The diffuse interface approximation of the Willmore functional in configurations with interacting phase boundaries

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The diffuse interface approximation of the Willmore functional in configurations with interacting phase boundaries

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Abstract

In this thesis we study a diffuse interface approximation of the sum of the area and Willmore functional for which Γ-convergence has already been established in the case of small space dimensions and smoothly bounded sets. We extend this result to a larger class of configurations with nonsmooth phase boundaries and explicitly allow intersecting boundary curves.

We also analyze the interaction of parallel planar phase fields and discuss their slow motion under the L^2-gradient flow of the diffuse Willmore functional. Moreover, we prove the existence of a new class of periodic entire solutions to the stationary Allen-Cahn equation in two dimensions.

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Contents

1 Introduction

Functionals which depend on the curvature of surfaces or curves play a major role in modern mathematical theory and numerous applications as for example in biology or computer science. This broad interest has led to enormous progress in the mathematical analysis of these energies and their gradient flows during the last decades.
In this thesis we will deal with the Willmore functional as a prototype of such energies together with the corresponding gradient flow. We examine its relation to a common diffuse interface approximation for which Γ-convergence already has been established in the case of smoothly bounded sets [RöSc06]. We extend this result to a new class of nonsmooth phase boundaries and thereby explain the occurrence of intersecting boundary curves in numerical simulations of the diffuse Willmore flow [EsRäRö14]. Moreover, we quantify the energy order of parallel planar phase fields and study the slow motion of these configurations under the gradient flow caused by interacting phase boundaries. As a further result we prove the existence of a new class of periodic entire solutions to the Allen-Cahn equation generalizing [DaFiPe92].

Consider a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. For an open set $E \subset \Omega$ with C^2-boundary ∂E we denote the inner unit normal vector field on ∂E by ν and the principal curvatures of ∂E with respect to ν by $\kappa_1, \ldots, \kappa_{n-1}$. The *Willmore energy* (or *Willmore functional*) [Wi93] of ∂E is given by

$$\mathcal{W}(\partial E) := \frac{1}{2} \int_{\partial E \cap \Omega} |\mathbf{H}_{\partial E}|^2 \, d\mathcal{H}^{n-1} \tag{1.1}$$

where $\mathbf{H}_{\partial E} := (\kappa_1 + \cdots + \kappa_{n-1})\, \nu$ is the mean curvature vector field of ∂E. Although named after Thomas Willmore (1919-2005) nowadays, the functional itself has already been proposed before by Poisson [Po1812] in 1812 and also later by Germain [Ge1821]. Since then many authors from differential geometry (see for example [Th23, Bl29]) and geometric measure theory [MaNe14] have contributed to its analysis. One of the most spectacular results in recent years has been the proof by Marques and Neves [MaNe14] of the Willmore conjecture [Wi65] on the minmial Willmore energy of immersed tori in $\mathbb{R}^3$.
(1.1) is the simplest type of a bending energy and therefore often serves as a representative for other more complicated curvature depending functionals. It naturally arises in biological models due to its connection to the Helfrich-Canham energy [He73] in the description of cell shapes [Ca70]. $\mathcal{W}$ also appears in computer science theory where it is used in image segmentation problems to control the appearance of noise.
We remark that in the case $n = 2$ the Willmore functional coincides with Euler's elastica

energy (see e.g. [Lo13], §263) which describes the bending of a rod and which also has been thoroughly investigated in many applications [LaSi84, Mu94]. However, we will always refer to $\mathcal{W}$ as the Willmore functional independently of the space dimension.
For convenience, we also define the Willmore energy of phase indicator functions $u = 2\chi_E - 1$ with E as above by

$$\mathcal{W}(u) := \mathcal{W}(\partial E).$$

The phase field model

Many applications make a numerical treatment of the Willmore energy and its gradient flow necessary. A common approach in this context is the approximation by diffuse interfaces modeling sharp phase boundaries by diffuse transitions. Thereby, a new space dimension (compared to the sharp interface) is added to the problem which yields an automatic treatment of topological changes as a major advantage to other approximation methods. The diffuse interface approximation is widely used in the simulation of geometric evolution equations and especially in the analysis of the gradient flow of $\mathcal{W}$.
We will briefly sketch the core of this theory. It originally goes back to thermodynamical studies of capillarity by van der Waals in the 1870s on the free energy of phase boundaries between two immiscible and incompressible fluids (see [Ro79] for an English translation). Contrary to the classical thermodynamical theory of capillarity by Gibbs [Gi1878] he argued that the transition between two phases is not given by a sharp (and discontinuous) interface but can rather be modeled as a continuous phenomenon happening on a thin layer which can be identified with the interface. To make things precise we slightly adapt the original notation and consider a spacial domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. In the classical sharp interface model, a configuration can be described by a function $u \in BV(\Omega; \{-1, 1\})$ (see Section 2.3 for a definition of BV functions) which takes the value 1 wherever the first fluid is present and becomes equal to -1 on sets occupied by the second fluid. For $E := \{u = 1\}$ we write $u = 2\chi_E - 1$. The free energy is then given by the perimeter functional as

$$P(u) := \mathrm{Per}_\Omega(E) = \frac{1}{2}\int_\Omega |\nabla u| = \mathcal{H}^{n-1}\big(\partial^* E \cap \Omega\big) \quad \text{if } u \in BV\big(\Omega; \{-1, 1\}\big)$$

and $P(u) := \infty$ if $u \in L^1(\Omega) \setminus BV(\Omega; \{-1, 1\})$ (see Section 2.3 for a definition of $\partial^* E$). For smooth phase boundaries, P describes the area of the common boundary of the phases.

In the diffuse interface model we allow mixtures of both fluids and describe their average volume densities by a function $u \in L^1(\Omega)$ with values in $[-1, 1]$. The values -1 and 1 can be interpreted as above while $u(x) = 0$ for example describes a point $x \in \Omega$ where both fluids are present with equal volume fraction. Van der Waals derived a free energy formulation for these interfaces which is given by

$$\mathcal{E}_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\,dx \quad \text{if } u \in H^1(\Omega) \tag{1.2}$$

and $\mathcal{E}_\varepsilon(u) := \infty$ otherwise. Here, $F(s) := \frac{1}{4}(s^2 - 1)^2$ is a prototype for an equal and

smooth double well potential (read the remark at the end of the introduction for a note on F) and $\varepsilon > 0$ is assumed to be small. The same functional was derived from basic laws of thermodynamics by Cahn and Hilliard in [CaHi58] as a (first order) approximation of an interface energy for mixtures of binary alloys. The structure of $\mathcal{E}_\varepsilon$ can be explained easily. While the second summand prefers large regions with u constant to -1 or 1 the first term penalizes steep changes of one phase to another. For small ε a minimizer of u forms phase transitions on a layer of size ε and becomes nearly constant elsewhere. This already yields strong evidence of the correlation between $\mathcal{E}_\varepsilon$ and Per. We will refer to (1.2) as the *Ginzburg-Landau energy* or *diffuse surface energy* in the following but many other names are common in the literature.
A rigorous connection between both models was proved by Modica and Mortola in the framework of Γ-convergence [MoMo77]. They could show that the functionals $\mathcal{E}_\varepsilon$ approximate Per as $\varepsilon \to 0$ in the sense that for $u \in L^1(\Omega)$

$$\Gamma(L^1) - \lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u) = \sigma P(u) \tag{1.3}$$

is satisfied (see Section 2.1 for a definition of Γ-convergence). The parameter $\sigma > 0$ describes the surface tension between both phases and can exactly be determined as $\sigma = \int_{-1}^{1} \sqrt{2F}\, ds$. Their proof strongly relies on the fact that energetically preferable phase transitions are shaped as the one-dimensional profile which connects the phases -1 and 1 in an energetic optimal way (see Section 2.4 for a detailed description of these profiles).

Based on a conjecture of De Giorgi [DeG91], Bellettini and Paolini formulated a diffuse interface approximation of the Willmore functional on $L^1(\Omega)$ in [BePa93] by

$$\mathcal{W}_\varepsilon(u) := \frac{1}{2\varepsilon} \int_\Omega \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} F'(u) \right)^2 dx \quad \text{if } u \in H^2(\Omega) \tag{1.4}$$

and $\mathcal{W}_\varepsilon(u) := \infty$ for $u \in L^1(\Omega) \backslash H^2(\Omega)$. We refer to Section 6.1 for a detailed description of De Giorgi's conjecture as well as its modification and only give a formal argument to motivate the relation between $\mathcal{W}_\varepsilon$ and $\mathcal{W}$ at this point. For a set E with smooth boundary ∂E the first variation of its energy is described by its mean curvature vector $\mathbf{H}_{\partial E}$ (see [Si83], §9). At the same time the L^2-gradient of (1.2) is given by $-\varepsilon\Delta u + \frac{1}{\varepsilon} F'(u)$ and can therefore be understood as a diffuse mean curvature. This makes $\mathcal{W}_\varepsilon$ a natural choice in order to approximate $\mathcal{W}$ in the framework of diffuse interfaces (the factor $\frac{1}{\varepsilon}$ in (1.4) expresses the size of the energy density in the transition layers, see Section 6.1). The argument also shows that the structures of $\mathcal{W}_\varepsilon$ and $\mathcal{E}_\varepsilon$ are related.
For a given open and C^2-bounded set $E \subset \Omega$ Bellettini and Paolini could show the lim sup inequality for the Γ-convergence of $\mathcal{W}_\varepsilon$. They used the idea from [MoMo77] to construct a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $H^2(\Omega)$ such that $\mathcal{W}_\varepsilon(u_\varepsilon)$ approximates $\mathcal{W}(\partial E)$ (again up to the constant σ). Since then several authors contributed in this field with partial results concerning the lim inf inequality (see [BeMu05, Mo05, HuTo00, To02, Sc09]). Finally, Röger and Schätzle [RöSc06] proved the Γ-convergence of $\mathcal{W}_\varepsilon$ to $\mathcal{W}$ for C^2-bounded sets in dimensions $n = 2, 3$.
For most applications it is convenient to consider the the perimeter and Willmore

functional simultaneously and we denote their sum by

$$\mathcal{F}(E) := \mathrm{Per}_\Omega(E) + \mathcal{W}(\partial E).$$

$\mathcal{F}$ can be extended to sets with nonsmooth boundary by its lower semicontinuous envelope

$$\overline{\mathcal{F}}(E) := \inf \left\{ \liminf_{k\to\infty} \mathcal{F}(E_k) : \partial E_k \in C^2 \text{ and } E_k \to E \text{ in } L^1(\Omega) \right\}. \tag{1.5}$$

This relaxation is natural from a variational point of view as it extends $\mathcal{F}$ to a lower semicontinuous functional.
$\overline{\mathcal{F}}$ has been analyzed in [BeDaPa93] where the authors find several conditions on a set E of finite perimeter such that $\overline{\mathcal{F}}(E)$ is finite. They show that this can only be the case if (after a possible change of E on a set of measure zero) there exists a unique non oriented tangent in *every* point of ∂E. This immediately shows that transversal intersections of the boundary always produce infinite energy $\overline{\mathcal{F}}$. On the other hand there exist sets E with nonsmooth boundary and $\overline{\mathcal{F}}(E) < \infty$: If ∂E is H^2-regular up to finitely many cusp points then $\overline{\mathcal{F}}(E) < \infty$ if and only if the number of cusps is even.
It is a natural question whether the Γ-convergence result of $\mathcal{F}_\varepsilon := E_\varepsilon + \mathcal{W}_\varepsilon$ can be transferred to $\overline{\mathcal{F}}$ for nonsmooth boundaries. Unfortunately the answer turns out to be negative in general as shown in [Mu13]. While transversal intersections of phase boundaries cause infinite relaxed energy $\overline{\mathcal{F}}$ as mentioned above it is possible to approximate these crossings with finite diffuse energy. This is strictly related to a class of saddle shaped entire smooth solutions u_ε of the stationary Allen-Cahn equation

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^2 \tag{1.6}$$

with a zero set consisting of both coordinate axes. The existence of such solutions has been shown in [DaFiPe92]. On an open set $\Omega \subset\subset \mathbb{R}^2$ with $0 \in \Omega$ these functions converge to an indicator function $u = 2\chi_E - 1$ with $E := \{x_1x_2 > 0\} \cap \Omega$ in $L^1(\Omega)$ and it can be shown that their diffuse surface energies in Ω remain bounded. Due to (1.6) the diffuse Willmore energy vanishes for every $\varepsilon > 0$ and hence,

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty = \overline{\mathcal{F}}(E).$$

We point out that in the example above the Γ-Limit of $\mathcal{F}_\varepsilon$ still exists. In this thesis we will identify the precise limit for a wide class of configurations with intersections of the boundary.

The phenomenon of intersecting boundary curves also appears in simulations of the L^2-gradient flow of $\mathcal{W}_\varepsilon$ that we briefly introduce now.
In the following we refer to the L^2-gradient flow of $\mathcal{W}$ as the *Willmore flow*. For $T > 0$ and a family of open and C^2-bounded sets $(E(t))_{t\in(0,T)}$ it determines the velocity v in the direction of ν by

$$v(t) = \Delta_{\partial E(t)} H_{\partial E(t)}(t) - \frac{1}{2} H^3_{\partial E(t)} + H_{\partial E(t)}(t)\, |A(t)|^2 \quad \text{on } \partial E(t) \tag{1.7}$$

where $H_{\partial E(t)} = \kappa_1 + \cdots + \kappa_{n-1}$ denotes the scalar mean curvature, $\Delta_{\partial E(t)}$ is the Laplace-Beltrami operator on $\partial E(t)$ and $|A(t)|^2 = \kappa_1^2 + \cdots + \kappa_{n-1}^2$ is the sum of the squared principal curvatures. The Willmore flow is well analyzed in the literature and we refer representatively to [DzKuSc02] and [KuSc01] for longtime existence results and a qualitative analysis.
The *diffuse Willmore flow* is analogously defined as the L^2-gradient flow of $\mathcal{W}_\varepsilon$ and given by

$$\varepsilon \partial_t u = \left(\varepsilon \Delta - \frac{1}{\varepsilon} F''(u)\right) \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} F'(u)\right) \tag{1.8}$$

for an evolving diffuse interface u. In [CoLa11] the authors show longtime existence of solutions to the flow under a volume constraint and in [CoLa12] under a volume and area constraint. We will adapt their proof in Section 5.1 to show the longtime existence of smooth solutions on periodic domains (without any further constraints) in up to three dimensions.
In general it is not clear that the gradient flows of Γ-converging functionals also converge to the gradient flow of the limit energy and there is no rigorous proof for the convergence of the diffuse Willmore flows as $\varepsilon \to 0$. However, Loreti and March [LoMa00] could prove on a formal level that the evolution of smooth surfaces under the Willmore flow can be approximated by solutions of (1.8) (rescaled in time) for small ε.

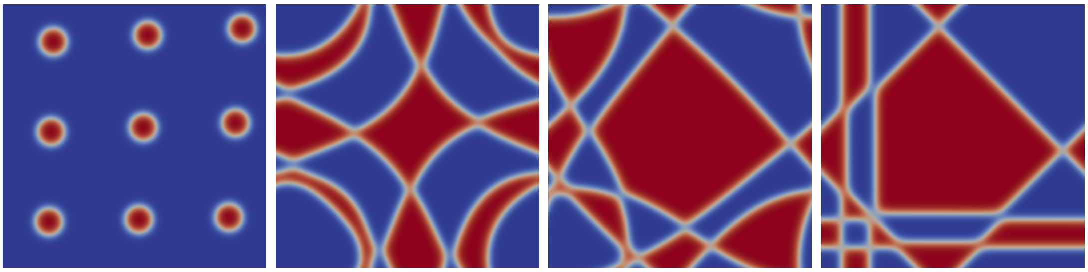


Figure 1.1: Evolution of a start configuration (left) under the diffuse Willmore flow with periodic boundary conditions. (The pictures have been taken from [EsRäRö14] with permission.) The occurring patterns show intersections of the phase boundary.

The classical Willmore flow (1.7) immediately terminates in singularities caused by touching or colliding interfaces. The evolution can be extended beyond those points by considering the gradient flow of the relaxed functional $\overline{\mathcal{W}}$. This evolution, however, cannot be approximated by the diffuse Willmore flow in general. In [EsRäRö14] the authors observe in several simulations that the diffuse flow (1.7) yields structures which correspond in the limit $\varepsilon \to 0$ to configurations with infinite relaxed energy $\overline{\mathcal{F}}$. In their paper they suggest a modified gradient flow which solves this issue and which always yields configurations with finite values of $\overline{\mathcal{F}}$. Figure 1.1 shows the evolution of an initial configuration (left picture) in several evolution snapshots. The gradient flow tends to produce transversal intersections of diffuse boundary curves (see also [BrMaOu15]).
Besides the occurrence of transversal intersection points another interesting observation can be made. After a short phase of energy relaxation the phase boundaries approximate

intersecting curves (as seen in the third picture) which then evolve into straight lines (right picture). In this state the system's energy is almost zero and the lines hardly move anymore.
We also refer to the difference between perpendicular and non perpendicular intersections of the approximated lines in the last picture. While the first type really consists of two crossing lines, the interface forms touching curves instead of true intersections in the non orthogonal case. In this thesis we point out that such structures are related to (4-ended) entire solutions of the stationary Allen-Cahn equation that have been investigated in depth over the last years. As mentioned above there exists a solution of (1.6) with its zero set given by two perpendicular lines due to [DaFiPe92]. In [PiKoPa10] a class of solutions has been introduced whose zero set is at least asymptotic to two intersecting lines at infinity (see also Section 2.5). In this case the actual zero set may look like two curves which nearly touch in the common point of both lines exactly as in Figure 1.1.

Main results

The described observations motivate the scope of this thesis in two different ways. On the one hand we will explain rigorously why configurations as in the right picture are energetically preferable states of the diffuse Willmore flow despite the transversal intersections of the interfaces. Consider the set E which is approximated by the red phase in the last picture and note that $\overline{\mathcal{F}}(\partial E) = \infty$ due to the intersection of boundaries. However, ∂E interpreted as an one-dimensional varifold (a generalized surface) has vanishing generalized curvature. This fact strongly suggests that $\mathcal{W}_\varepsilon$ approximates the Willmore energy generalized to varifolds for small $\varepsilon > 0$. We make this idea rigorous by proving the Γ-convergence of $\mathcal{F}_\varepsilon$ for a large set $\mathcal{S}$ of nonsmooth interfaces in Theorem 6.3. As we will see, cross intersections of the boundary do not contribute anything to the limit value of the energy which is purely determined by the Willmore energy and area of the boundary interpreted as a varifold. This result extends the work of Röger and Schätzle on the conjecture of De Giorgi [RöSc06] to nonsmooth limit sets in two dimensions.
The proof of Γ-convergence is basically divided into two parts. For the $\liminf$ inequality we rely on the results from [RöSc06]. As their measure geometric approach already uses varifold methods the ideas can easily be transferred from smooth boundaries to configurations in $\mathcal{S}$. To construct a recovery sequence and thereby showing the $\limsup$ estimate we make use of the above mentioned 4-ended solutions to the Allen-Cahn equation from [PiKoPa10] (Section 2.5) to approximate transversal intersections. We match these solutions with the approximation method from [BePa93] (see Section 2.6) which requires a careful analysis of the appearing error terms.

Another focus of this thesis lies on the described motion of the phase boundaries in Figure 1.1. In the last picture the boundary curves of the approximated sets seem to have fully relaxed and do not have any curvature. Nevertheless, it is not clear whether the gradient flow has reached a stationary state or only appears to be stationary. This behavior reminds a lot of the slow motion of other diffuse geometric flows as the Allen-Cahn [OtRe07] or the Cahn-Hilliard equation[OtWe14] (see also Section 5.2 for a detailed description). For the diffuse Willmore flow this phenomenon has not been studied yet and this work presents the first results in that direction.

More precisely, we will analyze the simple quasi one-dimensional situation of parallel stripes with different widths which interact with each other. We prove that diffuse interfaces which approximate these configurations still carry small amounts of energy if and only if the considered stripes are not perfectly symmetrically distributed. Under the absence of boundary curvature the minimal energy size is determined by the distance of neighboring phases which we will describe precisely by a scaling law for the energy order in terms of ε and the stripe widths (see Theorem 3.12). The proof requires a profound understanding of certain optimal arc solutions of the Allen-Cahn equation which describe the optimal way in which a single stripe can be approximated by diffuse interfaces (see Section 3.1).
The precise characterization of the minimal energies yields first results concerning the motion speed of evolving stripe configurations. Starting with small energy phase fields, we show that stripe solutions hardly move for exponentially long times. These statements are mainly due to the L^2-gradient structure and the energy estimates proved before (see Proposition 5.9). Moreover, we expect that the diffuse phase boundaries (i.e., the zeros of the phase fields) will distribute equally on an interval after a long time and we will show that they do not move asymptotically in the "wrong" direction.

The analysis of real two-dimensional configurations turns out to be much harder. Even for a slight modification of the stripe configurations a precise characterization of the energy order via a scaling law seems out of reach. We consider configurations of half stripes (see Figure 4.2) and prove the existence of zero energy states (periodic entire solutions of the Allen-Cahn equation (1.6)) with multiple and equi-distributed saddles on the x_1-axis. We show that for large values of x_2 these solutions are shaped as the optimal one-dimensional arc profiles from above and thereby find a connection between both classes of solutions. Apart from the relevance for our problem setting, the existence and analysis of those solutions extends the classical results of [DaFiPe92] on the entire solutions with only one saddle in the origin. The description and classification of entire solutions for the Allen-Cahn equation is a current research topic and our results also contributes in this field.

Outline

The thesis is structured as follows. Chapter 2 collects an overview of the mathematical concepts and theories which are used in the subsequent chapters. We especially give a brief introduction in the theory of varifolds (Section 2.2) and fix basic notations in preparation for Chapter 6.1.
In Chapter 3 we analyze the interaction of straight boundaries under the diffuse Willmore flow. Section 3.1 introduces the optimal arc profile $q_{\ell,\varepsilon}$ and we give a precise characterization of this class of solutions to the Allen-Cahn equation. While we quote most of the well known results we also prove some additional properties that we need later. In the following Section 3.2 we study the diffuse Willmore energy of quasi one-dimensional and periodic stripe configuration. Under suitable assumptions on ε and the stripe widths we show that there always exists an energy minimizing configuration. We are also able to quantify the energy (which is purely due to the interaction of phase boundaries) by a scaling law.
Chapter 4 yields a first step into the direction of real two-dimensional configurations

and their energy scaling. Corresponding to the one-dimensional optimal arc profiles $q_{\ell,\varepsilon}$ we prove the existence of a new class of entire solutions with multiple saddles to the Allen-Cahn equation.
In the fifth chapter we consider the dynamical problem. Section 5.1 is based on [CoLa11] and we prove the longtime existence of smooth solutions for the diffuse Willmore flow in up to three dimensions on a rectangular domain with periodic boundary conditions. In Section 5.2 we derive consequences of the scaling law from Section 3.2 for the one-dimensional diffuse Willmore flow. We analyze the time evolution of configurations with small energy and prove several results for the flow's velocity and the movement speed of layers.
Chapter 6 deals with the Γ-convergence of $\mathcal{F}_\varepsilon$ in nonsmooth limit points and may be the most relevant part of this thesis. Section 6.1 starts with a brief introduction into the history of De Giorgi's (modified) conjecture and we define a set $\mathcal{S}$ of configurations with nonsmooth interfaces. We extend both the perimeter functional and the Willmore functional to this set using the framework of varifolds. Afterwards, we prove the Γ-convergence of $\mathcal{F}_\varepsilon$ in $\mathcal{S}$ in Theorem 6.3. At the end of this chapter we give several examples of configurations in $\mathcal{S}$. This set is rather implicitly characterized and we briefly comment on its structure.

Notations and conventions

We will fix some basic notations and conventions for the rest of this work.

