc. Suppl.*, **183** (2008) 245–250.

[272] M. Steinhauser. `MATAD`: A program package for the computation of massive tadpoles. *Comput. Phys. Commun.*, **134** (2001) 335–364. (arXiv:hep-ph/0009029).

[273] K.G. Chetyrkin. Operator Expansions in the Minimal Subtraction Scheme. 1: The Gluing Method. *Theor.Math.Phys.*, **75** (1988) 346–356.

[274] K.G. Chetyrkin. Operator Expansions in the Minimal Subtraction Scheme. 2: Explicit Formulas for Coefficient Functions. *Theor.Math.Phys.*, **76** (1988) 809–817.

[275] F. V. Tkachov. Theory of asymptotic operation. a summary of basic principles. *Sov.J.Part.Nucl.*, **25** (1994) 649–681. (arXiv:hep-ph/9701272 [hep-ph]).

[276] S.G. Gorishnii and S.A. Larin. Coefficient Functions of Asymptotic Operator Expansions in Minimal Subtraction Scheme. *Nucl.Phys.*, **B283** (1987) 452–470.

[277] P. Nogueira. Automatic Feynman graph generation. *J. Comput. Phys.*, **105** (1993) 279–289.

[278] E.W. Barnes. A new development of the theory of the hypergeometric functions. *Proc. Lond. Math. Soc. (2)* **6**, (1908) 141–177.

[279] E.W. Barnes. A transformation of generalised hypergeometric series. *Quarterly Journal of Mathematics* **41** *(1910) 136*, (1910) 136–140.

[280] H. Mellin. Abriß einer einheitlichen theorie der gamma- und der hypergeometrischen funktionen. *Math. Ann.* **68***, no. 3*, (1910) 305337.

[281] E.T. Whittaker and G.N. Watson. A Course of Modern Analysis. (Cambridge University Press, Cambridge, 1927; reprinted 1996) 616 p.

[282] T. Huber and D. Maitre. `HypExp 2`, Expanding Hypergeometric Functions about Half-Integer Parameters. *Comput.Phys.Commun.*, **178** (2008) 755–776. (arXiv:0708.2443 [hep-ph]).

[283] L. Lewin. Dilogarithms and associated functions. (Macdonald, London, 1958).

[284] L. Lewin. Polylogarithms and associated functions. (North Holland, New York, 1981).

[285] A. Devoto and D.W. Duke. Table of Integrals and Formulae for Feynman Diagram Calculations. *Riv.Nuovo Cim.*, **7N6** (1984) 1–39.

[286] J. Ablinger and J. Blümlein. Harmonic Sums, Polylogarithms, Special Numbers, and their Generalizations. (2013). (arXiv:1304.7071 [math-ph]).

[287] J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, et al. 3-loop Massive $O(T_F^2)$ Contributions to the DIS Operator Matrix Element A_{gg}. (2014). (arXiv:1409.1435 [hep-ph]).

[288] E.C. Titchmarsh. Introduction to the Theory of Fourier Integrals. Introduction to the Theory of Fourier Integrals*, (Calendron Press, Oxford, 1937; 2nd Edition 1948).*

[289] Alexander Hasselhuhn. *3-Loop Contributions to Heavy Flavor Wilson Coefficients of Neutral and Charged Current DIS.* PhD thesis, TU Dortmund University, 2009.

[290] K. Hepp. Proof of the Bogolyubov-Parasiuk theorem on renormalization. *Commun.Math.Phys.*, **2** (1966) 301–326.

[291] M. Roth and A. Denner. High-energy approximation of one loop Feynman integrals. *Nucl.Phys.*, **B479** (1996) 495–514. (arXiv:hep-ph/9605420 [hep-ph]).

[292] T. Binoth and G. Heinrich. An automatized algorithm to compute infrared divergent multiloop integrals. *Nucl.Phys.*, **B585** (2000) 741–759. (arXiv:hep-ph/0004013 [hep-ph]).

[293] E. Panzer. On the analytic computation of massless propagators in dimensional regularization. *Nucl.Phys.*, **B874** (2013) 567–593. (arXiv:1305.2161 [hep-th]).

[294] E. Panzer. On hyperlogarithms and Feynman integrals with divergences and many scales. *JHEP*, **1403** (2014) 071. (arXiv:1401.4361 [hep-th]).

[295] S. Weinzierl. Tutorial on loop integrals which need regularisation but yield finite results. (2014). (arXiv:1402.4407 [hep-ph]).

[296] J. Ablinger, J. Blümlein, C. Raab, C. Schneider, and F. Wißbrock. Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms. *Nucl.Phys.*, **B885** (2014) 409–447. (arXiv:1403.1137 [hep-ph]).

