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Graphs with given diameter and a coloring problem

1. Introduction. Within the vast literature on colorability there are only a few papers concerning the following coloring problem. By an r-coloring relative to distance k of a graph G — throughout graphs are finite, undirected, without loops and multiple edges — we mean an assignment of at most r colors to the vertices of G so that the distance between any two vertices having the same color is greater than k. Let $\chi_k(G)$ denote the smallest number r such that G has an r-coloring relative to distance k, which we abbreviate r-coloring(k). Of course $\chi_1(G)$ is the usual chromatic number of G. r-colorings(k) have been considered by F.Kramer and H.Kramer [16], [17], [18]; especially they calculate the numbers $\chi_k(C_n)$ of circuit graphs C_n and characterize those graphs G which have $\chi_k(G) = k+1$ or $\chi_k(G) = \chi_{k+1}(G) = k+2$. In his forthcoming thesis C.Ivan [13] considers r-colorings(k) for cacti.

Now let $\mathcal{G}$ be a family of graphs; then we define

$$\chi_k(\mathcal{G}) := \sup\{\chi_k(G) \mid G \in \mathcal{G}\}.$$

If $\mathcal{G}$ contains graphs with arbitrarily high maximum degree, then $\chi_k(\mathcal{G}) = \infty$ for $k \geq 2$. In order to obtain nontrivial results we consider the families $\mathcal{G}_d$ [resp. $\overline{\mathcal{G}}_d$] of all graphs [resp. all planar graphs] with maximum degree not exceeding d. Obviously $\chi_k(G) = n$, if the graph G has n vertices and diameter $d(G) \leq k$. Therefore we have

(1) $\chi_k(\mathcal{G}) \geq n_k(\mathcal{G})$,

where $n_k(\mathcal{G})$ denotes the maximum number of vertices of those graphs in $\mathcal{G}$ whose diameter is not greater than k. Because of (1) it seems to be suitable to collect the results (some known and some new) on the numbers $n_k(\mathcal{G}_d)$ and $n_k(\overline{\mathcal{G}}_d)$. This is the aim of the next two sections; in section 4 we return to χ_k.

2. The numbers $n_k(\mathcal{G}_d)$. Trivially $n_1(\mathcal{G}_d) = d+1$ because of $K_{d+1} \in \mathcal{G}_d$ and and $n_k(\mathcal{G}_2) = 2k+1$ because of $C_{2k+1} \in \mathcal{G}_2$. Now suppose $k > 1$ and $d > 2$. We have

(2) $n_k(\mathcal{G}_d) \leq N(d,k) := 1 + d\,\dfrac{(d-1)^k - 1}{d-2}$

with equality iff a (d,k)-Moore graph exists, that is only if $k = 2$ (see H.D.Friedman [11], R.M.Damerell [8], E.Bannai - T.Ito [4]) and even then only for $d = 2, 3, 7$ and possibly $d = 57$ (A.J.Hoffman - R.R.Singleton [12]). In any other case we have $n_k(\mathcal{G}_d) < N(d,k)$ resp. even $n_k(\mathcal{G}_d) < N(d,k) - 1$, if d and $N(d,k)-1$ both are odd numbers (since a graph in $\mathcal{G}_d$ with diameter $\leq k$ and more than $\sum_{i=0}^{k}(d-1)^i = N(d,k) - \frac{(d-1)^k - 1}{d-2}$ vertices is necessarily regular of degree d, if such a graph exists at all; compare also B.Elspas [9]). General lower bounds have been given by H.D.Friedman [10] and I.Korn [15], overhauling the general bounds given by B.Elspas [9]:

(3) $n_{2h}(\mathcal{G}_d) \geq 2d\,\dfrac{(d-1)^h - 1}{d-2}$ [10]

(4) $n_{2h+1}(\mathcal{G}_d) \geq 2\,\dfrac{2(d-1)^{h+1} - d}{d-2}$ [15]

But these formulas don't yield useful estimates for small values of d and k. Especially for $k = d-1$ S.B.Akers [2] proved

(5) $$n_{d-1}(\mathcal{G}_d) \geq \binom{2d-1}{d}$$

which is in the cases $d \leq 12$ better than (3) and (4), but apart from $d = 2, 3$ and possibly $d = 4$ by no means best possible. First we give now an improvement of both (3) and (4).

Theorem 1. Let $h \geq 1$, $d > d_1 \geq 0$ and $k = k_1 + 2h$. Then

(6) $$n_k(\mathcal{G}_d) \geq n_{k_1}(\mathcal{G}_{d_1}) \frac{2(d-d_1)(d-1)^h + d(d_1-2)}{d-2} .$$

Remark: With $d_1 = k_1 = 0$ and $n_0(\mathcal{G}_0) = 1$ one gets Friedman's formula (3) and with $d_1 = k_1 = 1$ and $n_1(\mathcal{G}_1) = 2$ Korn's formula (4). But for $d > 4$ and suitable choice of k_1 and d_1 one gets with (6) better results than with (3) and (4).