- Constants will be denoted by C in this work and may change from line to line. The possible dependence of parameters is mentioned in postpositioned parantheses.
- Throughout this thesis, we will often consider families of functions which are indicated by a continuous parameter $\varepsilon > 0$. By abuse of notation, we still use the term "sequence" for these ordered families. In this context a subsequence denotes the part of the family which is indicated by $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \searrow 0$ as $n \to \infty$. We will often skip the index n, though.
- We write dx to indicate integration with respect to the n-dimensional Lebesgue measure whenever the dimension n is clear from the context. Further, the integration variable s is always one dimensional.
- $\mathcal{H}^k$ always denotes the k-dimensional Hausdorff-measure (see e.g. [Si83]).
- For a function $u : \Omega \to \mathbb{R}$ the expressions *diffuse Willmore* and *surface energy* of u will always refer to the integrals
$$\frac{1}{2\varepsilon}\int_\Omega \left(-\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u)\right)^2 dx \quad \text{and} \quad \int_\Omega \frac{\varepsilon}{2}\,|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\,dx.$$
The domain of integration is always implicitly given by the domain of u.
- The function $F : \mathbb{R} \to \mathbb{R}$ will always denote the quartic even *double well potential* (see Figure 1.2) defined by
$$F(s) := \frac{(s^2-1)^2}{4}, \quad s \in \mathbb{R}.$$

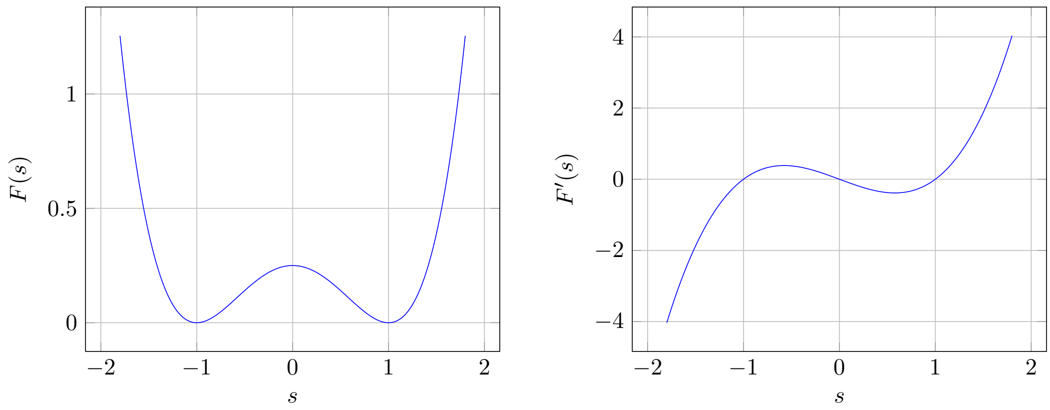


Figure 1.2: The double well potential $F(s) = \frac{1}{4}(s^2 - 1)^2$ and its derivative $F'(s) = s^3 - s$.

We point out that we only rely on certain qualitative properties of F (concerning regularity, growth, and the position of zeros for instance) instead of its actual definition for most of the results although we will sometimes make use of its precise form to shorten the proofs.
We will not comment on possible generalizations in the following (which is mostly not difficult) as this would not change the qualitative behavior that we are interested in.

2 Preliminaries

We pursue two different aims in this chapter. On the one hand we yield a brief introduction into the theories and concepts which will be used in the following. On the other hand we fix the basic notations we are going to use. For most of the proofs we will just refer to the literature.

2.1 Γ-convergence

Instead of finding a minimizer of a given functional it is a common approach in the calculus of variations to approximate it by a sequence of other functionals which for example are easier to describe or to minimize. However, this idea makes it necessary to find a proper definition of variational convergence which ensures that minimizers of the approximating sequence converge to minimizers of the limit functional. This property is not satisfied by the pointwise convergence of functionals and a more careful definition is necessary. This has been given by De Giorgi and Franzoni in [DeFr75] by introducing the notion of Γ-convergence of functionals.
We present the precise definition of Γ-convergence and its main properties in this section and refer to [Br06] or [AtBuMi14] for a detailed introduction into this topic.

Definition 2.1. Let X be a metric space and $(F_k)_{k\in\mathbb{N}}$ a sequence of functionals on X with $F_k : X \to \mathbb{R}\cup\{\infty\}$ for all $k \in \mathbb{N}$. We say that F_n Γ-*converges* to $F : X \to \mathbb{R}\cup\{\infty\}$ in a point $x \in X$ if the following two assertions hold:

i) (Lower bound inequality) For every sequence $(x_k)_{k\in\mathbb{N}}$ which converges to x we have

$$F(x) \leq \liminf_{k\to\infty} F_k(x_k).$$

ii) (Recovery sequence) There exists a sequence $(x_k)_{k\in\mathbb{N}}$ which converges to x with

$$F(x) = \lim_{k\to\infty} F_k(x_k).$$

In this situation we write

$$\Gamma - \lim_{k\to\infty} F_k(x) = F(x).$$

Remark. a) In some situations it is convenient to label the space in which the convergence of the sequence takes place. In this case we write

$$\Gamma(X) - \lim_{k\to\infty} F_k = F.$$

b) The Γ-limit is always a lower semicontinuous functional [Br06]. This especially implies that a constant sequence of functionals with $F_k = F$ for all $k \in \mathbb{N}$ does not Γ-converge towards F in general but to its *lower semi continuous envelope* (or *relaxation*) $\overline{F}$ which is defined as the largest lower semicontinuous functional not greater than F, i.e.

$$\overline{F}(x) := \liminf_{y\to x} F(y).$$

The Γ-convergence yields the desired convergence of minimizers in X.

Proposition 2.2. *Let $(F_k)_{k\in\mathbb{N}}$, $F_k : X \to \mathbb{R}\cup\{\infty\}$ for $k \in \mathbb{N}$, be a sequence of functionals on X which Γ-converges to $F : X \to \mathbb{R}\cup\{\infty\}$ as $k \to \infty$. Moreover, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X satisfying*

$$F_k(x_k) \le \inf_{x\in X} F_k(x) + \varepsilon_k$$

with $\varepsilon_k > 0$ for $k \in \mathbb{N}$ and $\varepsilon_k \to 0$ as $k \to \infty$. Then each cluster point x^* of $(x_k)_{k\in\mathbb{N}}$ is a minimizer of F and the convergence

$$\lim_{k\to\infty} \inf_{x\in X} F_k(x) = F(x^*)$$

holds.

Proof. See e.g. [AtBuMi14], Theorem 12.1.1. □

2.2 Rectifiable sets and varifolds

We introduce the theory of varifolds as a generalization of classical surfaces and give a short overview of well-known results in this field. In particular, we explain how the differential geometric concepts of first variation and mean curvature can be transferred to this context. For a wider introduction into this topic we refer to [Si83] or [Fe14] where also many of the results presented here are formulated more generally.
For the rest of this section we fix integers $k, n \in \mathbb{N}$ with $1 \le k < n$ and denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$ by $\mathcal{H}^k$.

Definition 2.3. The *Grassmannian* $G(n, k)$ is defined as the space of all k-dimensional linear subspaces of $\mathbb{R}^n$.
For a set $U \subset \mathbb{R}^n$ we further define

$$G_k(U) := U \times G(n, k).$$

In the following, we often identify a subspace $P \in G(n, k)$ with the $n \times n$ matrix representing the orthogonal projection of $\mathbb{R}^n$ onto P. This enables us to define a scalar product on $G(n,k)$ by

$$A\colon B := \operatorname{tr}(A^T B) = \sum_{1\leq i,j\leq n} A_{ij}B_{ij}, \quad \text{for } A,B \in G(n,k)$$

which induces a metric on $G(n,k)$. For $k = n-1$ a subspace P is (up to orientation) uniquely determined by its normal vector $\nu = (\nu_1, \ldots, \nu_n)$ and thus, we have

$$G(n, n-1) \cong S^{n-1}/\pm 1.$$

Particularly, the projection of $\mathbb{R}^n$ onto P then is given by the matrix

$$(\operatorname{Id} - \nu \otimes \nu) = (\delta_{ij} - \nu_i \nu_j)_{ij}.$$

Definition 2.4. A set $M \subset \mathbb{R}^n$ is called *countably k-rectifiable* if there exist $N_i \subset \mathbb{R}^n$, $i \in \mathbb{N} \cup \{0\}$ such that

$$M \subset N_0 \cup \bigcup_{i=1}^{\infty} N_i$$

where $\mathcal{H}^k(N_0) = 0$ and where N_i are k-dimensional C^1-submanifolds of $\mathbb{R}^n$ for $i \geq 1$.

Countably k-rectifiable sets can be characterized by the existence of approximate tangent spaces which are defined via the blow-ups $\eta_{x,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ given by $\eta_{x,\lambda}(y) := \lambda^{-1}(y - x)$ for $\lambda > 0$:

Definition 2.5. Let M be a $\mathcal{H}^k$-measurable subset of $\mathbb{R}^n$ and $\theta : M \to (0, \infty)$ a positive and locally $\mathcal{H}^k$-integrable function. A subspace $P \in G(n,k)$ is called the *approximate tangent space for M in $x \in \mathbb{R}^n$ with respect to θ* if

$$\lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda}(M)} \phi\theta(x + \lambda\cdot)\, d\mathcal{H}^k = \theta(x) \int_P \phi \, d\mathcal{H}^k, \quad \text{for all } \phi \in C^0_c(\mathbb{R}^n)$$

and we write $T_xM := P$ as in the case of classical tangent spaces.

Proposition 2.6. *Let $M \subset \mathbb{R}^n$ be an $\mathcal{H}^k$-measurable set. M is countably k-rectifiable if and only if for $\mathcal{H}^k$-almost every $x \in M$ there exists the approximate tangent space T_xM with respect to a positive and locally $\mathcal{H}^k$-integrable function $\theta : M \to (0, \infty)$.*

Proof. [Si83], Theorem 11.6. □

Remark. We can write every $\mathcal{H}^k$-measurable and countably k-rectifiable set M as the disjoint union of $\mathcal{H}^k$-measurable sets $\bigcup_{i=0}^{\infty} M_i$ where $\mathcal{H}^k(M_0) = 0$ and and $M_i \subset N_i$, $i \geq 0$, (with N_i as in Definition 2.4). Then, the approximate tangent spaces correspond to the classical ones almost everywhere as

$$T_xM = T_xN_i \quad \text{for } \mathcal{H}^k\text{-almost every } x \in M_i,\ i \geq 1.$$

We are now able to define general varifolds in $\mathbb{R}^n$. These have originally been introduced by Almgren who thereby generalized the concept of surfaces with methods from geometric measure theory (see [Al65] and [Al66]). Later, Allard was able to prove substantial regularity and compactness properties (see e.g. [Al72]) which led to the high mathematical relevance of this theory for example in variational calculus. In particular, varifolds yield a rather natural way to describe (measure theoretic) limits of smooth surfaces (see Proposition 2.14 and the remark thereafter).

Definition 2.7. Let $U \subset \mathbb{R}^n$. A *general k-varifold* V on U is a Radon measure on $G_k(U)$. The set of all general k-varifolds will be denoted by $\mathbb{V}_k(U)$.
For $V \in \mathbb{V}_k(U)$ the *weight measure* (or *mass*) $\|V\|$ of V is the measure on U given by

$$\|V\|(\phi) = \int_U \phi(x)\, d\|V\|(x) := \int_{G_k(U)} \phi(x)\, dV(x,S) \quad \text{for all } \phi \in C_c^0(U).$$

Example 2.8. Assume that $M \subset U \subset \mathbb{R}^n$ is a countably k-rectifiable set with $k = n-1$. Then M canonically induces a k-varifold denoted by $|M|$ according to

$$|M|(\phi) := \int_M \phi(x, T_x M)\, d\mathcal{H}^k(x) \quad \text{for all } \phi \in C_c^0(G_k(U)).$$

For its weight measure we obtain by definition with $\phi \in C_c^0(U)$

$$\int_U \phi(x)\, d\||M|\|(x) = \int_{G_k(U)} \phi(x)\, d|M|(x) = \int_M \phi(x)\, d\mathcal{H}^k(x)$$

and therefore, $\||M|\| = \mathcal{H}^k \lfloor M$ describes the area measure of M.

The example above shows that k-varifolds in $\mathbb{V}_k(U)$ can be understood as generalized surfaces. In the following, we will often concentrate on those $V \in \mathbb{V}_k(U)$ which are induced by k-rectifiable sets as in the example. This leads to the next definition.

Definition 2.9. A k-varifold $V \in \mathbb{V}_k(U)$ is called *rectifiable* if there exist an $\mathcal{H}^k$-measurable, countably k-rectifiable set $M \subset U$ and a function $\theta : U \to [0, \infty)$ which is locally $\mathcal{H}^k$-integrable on M and vanishes elsewhere such that for all $\phi \in C_c^0\big(G_k(U)\big)$

$$V(\phi) = \int_{G_k(U)} \phi(x,S)\, dV(x,S) = \int_M \phi(x, T_x M)\theta(x)\ d\mathcal{H}^k(x),$$

i.e., V is rectifiable if and only if its weight measure is given by

$$\|V\| = \theta\, \mathcal{H}^k \lfloor M$$

with M and θ as above. If further $\theta(x) \in \mathbb{N}$ for $\mathcal{H}^k$-almost every $x \in M$, we call V an *integral* varifold.
In view of the above given example, a k-varifold is called *unit density varifold* if $V \in \mathbb{V}_k(U)$ is integral with $\theta \equiv 1$ $\mathcal{H}^k$-almost everywhere on M (which especially means $V = |M|$).

Remark. For a rectifiable varifold $V \in \mathbb{V}_k(U)$ as above we always have

$$\theta(\|V\|, x) = \theta(x), \quad \text{for } \mathcal{H}^k\text{-almost every } x \in U.$$

The calculation of the first variation of a rectifiable varifold $V \in \mathbb{V}_k(U)$ is motivated by classical differential geometry for surfaces.
Consider a varifold $V = |M|$ on U for a countably k-rectifiable set $M \subset U$. We want to calculate the first variation of its area $\| |M| \| (U) = \mathcal{H}^k(M)$ with respect to a given vector field $X \in C_c^1(U;\mathbb{R}^n)$. Let $\varepsilon > 0$ and consider a family of diffeomorphisms

$$\Phi_t : U \to U, \quad t \in (-\varepsilon, \varepsilon)$$

with $\Phi_0 = \mathrm{Id}_U$ and such that there exists a compact subset $K \subset\subset U$ with $\Phi_t\big|_{U \setminus K} = \mathrm{Id}_{U\setminus K}$ for all $t \in (-\varepsilon,\varepsilon)$. We further prescribe $\frac{\partial \Phi_t}{\partial t}\big|_{t=0} = X$ and obtain

$$\frac{d}{dt}\Big|_{t=0} \mathcal{H}^k\big(\Phi_t(M) \cap K\big) = \int_M \mathrm{div}_M X \, d\mathcal{H}^k$$

(see for example [Si83], §16). Here, div_M denotes the surface divergence on M which for $\mathcal{H}^k$-almost every $x \in M$ can be written as

$$\mathrm{div}_M X(x) = T_x M : DX(x)$$

where we interpret $T_xM \in G(k,n)$ as the $n \times n$ projection matrix from $\mathbb{R}^n$ onto the approximate tangent space in x. This yields

$$\delta |M| (X) = \frac{d}{dt}\Big|_{t=0} \mathcal{H}^k\big(\Phi_t(M) \cap K\big) = \int_M T_x M : DX(x) \, d\mathcal{H}^k(x)$$

and motivates the definition of first variation of a general varifold $V \in \mathbb{V}_k(U)$:

Definition 2.10. For $V \in \mathbb{V}_k(U)$ and $X \in C_c^1(U;\mathbb{R}^n)$ the *first variation of V in direction of X* is defined by

$$\delta V(X) = \int_{G_k(U)} S : DX(x) \, dV(x,S).$$

Remark. i) With a similar argument as in the motivation above we obtain for every rectifiable varifold V with $\|V\| = \theta \, \mathcal{H}^k \lfloor M$

$$\delta V(X) = \int_U \mathrm{div}_M X \, d\|V\| = \int_M \mathrm{div}_M X \theta \, d\mathcal{H}^k \quad \text{for every } X \in C_c^1(U;\mathbb{R}^n).$$

Again we refer to [Si83], §16, where the exact calculation is performed.

ii) If in the motivation above M is a closed C^2-surface in U, the divergence theorem ([Si83], 7.6) yields for all $X \in C_c^1(U;\mathbb{R}^n)$

$$\begin{aligned} \delta |M| (X) &= \int_M \mathrm{div}_M X \, d\mathcal{H}^k = -\int_M X \cdot \mathbf{H}_M \, d\mathcal{H}^k \\ &= -\int_U X \cdot \mathbf{H}_M \, d \| |M| \| \end{aligned} \tag{2.1}$$

with $\mathbf{H}_M$ denoting the mean curvature vector field of M. This fact will motivate the definition of generalized mean curvature for general varifolds which we will give below.

Definition 2.11. For a general k-varifold $V \in \mathbb{V}_k(U)$ the first variation δV is called *locally bounded* in U if for all open, relatively compact subsets $\tilde{U} \subset\subset U$ there exists a constant $C = C(\tilde{U}) > 0$ such that

$$|\delta V(X)| \leq C \sup_{x \in \tilde{U}} |X(x)| \quad \text{for all } X \in C^1_c(\tilde{U}; \mathbb{R}^n). \tag{2.2}$$

Let $V \in \mathbb{V}_k(U)$ have locally bounded first variation δV. Since $C^1_c(\tilde{U}; \mathbb{R}^n)$ is dense in $C^0_c(\tilde{U}; \mathbb{R}^n)$ we can uniquely extend δV to a continuous linear functional on the entire space $C^0_c(\tilde{U}; \mathbb{R}^n)$ which satisfies inequality (2.2) for all $X \in C^0_c(\tilde{U}; \mathbb{R}^n)$. By the Riesz representation theorem A.5 it can be written as

$$\delta V(X) = \int_U X \cdot \nu \, d\,|\delta V|\,, \quad \text{for } X \in C^0_c(U, \mathbb{R}^n) \tag{2.3}$$

for the variation measure $|\delta V|$ and $\nu : U \to \mathbb{R}^n$ with $|\nu| = 1$ $|\delta V|$-almost everywhere. Now assume that $|\delta V| \ll \|V\|$. In this case the Radon-Nikodym derivative (see Proposition A.11) of $|\delta V|$ with respect to $\|V\|$ exists and we obtain from (2.3)

$$\delta V(X) = \int_U X \cdot \nu \frac{d\,|\delta V|}{d\,\|V\|} \, d\,\|V\|\,.$$

This motivates the definition of mean curvature for general varifolds in accordance with (2.1).

Definition 2.12. Let $V \in \mathbb{V}_k(U)$ have locally bounded first variation δV and assume

$$|\delta V| \ll \|V\|\,.$$

The *generalized mean curvature vector field of* V is defined by

$$\mathbf{H}_V := -\frac{d\,|\delta V|}{d\,\|V\|}\nu$$

with ν and $\frac{d|\delta V|}{d\|V\|}$ as above.

Example 2.13. Let M be a closed C^2-surface in U as above. Then its classical mean curvature vector $\mathbf{H}_M$ actually coincides with the generalized mean curvature vector of $|M|$ from Definition 2.12. As above, the first variation of $|M|$ is given by

$$\delta\,|M|\,(X) = -\int_M X \cdot \mathbf{H}_M \, d\mathcal{H}^k \quad \text{for all } X \in C^1_c(U; \mathbb{R}^n)$$

and is locally bounded. Indeed, for $\tilde{U} \subset\subset U$ and $X \in C^1_c(\tilde{U}; \mathbb{R}^n)$ we immediately obtain

$$\delta\,|M|\,(X) \leq \left(\int_M |\mathbf{H}_M| \; d\mathcal{H}^k \right) \sup_{x \in \tilde{U}} |X(x)|\,.$$

In order to show $|\delta |M|| \ll \||M|\| = \mathcal{H}^k \lfloor M$ let $A \subset U$ with $(\mathcal{H}^k \lfloor M)(A) = 0$. The outer regularity of $\mathcal{H}^k \lfloor M$ then implies

$$
\begin{aligned}
|\delta V|(A) = & \inf_{\substack{\tilde{U} \text{ open,} \\ A \subset \tilde{U}}} \sup \left\{ \int_U g \cdot \mathbf{H}_M \, d\mathcal{H}^k \ : \ g \in C_c^0(\tilde{U}; \mathbb{R}^n), |g| \leq 1 \right\} \\
\leq & C \inf_{\substack{\tilde{U} \text{ open,} \\ A \subset \tilde{U}}} (\mathcal{H}^k \lfloor M)(\tilde{U}) = 0
\end{aligned}
$$

and with the same arguments and notations as above we have

$$
\mathbf{H}_{|M|} = -\frac{d\,|\delta\,|M||}{d\,\||M|\|} \nu = \mathbf{H}_M .
$$

Remark. If in the situation above M is a C^2-surface with boundary $\partial M \neq \emptyset$ the divergence theorem yields

$$
\int_M \operatorname{div}_M X \, d\mathcal{H}^k = - \int_M X \cdot \mathbf{H}_M \, d\mathcal{H}^k + \int_{\partial M} X \cdot \nu \, d\mathcal{H}^{k-1} \quad \text{for all } X \in C_c^1(U; \mathbb{R}^n)
$$

where ν denotes the unit conormal vector field on ∂M which points into M. Since we have $\mathcal{H}^k(A) = 0$ for all sets $A \subset U$ with $\mathcal{H}^{k-1}(A) < \infty$, the measure $\nu \mathcal{H}^{k-1} \lfloor \partial M$ is not absolutely continuous with respect to $\|V\| = \mathcal{H}^k \lfloor M$.
In general, the absolute continuity in Definition 2.12 corresponds to the fact that $\operatorname{supp} \|V\|$ has no boundary.

The set of all integer k-varifolds satisfies the following important compactness property with respect to the weak convergence of measures.

Theorem 2.14 (Allard, '72)**.** *Let $(V_j)_{j \in \mathbb{N}}$ be a sequence of integer k-varifolds on $U \subset \mathbb{R}^n$ with locally bounded first variation which satisfies*

$$
\liminf_{j \to \infty} \left(\|V_j\| (\tilde{U}) + |\delta V_j| (\tilde{U}) \right) < \infty \quad \textit{for all } \tilde{U} \subset\subset U. \tag{2.4}
$$

Then there exists a subsequence $(V_{j_i})_{i \in \mathbb{N}}$ and an integer k-varifold V on U such that

$$
V_{j_i} \overset{*}{\rightharpoonup} V
$$

as $i \to \infty$.

Proof. See [Si83], 42.8, or [Al72]. □

Remark. i) The relevant part in the proof of Theorem 2.14 is to show that V indeed satisfies the integrality property. The mere existence of a limit varifold $V \in \mathbb{V}_k(U)$ is a direct consequence of the compactness result for general Radon measures from Proposition A.8.

ii) The statement of Theorem 2.14 is even non trivial for a sequence of smooth surfaces (or rather for their induced varifolds) which do not converge to classical surface in general:

Let $(M_j)_{j\in\mathbb{N}}$ be a sequence of smooth k-dimensional surfaces and consider the induced unit density varifolds $V_j = |M_j|$, $j \geq 1$. For $\tilde{U} \subset\subset U$ we have

$$\||M_j|\|(\tilde{U}) = \mathcal{H}^k(M_j \cap \tilde{U})$$

and

$$\begin{aligned} |\delta\,|M_j||\,(\tilde{U}) &= \sup\{\delta\,|M_j|\,(g) \,:\, g \in C_c^1(\tilde{U};\mathbb{R}^n),\, |g| \leq 1\} \\ &= \sup\{-\int_{M_j} g \cdot \mathbf{H}_{M_j}\, d\mathcal{H}^k \,:\, g \in C_c^1(\tilde{U};\mathbb{R}^n),\, |g| \leq 1\} \\ &\leq \left\|\mathbf{H}_{M_j}\right\|_{L^1(\mathcal{H}^k \lfloor M_j)} \end{aligned}$$

and thus, condition (2.4) is satisfied if

$$\liminf_{k\to\infty} \left(\mathcal{H}^k(M_j \cap \tilde{U}) + \left\|\mathbf{H}_{M_j}\right\|_{L^1(\mathcal{H}^k\lfloor M_j)}\right) < \infty \quad \text{for all } \tilde{U} \subset\subset U.$$

Theorem 2.14 then yields

$$|M_{j_i}| \stackrel{*}{\rightharpoonup} V = \theta\,|M|$$

as $i \to \infty$ for a countably k-rectifiable set M and a nonnegative integer valued function $\theta : U \to \mathbb{N}_0$. Notice, that V is not a unit density varifold in general.

2.3 Sets of finite perimeter and the reduced boundary

Let Ω denote an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. In this section we introduce the notion of BV functions and generalize the concept of C^1-boundaries of sets in a measure theoretic way.