[297] J. Ablinger, J. Blümlein, A. De Freitas, Al. Hasselhuhn, S. Klein, et al. New Results on the 3-Loop Heavy Flavor Wilson Coefcients in Deep-Inelastic Scattering. *PoS*, **ICHEP2012** (2013) 270. (arXiv:1212.5950 [hep-ph]).

[298] C. Bogner and S. Weinzierl. Feynman graph polynomials. *Int.J.Mod.Phys.*, **A25** (2010) 2585–2618. (arXiv:1002.3458 [hep-ph]).

[299] N. Nakanishi. Graph Theory and Feynman Integrals. (Gordon and Breach, New York, 1970).

[300] H. Cheng and T.T. Wu. Expanding Protons: Scattering at High Energies. (MIT Press, Cambridge, MA, 1987).

[301] J. A. Lappo-Danilevsky. Mémoires sur la théorie des systèmes des équations différentielles linéaires. (Chelsea Pub. Co., New York, 1953).

[302] J. Blümlein and A. Vogt. The Evolution of unpolarized singlet structure functions at small x. *Phys.Rev.*, **D58** (1998) 014020. (arXiv:hep-ph/9712546 [hep-ph]).

[303] J. Blümlein. Analytic continuation of Mellin transforms up to two-loop order. *Comput. Phys. Commun.*, **133** (2000) 76–104. (arXiv:hep-ph/0003100).

[304] J. Blümlein and S.-O. Moch. Analytic continuation of the harmonic sums for the 3-loop anomalous dimensions. *Phys. Lett.*, **B614** (2005) 53–61. (arXiv:hep-ph/0503188).

[305] A.V. Kotikov and V.N. Velizhanin. Analytic continuation of the Mellin moments of deep inelastic structure functions. (2005). (arXiv:hep-ph/0501274 [hep-ph]).

[306] F.V. Tkachov. A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions. *Phys.Lett.*, **B100** (1981) 65–68.

[307] K. G. Chetyrkin and F. V. Tkachov. Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops. *Nucl. Phys.*, **B192** (1981) 159–204.

[308] S. G. Gorishnii, S. A. Larin, L. R. Surguladze, and F. V. Tkachov. `MINCER`: Program for multiloop calculations in quantum field theory for the schoonschip system. *Comput. Phys. Commun.*, **55** (1989) 381–408.

[309] S. Laporta. High precision calculation of multiloop Feynman integrals by difference equations. *Int.J.Mod.Phys.*, **A15** (2000) 5087–5159. (arXiv:hep-ph/0102033 [hep-ph]).

[310] C. Schneider, A. De Freitas, and J. Blümlein. Recent Symbolic Summation Methods to Solve Coupled Systems of Differential and Difference Equations. *PoS*, LL2014 (2014) 017. (arXiv:1407.2537 [cs.SC]).

[311] J. Ablinger, J. Blümlein, and C. Schneider. Generalized Harmonic, Cyclotomic, and Binomial Sums, their Polylogarithms and Special Numbers. *J.Phys.Conf.Ser.*, **523** (2014) 012060. (arXiv:1310.5645 [math-ph]).

[312] Particle Data Group, J. Beringer et al. Review of Particle Physics (RPP). *Phys. Rev. D*, **86** (2012) 010001.

[313] T. van Ritbergen, A. N. Schellekens, and J. A. M. Vermaseren. Group theory factors for Feynman diagrams. *Int. J. Mod. Phys.*, **A14** (1999) 41–96. (arXiv:hep-ph/9802376).

[314] M. J. G. Veltman. Diagrammatica: The Path to Feynman rules. (Cambridge University Press, Cambridge, 1994).

[315] G. 't Hooft and M. J. G. Veltman. Diagrammar. CERN Yellow Report 73–9 (1973).

[316] E. G. Floratos, D. A. Ross, and C.T. Sachrajda. Higher Order Effects in Asymptotically Free Gauge Theories: The Anomalous Dimensions of Wilson Operators. Nucl. Phys. **B129**, (1977) 66-98; [Erratum-ibid.] **B139**, (1978) 545-546.

[317] R. Mertig and W. L. van Neerven. The Calculation of the two loop spin splitting functions $P_{ij}^{(1)}(x)$. *Z. Phys.*, **C70** (1996) 637–654. (arXiv:hep-ph/9506451).

[318] C. G. Bollini and J. J. Giambiagi. Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter. *Nuovo Cim.*, **B12** (1972) 20–25.

[319] E. R. Speer. Renormalization and Ward identities using complex space- time dimension. *J. Math. Phys.*, **15** (1974) 1–6.

[320] M. Abramowitz and I. A. Stegun. Handbook of Mathematical Functions. Handbook of Mathematical Functions, (Dover Publications Inc., New York, 1972), 1046 p.

[321] N. Nielsen. Handbuch der Theorie der Gammafunktion. (Chelsea Publishing Company, New York, 1965), 328 p; first published: (Teubner, Leipzig, 1906), 326 p.