Proof of (6): The construction is similar to that of Friedman and Korn. We start our construction with a graph G_1 of diameter k_1 and maximum degree d_1 having N vertices. Now we take a rooted tree with radius h, whose root has valency $d-d_1$ and whose further vertices other than endvertices have valency d. We identify each vertex of G_1 with the root of a copy

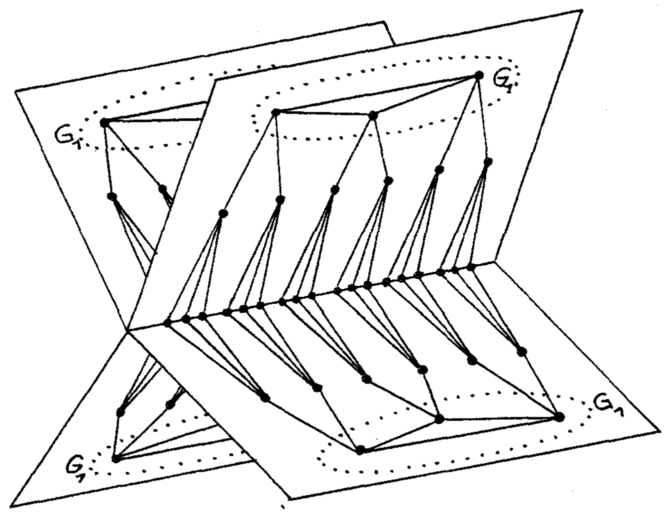

Fig. 1

of such a tree thus obtaining a graph with diameter $k_1 + 2h$ and

$$N + N(d-d_1) + N(d-d_1)(d-1) + \dots + N(d-d_1)(d-1)^{h-1}$$

vertices. It is easy to see that the diameter remains unchanged if we take d copies of this graph and identify their endvertices (see. figure 1). The resulting graph has maximum degree d and

$$N(d-d_1)(d-1)^{h-1} + d[N + N(d-d_1) + N(d-d_1)(d-1) + \dots + N(d-d_1)(d-1)^{h-2}] =$$
$$= \frac{N}{d-2}[2(d-d_1)(d-1)^h + d(d_1-2)]$$

vertices. //

For odd diameter we get bounds sometimes better than those arising from (6) by the following formula:

(7) $$n_{2k+1}(\mathcal{G}_{d+1}) \geq n_k(\mathcal{G}_d)\,[n_k(\mathcal{G}_d) + 1]$$

Proof. Let G be a graph with diameter k, maximum degree d and N vertices. Take $N+1$ copies of G and label them $0, 1, \dots, N$. Label the vertices of G_i with the same numbers omitting the number i , for each i . Then join the vertex of G_{i_1} labelled i_2 with the vertex of G_{i_2} labelled i_1 for every pair of numbers $i_1 \neq i_2$. Thus any two of the copies of G are joined by just one edge and it's clear that the resulting graph has diameter $2k+1$, maximum degree $d+1$ and $N(N+1)$ vertices. //

The application of both (6) and (7) needs good estimations for $n_k(\mathcal{G}_d)$ for small values of k and d . Now we shall collect the results for these values.

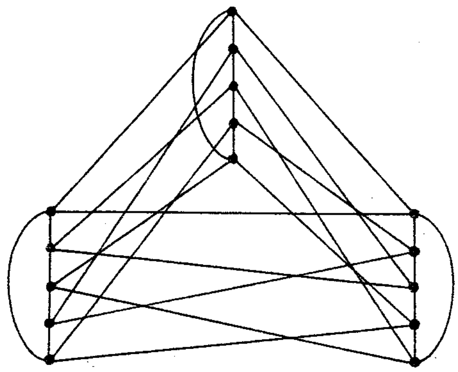

Fig. 2

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Fig. 3

$d-1$ copies of C_5 , pairwise connected by additional edges such that each pair yields a Peterson graph (see fig. 2 for $d = 4$), show

(8) $\quad n_2(\mathcal{G}_d) \geq 5d-5$.

In (8) we have equality not only for $d = 2, 3$, as is well known, but also for $d = 4$; this value has been given by B.Elspas [9] together with $n_2(\mathcal{G}_5) = 24$, both without proof. While it is not hard to prove $n_2(\mathcal{G}_4) = 15$, the inequality $n_2(\mathcal{G}_5) \geq 24$ in [9] is erroneously based upon a graph by M.W.Green, which does not have diameter 2 . Nevertheless the inequality is correct and figure 3 displays the adjacency matrix of a graph with 24 vertices, diameter 2 and degree 5 . — For $d = 6$ we have

(9) $\quad n_2(\mathcal{G}_6) \geq 32$.