Definition 2.15. A function $f \in L^1(\Omega)$ has *bounded variation* in Ω if the condition

$$\sup\left\{\int_\Omega f\,\nabla\cdot\varphi\,dx \,:\, \varphi \in C_c^1(\Omega;\mathbb{R}^n), |\varphi| \leq 1\right\} < \infty$$

is satisfied. We denote the set of all functions of bounded variation in Ω by $BV(\Omega)$.

The space $BV(\Omega)$ consists of those L^1-functions on Ω whose distributional derivative is given by a Radon measure.

Theorem 2.16. *For $f \in BV(\Omega)$ there exists a Radon measure μ on Ω and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that*

i) $|\sigma(x)| = 1$ *for μ-almost every $x \in \Omega$ and*

ii) $\int_\Omega f\,\nabla\cdot\varphi\,dx = -\int_\Omega \varphi\cdot\sigma\,d\mu$ *for all $\varphi \in C_c^1(\Omega;\mathbb{R}^n)$.*

μ is the variation measure induced by the distributional derivative of f and we therefore write $\mu =: |\nabla f|$ which is characterized by

$$|\nabla f|\,(U) = \sup\left\{\int_\Omega f\,\nabla\cdot\varphi\,dx \,:\, \varphi \in C_c^1(U;\mathbb{R}^n), |\varphi| \leq 1\right\}$$

for all open sets $U \subset \Omega$.

Proof. See [EvGa92], 5.1, Theorem 1. □

Definition 2.17. An $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ has *finite perimeter* in Ω if

$$\chi_E \in BV(\Omega).$$

Remark. Assume $E \subset \mathbb{R}^n$ is of finite perimeter in Ω. With $f = \chi_E$ the equation in ii) then reads

$$\int_E \nabla\cdot\varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu$$

for all $\varphi \in C_c^1(\Omega;\mathbb{R}^n)$ We write $\nu_E := -\sigma$ as well as $\mu = |\nabla\chi_E| := \|\partial E\|$ in this case and refer to them as the *generalized outer unit vector* field and the *boundary measure* of E. The notation is clearly motivated by the divergence theorem for sets with C^1-boundary for which we have $\|\partial E\| = \mathcal{H}^{n-1}\lfloor\partial E$ and $\nu_E = \nu$ with ν denoting the classical outer unit normal vector field of E.

Definition 2.18. Let $E \subset \mathbb{R}^n$ have bounded perimeter in $\mathbb{R}^n$ with ν_E and $\|\partial E\|$ as above. The *reduced boundary of* E is denoted by $\partial^* E$ and consists of all points $x \in \mathbb{R}^n$ with

i) $\|\partial E\| (B(x,r)) > 0$ for all $r > 0$,

ii) $⨍_{B(x,r)} \nu_E \, d\|\partial E\| \to \nu_E(x)$ as $r \to 0$,

iii) $|\nu_E(x)| = 1$.

Remark. By definition, the reduced boundary $\partial^* E$ of a set E with finite perimeter consists of those points of the topological boundary ∂E in which we can define an outer normal vector to E at least in a weak measure theoretic way.

We conclude this section with a characterization theorem for finite perimeter sets by De Giorgi. It basically states that these sets have a C^1-boundary measure theoretically.

Proposition 2.19 (De Giorgi)**.** *Let $E \subset \mathbb{R}^n$ have finite perimeter in $\mathbb{R}^n$. Then $\partial^* E$ is countably n-rectifiable and we have $\|\partial E\| = \mathcal{H}^{n-1}\lfloor\partial^* E$. In every point $x \in \partial^* E$ the approximate tangent space $T_x\partial^* E$ exists with multiplicity $\theta = 1$ and is given by*

$$T_x\partial^* E = \{y \in \mathbb{R}^n \ : \ y \cdot \nu_E(x) = 0\}$$

with ν_E from Definition 2.18.

Proof. See [Si83], Theorem 14.3. □

2.4 The optimal profile

The nonlinear reaction-diffusion equation (*Allen-Cahn* or *Ginzburg-Landau* equation)

$$\partial_t u = \varepsilon \Delta u - \frac{1}{\varepsilon} F'(u) \tag{2.5}$$

on a domain $\Omega \subset \mathbb{R}^n$ appears as the L^2-gradient flow of the diffuse surface energy functional

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} F(u)\, dx$$

and was originally chosen in [Ca60] to simulate phase boundary motion driven by surface tension in crystalline materials (see also [AlCa73]). Since then it has been used extensively in numerous other applications as population genetics or nerve pulse propagation [ArWe75] to describe phase transition phenomena between two modeled phases. Basically, stationary states $u \not\equiv 0$ in this context describe optimal configurations of $\mathcal{E}_\varepsilon$ while the (instable) stationary state $u \equiv 0$ corresponds to a perfect mixing of both modeled phases.
However, from our point of view, solutions u of

$$\varepsilon \Delta u - \frac{1}{\varepsilon} F'(u) = 0 \tag{2.6}$$

on appropriate n-dimensional spacial domains Ω appear in a different light as these functions naturally have vanishing diffuse Willmore energy $\mathcal{W}_\varepsilon(u)$. Even for $n = 1$ the problem stays interesting as in this case solutions of (2.6) can be seen as cross sections or optimal transition profiles of two-dimensional phase fields with energetically preferable shapes. A precise understanding of stationary solutions of the Allen-Cahn equation will be crucial throughout the whole thesis. In this section we introduce a solution which often appears in phase transition theory.

Before we define the specific problem we briefly remark that equation (2.6) naturally scales in terms of ε. Indeed, for a solution u of (2.6) on $\Omega \subset \mathbb{R}^n$ the function

$$\tilde{u} := u(\varepsilon \cdot)$$

obviously satisfies

$$\Delta \tilde{u} - F'(\tilde{u}) = 0 \quad \text{in } \varepsilon^{-1}\Omega$$

and hence, the solution theory for (2.6) is completely covered by the case $\varepsilon = 1$.

The problem

$$\begin{cases} \gamma'' - F'(\gamma) = 0 \\ \lim_{x \to \pm\infty} \gamma(x) = \pm 1, \quad \gamma(0) = 0. \end{cases} \tag{2.7}$$

is usually called the *optimal profile problem* and its solution describes the (in terms of $\mathcal{E}_\varepsilon$) energetically optimal phase transition connecting the states -1 and 1 without any disturbances caused by the domain size or prescribed boundary values. The existence of a solution of (2.7) can be shown easily by multiplying the equation with γ'. This first yields

$$\frac{1}{2}(\gamma')^2 - F(\gamma) = C$$

for a constant $C \in \mathbb{R}$ and we directly obtain $C = 0$ due to the prescribed behavior of γ at $\pm\infty$. Consequently, we can transform (2.7) to the first order ODE

$$\begin{cases} \gamma' = \sqrt{2F(\gamma)} \\ \gamma(0) = 0. \end{cases} \tag{2.8}$$

which yields a unique solution γ satisfying $-1 < \gamma < 1$ as $F \in C^1(\mathbb{R})$. Moreover, due to our special choice $F(s) = \frac{1}{4}(s^2 - 1)^2$ the solution is given by

$$\gamma(x) = \tanh\left(\frac{x}{\sqrt{2}}\right), \quad x \in \mathbb{R}.$$

We remark that the condition $\gamma(0) = 0$ ensures the uniqueness as otherwise every other function of the form $\gamma(\cdot - c)$, $c \in \mathbb{R}$ solves the equation with the same limit values in $\pm\infty$.
We will refer to γ and its ε scaled version $\gamma_\varepsilon = \gamma\left(\frac{\cdot}{\varepsilon}\right)$ as the *optimal profile* in the following.

2.5 4-ended solutions of the Allen-Cahn equation in two dimensions

We will give a short (and rather incomplete) introduction into the theory of 4-ended solutions of the Allen-Cahn equation in two dimensions

$$-\varepsilon^2 \Delta u_\varepsilon + F'(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^2. \tag{2.9}$$

Notice, that by rescaling we could assume $\varepsilon = 1$ without loss of generality. However, all results will directly be formulated for the ε-dependent equation which will turn out to be more convenient in Section 6.1 below.

In recent years there has been put much effort in the complete characterization of entire solutions to (2.9) in $n \in \mathbb{N}$ dimensions and their properties. This specific interest was initiated by a famous conjecture of De Giorgi in 1978 (see [DeG79]). Inspired by differential geometric considerations he assumed the following ([PiKoWe12]):

Let u be any bounded entire solution of (4.1) in $\mathbb{R}^n$ for $n \leq 8$ which is monotone in one direction $\xi \in \mathbb{R}^n$, i.e. $\partial_\xi u > 0$. Then all level sets of u are hyperplanes. Equivalently, u

has to be of the form

$$u(x) = \gamma\big((x-p)\cdot\xi\big)$$

for some point $p \in \mathbb{R}^n$ where γ denotes the optimal profile (see Section 3.1).

The conjecture has been proved for dimensions $n = 2, 3$ [AmCa00, GhGu98] and in the remaining dimensions under slightly stronger hypotheses [Sa09].
For the case $n = 2$ Dang, Fife and Peletier proved the existence of a nontrivial saddle shaped solution of (4.1) with a zero set consisting of 2 perpendicular lines [DaFiPe92]. The class of more general 4- or ($2k$-) ended solutions has been introduced in [PiKoPa10] and lot of effort has been spent in their description since then [KoLiPa12, PiKoPa13, KoLiPa14]. For a general overview of this theory we refer to the mentioned sources and the citations therein.

A 4-ended solution $u_\varepsilon \in C^2(\mathbb{R}^2)$ of (2.9) can roughly be described as an entire solution whose zero set consists, away from a compact set, of four curves which are asymptotic to four oriented half-lines at infinity. We call these half-lines the *ends* of the solution. Moreover, following these lines towards infinity, u_ε becomes approximately shaped like the one-dimensional optimal profile γ_ε (or $-\gamma_\varepsilon$, respectively) from Section 2.4.

To precise the motivation above and give a rigorous definition we first construct *approximate solutions* of equation (2.9) according to [KoLiPa12] or [PiKoPa13]. Note, that we directly restrict ourselves to symmetric (approximate) solutions (referred to as $\mathcal{M}_4^{\text{even}}$ in [KoLiPa12]) which makes the definition below slightly more restrictive in comparison. The existence result for solutions of (2.9), however, is not influenced by the restraint as those solutions only exist in the symmetric case (see [KoLiPa12] and [Gui12]).

Let $\varepsilon > 0$, $r \geq 0$, and $v \in S^1$. By reflecting v on both coordinate axes we obtain the four vectors

$$v_1 := v, \quad v_2 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} v, \quad v_3 := -v_1, \quad v_4 := -v_2.$$

Obviously, for all sufficiently large $R > 0$ there exists $s = s(R) > 0$ such that

$$\left| s v_i + (-1)^{i+1} r v_i^\perp \right| = R \quad \text{for } i = 1, \ldots, 4$$

and thus, $\varepsilon\big(s v_i + (-1)^{i+1} r v_i^\perp\big) \in \partial B(0, R\varepsilon)$. As always, $v_i^\perp$ denotes the rotation of v_i by $\frac{\pi}{2}$. We can define four disjoint oriented half-lines $G_1, \ldots, G_4$ included in $\mathbb{R}^2 \setminus B(0, R\varepsilon)$ by

$$G_i := \{t v_i + (-1)^{i+1} r\varepsilon v_i^\perp \,:\, t \geq s\varepsilon\} \quad \text{for } i = 1, \ldots, 4 \tag{2.10}$$

and for large radii R the distance between to distinct half-lines is greater than 4ε in accordance to [KoLiPa12], Chapter 2. Indeed, the minimal distance between G_i and

G_j, $1 \leq i, j \leq 4$ is determined by their points on $\partial B(0, R\varepsilon)$ and thus,

$$\begin{aligned}\operatorname{dist}(G_i, G_j) =& \varepsilon \left| s(v_i - v_j) - r\big((-1)^{j+1} v_j^{\perp} - (-1)^{i+1} v_i^{\perp}\big) \right| \\ \geq& \varepsilon \left| s\,|v_i - v_j| - r \left| (-1)^{j+1} v_j^{\perp} - (-1)^{i+1} v_i^{\perp} \right| \right| \\ \geq& 4\varepsilon\end{aligned}$$

yields the claim as s is strictly increasing in R and $|v_i - v_j| \geq C$ for all $1 \leq i, j \leq 4$. This implies that $\Omega_0 := B(0, \big(R+1\big)\varepsilon)$ and

$$\Omega_i := \big(\mathbb{R}^2 \setminus B\big(0, (R-1)\varepsilon\big)\big) \cap \{x \in \mathbb{R}^2 : \operatorname{dist}(x, G_i) < \operatorname{dist}(x, G_j) + 2\varepsilon,\ i \neq j\}$$

build an open covering of $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2 = \bigcup_{j=0}^{4} \Omega_j$$

such that G_i is contained in Ω_i for $i = 1, \ldots 4$ (see Figure 2.1).

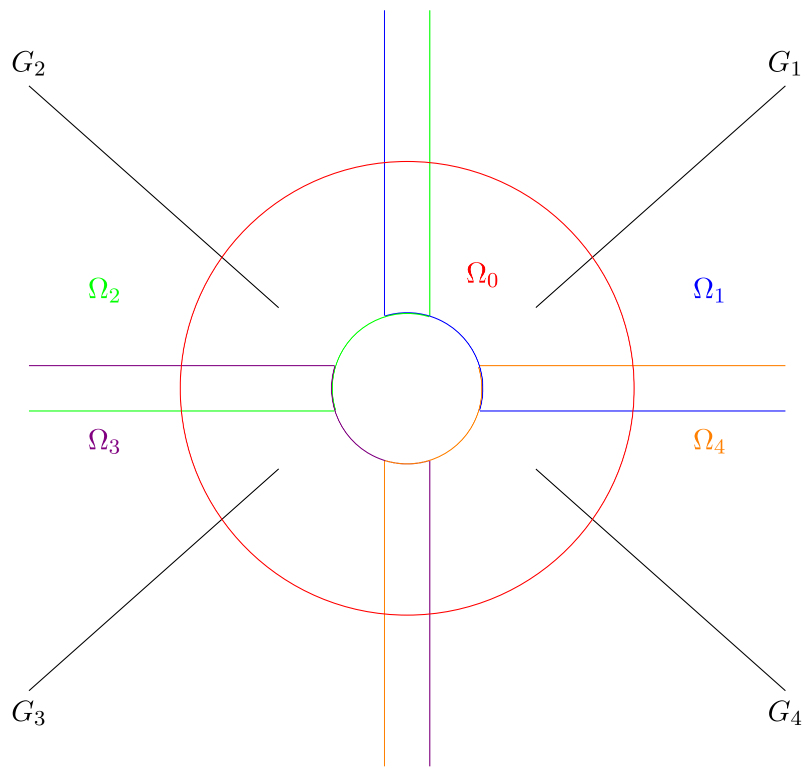


Figure 2.1: A schematic sketch of the open covering of $\mathbb{R}^2$ given by $\Omega_0, \ldots, \Omega_4$ with inner radius $(R-1)\varepsilon$ and outer radius $(R+1)\varepsilon$.

We can further choose a subordinate smooth partition of unity $\rho_0, \ldots, \rho_4$ of $\mathbb{R}^2$ and assume without loss of generality that

$$\rho_0 \equiv 1 \quad \text{in } B\big(0,(R-1)\varepsilon\big)$$

as well as

$$\rho_i \equiv 1 \quad \text{in } \big(\mathbb{R}^2 \setminus B\big(0,(R+1)\varepsilon\big)\big) \cap \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x,G_i) < \operatorname{dist}(x,G_j) - 2\varepsilon,\ i \neq j\}$$

for $j = 1,\ldots,4$.
With these notations we define

$$u_\varepsilon^G(x) := u_\varepsilon^{(G_1,G_2,G_3,G_4)}(x) := \sum_{j=1}^{4} (-1)^j \rho_j \gamma_\varepsilon\big(x \cdot v_j^\perp + (-1)^j r\big), \quad x \in \mathbb{R}^2$$

u_ε^G is an approximate solution of (6.19) in the sense that $-\varepsilon\Delta u_\varepsilon^G(x) + \frac{1}{\varepsilon}F'(u_\varepsilon^G(x))$ decays exponentially fast as $\frac{|x|}{\varepsilon} \to \infty$ (see [PiKoPa13], the remark after Definition 2.1 on p.726).

With the help of approximate solutions, we can now give a precise definition of a 4-ended solution of (2.9):

Definition 2.20. Let $\varepsilon > 0$. A solution $u_\varepsilon \in C^2(\mathbb{R}^2)$ of (2.9) is called a 4-*ended solution* if there exist $r \in \mathbb{R}$, $v \in S^1$, and a corresponding approximate solution u_ε^G as above such that (after a possible rotation and translation of u_ε^G)

$$u_\varepsilon - u_\varepsilon^G \in H^2(\mathbb{R}^2)$$

holds.

The following existence result from [KoLiPa12] will be crucial for the construction in Section 6.1, Theorem 6.3.

Proposition 2.21. *For every $\varepsilon > 0$ and $v \in S^1$ there exists $r \in \mathbb{R}$ and a* 4*-ended solution $u_\varepsilon \in C^2(\mathbb{R}^2)$ with $|u_\varepsilon| < 1$ and*

$$u_\varepsilon - u_\varepsilon^G \in H^2(\mathbb{R}^2) \tag{2.11}$$

where u_ε^G denotes an approximate solution of (2.9) *corresponding to v and r. Moreover for $\varepsilon \leq 1$, there exists a constant $\alpha > 0$ (independent of ε) such that*

$$\Big\| \varepsilon e^{\alpha \frac{|\cdot|}{2\varepsilon}} \left(\left|u_\varepsilon - u_\varepsilon^G\right| + \left|\nabla(u_\varepsilon - u_\varepsilon^G)\right| + \left|D^2(u_\varepsilon - u_\varepsilon^G)\right| \right) \Big\|_{L^2(\mathbb{R}^2)} \leq C \tag{2.12}$$

for a constant $C > 0$ uniform in ε and thus,

$$u_\varepsilon - u_\varepsilon^G \in \frac{1}{\varepsilon^2} e^{-\alpha \frac{|x|^2}{\varepsilon^2}} H^2(\mathbb{R}^2)$$

where $\frac{1}{\varepsilon^2} e^{-\alpha \frac{|x|^2}{\varepsilon^2}} H^2(\mathbb{R}^2)$ denotes the respective weighted H^2-space.

Proof. The existence part has been shown in [KoLiPa12], Theorem 2.7 for $\varepsilon = 1$. Then, by the usual rescaling we set

$$u_\varepsilon(x) := u_1\left(\frac{x}{\varepsilon}\right) \quad \text{for all } x \in \mathbb{R}^2$$

which yields (2.11) and consequently, u_ε is a 4-ended solution of (2.9).
From $0 \in \operatorname{Im}(u_\varepsilon)$ we directly obtain $|u_\varepsilon| < 1$, as every solution of (2.9) which attains the values $+1$ or -1 is known to be constant, e.g., by the estimate from [Mo85], Theorem 1. Further, [PiKoPa13], Theorem 2.1 yields the result about the *refined asymptotics* for $\varepsilon = 1$ and (2.12) follows again with the scaling for the ε-dependence. Indeed, with $v_\varepsilon := u_\varepsilon - u_\varepsilon^G$ and $\varepsilon \leq 1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(v_\varepsilon(x)^2 + |\nabla v_\varepsilon(x)|^2 + \left|D^2 v_\varepsilon(x)\right|^2\right) \varepsilon^2 e^{\alpha\frac{|x|^2}{\varepsilon^2}}\, dx \\
\leq &\int_{\mathbb{R}^2} \left(v_\varepsilon(x)^2 + \varepsilon^2 |\nabla v_\varepsilon(x)|^2 + \varepsilon^4 \left|D^2 v_\varepsilon(x)\right|^2\right) e^{\alpha\frac{|x|^2}{\varepsilon^2}} \frac{1}{\varepsilon^2}\, dx \\
= &\int_{\mathbb{R}^2} \left(v_1(x)^2 + |\nabla v_1(x)|^2 + \left|D^2 v_1(x)\right|^2\right) e^{C|x|^2}\, dx \leq C
\end{aligned}$$

with a change of variables in the last step. □

The proposition above particularly provides a statement about the distance between the zero set of u_ε and its ends. We will need this specific property later on.

Corollary 2.22. *In the situation of Proposition 2.21 there exists $R_0 > 0$ such that for all $\varepsilon \leq 1$ and $R > 0$ with $R \geq R_0\varepsilon$*

$$\operatorname{dist}\left(x, \{u_\varepsilon^G = 0\} \cap \{|y| \geq R\}\right) \leq \frac{C}{\varepsilon} e^{-\alpha\frac{R^2}{\varepsilon^2}} \tag{2.13}$$

is satisfied for all $x \in \{u_\varepsilon = 0\}$ with $|x| \geq R$ with $\alpha > 0$ from Proposition 2.21.

Proof. Let $\varepsilon \leq 1$. Due to (2.11) and the structure of approximate solutions of (2.9) we can choose $R_0 > 0$ such that outside of $B(0, R_0\varepsilon)$ the zero set of u_ε consists of four curves which are asymptotic to four distinct oriented half-lines at infinity (see [PiKoPa10] and also [PiKoPa13]). By the general Sobolev inequality (see, e.g., [AdFo03]) $H^2(\mathbb{R}^2)$ embeds continuously into $C^0(\mathbb{R}^2)$ and therefore, (2.12) implies

$$\left|u_\varepsilon(x) - u_\varepsilon^G(x)\right| \leq \frac{C}{\varepsilon^2} e^{-\alpha\frac{R^2}{\varepsilon^2}} \tag{2.14}$$

for all $x \in \mathbb{R}^2$ with $|x| \geq R$. To show (2.13), we can restrict ourselves to one of the four ends of u_ε and we assume it to be the positive part of the x_1-axis without loss of generality. Now let $R \geq R_0\varepsilon$ and $x = (x_1, x_2) \in \{u_\varepsilon = 0\}$ with $|x| \geq R$. Due to the assumption above, it suffices to consider the case $x_1 > 0$ and $|x_2| \leq 1$. From (2.14) and $\gamma_\varepsilon(x_2) = \tanh\left(\frac{x_2}{\sqrt{2}\varepsilon}\right)$ we obtain

$$\tanh\left(\frac{|x_2|}{\sqrt{2}\varepsilon}\right) = \left|\tanh\left(\frac{x_2}{\sqrt{2}\varepsilon}\right)\right| = \left|u_\varepsilon^G(x)\right| = \left|u_\varepsilon(x) - u_\varepsilon^G(x)\right| \leq \frac{C}{\varepsilon^2} e^{-\alpha\frac{R^2}{\varepsilon^2}},$$

which gives

$$|x_2| \leq \sqrt{2}\varepsilon \operatorname{artanh}\left(\frac{C}{\varepsilon^2} e^{-\alpha\frac{R^2}{\varepsilon^2}}\right) \leq \frac{C}{\varepsilon} e^{-\alpha\frac{R^2}{\varepsilon^2}},$$

where we have used that $\operatorname{artanh}(s) = s + O(s^3)$ as $s \to 0$ in the second inequality. Hence,

$$\operatorname{dist}\big(x, \{u_\varepsilon^G = 0\}\big) = \operatorname{dist}\Big(x, \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_1 > 0,\, y_2 = 0\}\Big) = |x_2| \leq \frac{C}{\varepsilon} e^{-\alpha\frac{R^2}{\varepsilon^2}}$$

yields (2.13). □

2.6 A classical approximation result for the Willmore and surface energy

In this part we want to comment briefly on the diffuse approximation of the Willmore and surface energy of a given set which has been discovered by Belletini and Paolini in [BePa93]. The relevant statement is formulated in the next theorem.

Theorem 2.23 (Bellettini, Paolini, '93)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open set and consider a bounded subset $E \subset \Omega$ with C^2 boundary. Then, for $u = 2\chi_E - 1$ there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $H^2(\Omega)$ which satisfies*

$$u_\varepsilon \longrightarrow u \quad \text{in } L^1(\Omega)$$

and

$$\begin{aligned}\mathcal{E}_\varepsilon(u_\varepsilon) + \mathcal{W}_\varepsilon(u_\varepsilon) &= \int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}F(u_\varepsilon)\,dx + \frac{1}{2\varepsilon}\int_\Omega \Big(\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)\Big)^2\,dx \\ &\longrightarrow \sigma\left(\mathcal{H}^{n-1}(\partial E \cap \Omega) + \mathcal{W}(u)\right).\end{aligned} \tag{2.15}$$

where $F(s) = \frac{1}{4}(s^2-1)^2$ denotes the standard symmetric double well potential as always and with $\sigma := \int_{-1}^1 \sqrt{2F}\,ds$.