The graph which proves this inequality is built up by the two subgraphs shown in figure 4 . Each vertex of the graph on the left hand — the graph of the dodecahedron together with its ten diagonals — has to be joined with two vertices of the graph on the right hand as indicated

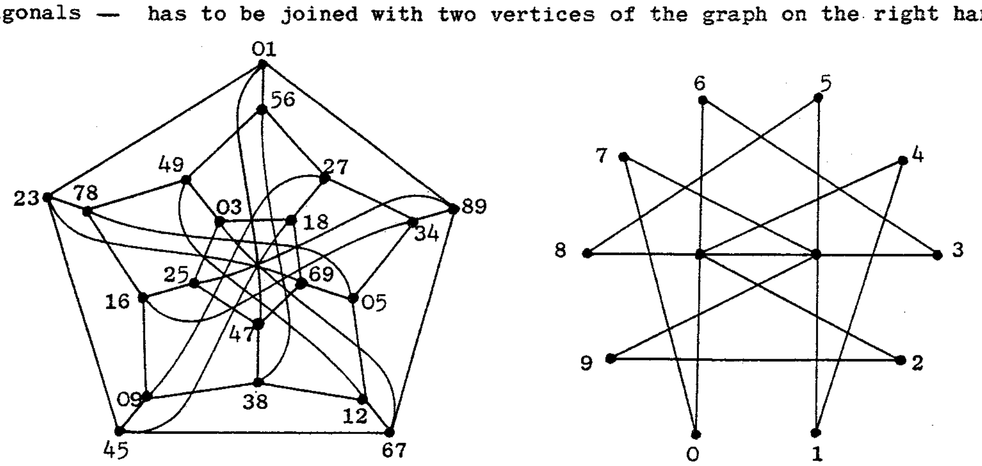

Fig. 4

by numbers. (The construction of this graph has been inspired by a 5-valent graph with girth 5 and 30 vertices given by N.Robertson [19]).

Next we consider $k = 3$. $n_3(\mathcal{G}_3) = 20$ has been proved by B.Elspas [9]. We draw this graph (which possibly is unique) in a somewhat different manner (figure 5) to exhibit its relationship to the Peterson graph.
From (5) we have $n_3(\mathcal{G}_4) \geq 35$, which is best possible up to now.
Further we have

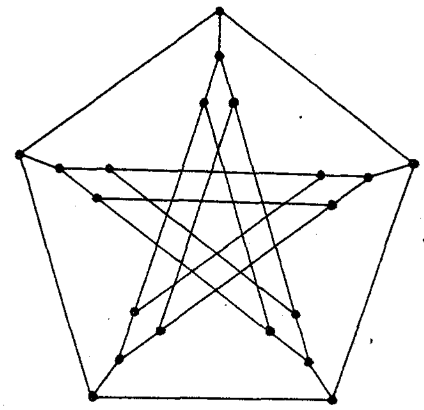

Fig. 5

$$(10) \qquad n_k(\mathcal{G}_d) \geq 2\,\frac{(d-1)^k - 1}{d-2} \qquad \text{for } d = 3, 4, 6 ; \; d-1 \text{ a prime power },$$

since the corresponding d-regular graphs with girth 2k given by F.Kárteszi [14], W.G.Brown [7] and C.T.Benson[5] also have diameter k (compare also R.Singleton [20]). In the case $k = 6$ this fact is not explicitly mentioned by Benson [5], but easy to prove by counting vertices: A d-regular graph with girth 2k and diameter $> k$ would have necessarily more than

$$1 + d + d(d-1) + \ldots + d(d-1)^{k-2} + (d-1)^{k-1} = 2\,\frac{(d-1)^k - 1}{d-2}$$

vertices. — For $k = 3$ (10) yields $n_3(\mathcal{G}_d) \geq 2d^2 - 2d + 2$, for $d-1$ a prime power, and concerning the other cases we have Elspas' result [9]

$$(11) \qquad n_3(\mathcal{G}_d) \geq 2d^2 - 3d + 1 .$$

Finally a graph showing

$$(12) \qquad n_5(\mathcal{G}_3) \geq 46$$

is given in figure 6.

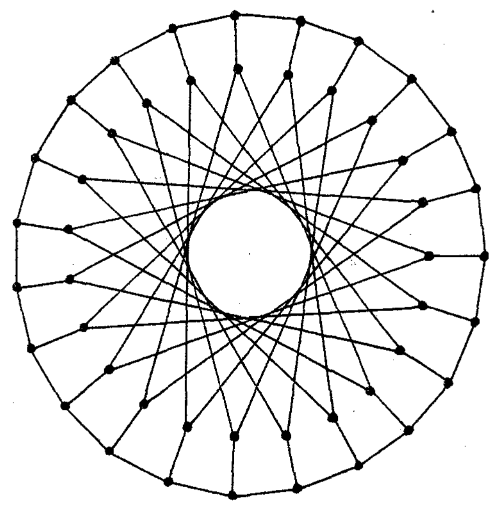

Fig. 6

Table 1 summarizes the results on $n_k(\mathcal{G}_d)$ for small values of k and d. The number in brackets indicates the formula from which this lower bound results.

Table 1

k \ d	2	3	4	5	6	7
1	3	4	5	6	7	8
2	5	10	15	24	36 / (9) / 32	50
3	7	20	52 / (5) / 35	104 / (10) / 42	186 / (10) / 62	300 / (11) / 78
4	9	44 / (10) / 30	160 / (10) / 80	424 / (10) / 170	936 / (10) / 312	1812
5	11	92 / (12) / 46	484 / (7) / 110	1704 / (7) / 240	4686 / (7) / 600	10884 / (7) / 1056
6	13	188 / (10) / 126	1456 / (10) / 728	8824 / (10) / 2730	23436 / (10) / 7812	65317

3. The numbers $n_k(\overline{\mathcal{G}}_d)$. Trivial upper bounds for $n_k(\overline{\mathcal{G}}_d)$ we get from the previous section since

(13) $\quad n_k(\overline{\mathcal{G}}_d) \leqq n_k(\mathcal{G}_d)$.

Thus we have $N(d,k)$ as an upper bound and since every planar graph contains vertices of degree < 5 one may improve this bound for $d > 5$ immediately to

(14) $\quad n_k(\overline{\mathcal{G}}_d) \leqq 1 + 5\,\frac{(d-1)^k - 1}{d-2} \qquad (d > 5)$.

Although this is a rather rough bound, it seems to be hard to give general improvements.

For $k = 1$ we have

(15) $\quad n_1(\overline{\mathcal{G}}_d) = \begin{cases} d+1 & \text{for } d \leqq 3 \\ 4 & \text{for } d > 3 \end{cases}$

because of the nonplanarity of K_{d+1} for $d > 4$. Of course $n_k(\overline{\mathcal{G}}_2) = n_k(\mathcal{G}_2) = 2k+1$ since $C_{2k+1} \in \overline{\mathcal{G}}_2$. For $k = 2$ we prove:

Theorem 2.

(16) $\quad \left[\frac{3d}{2}\right] + 1 \leqq n_2(\overline{\mathcal{G}}_d)$,

(17) $\quad \frac{3d}{2} + 8 \geqq n_2(\overline{\mathcal{G}}_d) \quad$ for $d \geqq 22$.

Inequality (16) is proved by the graph of figure 7 , where dotted lines may be added in the cases $d \geqq 4$ in order to obtain a 3-connected graph, if desired.

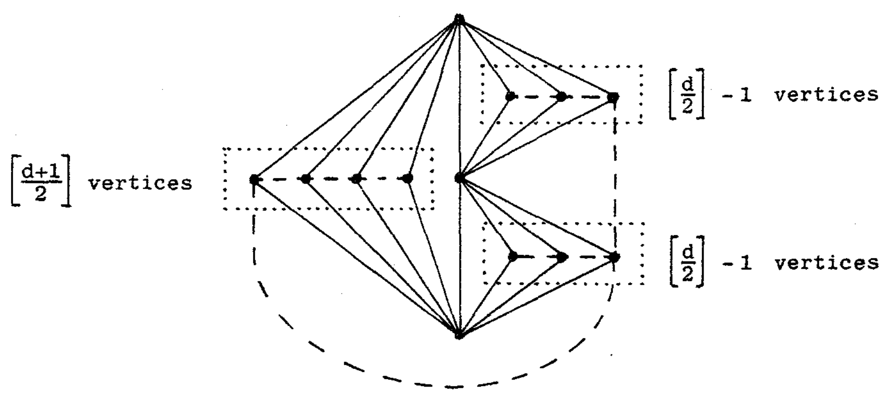

Fig. 7

In order to prove the second inequality we give a preparatory lemma.

Lemma. Let G be outerplanar and let be given a plane embedding of G with straight edges such that all vertices of G are situated on a circle C (this is always possible). Let A , B be two sets of vertices of G with the following properties: A and B are separable by some straight line, $|A| \geqq 4$, $|B| \geqq 4$ and any pair a , b of vertices $a \in A$, $b \in B$ has distance at most 2 in G . Then there exists a vertex x in G dominating both A and B .
(We say that a vertex x dominates A iff each vertex $a \in A$, $a \neq x$, is adjacent to x .)

Proof. Because of the separability we may assume that $a_1, \dots, a_m, b_1, \dots, b_n$ is a labelling of $A \cup B$ in counterclockwise order on C .

Case 1 : There exist $i \in \{2, \dots, m-1\}$ and $j \in \{2, \dots, n-1\}$ such that $a_i \sim b_j$. *)
Now the edge (a_i, b_j) separates a_1 and b_1 ; to ensure $\mathrm{dist}(a_1,b_1) \leqq 2$ we must have $a_1 \sim a_i \sim b_1$ or $a_1 \sim b_j \sim b_1$, say $a_1 \sim a_i \sim b_1$. Considering further pairs of vertices we see that a_i dominates $A \cup B$.

*) $a \sim b$ denotes adjacency of a and b .

Case 2 : None of $a_2, \dots, a_{m-1}$ is adjacent to any of the vertices $b_2, \dots, b_{n-1}$. Then there exists $x \notin \{a_2, \dots, a_{m-1}, b_2, \dots, b_{n-1}\}$ such that $a_2 \sim x \sim b_2$. By arguments similar to those above we see that x is a dominating vertex. //

Proof of (17) : Let $G \in \mathcal{G}_m$ have diameter 2 .