Remark. In Section 6.1 we will make extensive use of the result above. For convenience, we will always refer to the sequence $(u_\varepsilon)_{\varepsilon>0}$ in the theorem as the *standard approximation* of Willmore and surface energy by Bellettini and Paolini.

Instead of presenting all technical details of the proof (which can be found in [BePa93]), we rather describe the idea for the construction and give a heuristic argument for its convergence. For simplicity, we also restrict ourselves to the two-dimensional case, although the argument stays valid for arbitrary $n \geq 2$.
For small $\varepsilon > 0$ and $x \in \Omega$ we define the signed distance function of E by

$$d(x) := \operatorname{dist}(x, \Omega \setminus E) - \operatorname{dist}(x, E)$$

and set close to ∂E

$$u_\varepsilon(x) := \gamma_\varepsilon\big(d(x)\big).$$

We have

$$|u_\varepsilon| \approx 1 \tag{2.16}$$

outside a layer of size ε around ∂E and thereby, u_ε is an diffuse approximation of the indicator function u. Inside this narrow layer, the transition from one phase of u to another is shaped like the one-dimensional optimal profile γ_ε which will turn out to be energetically optimal. We remark that the gradient of d satisfies $|\nabla d| = 1$ (see [OsFe03]) and hence,

$$\nabla u_\varepsilon = \gamma_\varepsilon'(d)\nabla d \tag{2.17}$$

as well as

$$\Delta u_\varepsilon = \gamma_\varepsilon''(d)\,|\nabla d|^2 + \gamma_\varepsilon'(d)\Delta d = \gamma_\varepsilon''(d) + \gamma_\varepsilon'(d)\Delta d. \tag{2.18}$$

The convergence of the second summand $\mathcal{W}_\varepsilon(u_\varepsilon)$ in (2.15) can now be justified as follows. Due to (2.16) and (2.18) we obtain

$$\begin{aligned}\mathcal{W}_\varepsilon(u_\varepsilon) &\approx \frac{1}{2\varepsilon}\int_{\{|d|<\sqrt{\varepsilon}\}}\left(\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)\right)^2\,dx\\ &= \frac{1}{2\varepsilon}\int_{\{|d|<\sqrt{\varepsilon}\}}\varepsilon^2\left(\gamma_\varepsilon'(d)\right)^2(\Delta d)^2\,dx,\end{aligned} \tag{2.19}$$

where we have used that γ_ε is the rescaled solution of (2.7). Now, we introduce new tubular coordinates around ∂E and perform a change of variables in the integral: For small $\varepsilon > 0$ every $x \in \{|d| \le \sqrt{\varepsilon}\}$ has a unique representation

$$x = g(y,r) = y + r\nu_{\partial E}(y)$$

with $y \in \partial E$, $\nu_{\partial E}(y)$ denoting the outer unit normal vector of ∂E in y, and $|r| < \sqrt{\varepsilon}$. From [GiTr01], Lemma 14.16 we obtain

$$\det Dg = 1 - r\mathbf{H}(y)\cdot\nu_{\partial E}(y),$$

where $\mathbf{H}(y)$ denotes the curvature vector of ∂E in y. Hence, (2.19) reads

$$\begin{aligned}&\mathcal{W}_\varepsilon(u_\varepsilon)\\ \approx&\frac{\varepsilon}{2}\int_{\partial E}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}}\left(\gamma_\varepsilon'(r)\right)^2(\Delta d)^2\big(y + r\nu_{\partial E}(y)\big)\big(1 - r\mathbf{H}(y)\cdot\nu_{\partial E}(y)\big)\,dr\,d\mathcal{H}^1(y)\\ =&\frac{1}{2}\int_{\partial E}\int_{-\frac{1}{\sqrt{\varepsilon}}}^{\frac{1}{\sqrt{\varepsilon}}}\left(\gamma'(s)\right)^2(\Delta d)^2\big(y + \varepsilon s\nu_{\partial E}(y)\big)\big(1 - \varepsilon s\mathbf{H}(y)\cdot\nu_{\partial E}(y)\big)\,ds\,d\mathcal{H}^1(y)\\ \longrightarrow&\left(\int_{-\infty}^{\infty}\left(\gamma'(s)\right)^2\,ds\right)\int_{\partial E\cap\Omega}\frac{1}{2}(\Delta d)^2(y)\,d\mathcal{H}^1(y)\\ =&\sigma\mathcal{W}(\partial E) = \sigma\mathcal{W}(u)\end{aligned}$$

as $\varepsilon \to 0$ since $\Delta d = -\mathbf{H} \cdot \nu_{\partial E}$ on ∂E and

$$\int_{-\infty}^{\infty} (\gamma')^2 \, ds = \int_{-\infty}^{\infty} \sqrt{2F(\gamma)}\gamma' \, ds = \int_{-1}^{1} \sqrt{2F(\gamma)} \, d\gamma = \sigma.$$

The convergence of the first summand $\mathcal{E}_\varepsilon(u_\varepsilon)$ has already been proven in [MoMo77] (or [Mo87]). It can be motivated by (2.17) and the coarea formula since

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &\approx \int_{\{|d|<\sqrt{\varepsilon}\}} \left(\frac{\varepsilon}{2} \left|\gamma_\varepsilon'(d)\right|^2 + \frac{1}{\varepsilon} F\big(\gamma_\varepsilon(d)\big) \right) |\nabla d| \; dx \\ &= \mathcal{H}^1(\partial E \cap \Omega) \left(\int_{-\frac{1}{\sqrt{\varepsilon}}}^{\frac{1}{\sqrt{\varepsilon}}} \frac{1}{2} \left|\gamma'\right|^2 + F(\gamma) \, dr \right) \\ &\longrightarrow \sigma \mathcal{H}^1(\partial E \cap \Omega) \end{aligned}$$

as $\varepsilon \to 0$ since $\frac{1}{2}(\gamma')^2 + F(\gamma) = \sqrt{F(\gamma)}\gamma'$.

Remark. Instead of $\sqrt{\varepsilon}$ we could have chosen any other function $h(\varepsilon)$ with $\varepsilon^{-1}h(\varepsilon) \to 0$ as $\varepsilon \to 0$. This is an immediate consequence of the rigorous proof in [BePa93]. Indeed, the authors choose u_ε to be constant ± 1 outside a neighborhood of ∂E twice as large as the considered layer above and connect the constant states with the optimal profile on the narrow gap such that the resulting function is smooth (see Section 6.1 for the precise construction). The energy contribution of the appearing error terms vanishes if we ensure that the region where u_ε is not constant shrinks faster than ε. This will turn out to be important in Section 6.1.

3 Energy scalings of stripe configurations

3.1 The optimal arc profile

In this section we will introduce another relevant class of solutions to the one-dimensional stationary Allen-Cahn equation (2.6). While the phase transition performed by γ (Section 2.4) takes place on the whole real line it will also be crucial to understand and characterize the shape of solutions with prescribed boundary data on a bounded interval $[0, \ell]$ for an $\ell > 0$.

We consider the Dirichlet problem

$$\begin{cases} -\varepsilon q''_{\ell,\varepsilon} + \dfrac{1}{\varepsilon} F'(q_{\ell,\varepsilon}) = 0 \text{ in } (0,\ell) & (3.1a) \\ 0 < q_{\ell,\varepsilon} < 1 \qquad \text{in } (0,\ell) & (3.1b) \\ q_{\ell,\varepsilon} = 0 \qquad \text{in } \{0,\ell\}. & (3.1c) \end{cases}$$

and show existence of such solutions together with some elementary properties in the following proposition. The results are well known and can be found more or less explicitly for instance in [CaPe89, OtRe07, BeNaNo15], although the proofs are often skipped. For the reader's convenience we include a complete proof of the presented statements.

Proposition 3.1. *There exists a constant $\ell_1 > 0$ such that for all ratios $\frac{\ell}{\varepsilon} > \ell_1$ (3.1) has a unique solution $q_{\ell,\varepsilon} \in C^\infty((0,\ell))$ which only depends of ε and ℓ by the ratio $\frac{\ell}{\varepsilon}$. Moreover, $q_{\ell,\varepsilon}$ is symmetric with respect to the interval midpoint $\frac{\ell}{2}$ and satisfies*

$$q'_{\ell,\varepsilon} = \begin{cases} \dfrac{1}{\varepsilon}\sqrt{2\big(F(q_{\ell,\varepsilon}) - F(\bar{q}_{\ell,\varepsilon})\big)} & \text{in } \left[0, \dfrac{\ell}{2}\right] \\ -\dfrac{1}{\varepsilon}\sqrt{2\big(F(q_{\ell,\varepsilon}) - F(\bar{q}_{\ell,\varepsilon})\big)} & \text{in } \left(\dfrac{\ell}{2}, \ell\right] \end{cases} \tag{3.2}$$

where $\bar{q}_{\ell,\varepsilon} = q_{\ell,\varepsilon}\left(\frac{\ell}{2}\right)$ denotes the maximum of $q_{\ell,\varepsilon}$ in $(0,\ell)$. We will refer to $q_{\ell,\varepsilon}$ as the optimal arc profile *corresponding to ℓ and ε in the following.*

Proof. By the usual scaling $q_{\ell,\varepsilon}(x) = q_{\frac{\ell}{\varepsilon},1}\left(\frac{x}{\varepsilon}\right)$ we immediately see that it suffices to consider the case $\varepsilon = 1$ in (3.1a) which already shows the desired dependence of $q_{\ell,\varepsilon}$ on this ratio. For the sake of notation we skip the indices of $q_{\ell,1}$ in the following.

The existence of a solution of (3.1) follows easily by the direct method of the calculus of variations. We choose a minimizing sequence $(q_k)_{k\in\mathbb{N}}$ of $\mathcal{E}_1$ in $H_0^1\big((0,\ell)\big)$ and without loss of generality we can assume that for all $k \in \mathbb{N}$ we have $0 \le q_k \le 1$. Otherwise, we can choose $\tilde{q}_k := \min\{|u_k|, 1\}$ which also minimizes $\mathcal{E}_1$ since F is even and $F(1) = 0$. Therefore, we obtain a minimizer $0 \le q \le 1$ which solves (3.1a) and standard regularity theory for ODEs directly yields $q \in C^\infty((0,\ell))$.
To see that u satisfies (3.1b) for sufficiently large ℓ we first remark that the function $v \in H_0^1\big((0,\ell)\big)$

$$v(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x < \ell - 1 \\ -x + \ell & \text{if } \ell - 1 \le x \le \ell \end{cases}$$

satisfies

$$\mathcal{E}_1(q) = \min_{u \in H_0^1((0,\ell))} \mathcal{E}_1(u) \le \mathcal{E}_1(v) = \frac{34}{15} < \frac{\ell}{4} = \mathcal{E}_1(0)$$

for $\ell > 0$ sufficiently large and hence, we conclude $q \not\equiv 0$. Now we can rewrite (3.1a) as

$$-q'' + q \fint_0^q F''(r)\, dr = 0$$

due to $F'(0) = 0$, and the strong maximum principle ([GiTr01], Theorem 3.5 and the remark thereafter) yields $q > 0$ in $(0, \ell)$. With the same argument we also obtain $q < 1$ which proves (3.1b).
Next, we deduce from (3.1a) that q is a strict concave function and hence, attains its maximum in exactly one point $\bar{x} \in (0, \ell)$, i.e., $q(\bar{x}) = \bar{q}$. We multiply (3.1a) by q' for

$$\big((q')^2\big)' = \big(F(q)\big)'$$

and integrating over $(x, \bar{x})$ with $x < \bar{x}$ (over $(\bar{x}, x)$ with $x > \bar{x}$, respectively) yields

$$q' = \begin{cases} \sqrt{2\big(F(q) - F(\bar{q})\big)} & \text{in } [0, \bar{x}) \\ -\sqrt{2\big(F(q) - F(\bar{q})\big)} & \text{in } (\bar{x}, \ell] \end{cases}$$

which implies $\bar{x} = \frac{\ell}{2}$ and therefore the symmetry of q and (3.2). Finally, from this first order ODE and the prescribed boundary values we obtain the uniqueness of q in $\big(0, \frac{\ell}{2}\big)$ and $\big(\frac{\ell}{2}, \ell\big)$ as the right hand side of (3.2) is locally Lipschitz continuous away from $\frac{\ell}{2}$. This yields the uniqueness of q by continuation and concludes the proof. □

An exact description of the size of $\bar{q}_{\ell,\varepsilon} < 1$ (which only depends on the ratio $\frac{\ell}{\varepsilon}$) will be necessary in Section 3.2.

Lemma 3.2. *As $\frac{\ell}{\varepsilon} \to \infty$ we have $\bar{q}_{\ell,\varepsilon} \to 1$ and there exist $\ell_2 \ge \ell_1 > 0$ such that for* *ratios* $\frac{\ell}{\varepsilon} \geq \ell_2$

$$1 - \bar{q}_{\ell,\varepsilon} = K e^{-\alpha \frac{\ell}{2\varepsilon}} \left[1 + O\left(\frac{\ell}{\varepsilon} e^{-\alpha \frac{\ell}{2\varepsilon}} \right) \right]$$

is satisfied with explicitly given constants $\alpha := \sqrt{F''(1)} = \sqrt{2}$ *and*

$$K := 2 \exp\left(\int_0^1 \frac{\alpha}{\sqrt{2F(t)}} - \frac{1}{1-t} \, dt \right) = 4.$$

Particularly, we have

$$F(\bar{q}_{\ell,\varepsilon}) = \frac{1}{2} K^2 \alpha^2 e^{-\alpha \frac{\ell}{\varepsilon}} \left[1 + O\left(\frac{\ell}{\varepsilon} e^{-\alpha \frac{\ell}{2\varepsilon}} \right) \right]$$

in this case.

Proof. The proof can be found in [CaPe89], Proposition 3.4. □

The profiles $q_{\ell,\varepsilon}$ and γ_ε both solve the same differential equation with different boundary conditions. For large interval lengths ℓ the influence of the right boundary condition $q_{\ell,\varepsilon}(\ell) = 0$ on the shape of the monotone increasing half of $q_{\ell,\varepsilon}$ declines and we can show that $q_{\ell,\varepsilon}$ approximates γ_ε in this case.

Lemma 3.3. *On every bounded interval* $[0, R]$*,* $R > 0$*, the optimal arcs* $q_{\ell,1}$ *converge uniformly towards* γ*.*

Proof. Fix an interval length R and consider the sequence $(q_{\ell,1})_{\ell > R}$ which satisfies $0 \leq q_{\ell,1} < 1$ for all $\ell > R$ due to (3.1b). By (3.1a), (3.2) and the fact that $F(\bar{q}_{\ell,1}) \to 0$ as $\ell \to \infty$ from Lemma 3.2, we immediately obtain that $(q_{\ell,1})_{\ell > R}$ is bounded in $H^2((0, R))$ and hence we can extract a subsequence which converges weakly in $H^2((0, L))$ and strongly in $C^1([0, R])$ (by the general Sobolev inequality, e.g. [Eva10], 5.6.3, Theorem 6) towards a function $q \in H^2((0, R))$. Passing to the limit in (3.1a) then shows that q solves (2.8) and hence, $q = \gamma$ as the solution is unique. □

As $\gamma(x) \to 1$ with $x \to \infty$ the foregoing lemma especially implies that $q_{\ell,\varepsilon}$ approximately looks like $\chi_{(0,\ell)}$ for small $\varepsilon > 0$:

Corollary 3.4. *For fixed* $\ell > 0$ *we have*

$$q_{\ell,\varepsilon}(x) \to 1 \quad \textit{for all } x \in (0, \ell) \tag{3.3}$$

as $\varepsilon \to 0$*.*

Proof. Due to the symmetry of $q_{\ell,\varepsilon}$ it suffices to consider $x \leq \frac{\ell}{2}$. For $\delta > 0$ small and $N > 0$ we have for $x \in (\delta, \frac{\ell}{2})$

$$q_{\ell,\varepsilon}(x) = q_{\frac{\ell}{\varepsilon},1}\left(\frac{x}{\varepsilon} \right) \geq q_{\frac{\ell}{\varepsilon},1}(N)$$

if $\varepsilon < \varepsilon_0(\delta, N)$. Sending $\varepsilon \to 0$ gives

$$\liminf_{\varepsilon\to 0} q_{\ell,\varepsilon}(x) \geq \gamma(N)$$

by Lemma 3.3 and with $N \to \infty$ we conclude

$$\liminf_{\varepsilon\to 0} q_{\ell,\varepsilon}(x) \geq 1.$$

Thus, (3.3) follows due to (3.1b) and since δ was arbitrary. □

The specific form of F allowed us to find a precise representation of γ (which is not possible in general for other double well potentials) by the hyperbolic tangent. A similar description is available for $q_{\ell,\varepsilon}$ using the Jacobian elliptic sine function Sn (see [DLMF], §22 for a definition). We include the precise form in the next proposition for completeness, though we will not make use of it later on.

Proposition 3.5. *The optimal arc profile $q_{\ell,\varepsilon}$ has the representation*

$$q_{\ell,\varepsilon}(x) := k\sqrt{\frac{2}{k^2+1}}\operatorname{Sn}\left(\frac{x}{\varepsilon\sqrt{k^2+1}}, k\right) \tag{3.4}$$

with $k = k\left(\frac{\ell}{\varepsilon}\right) \in (0,1)$ chosen such that the equation

$$\frac{\ell}{\varepsilon} = 2\bar{K}(k)\sqrt{k^2+1}$$

holds with

$$\bar{K}(k) := \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2(t)}}$$

denoting the complete Jacobian elliptic integral*.*

Proof. We remark that Sn satisfies the second order ODE

$$\operatorname{Sn}'' = -(1+k^2)\operatorname{Sn} + 2k^2\operatorname{Sn}^3$$

(see [DLMF], §22.13.13) and calculate

$$\begin{aligned}
&-\varepsilon^2 q''_{\ell,\varepsilon} + F'(q_{\ell,\varepsilon}) = -\varepsilon^2 q''_{\ell,\varepsilon} + (q_{\ell,\varepsilon})^3 - q_{\ell,\varepsilon}\\
=&-k\sqrt{\frac{2}{(k^2+1)^3}}\operatorname{Sn}''\left(\frac{x}{\varepsilon\sqrt{k^2+1}}, k\right) + 2k^3\sqrt{\frac{2}{(k^2+1)^3}}\operatorname{Sn}^3\left(\frac{x}{\varepsilon\sqrt{k^2+1}}, k\right)\\
&\quad - k\sqrt{\frac{2}{k^2+1}}\operatorname{Sn}\left(\frac{x}{\varepsilon\sqrt{k^2+1}}, k\right)\\
=&k\sqrt{\frac{2}{(k^2+1)^3}}\Bigg(-\operatorname{Sn}''\left(\frac{x}{\varepsilon\sqrt{k^2+1}}, k\right) + 2k^2\operatorname{Sn}^3\left(\frac{x}{\varepsilon\sqrt{k^2+1}}, k\right)
\end{aligned}$$

$$
- (k^2+1)\,\mathrm{Sn}\left(\frac{x}{\varepsilon\sqrt{k^2+1}},k\right)\Bigg)
$$
$$
=0.
$$

Since $\mathrm{Sn}(0,k) = \mathrm{Sn}(2\bar{K}(k),k) = 0$ and $\mathrm{Sn}(\cdot,k) > 0$ in $(0, 2\bar{K}(k))$ the statement follows as the solution of (3.1) is unique. □

Remark. The choice of k in the representation of $q_{\ell,\varepsilon}$ above is only possible for large values of $\frac{\ell}{\varepsilon}$ in accordance to the requirement in Proposition 3.1 and we have $k \nearrow 1$ as $\frac{\ell}{\varepsilon} \to \infty$.
As a byproduct, Proposition 3.5 yields an precise expression for $\bar{q}_{\ell,\varepsilon}$. As Sn has its maximum at value 1, we deduce from (3.4) that

$$
\bar{q}_{\ell,\varepsilon} = k\sqrt{\frac{2}{k^2+1}}.
$$

We also point out that we could have shorten the proof of Proposition 3.1 in some points by using the representation of $q_{\ell,\varepsilon}$. However, we decided for the general approach which stays valid for more general double well potentials.

We conclude this section with two technical lemmas on $q_{\ell,\varepsilon}$ which we will need in Section 3.2.

Lemma 3.6. *For $\ell,\varepsilon > 0$ with $\frac{\ell}{\varepsilon} > \ell_2$ the corresponding optimal arc $q_{\ell,\varepsilon}$ satisfies*

$$
\int_0^\ell \frac{\varepsilon}{2}(q_{\ell,\varepsilon}')^2 + \frac{1}{\varepsilon}F(q_{\ell,\varepsilon})\,dx \le \sigma + \frac{\ell}{\varepsilon}F(\bar{q}_{\ell,\varepsilon}) \le \sigma + C\frac{\ell}{\varepsilon}e^{-\alpha\frac{\ell}{\varepsilon}} \tag{3.5}
$$

with $\sigma = \int_{-1}^1 \sqrt{2F(r)}\,dr$.

Proof. We rearrange (3.2) for

$$
\frac{1}{\varepsilon}F(q_{\ell,\varepsilon}) = \frac{\varepsilon}{2}(q_{\ell,\varepsilon}')^2 + \frac{1}{\varepsilon}F(\bar{q}_{\ell,\varepsilon}) \quad \text{in } \left[0,\frac{\ell}{2}\right].
$$

and use the symmetry of $q_{\ell,\varepsilon}$ to obtain the first inequality of (3.5) by

$$
\begin{aligned}
&\int_0^\ell \frac{\varepsilon}{2}(q_{\ell,\varepsilon}')^2 + \frac{1}{\varepsilon}F(q_{\ell,\varepsilon})\,dx = 2\int_0^{\frac{\ell}{2}} \frac{\varepsilon}{2}(q_{\ell,\varepsilon}')^2 + \frac{1}{\varepsilon}F(q_{\ell,\varepsilon})\,dx \\
=&2\int_0^{\frac{\ell}{2}} \varepsilon(q_{\ell,\varepsilon}')^2 + \frac{1}{\varepsilon}F(\bar{q}_{\ell,\varepsilon})\,dx = 2\int_0^{\frac{\ell}{2}} \sqrt{2\big(F(q_{\ell,\varepsilon}) - F(\bar{q}_{\ell,\varepsilon})\big)}q_{\ell,\varepsilon}'\,dx + \frac{\ell}{\varepsilon}F(\bar{q}_{\ell,\varepsilon}) \\
=&2\int_0^{\bar{q}_{\ell,\varepsilon}} \sqrt{2\big(F(r) - F(\bar{q}_{\ell,\varepsilon})\big)}\,dr + \frac{\ell}{\varepsilon}F(\bar{q}_{\ell,\varepsilon}) \\
\le&\int_0^1 \sqrt{2F(r)}\,dr + \frac{\ell}{\varepsilon}F(\bar{q}_{\ell,\varepsilon}) \\
=&\,\sigma + \frac{\ell}{\varepsilon}F(\bar{q}_{\ell,\varepsilon}).
\end{aligned}
$$

Then, the remaining part of (3.5) follows with an application of Lemma 3.2. □

Lemma 3.7. *Let $\varepsilon, \ell > 0$ with $\frac{\ell}{\varepsilon} > \ell_1$. Then $q_{\ell,\varepsilon}$ satisfies*

$$\int_0^{\frac{\ell}{2}} \left|q'_{\ell,\varepsilon}\right|^2 \, dx \leq \frac{C}{\varepsilon}, \tag{3.6}$$

$$\int_0^{\frac{\ell}{2}} \left|q''_{\ell,\varepsilon}\right|^2 \, dx \leq \frac{C}{\varepsilon^3} \tag{3.7}$$

with a constant $C = C(\ell_1) > 0$.