Case 1 : There exists a separating set T of at most three vertices a_i. Because of $\operatorname{diam} G = 2$ T is a dominating set in G. Let R, S be the two sets of vertices separated by $T = \{a_1, a_2, a_3\}$. and $r := |R|$, $s := |S|$, $n := |\operatorname{vert} G|$. By R_i [resp. R_{ij}] we denote the set of vertices of R having in T only a_i [resp. a_i and a_j] as neighbour, likewise S_i and S_{ij}; as above $r_i := |R_i|$ and so on. The numbers r_{123} and s_{123} of vertices of R and S adjacent to all three vertices of T is 0 or 1 .

Case 1.1 : Each vertex of $R \cup S$ is adjacent to at least two vertices of T. Then $2(r+s) \leq 3d$ and thus $n = 3+r+s \leq 3+\frac{3d}{2}$.

Case 1.2 : There exist vertices in $R \cup S$ adjacent to just one vertex in T, say $r_1 \neq 0$. Then $r_{23} \leq 2$ [and $s_2 = s_{23} = s_3 = 0$] and

(*) $\quad r_1 + r_{12} + r_{13} + r_{123} + s \leq d$

since any vertex of S must be adjacent to a_1.

Case 1.2.1 : $r_2 + r_{23} + r_3 \leq d$. Then $n = 3 + r + s \leq 3 + \frac{3d}{2}$.

Case 1.2.2 : $r_2 + r_{23} + r_3 > d$. Then $r_2 + r_3 > \frac{d}{2} - 2$; since $d > 16$ we may assume $r_2 \geq 4$ and so $r_{13} \leq 2$.
If also $r_3 \neq 0$, then $r_{12} \leq 2$, $s = s_{123} = 1$ and $r_1 + r_2 + r_3 \leq d + 5$ in view of the lemma : Assume $r_1 + r_2 + r_3 > d + 5$; since $r_i \leq d - 1$ at least two of these numbers are greater than 3 , say r_1 and r_2, taking $A = R_1$ and $B = R_2 \cup R_3$ we see that there exist a dominating vertex of degree $\geq r_1 + r_2 + r_3 - 1$, which is impossible. Thus

$$n = 3+r_1+r_2+r_3+r_{12}+r_{13}+r_{23}+r_{123}+s \leq 3+d+5+2+2+2+1+1 = d + 16 \leq \frac{3d}{2} + 5$$

for $d \geq 22$.
Now assume $r_3 = 0$. Then $r_2 > \frac{d}{2} - 2$ and with the help of the lemma $r_1 + r_{13} \leq \frac{d}{2} + 2$ (take $A = R_2$ and $B = R_1 \cup R_{13}$), thus $n \leq \frac{3d}{2} + 5$ since similar to (*)
$r_2 + r_{12} + r_{23} + r_{123} + s \leq d$.

Case 2 : Any separating set of vertices has at least 4 vertices. Then G cannot contain vertices of degree ≤ 3. Let x be a vertex of minimum degree k ($k = 4$ or 5) and $y_1, \dots, y_k$ its neighbours labelled according to their plane cyclical order. Any further vertex of G is adjacent to at least one of $y_1, \dots, y_k$. y_i cannot be adjacent to y_j unless $j \equiv i \pm 1 \pmod k$ otherwise $\{x, y_i, y_j\}$ would be a separating set.

Case 2.1 : Two of $y_1, \dots, y_k$, say y_j, y_l, that are not cyclically neighboured, have a common neighbour $z \neq x$. Omitting x and adding edges (y_i, y_{i+1}) $(i \bmod k)$ so far $y_i \not\sim y_{i+1}$ we get a graph $G' \in \mathcal{G}_{d+1}$ with $\operatorname{diam} G' \leq 2$ having the separating set $\{y_j, y_l, z\}$ and so $n \leq 1 + \frac{3(d+1)}{2} + 5$ according to case 1.

Case 2.2 : Any further vertex belongs to some set R_i of vertices adjacent to y_i only or to some set $R_{i,i+1}$ of vertices adjacent to both y_i and y_{i+1} $(i \bmod k)$. We have $|R_{i,i+1}| \leq 1$, otherwise we would have a separating triple. Now with $A_i = R_i \cup R_{i,i+1} \cup R_{i+1}$ and $B = R_{i+2} \cup R_{i+2,i+3} \cup \dots \cup R_{i-1}$ $(i \bmod k)$ we may apply the lemma. Thus either there exists i such that both $|A_i| \geq 4$ and $|B_i| \geq 4$ and then according to the lemma $|A_i| + |B_i| \leq d$ and so $n \leq d + 2 + 5 + 1 = d + 8$, or we have for each i either $|A_i| \leq 3$ or $|B_i| \leq 3$. But then there is at most one j such that $|A_i| \geq 4$, on the other hand we have $|R_{j,j-1} \cup R_j \cup R_{j,j+1}| \leq d - 1$ and so $n \leq d-1+3+3+1+k+1 \leq d+12$. //

For $3 \leq d \leq 5$ we have

(18) $n_2(\overline{\mathcal{G}}_3) = 7$, $n_2(\overline{\mathcal{G}}_4) = 9$, $n_2(\overline{\mathcal{G}}_5) = 10$.

Graphs showing "$\geq$" are given in figure 8 , the proofs of "$\leq$" are elementary, but tedious; we omit details.

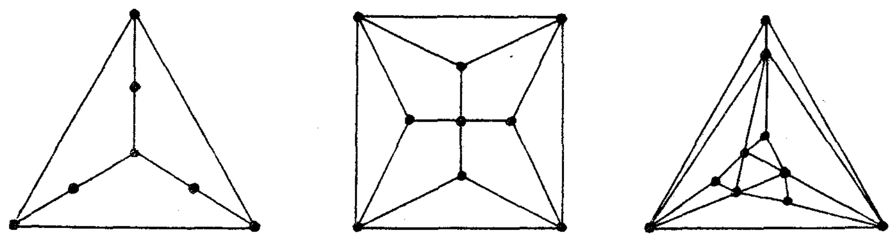

Fig. 8

It is worth noting that the first graph of figure 8 is not 3-connected. Indeed for the class $\mathcal{P}_3$ of 3-valent, 3-connected planar graphs (i.e. the graphs of simple 3-polytopes) we have $n_2(\mathcal{P}_3) = 6$.

Figure 9 shows

(19) $n_2(\overline{\mathcal{G}}_6) \geq 11$, $n_2(\overline{\mathcal{G}}_7) \geq 12$

and we conjecture:

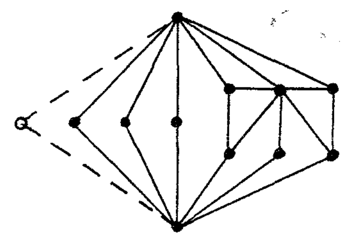

Fig. 9

<u>Conjecture.</u> $n(\overline{\mathcal{G}}_d) = d + 5$ for $d = 6 , 7$

$n(\overline{\mathcal{G}}_d) = \left[\frac{3d}{2}\right] + 1$ for $d \geq 8$

Finally we give by some easy constructions shown for $r = 2$ in figures 10 - 12 general lower bounds and it seems very likely that these bounds are close by the exact values.

(20) $n_{2r+1}(\overline{\mathcal{G}}_d) \geq 3(d-1)^r + 4\frac{(d-1)^r - 1}{d-2}$ for $d = 3 , 4$ (see figure 10) .

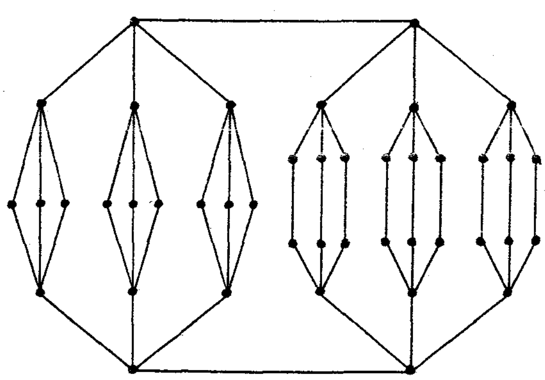

Fig. 10

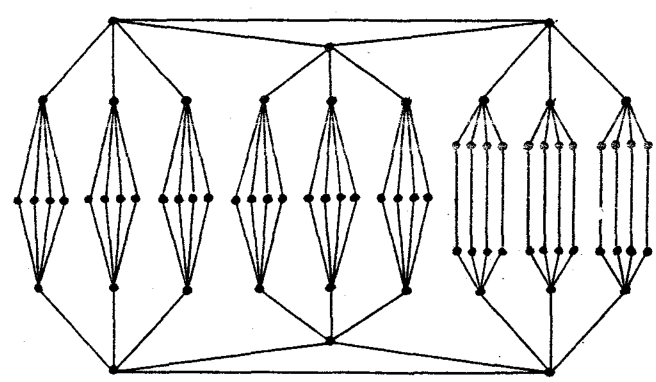

Fig. 11

(21) $n_{2r+1}(\overline{\mathcal{G}}_d) \geq (4d-2)(d-1)^{r-1}$ for $d > 4$ (see figure 11) .

(22) $n_{2r}(\overline{\mathcal{G}}_d) \geq \frac{1}{d-2}[(d+2)(d-1)^r - 4]$ (see figure 12) .

In general these inequalities will not be best possible, for instance we have

(23) $n_3(\overline{\mathcal{G}}_3) \geq 12$ (see figure 13) .