Proof. (3.6) follows directly from the first inequality in (3.5) as

$$\int_0^{\frac{\ell}{2}} \frac{\varepsilon}{2} \left|q'_{\ell,\varepsilon}\right|^2 + \frac{1}{\varepsilon} F(q_{\ell,\varepsilon}) \, dx \leq C.$$

For (3.7) an integration by parts yields

$$\begin{aligned}\int_0^{\frac{\ell}{2}} \left(q''_{\ell,\varepsilon}\right)^2 \, dx &= \frac{1}{\varepsilon^2} \int_0^{\frac{\ell}{2}} q''_{\ell,\varepsilon} F'(q_{\ell,\varepsilon}) \, dx \\ &= \frac{1}{\varepsilon^2} \Big[q'_{\ell,\varepsilon} F'(q_{\ell,\varepsilon})\Big]_0^{\frac{\ell}{2}} - \frac{1}{\varepsilon^2} \int_0^{\frac{\ell}{2}} F''(q_{\ell,\varepsilon}) \left|q'_{\ell,\varepsilon}\right|^2 \, dx \\ &\leq \frac{C}{\varepsilon^2} \int_0^{\frac{\ell}{2}} \left|q'_{\ell,\varepsilon}\right|^2 \, dx \leq \frac{C}{\varepsilon^3}\end{aligned}$$

where we have used (3.6) in the last step. □

3.2 Quasi one-dimensional configurations and their minimal energy scaling

To understand the driving forces which cause the observed slow evolution in the simulations from [EsRäRö14] we study a corresponding stationary problem. In this section we consider configurations with only straight phase boundaries and examine whether these interfaces still carry diffuse Willmore energy in contrast to the sharp interface limit. Moreover, we restrict ourselves to simple phase fields with a quasi one-dimensional structure. As we will see below, these configurations already show an interesting behavior.

In the following, we consider the whole space $\mathbb{R}^2$ divided into vertical stripes with periodically repeating widths $\ell, r > 0$ (see Figure 3.1) and denote the (fixed) period length by $L := \ell + r$.

Defining $E \subset \mathbb{R}^2$ by

$$E := \{x \in \mathbb{R}^2 \,:\, kL < x_1 < kL + \ell \text{ for some } k \in \mathbb{Z}\}$$

the corresponding indicator function $u = 2\chi_E - 1$ then obviously satisfies $\mathcal{W}(u) = 0$ as ∂E has no curvature. Consequently, u is a global minimizer of $\mathcal{W}$ and a (stable) stationary state of the Willmore flow independent of ℓ and r.

In this section we will investigate the same situation for the diffuse Willmore functional

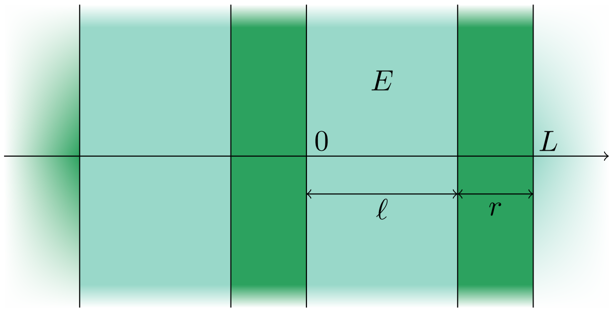


Figure 3.1: $\mathbb{R}^2$ periodically divided into vertical stripes of widths ℓ and r.

$\mathcal{W}_\varepsilon$ and compare it to the sharp interface limit described above. For $\varepsilon > 0$ small we only choose phase fields which are constant in x_2-direction and therefore reduce the problem to one dimension (see Figure 3.2).

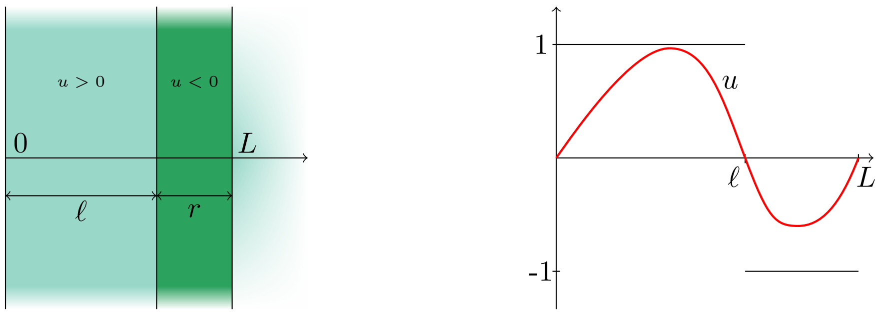


Figure 3.2: Dimension reduction.

More specific, we consider $u \in H^2_{\text{per}}(U)$ with

$$\begin{cases} u > 0 \text{ in } (0, \ell) & (3.8a) \\ u < 0 \text{ in } (\ell, L) & (3.8b) \\ u = 0 \text{ in } \{0, \ell, L\} & (3.8c) \end{cases}$$

and constrained to

$$\mathcal{E}_\varepsilon(u) := \int_0^L \frac{\varepsilon}{2} \left|u'\right|^2 + \frac{1}{\varepsilon} F(u)\, dx \leq \frac{F(0)}{2} \ell_4 \tag{3.9}$$

for a sufficiently large constant $\ell_4 > 0$ chosen below in the proof of Theorem 3.8. We briefly remark at this point that the pointwise conditions in (3.8) have a proper meaning as $H^2_{\text{per}}\big((0, L)\big)$ embeds continuously into $C^1_{\text{per}}([0, L])$.
We will always assume $\frac{\ell}{\varepsilon}, \frac{r}{\varepsilon} \geq \ell_4$ such that condition (3.9) excludes functions which are "too close" to the zero function since

$$\mathcal{E}_\varepsilon(0) = \frac{L}{\varepsilon} F(0) = \frac{\ell + r}{\varepsilon} F(0) > 2F(0)\ell_4.$$

For $\varepsilon, \ell > 0$ with $\frac{\ell}{\varepsilon} > \ell_4$ we set

$$M_\ell^\varepsilon := \left\{ u \in H^2_{\mathrm{per}}\big((0,L)\big) \ : \ u \text{ satisfies (3.8) and (3.9)} \right\}$$

for convenience and also define the diffuse Willmore functional $\mathcal{W}_\varepsilon$ for $\varepsilon > 0$ in one dimension by

$$\mathcal{W}_\varepsilon(u) := \frac{1}{2\varepsilon} \int_0^L \left(-\varepsilon u'' + \frac{1}{\varepsilon} F'(u) \right)^2 dx, \quad u \in H^2_{\mathrm{per}}\big((0,L)\big).$$

In order to understand the differences between $\mathcal{W}$ and $\mathcal{W}_\varepsilon$ for the considered configurations we are interested in two major issues:

1) Does a minimizer u_ε of $\mathcal{W}_\varepsilon$ exist in M_ℓ^ε?

2) Do configurations in M_ℓ^ε always carry positive energy and (if the answer is positive) how does the minimal energy scale in terms of ε and ℓ?

Before we answer these questions in general, we remark that both are completely trivial for $\ell = r = \frac{L}{2}$. In this case we have

$$\min_{w \in M_\ell^\varepsilon} \mathcal{W}_\varepsilon(w) = \mathcal{W}_\varepsilon(u_\varepsilon) = 0$$

with u_ε given by

$$u_\varepsilon = \begin{cases} q_{\ell,\varepsilon} & \text{in } (0,\ell] \\ -q_{\ell,\varepsilon}(\cdot - \ell) & \text{in } (\ell, L) \end{cases}$$

where $q_{\ell,\varepsilon}$ denotes the optimal arc profile (see Section 3.1) on $[0,\ell]$. For $\ell \neq r$ this simple construction does not apply anymore as optimal arc profiles corresponding to different interval lengths have different derivatives in 0. Therefore, u_ε as constructed above would not be differentiable and hence not in $H^2_{\mathrm{per}}\big((0,L)\big)$. Nevertheless, the existence of a minimizer in M_ℓ^ε and its diffuse energy are strongly connected to the corresponding optimal arc profiles as we will see below.

Let us briefly outline the structure of this section: Our main result and the answer to both questions above is presented in Theorem 3.12 where we show that a minimizer of $\mathcal{W}_\varepsilon$ in M_ℓ^ε exists with the direct method of the calculus of variations. Moreover, we prove a scaling law for the minimal energy which especially implies that for the case $\ell \neq r$ the minimizer still has (exponentially small) positive energy in contrast to the sharp interface limit. We end the section with a presentation of numerical results which show the convex dependence of $\min \mathcal{W}_\varepsilon$ on the zero position around $\frac{L}{2}$.

Crucial for the analysis in this section is a result by Otto and Westdickenberg [OtRe07] which says that in $(0,\ell)$ ((ℓ, L), respectively) the H^2-difference of a function $u \in M_\ell^\varepsilon$ and $q_{\ell,\varepsilon}$ ($-q_{r,\varepsilon}(\cdot - \ell)$, respectively) is controlled by the diffuse Willmore energy of u. We give the precise formulation in the following theorem.

Theorem 3.8 (Otto,Westdickenberg, '07)**.** *There exist positive constants $C > 0$ and $\ell_3 \geq \ell_1 > 0$ such that for every $\ell \geq \ell_3$ and every $u \in H^2\big((0,\ell)\big)$ with*

$$u \geq 0, \ \textit{in } (0,\ell), \tag{3.10}$$
$$u(0) = u(\ell) = 0 \ \textit{and} \tag{3.11}$$
$$\mathcal{E}_1(u) \leq \frac{F(0)}{2}\ell_3 \tag{3.12}$$

we have

$$\|u - q_{\ell,1}\|^2_{H^2([0,\ell])} \leq C \int_0^\ell \big(-u'' + F'(u)\big)^2 \, dx = C\mathcal{W}_1(u).$$

Proof. The proof can be found in [OtRe07] for smooth functions u. A simple approximation argument then yields the result for general $H^2_{\text{per}}\big((0,L)\big)$ functions. □

Remark. Condition (3.12) merely ensures that u is bounded away from 0. It follows directly from the proof in [OtRe07] that the statement of Theorem 3.8 also holds for all bigger constants $\tilde{\ell}_3 > \ell_3$.

We continue by proving an ε-scaled version of the above theorem.

Theorem 3.9. *Let $\ell_3, C > 0$ as in Theorem 3.8. For every $\ell, \varepsilon > 0$ satisfying $\frac{\ell}{\varepsilon} \geq \ell_3$ and every $u \in H^2\big((0,\ell)\big)$ with*

$$u \geq 0, \ \textit{in } (0,\ell) \ \textit{and}$$
$$u(0) = u(\ell) = 0$$
$$\mathcal{E}_\varepsilon(u) \leq \frac{F(0)}{2}\ell_3$$

we have

$$\|u - q_{\ell,\varepsilon}\|^2_{L^2((0,\ell))} \leq C\varepsilon^2 \int_0^\ell \left(-\varepsilon u'' + \frac{1}{\varepsilon}F'(u)\right)^2 dx = C\varepsilon^3 \mathcal{W}_\varepsilon(u), \tag{3.13}$$
$$\|u - q_{\ell,\varepsilon}\|^2_{H^1((0,\ell))} \leq C \int_0^\ell \left(-\varepsilon u'' + \frac{1}{\varepsilon}F'(u)\right)^2 dx = C\varepsilon \mathcal{W}_\varepsilon(u), \tag{3.14}$$
$$\|u - q_{\ell,\varepsilon}\|^2_{H^2((0,\ell))} \leq \frac{C}{\varepsilon^2} \int_0^\ell \left(-\varepsilon u'' + \frac{1}{\varepsilon}F'(u)\right)^2 dx = \frac{C}{\varepsilon}\mathcal{W}_\varepsilon(u). \tag{3.15}$$

Proof. We prove (3.15) first. For u, ε and ℓ with the mentioned properties we define

$$\tilde{u}(x) = u(\varepsilon x)$$

and observe

$$\tilde{u}(0) = \tilde{u}\left(\frac{\ell}{\varepsilon}\right) = 0$$

as well as

$$\tilde{u}'(x) = \varepsilon u'(\varepsilon x) \quad , \quad \tilde{u}''(x) = \varepsilon^2 u''(\varepsilon x).$$

We obtain

$$\mathcal{E}_1(\tilde{u}) = \int_0^{\frac{\ell}{\varepsilon}} \frac{1}{2} \left|\tilde{u}'\right|^2 + F(\tilde{u})\, dx = \int_0^{\ell} \frac{\varepsilon}{2} \left|u'\right|^2 + \frac{1}{\varepsilon} F(u)\, dx = \mathcal{E}_\varepsilon(u) \leq \frac{F(0)}{2}\ell_3$$

and together with the scaling behavior of $q_{\ell,\varepsilon}$ (Section 3.1) this yields

$$\begin{aligned}
&\|u - q_{\ell,\varepsilon}\|^2_{H^2((0,\ell))} \\
=&\int_0^{\ell} (u - q_{\ell,\varepsilon})^2 + (u' - (q_{\ell,\varepsilon})')^2 + (u'' - (q_{\ell,\varepsilon})'')^2\, dx \\
=&\int_0^{\frac{\ell}{\varepsilon}} \varepsilon(\tilde{u} - q_{\ell,1})^2 + \frac{1}{\varepsilon}(\tilde{u}' - (q_{\ell,1})')^2 + \frac{1}{\varepsilon^3}(\tilde{u}'' - (q_{\ell,1})'')^2\, dy \\
\leq&\frac{1}{\varepsilon^3} \|\tilde{u} - q_{\ell,1}\|^2_{H^2((0,\frac{\ell}{\varepsilon}))} \\
\leq&\frac{C}{\varepsilon^3} \int_0^{\frac{\ell}{\varepsilon}} \left(-\tilde{u}'' + F'(\tilde{u})\right)^2\, dy \\
=&\frac{C}{\varepsilon^2} \int_0^{\ell} \left(-\varepsilon u'' + \frac{1}{\varepsilon} F'(u)\right)^2\, dx \\
=&\frac{C}{\varepsilon} \mathcal{W}_\varepsilon(u)
\end{aligned}$$

where we have applied Theorem 3.8 in the fourth step.
The proof shows that the factor $\frac{1}{\varepsilon^2}$ in (3.15) originates from the scaling in the second derivative of $(u - q_{\ell,\varepsilon})$ and that lower derivatives therefore behave better in terms of ε. Thus, we especially obtain (3.13) and (3.14) by the same calculation. □

In the following theorem we characterize the energy order of $\inf \mathcal{W}_\varepsilon$ in M_ℓ^ε in terms of ℓ, r and ε. While we rely on Theorem 3.9 for the lower bound, we prove the upper bound by explicitly constructing a competitor function with the desired energy scale. The arguments for both inequalities require a good knowledge of the qualitative behavior of the optimal arc profiles $q_{\ell,\varepsilon}$ and $q_{r,\varepsilon}$ (see especially Proposition 3.2).

Theorem 3.10 (Scaling Law)**.** *There exist constants C_1, C_2 only depending on L and a constant $\ell_4 > 0$ such that for all $\varepsilon, \ell > 0$ with $\frac{\ell}{\varepsilon}$, $\frac{r}{\varepsilon} \geq \ell_4$ we have*

$$C_1 \frac{1}{\varepsilon^2} \left| e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{r}{\varepsilon}} \right|^2 \leq \inf_{u \in M_\ell^\varepsilon} \mathcal{W}_\varepsilon(u) \leq C_2 \frac{1}{\varepsilon^2} \left| e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{r}{\varepsilon}} \right|^2 \tag{3.16}$$

where $\alpha := \sqrt{F''(1)} = \sqrt{2}$.

Proof. We choose $\ell_4 \geq \max\{\ell_2, \ell_3\}$with ℓ_2, ℓ_3 from Lemma 3.2 and Theorem 3.8 and such that additionally inequalities (3.17) and (3.18) hold.

Lower bound: For $u \in M_\ell^\varepsilon$ and the optimal arc profiles $q_{\ell,\varepsilon}$ and $q_{r,\varepsilon}$ we have

$$\left|q'_{\ell,\varepsilon}(0) - q'_{r,\varepsilon}(0)\right| \le \left|q'_{\ell,\varepsilon}(0) - u'(0)\right| + \left|q'_{r,\varepsilon}(0) - u'(0)\right|$$

and since both summands can be treated analogously we concentrate on the first one. As $q'_{\ell,\varepsilon}\left(\frac{\ell}{2}\right) = 0$ we obtain

$$\begin{aligned}
&\left|q'_{\ell,\varepsilon}(0) - u'(0)\right| \\
=&\left|\left(q'_{\ell,\varepsilon}(0) - u'(0)\right)\frac{q'_{\ell,\varepsilon}(0)}{q'_{\ell,\varepsilon}(0)} - \left(q'_{\ell,\varepsilon}\left(\frac{\ell}{2}\right) - u'\left(\frac{\ell}{2}\right)\right)\frac{q'_{\ell,\varepsilon}\left(\frac{\ell}{2}\right)}{q'_{\ell,\varepsilon}(0)}\right| \\
=&\frac{1}{q'_{\ell,\varepsilon}(0)}\left|\int_0^{\frac{\ell}{2}}\left(\left(q'_{\ell,\varepsilon} - u'\right)q'_{\ell,\varepsilon}\right)' dx\right| \\
=&\frac{1}{q'_{\ell,\varepsilon}(0)}\left|\int_0^{\frac{\ell}{2}}\left(q''_{\ell,\varepsilon} - u''\right)q'_{\ell,\varepsilon} + \left(q'_{\ell,\varepsilon} - u'\right)q''_{\ell,\varepsilon}\, dx\right| \\
\le&\frac{1}{q'_{\ell,\varepsilon}(0)}\left(\int_0^{\frac{\ell}{2}}\left|q''_{\ell,\varepsilon} - u''\right|^2 dx\right)^{\frac{1}{2}}\left(\int_0^{\frac{\ell}{2}}\left|q'_{\ell,\varepsilon}\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{q'_{\ell,\varepsilon}(0)}\left(\int_0^{\frac{\ell}{2}}\left|q'_{\ell,\varepsilon} - u'\right|^2 dx\right)^{\frac{1}{2}}\left(\int_0^{\frac{\ell}{2}}\left|q''_{\ell,\varepsilon}\right|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Both first factors can now be estimated by (3.15) and (3.14), respectively. Together with (3.6) and (3.7) and $q'_{\ell,\varepsilon}(0) \ge \frac{C}{\varepsilon}$ we deduce

$$\left|q'_{\ell,\varepsilon}(0) - u'(0)\right| \le C\big(\mathcal{W}_\varepsilon(u)\big)^{\frac{1}{2}}$$

and this proves

$$\mathcal{W}_\varepsilon(u) \ge C\left|q'_{\ell,\varepsilon}(0) - q'_{r,\varepsilon}(0)\right|^2.$$

The lower bound estimate follows with the qualitative description of $\bar{q}_{\ell,\varepsilon} = \max q_{\ell,\varepsilon}$ from Lemma 3.2 as

$$\begin{aligned}
&\left|q'_{\ell,\varepsilon}(0) - q'_{r,\varepsilon}(0)\right|^2 \\
=&\frac{1}{\varepsilon^2}\left|\sqrt{F\big(q_{\ell,\varepsilon}(0)\big) - F(\bar{q}_{\ell,\varepsilon})} - \sqrt{F\big(q_{r,\varepsilon}(0)\big) - F(\bar{q}_{r,\varepsilon})}\right|^2 \\
=&\frac{1}{\varepsilon^2}\left|\sqrt{F(0) - F(\bar{q}_{\ell,\varepsilon})} - \sqrt{F(0) - F(\bar{q}_{r,\varepsilon})}\right|^2 \\
\ge&\frac{C}{\varepsilon^2}\left|F(\bar{q}_{r,\varepsilon}) - F(\bar{q}_{\ell,\varepsilon})\right|^2 \\
\ge&\frac{C_1}{\varepsilon^2}\left|e^{-\frac{\alpha\ell}{\varepsilon}} - e^{-\frac{\alpha r}{\varepsilon}}\right|^2
\end{aligned}$$

where we have also used the local Lipschitz continuity of $y \mapsto y^2$ on $\mathbb{R}$ in the second last step.

Upper Bound: For the second inequality in (3.16) we explicitly construct an element $u^* \in M_\ell^\varepsilon$ with small energy. The first idea is to choose u^* analogously to the case $\ell = r$ from above as the optimal arc profiles corresponding to ℓ and r in $[0,\ell]$ and $(\ell, L]$, respectively. This would make $\mathcal{W}_\varepsilon(u^*)$ vanish on both parts of the interval but as their derivatives differ in 0, the resulting function would not be in $C^1([0,L])$ and consequently not in $H^2(0,L)$. We solve this problem by modifying the arc on (ℓ, L) and moreover, by keeping the correction small away from the zeros in ℓ and L. We make the ansatz

$$u^* := \begin{cases} q_{\ell,\varepsilon} & \text{in } [0,\ell] \\ -\Big(q_{r,\varepsilon}(\cdot - \ell) + dF'(q_{r,\varepsilon}(\cdot-\ell)\Big) & \text{in } (\ell, L] \end{cases}$$

where we choose d such that

$$-q'_{\ell,\varepsilon}(0) = q'_{\ell,\varepsilon}(\ell) = -q'_{r,\varepsilon}(0) - dF''\big(q_{r,\varepsilon}(0)\big)q'_{r,\varepsilon}(0) = (d-1)q'_{r,\varepsilon}(0)$$

i.e.

$$d := \frac{q'_{r,\varepsilon}(0) - q'_{\ell,\varepsilon}(0)}{q'_{r,\varepsilon}(0)}.$$

Together with $q''_{\ell,\varepsilon}(0) = q''_{r,\varepsilon}(0) = 0$ this proves that u^* satisfies (3.8). Notice, that $d = 0$ for $\ell = r = \frac{L}{2}$ and thus, u^* is the minimizer of $\mathcal{W}_\varepsilon$ in M_ℓ^ε from above in this case.
We will see below, that the size of d mainly determines the energy of u^*. We remark that

$$\big|dF'(q_{r,\varepsilon})\big| \leq C\left|\frac{q'_{r,\varepsilon}(0) - q'_{\ell,\varepsilon}(0)}{q'_{r,\varepsilon}(0)}\right| \leq C\varepsilon\left|q'_{r,\varepsilon}(0) - q'_{\ell,\varepsilon}(0)\right| \tag{3.17}$$

holds as $q_{r,\varepsilon}$ is bounded and that $q'_{r,\varepsilon}(0) \geq \frac{1}{2\varepsilon}$ for $\frac{r}{\varepsilon}$ sufficiently large.
Before we show that (3.9) holds for u^* which then implies $u^* \in M_\ell^\varepsilon$ it is convenient to determine its diffuse Willmore energy. As $u^* = q_{\ell,\varepsilon}$ on $[0,\ell]$, there is no energy contribution on this part of the interval and hence, we observe

$$\begin{aligned}
&2\varepsilon\mathcal{W}_\varepsilon(u^*) \\
&= \int_\ell^L \Big[-\varepsilon\Big(q_{r,\varepsilon}(\cdot-\ell) + dF'\big(q_{r,\varepsilon}(\cdot-\ell)\big)\Big)'' \\
&\qquad + \frac{1}{\varepsilon}F'\Big(q_{r,\varepsilon}(\cdot-\ell) + dF'\big(q_{r,\varepsilon}(\cdot-\ell)\big)\Big)\Big]^2\,dx \\
&= \int_0^r \Big[-\varepsilon\big(q_{r,\varepsilon} + dF'(q_{r,\varepsilon})\big)'' + \frac{1}{\varepsilon}F'\big(q_{r,\varepsilon} + dF'(q_{r,\varepsilon})\big)\Big]^2\,dx \\
&= \int_0^r \Big[-\varepsilon q''_{r,\varepsilon} - \varepsilon^3 d q^{(4)}_{r,\varepsilon} + \frac{1}{\varepsilon}\Big(F'(q_{r,\varepsilon}) + F''(q_{r,\varepsilon})dF'(q_{r,\varepsilon}) \\
&\qquad + F'''(q_{r,\varepsilon})\big(dF'(q_{r,\varepsilon})\big)^2 + F^{(4)}(q_{r,\varepsilon})\big(dF'(q_{r,\varepsilon})\big)^3\Big)\Big]^2\,dx \\
&= \int_0^r \Big[-\varepsilon^3 d q^{(4)}_{r,\varepsilon} + \frac{1}{\varepsilon}\Big(\varepsilon^2 F''(q_{r,\varepsilon})dq''_{r,\varepsilon} + 6q_{r,\varepsilon}\big(dF'(q_{r,\varepsilon})\big)^2 + 6\big(dF'(q_{r,\varepsilon})\big)^3\Big)\Big]^2\,dx
\end{aligned}$$

$$\leq C\int_0^r \varepsilon^2 d^2\big(\varepsilon^2 q_{r,\varepsilon}^{(4)} - F''(q_{r,\varepsilon})q_{r,\varepsilon}''\big)^2 + \frac{C}{\varepsilon^2}\big(dF'(q_{r,\varepsilon})\big)^4 + + \frac{C}{\varepsilon^2}\big(dF'(q_{r,\varepsilon})\big)^6\,dx$$

where we have done a complete Taylor expansion of $F'\big(q_{r,\varepsilon} + dF'(q_{r,\varepsilon})\big)$ in $d = 0$. Taking the second derivative of $-\varepsilon^2 q_{r,\varepsilon}'' + F'(q_{r,\varepsilon}) = 0$ yields

$$-\varepsilon^2 q_{r,\varepsilon}^{(4)} + F''(q_{r,\varepsilon})q_{r,\varepsilon}'' + F'''(q_{r,\varepsilon})(q_{r,\varepsilon}')^2 = 0$$

and thus

$$\varepsilon^2 q_{r,\varepsilon}^{(4)} - F''(q_{r,\varepsilon})q_{r,\varepsilon}'' = F'''(q_{r,\varepsilon})(q_{r,\varepsilon}')^2 = 6q_{r,\varepsilon}(q_{r,\varepsilon}')^2$$

which we plug in the calculation above together with (3.17). Almost analogously to the arguments for the lower bound above this yields

$$\begin{aligned}
&2\varepsilon\mathcal{W}_\varepsilon(u^*)\\
\leq& C\int_0^r \varepsilon^2 d^2 q_{r,\varepsilon}^2 (q_{r,\varepsilon}')^4 + \varepsilon^2\left|q_{r,\varepsilon}'(0) - q_{\ell,\varepsilon}'(0)\right|^4 + \varepsilon^4\left|q_{r,\varepsilon}'(0) - q_{\ell,\varepsilon}'(0)\right|^6\,dx\\
\leq& C\int_0^r \varepsilon^2 \frac{\left|q_{r,\varepsilon}'(0) - q_{\ell,\varepsilon}'(0)\right|^2}{(q_{r,\varepsilon}'(0))^2}(q_{r,\varepsilon}')^4 + \varepsilon^2\left|q_{r,\varepsilon}'(0) - q_{\ell,\varepsilon}'(0)\right|^4\\
&\quad + \varepsilon^4\left|q_{r,\varepsilon}'(0) - q_{\ell,\varepsilon}'(0)\right|^6\,dx\\
\leq& C\int_0^r \varepsilon^2\left|q_{r,\varepsilon}'(0) - q_{\ell,\varepsilon}'(0)\right|^2 (q_{r,\varepsilon}')^2\,dx\\
\leq& \frac{C}{\varepsilon}\left|\sqrt{F(0) - F(\bar{q}_{\ell,\varepsilon})} - \sqrt{F(0) - F(\bar{q}_{r,\varepsilon})}\right|^2\\
\leq& \frac{C}{\varepsilon}\left|F(\bar{q}_{r,\varepsilon}) - F(\bar{q}_{\ell,\varepsilon})\right|^2\\
\leq& \frac{C_2}{\varepsilon}\left|e^{-\frac{\alpha\ell}{\varepsilon}} - e^{-\frac{\alpha r}{\varepsilon}}\right|^2.
\end{aligned}$$

Remark that due to $0 \approx F(q_{\ell,\varepsilon}), F(q_{r,\varepsilon}) \ll F(0) = \frac{1}{4}$ the arguments in the square roots stay away from 0 and hence the estimate follows by the local Lipschitz continuity of $y \mapsto \sqrt{y}$ on $(0,\infty)$. This proves

$$\mathcal{W}_\varepsilon(u^*) \leq \frac{C_2}{\varepsilon^2}\left|e^{-\frac{\alpha\ell}{\varepsilon}} - e^{-\frac{\alpha r}{\varepsilon}}\right|^2$$

and it remains to show that (3.9) holds for u^*, which then implies that $u^* \in M_\ell^\varepsilon$. As the integrand of $\mathcal{E}_\varepsilon$ is always positive, we can estimate the integrals over $(0,\ell)$ and (ℓ, L) separately. From Lemma 3.6 we directly obtain

$$\begin{aligned}
&\int_0^\ell \frac{\varepsilon}{2}\big((u^*)'\big)^2 + \frac{1}{\varepsilon}F(u^*)\,dx\\
=&\int_0^\ell \frac{\varepsilon}{2}(q_{\ell,\varepsilon}')^2 + \frac{1}{\varepsilon}F(q_{\ell,\varepsilon})\,dx
\end{aligned}$$

$$\leq C + C\frac{\ell}{\varepsilon}e^{-\alpha\frac{\ell}{\varepsilon}}.$$

For the remaining integral we have similarly as above

$$\begin{aligned}
&\int_\ell^L \frac{\varepsilon}{2}\big((u^*)'\big)^2 + \frac{1}{\varepsilon}F(u^*)\,dx \\
=&\int_0^r \frac{\varepsilon}{2}\big((1+dF''(q_{r,\varepsilon}))q'_{\ell,\varepsilon}\big)^2 + \frac{1}{\varepsilon}F\big(q_{r,\varepsilon}+dF'(q_{r,\varepsilon})\big)\,dx \\
\leq& C\int_0^r \varepsilon(q'_{\ell,\varepsilon})^2 + \frac{1}{\varepsilon}F(q_{r,\varepsilon}) + \frac{1}{\varepsilon}F'(q_{r,\varepsilon})\big(dF'(q_{r,\varepsilon})\big)\,dx \\
\leq& C\left(\int_0^r \frac{\varepsilon}{2}(q'_{r,\varepsilon})^2 + \frac{1}{\varepsilon}F(q_{r,\varepsilon})dx + \varepsilon r\left|q'_{r,\varepsilon}(0)-q'_{\ell,\varepsilon}(0)\right|^2\right) \\
\leq& C\left(1+\frac{r}{\varepsilon}e^{-\alpha\frac{r}{\varepsilon}}\right)
\end{aligned}$$

by Lemma 3.6 and (3.17). We combine the inequalities above for

$$\mathcal{E}_\varepsilon(u^*) \leq C\left(1+\frac{\ell}{\varepsilon}e^{-\alpha\frac{\ell}{\varepsilon}}+\frac{r}{\varepsilon}e^{-\alpha\frac{r}{\varepsilon}}\right) \leq \frac{F(0)}{2}\ell_4 \tag{3.18}$$

by the requirements for ℓ_4. □

Remark. i) The competitor function u^* for the upper bound inequality in the proof of Theorem 3.10 has been constructed to have vanishing diffuse Willmore energy on $[0,\ell]$ and as shown above, the value of $\mathcal{W}_\varepsilon(u^*)$ is determined by the correction which ensured $u^* \in H^2_{\text{per}}\big((0,L)\big)$. Although this leads to the least possible energy scaling in terms of the exponential decay with ε, u^* is certainly not a minimizer of $\mathcal{W}_\varepsilon$ in M^ε_ℓ. A refined construction by modifying both arcs could be a possible way to improve the constant C_2. Still this will not be sufficient to make the constants C_1, C_2 equal in (3.16) as C_1 is only implicitly characterized by [OtRe07].

ii) The mere exponential smallness of $\min_{M^\varepsilon_\ell}\mathcal{W}_\varepsilon$ can be shown with much simpler constructions in the upper bound equation. For example, the choice

$$u^{**} := \begin{cases} q_{\ell,\varepsilon} & \text{in } [0,\ell] \\ \beta q_{r,\varepsilon}(\cdot-\ell) & \text{in } (\ell,L] \end{cases}$$

with $\beta := \frac{q'_{\ell,\varepsilon}(0)}{q'_{r,\varepsilon}(0)}$ yields

$$\mathcal{W}_\varepsilon(u^{**}) \leq c\frac{1}{\varepsilon^3}\left|e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{r}{\varepsilon}}\right|^2$$

which qualitatively differs from (3.20) only in a factor $\frac{1}{\varepsilon}$.

The next lemma is formulated for arbitrary space dimensions. It is only used for $n=1$ in this chapter, but we will need the higher dimensional version in Section 5.1.

Lemma 3.11. *For $n \geq 1$ let U be an n-dimensional rectangle, i.e.,*

$$U := \prod_{k=1}^{n} (a_k, b_k) \subset \mathbb{R}^n$$

for $a_k, b_k \in \mathbb{R}$ and $a_k < b_k$, $1 \leq k \leq n$. Then, every $u \in H^2_{per}(U)$ satisfies

$$\|u\|^2_{H^2(U)} \leq C(\varepsilon, U)\big(1 + \mathcal{W}_\varepsilon(u)\big).$$

Proof. The periodic boundary conditions on u allow integrations by part without occurring boundary terms and Young's inequality yields the interpolation inequality

$$\int_U |\nabla u|^2 \, dx = \int_U -u\Delta u \, dx \leq \delta \int_U (\Delta u)^2 \, dx + \frac{1}{4\delta} \int_U u^2 \, dx, \tag{3.19}$$

where $\delta > 0$ is arbitrary. Since F'' is bounded from below, we have

$$\begin{aligned} 2\varepsilon \mathcal{W}_\varepsilon(u) &= \int_U \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} F'(u) \right)^2 dx \\ &= \int_U \varepsilon^2 (\Delta u)^2 - 2F'(u)\Delta u + \frac{1}{\varepsilon^2} F'(u)^2 \, dx \\ &= \int_U \varepsilon^2 (\Delta u)^2 + 2F''(u) \, |\nabla u|^2 + \frac{1}{\varepsilon^2} F'(u)^2 \, dx \\ &\geq \int_U \varepsilon^2 (\Delta u)^2 - C \, |\nabla u|^2 + \frac{1}{\varepsilon^2} F'(u)^2 \, dx \\ &\geq \int_U \varepsilon^2 (\Delta u)^2 - \frac{\varepsilon^2}{2} (\Delta u)^2 - \frac{C}{2\varepsilon^2} u^2 + \frac{1}{\varepsilon^2} F'(u)^2 \, dx \\ &= \int_U \frac{\varepsilon^2}{2} (\Delta u)^2 + \frac{1}{2\varepsilon^2} \left(-Cu^2 + F'(u)^2 \right) + \frac{1}{2\varepsilon^2} F'(u)^2 \, dx \end{aligned}$$

where we have used (3.19) with $\delta = \frac{\varepsilon^2}{2C}$ in the second last step. The term in parentheses is bounded from below since $F'(u)^2 = (u^3 - u)^2$ which yields

$$2\varepsilon \mathcal{W}_\varepsilon(u) \geq \int_U \frac{\varepsilon^2}{2} (\Delta u)^2 - \frac{C}{\varepsilon^2} + \frac{1}{2\varepsilon^2} F'(u)^2 \, dx$$

and therefore,

$$\int_U (\Delta u)^2 + F'(u)^2 \, dx \leq C(\varepsilon, U)\big(1 + \mathcal{W}_\varepsilon(u)\big).$$

Since $F'(u)^2 \geq C(u^2 - 1)$ for a small positive constant $C > 0$, we obtain

$$\int_U (\Delta u)^2 + u^2 \, dx \leq C(\varepsilon, U)\big(1 + \mathcal{W}_\varepsilon(u)\big)$$

and the desired estimate follows by the fact that

$$\int_U \left|D^2 v\right| \, dx = \int_U (\Delta v)^2 \, dx \quad \text{for all } v \in H^2_{\text{per}}(U)$$

and another application of (3.19) with δ chosen equal to $\frac{1}{2}$. □

Theorem 3.12 (Existence of minimizer). *There exists $\ell^* > \ell_4 > 0$ such that for $\varepsilon, \ell > 0$ with $\frac{\ell}{\varepsilon}, \frac{r}{\varepsilon} > \ell^*$ there is a minimizer u of $\mathcal{W}_\varepsilon$ in M^ε_ℓ which satisfies*

$$C_1 \frac{1}{\varepsilon^2} \left| e^{-\alpha \frac{\ell}{\varepsilon}} - e^{-\alpha \frac{r}{\varepsilon}} \right|^2 \leq \mathcal{W}_\varepsilon(u) = \min_{w \in M} \mathcal{W}_\varepsilon(w) \leq C_2 \frac{1}{\varepsilon^2} \left| e^{-\alpha \frac{\ell}{\varepsilon}} - e^{-\alpha \frac{r}{\varepsilon}} \right|^2 \tag{3.20}$$

with constants $C_1, C_2 > 0$ and where $\alpha := \sqrt{F''(1)} = \sqrt{2}$.

Proof. We only have to show that a minimizer exists in M^ε_ℓ. Then (3.20) follows directly from (3.16).
Since $\mathcal{W}_\varepsilon$ is obviously bounded from below by 0, there exists a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in M^ε_ℓ such that

$$W_\varepsilon(u_k) \longrightarrow \inf_{w \in M^\varepsilon_\ell} \mathcal{W}_\varepsilon(w) =: m$$

as $k \to \infty$. Lemma 3.11 yields an H^2-bound for u_k uniform in k, since

$$\|u_k\|_{H^2((0,L))} \leq C(\varepsilon, U)\big(1 + \sqrt{\mathcal{W}_\varepsilon(u_k)}\big) < \infty.$$

The weak precompactness of $H^2_{\text{per}}\big((0,L)\big)$ implies the existence of a subsequence again denoted by $(u_k)_{k\in\mathbb{N}}$ and of $u \in H^2_{\text{per}}\big((0,L)\big)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } H^2\big((0,L)\big) \tag{3.21}$$

and with an application of the general Sobolev inequality (e.g. [Eva10], 5.6.3, Theorem 6) we can assume that

$$u_k \longrightarrow u \quad \text{in } C^{1,\alpha}([0,L]) \text{ for } 0 \leq \alpha < \frac{1}{2} \tag{3.22}$$

as $k \in \mathbb{N}$. We have to prove that u satisfies (3.8) and (3.9). Due to (3.22), it follows immediately that

$$u(0) = u(\ell) = u(L) = 0 \quad \text{and} \quad u'(0) = u'(L)$$

and

$$u(x) \begin{cases} \geq 0, & x \in [0, \ell] \\ \leq 0, & x \in [\ell, L]. \end{cases}$$

However, to see that u actually satisfies (3.8) we require a further argument: Assume that there exists another point $x_0 \in (0, L)$, $x \neq \ell$ with $u(x_0) = 0$. Without loss of generality we can restrict to the case $x_0 \in (0, \ell)$. Since u is nonnegative on this part of the interval, x_0 has to be a local minimum of u and hence $u'(x) = 0$. We combine Theorem 3.9 with Theorem 3.10 for

$$\begin{aligned}|q_{\ell,\varepsilon}(x_0)| + \left|q'_{\ell,\varepsilon}(x_0)\right| &= |q_{\ell,\varepsilon}(x_0) - u(x_0)| + \left|q'_{\ell,\varepsilon}(x_0) - u'(x_0)\right| \\ &\le \frac{C}{\sqrt{\varepsilon}}\sqrt{m} \le \frac{C}{\varepsilon^{\frac{3}{2}}}\left|e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{r}{\varepsilon}}\right| \end{aligned} \tag{3.23}$$

which is a contradiction as there is no point in $(0, \ell)$ for which $q_{\ell,\varepsilon}$ and $q'_{\ell,\varepsilon}$ both become (exponentially) small at the same time. Indeed, (3.23) is obviously not satisfied for $x_0 \in \left\{q_{\ell,\varepsilon} > \frac{1}{2}\right\}$ and $\frac{\ell}{\varepsilon}, \frac{r}{\varepsilon} < \ell^*$ with ℓ^* sufficiently large. On the other hand, for $x \in \left\{q_{\ell,\varepsilon} \le \frac{1}{2}\right\}$ we have

$$\begin{aligned}\left|q'_{\ell,\varepsilon}(x_0)\right| &= \frac{\sqrt{2}}{\varepsilon}\sqrt{F\big(q_{\ell,\varepsilon}(x_0)\big) - F(\bar{q}_{\ell,\varepsilon})} \ge \frac{\sqrt{2}}{\varepsilon}\sqrt{F\left(\frac{1}{2}\right) - F(\bar{q}_{\ell,\varepsilon})} \\ &\ge \frac{C}{\varepsilon}\sqrt{1 - Ce^{-\alpha\frac{\ell}{\varepsilon}}}\end{aligned}$$

which also contradicts (3.23) for large $\ell^* > 0$.

To obtain (3.9) we observe that both u_k and u'_k converge uniformly on $[0, L]$ by (3.22) and hence,

$$\begin{aligned}F(u_k) &\longrightarrow F(u), \\ \left|u'_k\right|^2 &\longrightarrow \left|u'\right|^2\end{aligned}$$

uniformly on $[0, L]$ as $k \to \infty$. This implies

$$E_\varepsilon(u) = \int_0^L \frac{\varepsilon}{2}\left|u'\right|^2 + \frac{1}{\varepsilon}F(u)\,dx = \lim_{k\to\infty}\int_0^L \frac{\varepsilon}{2}\left|u'_k\right|^2 + \frac{1}{\varepsilon}F(u_k)\,dx \le \frac{F(0)}{2}\ell_4$$

which is (3.9) for u and thus, $u \in M^\varepsilon_\ell$.
It remains to show that u is a minimizer of $\mathcal{W}_\varepsilon$ in M^ε_ℓ. (3.21) and (3.22) imply that

$$-\varepsilon u''_k + \frac{1}{\varepsilon}F'(u_k) \rightharpoonup -\varepsilon u'' + \frac{1}{\varepsilon}F'(u) \quad \text{weakly in } L^2\big((0,\ell)\big)$$

and by the weak lower semicontinuity of the norm we have

$$\begin{aligned}\liminf_{k\to\infty}\mathcal{W}_\varepsilon(u_k) &= \frac{1}{2\varepsilon}\liminf_{k\to\infty}\left\|-\varepsilon u''_k + \frac{1}{\varepsilon}F'(u_k)\right\|^2_{L^2\left((0,\ell)\right)} \\ &\ge \frac{1}{2\varepsilon}\left\|-\varepsilon u'' + \frac{1}{\varepsilon}F'(u)\right\|^2_{L^2\left((0,\ell)\right)} = \mathcal{W}_\varepsilon(u)\end{aligned}$$

This is the weak lower semicontinuity of $\mathcal{W}_\varepsilon$ and we finally obtain

$$m \le \mathcal{W}_\varepsilon(u) \le \liminf_{k\to\infty}\mathcal{W}_\varepsilon(u_k) = m$$

and therefore,

$$\mathcal{W}_\varepsilon(u) = \min_{w \in M_\ell^\varepsilon} \mathcal{W}_\varepsilon(w)$$

which completes the proof. □

Theorem 3.12 only describes the minimal energy scale of W_ε for a given zero position ℓ and we are not able to determine the constants C_1 and C_2 explicitly. Observe that in case $\tilde{C} = C_1 = C_2$ the minimal energy is given by the expression

$$\min_{M_\ell^\varepsilon} \mathcal{W}_\varepsilon = \frac{\tilde{C}}{\varepsilon^2} \left| e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{r}{\varepsilon}} \right|^2$$

which is strictly convex in terms of $\ell - \frac{L}{2}$.
To conclude this section we present numerical results for the minimal energies corresponding to different zero positions $\ell \in (0, L)$.
Let $n \in \mathbb{N}$ denote the number of equidistant grid points in the interval $[0, L]$ and set $h := \frac{L}{n-1}$. For $\varepsilon > 0$ and a discrete phase field $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ we replace the second derivative in $\mathcal{W}_\varepsilon$ by a difference quotient and use a simple rectangle rule to approximate the integral. Precisely, we define the discretization of $\mathcal{W}_\varepsilon$ by

$$\mathcal{W}_{\text{dis}}(u) := \frac{h}{2\varepsilon} \sum_{i=1}^{n} \left(-\varepsilon (D^2 u)_i + \frac{1}{\varepsilon} F'(u_i) \right)^2$$

where D^2 is the $n \times n$ second order finite difference matrix for periodic boundary values

$$D^2 := \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & & 0 & 1 \\ 1 & -2 & 1 & & & & 0 \\ 0 & 1 & -2 & & \ddots & & \\ \vdots & & & \ddots & & & \vdots \\ & & \ddots & & -2 & 1 & 0 \\ 0 & & & & 1 & -2 & 1 \\ 1 & 0 & & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

We consider discrete periodic phase fields with vanishing boundary values and a further prescribed zero position $j_0 \in \{3, \ldots, n-2\}$, i.e. $u_1 = u_{j_0} = u_n = 0$, which satisfy

$$\begin{aligned} &u_j > 0 \text{ if } 2 \le j \le j_0 - 1 \qquad \text{and} \\ &u_j < 0 \text{ if } j_0 + 1 \le j \le n - 1. \end{aligned}$$

To determine a minimizer of $\mathcal{W}_{\text{dis}}$ and its energy among all these configurations we apply a classical Newton method for its derivative $\nabla \mathcal{W}_{\text{dis}}$. The resulting iteration scheme for a minimizing sequence $(u^{(k)})_{k\in\mathbb{N}}$ is as follows. With $u^{(0)}$ given by

$$u_j^{(0)} = \begin{cases} 1 & \text{if } 2 \le j \le j_0 - 1 \\ -1 & \text{if } j_0 + 1 \le j \le n - 1 \\ 0 & \text{if } j = 1, j_0, n \end{cases}$$

we define $\tilde{u}^{(k)} \in \mathbb{R}^n$ for $k \ge 0$ as the solution of

$$\nabla^2 \mathcal{W}_{\text{dis}}(u^{(k)})\tilde{u}^{(k)} = -\nabla \mathcal{W}_{\text{dis}}(u^{(k)}). \tag{3.24}$$

and set

$$u^{(k+1)} = u^{(k)} - \tilde{u}^{(k)}, \quad k \ge 0$$

afterwards. Here, the derivatives of $\mathcal{W}_{\text{dis}}$ are given by

$$\nabla \mathcal{W}_{\text{dis}} = \frac{h}{\varepsilon}\left(-\varepsilon D^2 + \frac{1}{\varepsilon}\Big(F''(u_i)\delta_{ij}\Big)_{ij}\right)\left(-\varepsilon D^2 u + \frac{1}{\varepsilon}\big(F'(u_i)\big)_i\right) \in \mathbb{R}^n$$

and

$$\begin{aligned}\nabla^2 \mathcal{W}_{\text{dis}} =& \frac{h}{\varepsilon}\Bigg(\varepsilon^2 \big(D^2\big)^2 - D^2\Big(F''(u_i)\delta_{ij}\Big)_{ij} - \Big(F''(u_i)\delta_{ij}\Big)_{ij} D^2 \\ & - \Big(F'''(u_i)\big(D^2 u\big)_i \delta_{ij}\Big)_{ij} + \frac{1}{\varepsilon^2}\Big(\big(F'''(u_i)F'(u_i) + F''(u_i)^2\big)\delta_{ij}\Big)_{ij}\Bigg) \\ & \in \mathbb{R}^{n\times n}.\end{aligned}$$

However, to ensure that each $u^{(k)}$ satisfies the imposed conditions $u_1^{(k)} = u_{j_0}^{(k)} = u_n^{(k)} = 0$ we have to replace the entries of the corresponding three lines and columns by 0 before solving (3.24).
As the energy values are expected to be exponentially small, a major challenge in the calculations is to keep rounding errors as small as possible. For that purpose we use variable-precision floating-point arithmetic (VPA) provided by MATLAB to apply the iteration scheme above with the highest possible accuracy.

For the performed calculations we have chosen $\varepsilon = 0.1$, $L = 10$ and $n = 301$. Figure 3.3 shows the minimal energy values of $\mathcal{W}_{\text{dis}}$ for prescribed zero positions in each grid point between 141 and 161 while the numerical values around $j = 151$ (corresponding to the case $\ell = \frac{L}{2}$) are also contained in Figure 3.4.