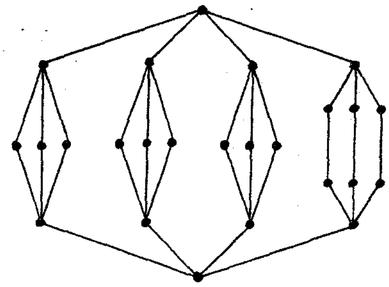

Fig. 12

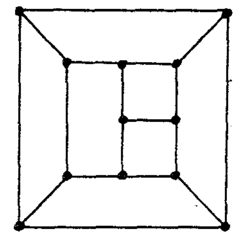

Fig. 13

4. The coloring numbers $\chi_k(\mathcal{G}_d)$ and $\chi_k(\overline{\mathcal{G}}_d)$. As noted in section 1 we have $\chi_k(\mathcal{G}_d) \geqq n_k(\mathcal{G}_d)$ and $\chi_k(\overline{\mathcal{G}}_d) \geqq n_k(\overline{\mathcal{G}}_d)$. For $d = 2$ we have equality in both cases, so we restrict our attention in the following to $d \geqq 3$. And as just used in [1] the problem of finding an r-coloring(k) of a graph G may be reduced to the problem of finding an ordinary coloring by considering the k-th power of G: $\chi_k(G)$ equals the ordinary chromatic number $\chi(G^k)$ of G^k. If we define the clique number $\sigma_k(G)$ relative to distance k to be the maximum number of vertices of subgraphs G' of G with $\operatorname{diam} G' \leqq k$, then we have similarly $\sigma_k(G) = \sigma(G^k)$, where σ denotes the usual clique number. Of course $\chi_k(G) \geqq \sigma_k(G)$. If G has maximum degree d, then G^k has maximum degree $N(d,k)-1$. Thus according to a wellknown theorem $\chi(G^k) \leqq N(d,k)$ for every $G \in \mathcal{G}_d$ and we have

(24) $\quad n_k(\mathcal{G}_d) = \sigma_k(\mathcal{G}_d) = \chi_k(\mathcal{G}_d) = N(d,k)$ *)

whenever a (d,k)-Mooregraph exists (this cases are listed in section 2, now including $k = 1$), while in any other case we get using a theorem of Brooks [5]

(25) $\quad n_k(\mathcal{G}_d) \leqq \sigma_k(\mathcal{G}_d) \leqq \chi_k(\mathcal{G}_d) < N(d,k)$.

The difficulty to prove further restrictions on $\chi_k(\mathcal{G}_d)$ becomes evident if we now consider the case of planar graphs. Again we have $\chi_k(\overline{\mathcal{G}}_2) = n_k(\overline{\mathcal{G}}_2) = 2k+1$ and for $d \geqq 3$ similarly to (25)

(26) $\quad n_k(\overline{\mathcal{G}}_d) \leqq \sigma_k(\overline{\mathcal{G}}_d) \leqq \chi_k(\overline{\mathcal{G}}_d) < N(d,k)$

for any $d \geqq 3$ and $k \geqq 2$, since all the Moore graphs in question are not planar. For $k = 1$ we know $\chi_1(\overline{\mathcal{G}}_d) = 4$ for $3 \leqq d \leqq 5$ (see J.M.Aarts - J.de Groot [1]), but the question whether $\chi_1(\overline{\mathcal{G}}_d) = 4$ holds for all $d \geqq 3$ is precisely the famous and long standing four color problem, which just has been solved by K.Appel and W.Haken with a proof that is very long and depends heavily on extensive use of computers (see K.Appel - W.Haken [3]).

In order to stimulate further research we venture a challenging conjecture:

Conjecture: For any $d \geqq 3$, $k \geqq 1$

$$n_k(\mathcal{G}_d) = \sigma_k(\mathcal{G}_d) = \chi_k(\mathcal{G}_d) \quad \text{and}$$
$$n_k(\overline{\mathcal{G}}_d) = \sigma_k(\overline{\mathcal{G}}_d) = \chi_k(\overline{\mathcal{G}}_d) .$$

As noted above one cannot expect a general answer but it would be interesting to settle some cases.As a first step in this direction we prove $\chi_2(\overline{\mathcal{G}}_3) \leqq 8$ and it remains open whether $\chi_2(\overline{\mathcal{G}}_3) = 7$ or $\chi_2(\overline{\mathcal{G}}_3) = 8$.

Theorem 3 . $\quad \chi_2(\overline{\mathcal{G}}_3) \leqq 8$.

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*) where $\sigma_k(\mathcal{G}) := \sup\{\sigma_k(G) \mid G \in \mathcal{G}\}$.

Proof. Let G be a graph of $\overline{\mathcal{G}}_3$ with $\chi_2(G) \geq 9$ and minimum number of vertices. We prove by contradiction that such a graph cannot exist. In order to do this we first deduce some properties of G .

(a) G is regular of degree 3 and does not contain 3-circuits or pairs of 4-circuits with an edge in common.

Otherwise let v be a vertex of degree < 3 or a vertex of some 3-circuit or a vertex of an edge belonging to two 4-circuits. The antistar G' of v in G is 8-colorable(2) by minimality of G . But this coloring can be extended to G since v has at most 7 neighbours of first and second order, a contradiction. /

(b) G is 3-connected.