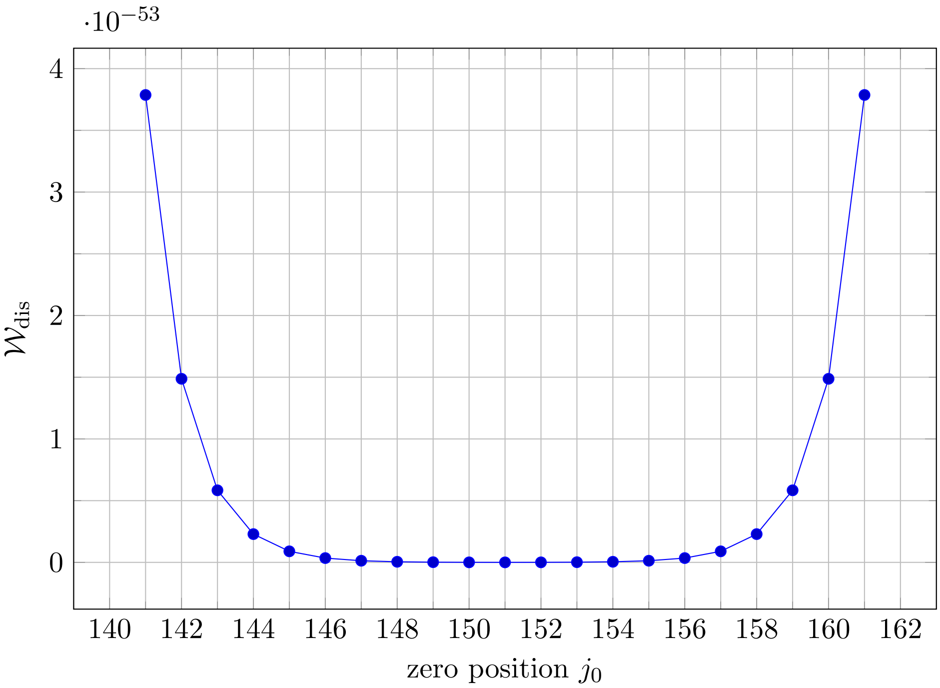


Figure 3.3: The minimal energies for prescribed zero positions k show a strictly convex behavior around $j = 151$ corresponding to $\ell = \frac{L}{2}$.

One can directly observe its symmetric and strictly convex structure around $j_0 = 151$ in accordance to the assumed behavior from Theorem 3.12. Moreover, there is a rather drastic drop of the energy values (of order 10^{-17}) in $j_0 = 151$ which corresponds to the exact value $\mathcal{W}_\varepsilon = 0$ for $\ell = \frac{L}{2}$.

j_0	$\mathcal{W}_{\text{dis}}$
146	3.478304e-55
147	1.326893e-55
148	4.827121e-56
149	1.536922e-56
150	3.112267e-57
151	1.879347e-74
152	3.112267e-57
153	1.536922e-56
154	4.827121e-56
155	1.326893e-55
156	3.478304e-55

Figure 3.4: Minimal values of $\mathcal{W}_{\text{dis}}$ for j_0 around 151.

For the sake of completeness, we include a plot of a generic minimizer for $j_0 = 181$ in Figure 3.5. Away from its zeros it is almost constant to 1 (or -1 respectively) and does the transition from one phase to another in a small interval around its zeros.

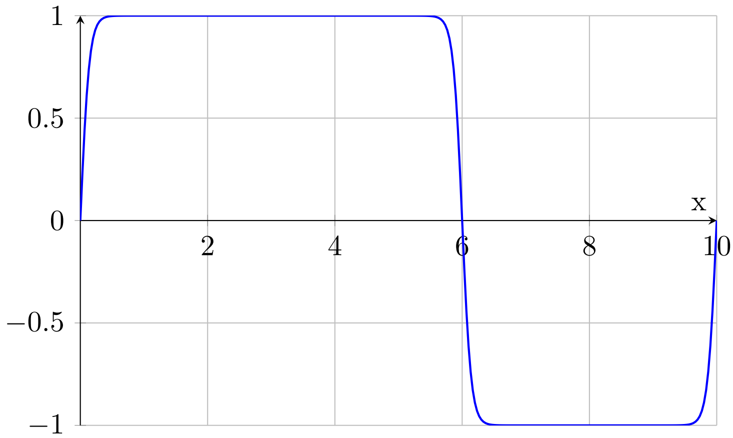


Figure 3.5: The obtained minimizer of $\mathcal{W}_{\text{dis}}$ for $j_0 = 181$ corresponding to $\ell = 6$.

4 Existence and qualitative properties of multisaddle solutions

In Chapter 3 we analyzed quasi one-dimensional configurations of periodic stripes and characterized their Willmore energy. For the analysis we heavily relied on a profound understanding of the optimal arc profiles $q_{\ell,\varepsilon}$ which describe the energetically optimal way to approximate a single stripe by diffuse interfaces. We have seen that a configuration of parallel stripes has no diffuse Willmore energy if and only if the stripes are perfectly symmetrically distributed. In this case the corresponding optimal arcs can be extended to a entire solution of the stationary Allen-Cahn equation on $\mathbb{R}$ by odd reflecting and periodic continuation.
However, these results only describe the simplest type of configurations in two dimensions and it is natural to ask whether the energy of real two-dimensional interfaces can be described similarly (e.g., by a scaling law to determine the energy order).

In this chapter we will consider a modification of stripe configurations which cannot be reduced to one dimension anymore. We start out from a configuration of parallel stripes as before and add a further line perpendicularly intersecting the stripes to the phase boundary. The resulting configuration $E \subset \mathbb{R}^2$ consists of semi infinite rectangles (see Figure 4.1).

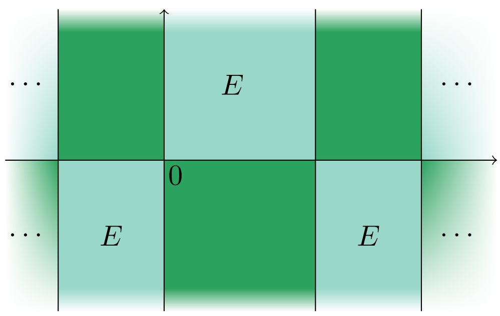


Figure 4.1: A configuration for E consisting of semi infinite rectangles.

Even in this situation an energy characterization similar to Section 3.2 turns out to be difficult. Here, we will prove the existence of two-dimensional entire solutions of the stationary Allen-Cahn equation whose zero set is given by the boundaries of the semi infinite rectangles. These solutions can be seen as a two-dimensional analogue of the (reflected) optimal arc profiles $q_{\ell,\varepsilon}$ and thus, our results are a first step into the energy quantification of those configurations. In correspondence with the quasi one-dimensional case, such solutions only exist for the symmetric case of rectangles with the same (and sufficiently large) width (see Theorem 4.2 and Corollary 4.3 below). On the other hand, this implies that phase fields which approximate E always have positive diffuse Willmore energy if the stripes have different widths.

In this chapter we will write f instead of F' for the sake of notation and therefore, the Allen-Cahn equation reads

$$-\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^2 \tag{4.1}$$

with $f(s) = s^3 - s$ for $s \in \mathbb{R}$.

The solutions we construct below generalize the result of [DaFiPe92] (see Section 2.5) in the sense that they have countably many saddles instead of one. Moreover, their shape far away from the x_1-axis approximates the optimal arc profile which yields a connection between configurations of infinite rectangles and the quasi one-dimensional case. To the best of our knowledge, entire solutions with more than one saddle have not been studied so far.
We begin with a general definition of the considered *multisaddle solutions*.

Definition 4.1 (multisaddle solution)**.** An entire solution $u \in C^2(\mathbb{R}^2)$ of (4.1) with $-1 < u < 1$ in $\mathbb{R}^2$ is called *multisaddle solution* if it satisfies (after a possible translation and rotation) the following properties (see also Figure 4.2):

i) The zero set of u satisfies

$$\{u = 0\} = \big(\mathbb{R} \times \{0\}\big) \cup \left(\bigcup_{j \in I} \Big(\{z_j\} \times \mathbb{R} \Big) \right)$$

for an consecutive index set $I \subset \mathbb{Z}$ with $0 \in I$ and such that

$$z_0 = 0 \quad \text{and} \quad z_j < z_k \quad \text{for all } j, k \in I \text{ with } j < k.$$

ii) u changes its sign when crossing any line in $\{u = 0\}$ in orthogonal direction and is positive in

$$\big\{ x = (x_1, x_2) \in \mathbb{R}^2 \, : \, 0 < x_1 < z_1, \, x_2 > 0 \big\}$$

or in

$$\big\{ x = (x_1, x_2) \in \mathbb{R}^2 \, : \, 0 < x_1, \, x_2 > 0 \big\},$$

respectively, if $1 \notin I$.

Note, that in the case $I = \{0\}$ the corresponding multisaddle solution u is exactly the

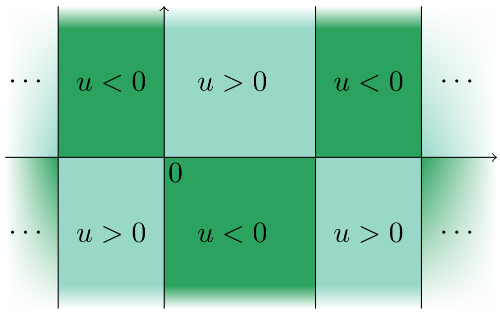


Figure 4.2: A multisaddle solution u with constant signs on the semi infinite rectangles and with zero set equal to the x_1-axis and countable many lines parallel to the x_2-axis.

solution constructed in [DaFiPe92].

Our main result is included in the following theorem.

Theorem 4.2. *There is a constant $\ell_{min} > 0$ such that for all $\ell > \ell_{min}$ there exists a unique multisaddle solution $u \in C^\infty(\mathbb{R}^2)$ as in Definition 4.1 with $I = \mathbb{Z}$ and*

$$x_i = i\ell \quad \text{for } i \in \mathbb{Z}.$$

u is 2ℓ-periodic in x_1-direction and satisfies $0 < u < 1$ in $(0,\ell) \times (0,\infty)$. Moreover, we have

$$\begin{aligned} u(\cdot, x_2) &\longrightarrow \pm q_\ell \quad \text{as } x_2 \longrightarrow \pm\infty \quad \text{in } (0,\ell) \\ u(\cdot, x_2) &\longrightarrow \mp q_\ell \quad \text{as } x_2 \longrightarrow \pm\infty \quad \text{in } (\ell, 2\ell) \end{aligned}$$

where q_ℓ denotes the odd extension and periodic continuation of the optimal arc profile $q_{\ell,1}$ from Section 3.1 to a function on $\mathbb{R}$.

Remark. A solution u in Theorem 4.2 can instantly be transformed by $u_\varepsilon := u\left(\frac{\cdot}{\varepsilon}\right)$ to a solution of

$$-\varepsilon^2 \Delta u_\varepsilon + f(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^2. \tag{4.2}$$

Therefore, Theorem 4.2 is also true for solutions of (4.2) and sufficiently large ratios $\frac{\ell}{\varepsilon} > \ell_{\min}$.

Before we prove the theorem we present a simple consequence which shows that there cannot exist other multisaddle solutions than the symmetric ones at least in the case of sufficiently wide stripes.

Corollary 4.3. *Assume a consecutive index set $I \subset \mathbb{Z}$ as in Definition 4.1 and*

$$|z_i - z_j| > \ell_{min} \quad \text{for all } i, j \in I \text{ with } i \neq j.$$

Then there exists a corresponding multisaddle solution of (4.1) if and only if $I = \{0\}$ or

$I = \mathbb{Z}$ with

$$z_i = i\ell \quad \text{for } i \in \mathbb{Z}.$$

for a constant $\ell > \ell_{min}$.

We prove Corollary 4.3 at the end of this chapter.

The proof of Theorem 4.2 is divided into several propositions. For the rest of this chapter we fix the notation

$$R^\ell := (0, \ell) \times (0, \infty) \tag{4.3}$$

for the semi infinite stripe with width $\ell > 0$. In some proofs we also consider bounded subsets of R^ℓ. For $s > 0$ we define

$$R^\ell_s := (0, \ell) \times (0, s). \tag{4.4}$$

We will first find a classical solution of (4.1) on R^ℓ with $u = 0$ on ∂R^ℓ and $u > 0$ in R^ℓ by applying the method of sub- and supersolutions. Although this result is fairly standard, there seems to be no exact reference for the case of unbounded domains. For the sake of completeness, we give a detailed proof below.
By odd reflection we then extend u to $\mathbb{R}^2$ and prove that the thereby constructed function is indeed a multisaddle solution of (4.1).

Definition 4.4. We call a function u a *weak supersolution* (*subsolution* resp.) of (4.1) in a (possibly unbounded) domain Ω if the following conditions hold:

i) $u\varphi \in H^1(\Omega)$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$

ii) $\int_\Omega \nabla u \cdot \nabla \varphi + f(u)\varphi \geq 0$ (≤ 0 resp.) for all $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$ in Ω

iii) $u \geq 0$ (≤ 0 resp.) on $\partial\Omega$ in the trace sense.

Furthermore, we call u a *weak solution* of (4.1) in Ω if u is a weak sub- and supersolution of (4.1).

The specific result we need for the proof of Theorem 4.2 reads

Proposition 4.5. *For $\ell > 0$ let $\underline{v}$, $\overline{v}$ be a weak sub- and supersolution on R^ℓ as in Definition 4.4 with*

$$0 \leq \underline{v} \leq \overline{v} \leq 1 \quad \text{a.e. in } R^\ell$$

and $\underline{v} = \overline{v} = 0$ on ∂R^ℓ. Then there exists a weak solution u of (4.1) in R^ℓ with

$$\underline{v} \leq u \leq \overline{v} \quad \text{a.e. in } R^\ell$$

and $u = 0$ on ∂R^ℓ.

Proof. Step 1: For $L > 0$ we restrict ourselves to the finite rectangle R^ℓ_L as in (4.4). Since $\underline{v}$ and $\overline{v}$ restricted to R^ℓ_L are sub- and supersolution for (4.1) on R^ℓ_L with $\underline{v}, \overline{v} \geq 0$

on $\{x_2 = L\} \subset \partial R^\ell_L$, we can use the method of weak sub- and supersolutions (see [DaSw89]) to find for every $L > 0$ a weak solution $u_L \in H^1(R^\ell_L)$ of

$$\begin{cases} -\Delta u_L + f(u_L) = 0 & \text{in } R^\ell_L \\ \qquad\qquad u_L \; = 0 & \text{on } \partial R^\ell_L \cap \partial R^\ell \end{cases} \tag{4.5}$$

which satisfies $0 \le \underline{v} \le u_L \le \overline{v} \le 1$ on R^ℓ_L. Precisely, we extend $\underline{v}$ and $\overline{v}$ to a weak sub- and supersolution on the domain $\tilde{R}^\ell_L := (0,\ell) \times (0, 2L)$ by an even reflection on the line $\{x_2 = L\}$. We note that $\underline{v} = \overline{v} = 0$ on $\partial \tilde{R}^\ell_L$ and apply the mentioned technique to show the existence of a weak solution between $\underline{v}$ and $\overline{v}$ on $\tilde{R}^\ell_L$ with vanishing boundary values. A restriction on R^ℓ_L then yields the desired function u_L.

Step 2: ($L \to \infty$) We fix some some $\tilde{L} > 1$ and show that u_L with $L \ge \tilde{L}$ is uniformly bounded in $H^1(R^\ell_{\tilde{L}-1})$.
Let $\eta \in C^\infty(\overline{R^\ell})$ be a smooth cutoff function with

$$\eta(x) = \begin{cases} 1, & x_2 \le \tilde{L} - 1 \\ 0, & x_2 \ge \tilde{L} \end{cases}$$

satisfying $0 \le \eta \le 1$ and $|\nabla \eta| \le 2$ in R^ℓ. We multiply (4.5) by $\eta^2 u_L \in H^1_0(R^\ell_{\tilde{L}})$ and integrate over $R^\ell_{\tilde{L}}$ to obtain

$$\begin{aligned} 0 &= \int_{R^\ell_{\tilde{L}}} \nabla u_L \cdot \nabla \left(\eta^2 u_L\right) + f(u_L)\eta^2 u_L \, dx \\ &= \int_{R^\ell_{\tilde{L}}} |\nabla u_L|^2 \eta^2 + 2 u_L \eta \nabla u_L \cdot \nabla \eta + f(u_L) u_L \eta^2 \, dx \end{aligned}$$

and hence

$$\begin{aligned} \int_{R^\ell_{\tilde{L}}} |\nabla u_L|^2 \eta^2 \, dx &= - \int_{R^\ell_{\tilde{L}}} 2 u_L \eta \nabla u_L \cdot \nabla \eta + f(u_L) u_L \eta^2 \, dx \\ &\le \int_{R^\ell_{\tilde{L}}} \frac{1}{2} |\nabla u_L|^2 \eta^2 + 2 u_L^2 \left|\nabla \eta^2\right|^2 + |f(u_L) u_L| \, \eta^2 \, dx \end{aligned}$$

where we have applied Young's inequality in the last line. Rearranging the terms on both sides and using the prescribed bounds on η gives us

$$\begin{aligned} \int_{R^\ell_{\tilde{L}}} \frac{1}{2} |\nabla u_L|^2 \eta^2 \, dx \le &\int_{R^\ell_{\tilde{L}}} C u_L^2 + |f(u_L) u_L| \, dx \\ \le &C(\tilde{L}) \end{aligned}$$

since $|u_L| \le 1$ and therefore also $|f(u_L)| \le 1$ almost everywhere in R^ℓ. As $\eta \equiv 1$ in

$R^\ell_{\tilde{L}-1}$ this particularly implies

$$\int_{R^\ell_{\tilde{L}-1}} |\nabla u_L|^2 \, dx \le C(\tilde{L})$$

and thus, u_L is bounded in $H^1(R^\ell_{\tilde{L}-1})$ uniformly for all $L > \tilde{L}$. Hence, for every given $\tilde{L} > 1$ and every sequence $(u_{L_k})_{k\in\mathbb{N}}$ with $L_k \to \infty$ as $k \to \infty$ we can find a subsequence denoted by $(u_k^{\tilde{L}})_{k\in\mathbb{N}}$ which converges weakly in $H^1(R^\ell_{\tilde{L}-1})$. Now we extract a diagonal sequence $(u_k^k)_{k\in\mathbb{N}}$ which converges weakly in $H^1(R^\ell_{\tilde{L}-1})$, strongly in $L^2(R^\ell_{\tilde{L}-1})$ and pointwise almost everywhere in $R^\ell_{\tilde{L}-1}$ to a function $u : R^\ell \to \mathbb{R}$ for *every* $\tilde{L} > 0$ (by the uniqueness of limits).
Choose an arbitrary test function $\varphi \in C_c^\infty(\mathbb{R}^2)$. From the convergence of the diagonal sequence we obtain

$$u_k^k \varphi \rightharpoonup u\varphi \quad \text{in } H^1(R^\ell)$$

as there exists $\tilde{L} > 0$ such that $\operatorname{supp} \varphi \cap \{x_2 > \tilde{L} - 1\} = \emptyset$. This especially proves

$$u\varphi \in H^1(R^\ell) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^2).$$

It remains to show that u is indeed a weak solution on R^ℓ. Therefore, let $\varphi \in C_c^\infty(R^\ell)$ be any testfunction. Then there exists $k_0 \in \mathbb{N}$ such that $\operatorname{supp}(\varphi) \subset R^\ell_{k_0}$ and hence,

$$\int_{R^\ell} \nabla u_k^k \cdot \nabla\varphi + f(u_k^k)\varphi \, dx = \int_{R^\ell_{k_0}} \nabla u_k^k \cdot \nabla\varphi + f(u_k^k)\varphi \, dx = 0 \tag{4.6}$$

for all $k \ge k_0$. The first term of the integrand now converges due to the weak convergence of $(u_k^k)_{k\in\mathbb{N}}$ in $H^1(R^\ell_{k_0})$. For the second term we observe that

$$\int_{R^\ell_{k_0}} \left| f(u_k^k) - f(u) \right|^2 dx \le C \int_{R^\ell_{k_0}} |u_k - u|^2 \, dx \longrightarrow 0$$

as $k \to \infty$ where we have used that f is locally Lipschitz. This yields

$$f(u_k^k) \to f(u) \quad \text{in } L^2(R^\ell_{k_0})$$

and we can therefore pass to the limit $k \to \infty$ in (4.6). This gives

$$\int_{R^\ell} \nabla u \cdot \nabla\varphi + f(u)\varphi \, dx = 0$$

and u is a weak solution on R^ℓ. Due to the pointwise convergence of $(u_k^k)_{k\in\mathbb{N}}$ almost everywhere we immediately deduce

$$\underline{v}(x) \le u(x) \le \overline{v}(x)$$

for a.e. $x \in R^\ell$ which completes the proof. □

Before we continue and apply the above proven method to show the existence of solutions on R^ℓ we prove a short regularity result for solutions of the Allen-Cahn equation.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^2$ be an open domain and assume that there exists a weak solution of* (4.1) *in Ω satisfying $-1 < u < 1$. Then we have $u \in C^\infty(\Omega)$ and for all subsets $\tilde{\Omega} \subset\subset \Omega$*

$$\sup_{x\in\tilde{\Omega}} \left|\partial^\beta u\right|(x) \leq C_k \quad \text{for all } |\beta| \leq k,\, k \geq 0 \tag{4.7}$$

with constants $C_k > 0$ only depending on Ω and $\tilde{\Omega}$.

Proof. For $\tilde{\Omega} \subset\subset \Omega$ we choose a sequence of open and smoothly bounded sets $U_i \subset\subset \Omega$ with $\tilde{\Omega} \subset\subset U_i$ for all $i \in \mathbb{N}$ and such that

$$U_{i+1} \subset\subset U_i \quad \text{for all } i \in \mathbb{N}.$$

u is a weak solution of the Poisson equation $\Delta u = g$ on Ω where $g := f(u)$ is $L^2(\Omega)$ since u was assumed to be bounded. Hence, we can apply standard theory for interior elliptic regularity ([GiTr01], Theorem 8.8), to conclude $u \in H^2(U_1)$. This implies $g \in H^1(U_1)$ as f is smooth and we obtain $u \in H^3(U_2)$. Continuing this argument finally yields $u \in H^k(\tilde{\Omega})$ for all $k \geq 0$ and hence, by the general Sobolev embedding theorem ([Eva10], 5.6.3, Theorem 6) it follows that for any multiindex $\beta \in (\mathbb{N}_0)^2$ with $|\beta| \leq k$

$$\sup_{x\in\tilde{\Omega}} \left|\partial^\beta u\right|(x) \leq \|u\|_{C^{k,\frac{1}{2}}(\bar{\tilde{\Omega}})} \leq C(k,\Omega,\tilde{\Omega})\, \|u\|_{H^{k+2}(\tilde{\Omega})}\,.$$

This proves (4.7) and as k and $\tilde{\Omega}$ were arbitrary also $u \in C^\infty(\Omega)$. □

Proposition 4.7. *There exists a constant $\tilde{\ell}_{\underline{min}} > 0$ such that for all $\ell > \tilde{\ell}_{min}$ sufficiently large there exists a smooth solution $u \in C^\infty(\overline{R^\ell})$ of* (4.1) *on R^ℓ with $u = 0$ on ∂R^ℓ and $0 < u < 1$ in R^ℓ.*

Proof. We start by proving the existence of a solution on R^ℓ from (4.3). Due to Proposition 4.5 it is sufficient to find a weak sub- and supersolution $\underline{v}, \overline{v}$ of (4.1).
In [DaFiPe92], Lemma 1 the existence of radial solutions U_b, $b \in (0,1)$ of (4.1) was shown which attend their positive maximum value b in the origin. Furthermore, for each b there exists a radius $r_b > 0$ (depending continuously on b) with $U_b > 0$ in $[0, r_b)$ and $U_b(r_b) = 0$. We choose $\tilde{\ell}_{\min} = 2\inf\{r_b : b \in (0,1)\}$ and hence for $\ell > \tilde{\ell}_{\min}$ there exists $b \in (0,1)$ such that

$$B\left(\left(\frac{\ell}{2}, \frac{\ell}{2}\right), r_b\right) \subset R^\ell.$$

We now set

$$\underline{v}(x) := \max\left\{U_b\left(\left|x - \left(\frac{\ell}{2}, \frac{\ell}{2}\right)\right|\right), 0\right\}, \quad x \in R^\ell$$

which as a maximum of two solutions is a subsolution of (4.1).