Clearly G is connected. Assume that G is not 3-connected and let $e_1 = (v_1', v_1'')$ and $e_2 = (v_2', v_2'')$ be two edges separating G into two components G' and G'' with $v_i' \in G'$ and $v_i'' \in G''$ [omit e_2 in the case of 1-connectedness] *) . We are able to color G' rel. to distance two with 8 colors — this coloring may be described by a function $f : \text{vert } G' \longrightarrow \{1,2,\ldots,8\}$ — such that $f(v_1'), f(v_2') \in \{1,2\}$ and none of the neighbours of v_1' and v_2' has color 3 or 4 (since there are at most 4 neighbours). Likewise we color G'' such that $f(v_1''), f(v_2'') \in \{3,4\}$ and none of the neighbours of v_1'', v_2'' has color 1 or 2. Obviously both colorings may be fitted together to yield an 8-coloring(2) of G . /

It is worth noting that in so far we didn't make use of the planarity of G .

(c) G cannot contain 4-circuits.

Assume that $x_1, \ldots, x_4$ are the vertices of some 4-circuit C of G . Because of (a) and (b) each x_i has a neighbour $y_i \notin C$ and all y_i are different and nonadjacent ($y_1 \not\sim y_3$ and $y_2 \not\sim y_4$ involve together with (b) the planarity of G).
Omitting C and the edges incident with C we get a graph G' (see figure 13) which has an 8-coloring(2). We try to extend this coloring to G . Consider one fixed x_i ; coloring this x_i we have to avoid the colors of five vertices of G'. Thus in view of G' we can assign to each x_i a set A_i of at least three admissible colors. Now it is possible to choose for all x_i different admissible colors provided that not all A_i consist of the same three colors, say the colors 6 , 7 , 8. In that case we change the coloring of y_1 in G'. First note that the colors assigned to y_1, y_2, y_4, u, v (see figure 14) all are different, otherwise at least four colors would be admissible for x_1. Further at least one of the colors $f(y_2), f(y_4), 6, 7, 8$ does not occur within the colors of the (at most four) neighbours of second order of y_1 (among which may be some y_i). Recoloring y_1 with that color x_1 has four admissible colors or an admissible triple different from that of x_3 , which remains unchanged. After recoloring we have the general case of above. /

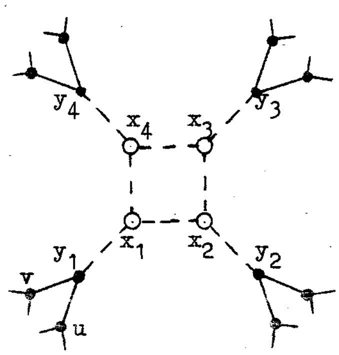

Fig. 14

In the last step of the proof we show that G cannot contain 5-circuits. Since every 3-connected, planar graph contains n-circuits with $n < 6$ this proves the nonexistence of G .

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*) In the case of cubic graphs edge-connectivity coincides with vertex-connectivity.

The procedure in this last step is the same as in the proof of (c). Let $x_1, \dots, x_5$ be the vertices of a 5-circuit C of G; each x_i has a neighbour $y_i \notin C$ and all y_i are different and nonadjacent. Let be given an 8-coloring(2) of the antistar G' of C in G. As in (c) denote by A_i the set of colors admissible for x_i (in view of G'). In any way we have up to permutation of vertices or colors one of the following cases.

Case 1: $f(y_1) = f(y_2) = f(y_3)$. Then $|A_2| \geqq 5$, $|A_1| \geqq 4$ and $|A_3| \geqq 4$ and the A_i have in either case a transversal, which means we can assign to all x_i different admissible colors.

In the following we consider only the "critical cases" where such a transversal does not necessarily exist and we indicate which vertex of G' should be recolored in that case.

Case 2: $f(y_1) = f(y_2) = f(y_4)$. The critical case is $|\bigcup_{i=1}^{5} A_i| = 4$. In that case recolor y_4 !

Case 3: $f(y_1) = f(y_2)$, but none of the cases above. Then $|A_1| \geqq 4$ and $|A_2| \geqq 4$.
The critical case is $A_3 \cup A_4 \cup A_5 \subseteq A_1 = A_2$. Then $f(y_3), f(y_5) \notin A_1$ and say $f(y_3) \neq f(y_4)$ ($f(y_3)$ and $f(y_5)$ cannot both equal $f(y_4)$, otherwise case 1). Then recolor y_3 !

Case 4: $f(y_2) = f(y_5)$, but none of the cases above. Then $f(y_3) \neq f(y_4)$, $|A_1| \geqq 4$ and the critical case is: $|A_1| = 4$ and $A_2 \cup A_3 \cup A_4 \cup A_5 \subseteq A_1$.
Then not both $f(y_3)$, $f(y_4) \in A_1$, say $f(y_3) \notin A_1$. Recolor y_3 !

Case 5: All colors $f(y_i)$ are pairwise different. Then we have two critical cases: $|\bigcup_{i=1}^{5} A_i| \leqq 4$ or some four of the A_i consist of the same triple of colors.
If in that cases for some i $y_i \in A_{i+2}$ or $y_i \in A_{i-2}$ ($i \bmod 5$), say $y_1 \in A_3$, then recolor y_5! Otherwise necessarily all A_i consist of the same triple of colors and recoloring any of y_i reduces also that case to one of the cases above. //

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