As a supersolution we set

$$\overline{v}(x) = q_\ell(x_1)\gamma(x_2) > 0, \quad x = (x_1, x_2) \in R^\ell$$

where $q_\ell := q_{\ell,1}$ denotes the optimal arc profile from Section 3.1. Indeed, we have

$$\begin{aligned}
-\Delta\overline{v} + f(\overline{v}) &= -q_\ell''\gamma - q_\ell\gamma'' + q_\ell^3\gamma^3 - q_\ell\gamma \\
&= (-q_\ell^3 + q_\ell)\gamma + q_\ell(-\gamma^3 + \gamma) + q_\ell^3\gamma^3 - q_\ell\gamma \\
&= (q_\ell^3 - q_\ell)(\gamma^3 - \gamma) \\
&> 0
\end{aligned}$$

since q_ℓ and γ both take values in $(0,1)$ in the considered region.
We can now apply Proposition 4.5 and find a weak solution u of (4.1) on R^ℓ with $u\varphi \in H^1(R^\ell)$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and satisfying

$$0 \leq \underline{v} \leq u \leq \overline{v} < 1 \quad \text{in } R^\ell$$

and $u = 0$ on ∂R^ℓ.
By three odd reflections of u at the x_1-axis, the x_2-axis and the line $\{x_1 = \ell\}$ one immediately obtains a weak solution of (4.1) in $(-\ell, 2\ell) \times \mathbb{R}$ with 0 boundary values and satisfying $-1 < u < 1$. Indeed, the thereby constructed function u solves (4.1) since f is odd. Now, Lemma 4.6 directly implies $u \in C^\infty(\overline{R^\ell})$ as we can cover $\overline{R^\ell}$ by open and smoothly bounded sets $\Omega \subset (-\ell, 2\ell) \times \mathbb{R}$.
Using the mean value theorem of integral calculus in (4.1) together with $f(0) = 0$ we deduce

$$-\Delta u + cu = 0 \quad \text{in } R^\ell$$

with $c := \fint_0^u f'(s)\,ds$ and the strong maximum principle (see Theorem 3.5 in [GiTr01] and the remark thereafter) implies

$$u > 0 \quad \text{in } R^\ell.$$

This completes the proof. □

Remark. Instead of using the theory of sub- and supersolutions in the proof of Proposition 4.7 it is also possible to apply energy based methods to find a minimizer of the Ginzburg-Landau-energy (1.2) with the desired form which satisfies the corresponding Euler Lagrange equation (4.1). This alternative method yields the same result in two dimensions but can easier be generalized for higher (even) dimensions. However, it then becomes quite technical to show that the thereby constructed solution is not identical to 0 since you do not have a subsolution for a comparison anymore. We refer the reader to [CaTe09] where the authors apply this method to prove the existence of saddle-shaped solutions in all even dimensions.

We continue by describing the qualitative behavior of the solution u which has been constructed in Proposition 4.7 above. We show that it increases in x_2-direction and converges uniformly in C^2 towards the optimal arc profile q_ℓ for $x_2 \to \infty$. This yields a strong connection between entire solutions of the stationary Allen-Cahn equation in one and two dimensions.

To show the monotonicity of *any* bounded solution u of (4.1) on the infinite rectangle R^ℓ we apply the *Method of Moving Planes*. In its origins this technique has been developed by Alexandroff [AL56]. In [GNN79] and later in [GNN81] the authors have applied the method to show monotonicity and resulting symmetry properties of solutions of equations similar to (4.1) on bounded domains and in the whole of $\mathbb{R}^n$. Since then many generalizations and applications have been made (see, e.g., [Li91, BeNi88, BeNi91]).
The key idea is to introduce a hyperplane T which cuts the domain in two parts and compare the values of u in one point and its reflection at T. Then the hyperplane is moved in one direction up to a critical value where u attains a larger value than its reflection. Usually a maximum principle as the key ingredient for this technique then yields a contradiction which proves the monotonicity. But exactly this step requires a careful analysis since the maximum principle does not hold in every domain. For unbounded domains irregular boundaries (with corners e.g.) it is often difficult to apply the technique directly.
Berestycki and Nirenberg generalized the method to a wider class of domains in [BeNi88] and [BeNi91] by proving several maximum principles and adapting the way of argumentation. First, a weak maximum principle for arbitrary narrow domains (see Proposition A.1 or [BeNi91]) yields the assertion for a small part of the domain. Afterwards, another weak maximum principle for subdomains of small measure (Proposition A.2 or [BeNi91]) is used to enlarge the part of the domain where the statement still holds.
The proposition and its proof below rely heavily on the techniques presented in [BeNi91] although we will consider the unbounded domain R^ℓ instead of a bounded one.

Proposition 4.8. *Let $\ell > 0$ be arbitrary and R^ℓ given by* (4.3). *Every solution $u \in C^2(\overline{R^\ell})$ of* (4.1) *which satisfies*

$$\begin{cases} \Delta u - f(u) = 0 & \text{in } R^\ell \\ u = 0 & \text{on } \partial R^\ell \\ 0 < u < 1 & \text{in } R^\ell \end{cases}$$

is strictly increasing in x_2-direction for fixed $x_1 \in (0, \ell)$.

Proof. For $\lambda > 0$ we define

$$T_\lambda := \{x_2 = \lambda\},$$
$$\Sigma(\lambda) := \{x \in R^\ell \,:\, x_2 < \lambda\},$$

(see Figure 4.3) and set

$$v(x) := v(x, \lambda) := u(x_1, 2\lambda - x_2),$$
$$w(x, \lambda) := v(x) - u(x),$$

for $x = (x_1, x_2) \in \Sigma(\lambda)$. Defined in this way v satisfies (4.1) and coincides with u at the line T_λ. The strict monotonicity of u in x_2-direction is therefore equivalent to the positivity of w for all $\lambda > 0$ and $x \in \Sigma(\lambda)$. Indeed, for fixed $x_1 \in (0, \ell)$ and $0 < x_2 < \tilde{x}_2$ we choose $\tilde{\lambda} = \frac{1}{2}(\tilde{x}_2 - x_2)$. Thus, $u(x_1, x_2) < u(x_1, \tilde{x}_2)$ corresponds to $w((x_1, x_2), \tilde{\lambda}) > 0$ and vice versa.

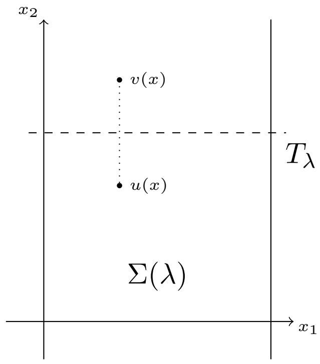


Figure 4.3

We will use various classical maximum principles to show that w is always greater than 0. We begin with the observation that for $u \neq v$ we have

$$\Delta w = \Delta v - \Delta u = f(v) - f(u) = \frac{f(v) - f(u)}{v - u}(v - u)$$

and in the case $u = v$

$$\Delta w = 0.$$

We therefore define

$$-c(\lambda) := -c(x, \lambda) := \begin{cases} \frac{f(v(x,\lambda)) - f(u(x))}{v(x.\lambda) - u(x)}, & v(x, \lambda) \neq u(x) \\ 0, & v(x, \lambda) = u(x) \end{cases}$$

and note that hence $c(\lambda)$ is a bounded function since f is locally Lipschitz. Consequently, w satisfies

$$\begin{cases} \Delta w(\cdot, \lambda) + c(\lambda) w(\cdot, \lambda) = 0 & \text{in } \Sigma(\lambda) \\ w(\cdot, \lambda) \geq 0 & \text{on } \partial\Sigma(\lambda). \end{cases}$$

Observe that $w(\cdot, \lambda) = 0$ on $\partial\Sigma(\lambda) \setminus \{x_2 = 0\}$ and $w((x_1, 0), \lambda) = v((x_1, 0), \lambda) = u(x_1, 2\lambda) > 0$ for all $\lambda > 0$ and $x_1 \in (0, \ell)$. Thus, $w(\cdot, \lambda)$ does not vanish completely on $\partial\Sigma(\lambda)$.
For small $\lambda > 0$, $\Sigma(\lambda)$ is a narrow domain in the sense of Proposition A.1 and therefore, the maximum principle holds. We conclude that $w \geq 0$ in $\Sigma(\lambda)$ and the strong maximum principle (see [GiTr01], Theorem 3.5 and the remark thereafter) even yields

$$w(\cdot, \lambda) > 0 \quad \text{in } \Sigma(\lambda) \tag{4.8}$$

since $w(\cdot, \lambda)$ is not constant.

Now, we want to show that (4.8) remains true for all $\lambda > 0$. We shift the plane T_λ in vertical direction and argue by contradiction: Suppose there exists $\mu > 0$ such that $(0, \mu)$ is the largest interval with $w(\cdot, \lambda) > 0$ in $\Sigma(\lambda)$ for all $\lambda \in (0, \mu)$. Since w obviously is continuous in λ, we deduce $w(\cdot, \mu) \geq 0$ in $\Sigma(\mu)$.
Again we use the strong maximum principle to see that

$$w(\cdot, \mu) > 0 \quad \text{in } \Sigma(\mu). \tag{4.9}$$

Indeed, if there exists $x_0 \in \Sigma(\mu)$ with $w(x_0, \mu) = 0$, $w(\cdot, \mu)$ attains its minimum in an interior point and thus is constant which contradicts the non vanishing boundary values of $w(\cdot, \mu)$.
We derive the desired contradiction by proving

$$w(\cdot, \mu + \varepsilon) > 0 \quad \text{in } \Sigma(\mu + \varepsilon)$$

for sufficiently small $\varepsilon > 0$. For that purpose choose $\delta > 0$ as in Proposition A.2 and let $K \subset \Sigma(\mu)$ be a closed subset such that

$$|\Sigma(\mu) \setminus K| \leq \frac{\delta}{2}.$$

Obviously, by (4.9) we have

$$w(\cdot, \mu) > 0 \quad \text{in } K. \tag{4.10}$$

As above, by the fact that $w(\cdot, \mu)$ is continuous in μ by definition, there exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ we have

$$|\Sigma(\mu + \alpha) \setminus K| \leq \delta \qquad \text{and} \qquad w(\cdot, \mu + \alpha) > 0 \quad \text{in } K.$$

For any such α we set $\tilde{\Sigma} := \Sigma(\mu + \alpha) \setminus K$ and remark that hence

$$\partial\tilde{\Sigma} = \partial K \cup \partial R.$$

Therefore, $w(\cdot, \mu + \alpha)$ satisfies

$$\begin{cases} \Delta w(\cdot, \mu + \alpha) + c(\mu + \alpha) w(\cdot, \mu + \alpha) & = 0 \quad \text{in } \tilde{\Sigma} \\ w(\cdot, \mu + \alpha) & \geq 0 \quad \text{on } \partial\tilde{\Sigma} \end{cases}$$

where we have used that $w(\cdot, \mu + \alpha) \geq 0$ on $\partial\Sigma(\mu + \alpha)$ and $w(\cdot, \mu + \alpha) > 0$ on $\partial K \subset K$ by (4.10) (see Figure 4.4).

By Proposition A.2 the weak maximum principle for w holds in $\tilde{\Sigma}$ and we obtain

$$w(\cdot, \mu + \alpha) \geq 0 \quad \text{in } \tilde{\Sigma}$$

and therefore

$$w(\cdot, \mu + \alpha) \geq 0 \quad \text{in } \Sigma(\mu + \alpha) = \tilde{\Sigma} \cup K.$$

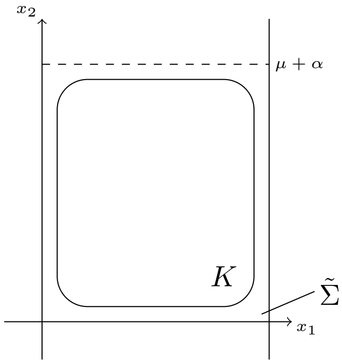


Figure 4.4

Once again the strong maximum principle finally yields the contradiction

$$w(\cdot, \mu + \alpha) > 0 \quad \text{in } \Sigma(\mu + \alpha)$$

since μ was chosen to be maximal.
This implies $w(\cdot, \lambda) > 0$ for all $\lambda > 0$ and thus, the strict monotonicity in x_2-direction as already mentioned above. □

In the following, we will show that far away from $\{x_2 = 0\}$ any positive bounded solution of (4.1) on R^ℓ is shaped like the optimal arc profile q_ℓ. This can be explained heuristically as for sufficiently large values of x_2 the influence of the prescribed zero boundary values at $\{x_2 = 0\}$ vanishes and u approaches the unique positive solution of the one-dimensional Dirichlet problem which is q_ℓ as stated in Section 3.1.
The boundedness and monotonicity of u immediately yield pointwise convergence of $u(x_1, \cdot)$ for fixed $x_1 \in [0, \ell]$ as $x_2 \to \infty$. To pass to the limit in (4.1), however, we need a stronger convergence result. For that purpose we fix real numbers $0 < a < b$ and set

$$A := [0, \ell] \times [a, b] \subset \overline{R^\ell}.$$

Then we define for $\xi \geq 0$ a vertically shifted version of u by

$$\tilde{u}_\xi(x, y) = u(x, y + \xi), \quad (x, y) \in R^\ell.$$

This shift does not change the derivative of u and consequently, it satisfies

$$\begin{cases} \Delta \tilde{u}_\xi - f(\tilde{u}_\xi) &= 0 \quad \text{in } A \\ \qquad\qquad \tilde{u}_\xi &= 0 \quad \text{on } \{0, \ell\} \times [a, b]. \end{cases}$$

By this construction we keep the domain A fixed but as varying ξ is equivalent to a slide of the domain in vertical direction through R^ℓ we can still study the limit $x_2 \to \infty$. We prepare the convergence result in the following lemma.

Lemma 4.9. *For $\ell > 0$ let $u \in C^2(\overline{R^\ell})$ be a solution of* (4.1) *satisfying* $0 < u < 1$ *in*

R^ℓ *and* $u = 0$ *on* ∂R. *Then there exists a constant* $C = C(A)$ *such that*

$$\|\tilde{u}_\xi\|_{C^3(A)} \leq C \tag{4.11}$$

uniformly in $\xi \geq 0$ *where* $\tilde{u}_\xi$ *is defined as above.*

Proof. This follows directly from Lemma 4.6. By odd reflections we extend u to a (weak) solution of (4.1) on $(-\ell, 2\ell) \times (0, \infty)$ and (4.7) gives (4.11) by choosing A as the compact subset. □

We are now able to give the precise characterization of the limit $x_2 \to \infty$ ($\xi \to \infty$, respectively). The idea is to use the uniform bound from Lemma 4.9 for a compactness argument and afterwards pass to the limit in equation (4.1).

Proposition 4.10. *Let* $\ell_{min} := \max\{\tilde{\ell}_{min}, \ell_1\}$ *(with* ℓ_1 *from Proposition 3.1) and* $\ell > \ell_{min}$. *Every solution* $u \in C^2(\overline{R^\ell})$ *of* (4.1) *with* $0 < u < 1$ *and* $u = 0$ *on* ∂R^ℓ *converges uniformly towards* q_ℓ *in* $C^2([0, \ell])$ *as* $x_2 \to \infty$.

Proof. As $u(x_1, \cdot)$ is increasing and bounded for every $x_1 \in [0, \ell]$ it converges pointwise and we set

$$\lim_{x_2 \to \infty} u(x_1, x_2) := u_\infty(x_1), \quad x_1 \in [0, \ell]. \tag{4.12}$$

Now, let $(\xi_j)_{j \in \mathbb{N}}$ be an arbitrary nonnegative sequence with $\xi_j \to \infty$ as $j \to \infty$. Due to Lemma 4.9 the set $\{u_{\xi_j} : j \in \mathbb{N}\}$ is uniformly bounded in $C^3(A)$ and hence equicontinuous in $C^2(A)$ with u_{ξ_j} defined as above. Thus, by the Arzelá-Ascoli theorem we can find a subsequence $j_k \to \infty$ as $k \to \infty$ such that

$$\tilde{u}_{\xi_{j_k}} = u(\cdot, \cdot + \xi_{j_k}) \longrightarrow \tilde{u} \quad \text{in } C^2\big([0, \ell]\big)$$

for some $\tilde{u} \in C^2([0, \ell])$. But as the uniform convergence implies pointwise convergence of this subsequence we immediately deduce

$$\tilde{u} = u_\infty$$

by (4.12). This means that for every sequence $(\xi_j)_{j \in \mathbb{N}}$ tending to infinity we can find a subsequence which converges towards u_∞ in $C^2([0, \ell])$. Hence, this implies

$$u(\cdot, x_2) \longrightarrow u_\infty \quad \text{in } C^2\big([0, \ell]\big)$$

as $x_2 \to \infty$.
The convergence allows us to pass to the limit in (4.1) to obtain

$$-u_\infty'' + f(u_\infty) = 0 \quad \text{in } [0, \ell]$$

as well as

$$u_\infty(0) = u_\infty(\ell) = 0, \qquad u_\infty > 0 \text{ in } (0, \ell).$$

This identifies u_∞ as the unique solution of the one-dimensional problem (Proposition 3.1) and we finally conclude

$$u_\infty = q_\ell. \qquad \square$$

For later use we show another statement which follows directly from the proof of Proposition 4.10.

Corollary 4.11. *For $\ell > \ell_{min}$ every solution $u \in C^2(\overline{R^\ell})$ with $0 < u < 1$ and $u = 0$ on ∂R^ℓ satisfies*

$$\int_0^\ell |\partial_2 u(x_1, x_2)| \, dx_1 \longrightarrow 0 \quad , x_2 \to \infty. \tag{4.13}$$

Proof. By Proposition 4.10 u and $\partial_2 u$ converge uniformly as $x_2 \to \infty$. This already implies that $\partial_2 u(x_1, x_2) \to 0$ as $x_2 \to \infty$ for all $x_1 \in [0, \ell]$ since any other limit value would contradict the convergence of u. Now (4.13) follows directly with

$$\begin{aligned} \lim_{x_2 \to \infty} \int_0^\ell |\partial_2 u(x_1, x_2)| \, dx_1 &= \lim_{x_2 \to \infty} \int_0^\ell \partial_2 u(x_1, x_2) \, dx_1 \\ &= \int_0^\ell \lim_{x_2 \to \infty} \partial_2 u(x_1, x_2) \, dx_1 \\ &= 0. \end{aligned} \qquad \square$$

The last remaining point to show is the uniqueness of the solution u. The proof presented here goes back to [DaFiPe92], Theorem 1, where the authors use the same technique to show uniqueness for solutions of (4.1) with one saddle.

Proposition 4.12. *For $\ell > \ell_{min}$ there exists a unique solution of (4.1) on R^ℓ with $u = 0$ on ∂R^ℓ satisfying $0 < u < 1$.*

Proof. It remains to show the uniqueness of solutions. Assume there exist two solutions $u_1, u_2 \in C^2(\overline{R^\ell})$ with $0 < u_1, u_2 < 1$ and $u_1, u_2 \not\equiv 0$. Without loss of generality we can assume that $u_1 < u_2$. Otherwise $\max(u_1, u_2)$ is a weak subsolution and there exists a solution u of (4.1) with

$$\max(u_1, u_2) \le u \le 1 \quad \text{in } R^\ell.$$

As above we can apply the strong maximum principle from [GiTr01], Theorem 3.5 and the remark thereafter, to conclude

$$u > u_1 \quad \text{and} \quad u > u_2.$$

We restrict ourselves to finite rectangles $R_s^\ell = (0, \ell) \times (0, s)$ and obtain by (4.1) for all $s \ge 1$

$$\int_{R_s^\ell} u_1 \Delta u_2 - u_2 \Delta u_1 \, dx = \int_{R_s^\ell} u_1 u_2 \left[\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right] dx > C \tag{4.14}$$

for a constant $C > 0$ independent of s. The inequality follows from the fact that on the one hand $\frac{f(h)}{h} = h^2 - 1$ is strictly monotonically increasing in h and on the other hand that u_1 and u_2 are strictly positive in R_s^ℓ.
We apply Green's Formula on the left hand side of (4.14) and thus obtain

$$\begin{aligned} C &< \int_{\partial R_s^\ell} u_1 \partial_\nu u_2 - u_2 \partial_\nu u_1 \, d\mathcal{H}^1 \\ &= \int_0^\ell u_1(x_1, s)\partial_2 u_2(x_1, s) - u_2(x_1, s)\partial_2 u_1(x_1, s)\, dx_1 \\ &\leq \int_0^\ell |\partial_2 u_2(x_1, s)| \, dx_1 + \int_0^\ell |\partial_2 u_1(x_1, s)| \, dx_1 \end{aligned}$$

where we have used that u_1 and u_2 vanish on everywhere of the boundary ∂R_s^ℓ except on the upper portion. By Corollary 4.11 the right hand side tends to 0 as $s \to \infty$. This yields the desired contradiction and the uniqueness is proven. □

A combination of the propositions above now yields a proof for Theorem 4.2 and Corollary 4.3.

Proof of Theorem 4.2. We extend the solution u on R^ℓ from above via odd reflections and periodic continuation to a (weak) solution of (4.1) on $\mathbb{R}^2$. By the local regularity result from Lemma 4.6 u is smooth on $\mathbb{R}^2$ and hence is a multisaddle solution in the sense of Definition 4.1 with $I = \mathbb{Z}$ and $z_i = \ell i$ for all $i \geq 0$.
The uniqueness of u follows directly from Proposition 4.12. Indeed, if there exists another multisaddle solution as in the theorem we can restrict it to one semi infinite rectangle where it differs from u. This already contradicts the uniqueness of solutions on R^ℓ. □

Proof of Corollary 4.3. Consider a multisaddle solution of (4.1) and assume that two consecutive rectangles have finite widths $\ell, \tilde{\ell} > \ell_{\min}$. Without loss of generality we can choose them to be $(0, \ell) \times (0, \infty)$ and $(\ell, \ell + \tilde{\ell}) \times (0, \infty)$ and assume u to be positive inside the first and negative inside the second one. Due to Lemma 4.6, u is smooth and by Proposition 4.12 u is determined uniquely on both separate stripes. Moreover, we have

$$\begin{aligned} &u(\cdot, x_2) \longrightarrow q_\ell && \text{in } C^2\big([0, \ell]\big), \\ &u(\cdot, x_2) \longrightarrow -q_{\tilde{\ell}}(\cdot - \ell) && \text{in } C^2\big([\ell, \ell + \tilde{\ell}]\big) \end{aligned}$$

as $x_2 \to \infty$. Hence, we can choose $\bar{x}_2 > 0$ such that

$$\left|u'(\ell, \bar{x}_2) - q_\ell'(\ell)\right| \leq \frac{1}{4}\left|q_\ell'(0) - q_{\tilde{\ell}}'(0)\right|$$

and

$$\left|u'(\ell, \bar{x}_2) + q_{\tilde{\ell}}'(0)\right| \leq \frac{1}{4}\left|q_\ell'(0) - q_{\tilde{\ell}}'(0)\right|$$

hold. We conclude

$$\begin{aligned}\left|q'_\ell(0) - q'_{\tilde\ell}(0)\right| &= \left|q'_\ell(\ell) + q'_{\tilde\ell}(0)\right| \le \left|u'(\ell,\bar x_2) - q'_\ell(\ell)\right| + \left|u'(\ell,\bar x_2) + q'_{\tilde\ell}(0)\right| \\ &\le \frac{1}{2}\left|q'_\ell(0) - q'_{\tilde\ell}(0)\right|\end{aligned}$$

and hence, $q'_\ell(0) = q'_{\tilde\ell}(0)$ which is equivalent to $\ell = \tilde\ell$ by Proposition 3.1.
Analogously, the argument holds for infinite stripe widths and the optimal profile γ instead of the optimal arcs. For the corresponding convergence result see [DaFiPe92]. This proves that multisaddle solutions with different stripe widths cannot exist. □

Remark. We want to point out that the concept of multisaddle solutions is just one possible class of entire solutions of (4.1) when generalizing the quasi one-dimensional pattern from Section 3.2. One could also think of classical checkerboard patterns or pavings of the plane with arbitrary rectangles. Having the results of this chapter in mind, it seems natural (at least for large length scales) that entire solutions of the shapes described above can only exist if the pattern is completely symmetric and all panels have the same size. We omit a proof here, though the methods to study the existence and qualitative properties of such solutions are identical to the tools used in this chapter.

5 The L^2-gradient flow of $\mathcal{W}_\varepsilon$: Existence and qualitative behavior

5.1 Longtime existence

In this part we consider the existence and uniqueness of periodic solutions for the L^2-gradient flow of $\mathcal{W}_\varepsilon$. Although we are only interested in the one-dimensional case hereinafter, the contemplation can be generalized for space dimensions $n \leq 3$ without any problems.
Throughout the rest of the section let $1 \leq n \leq 3$ and let U denote a nonempty n-dimensional rectangle, i.e.,

$$U := \prod_{k=1}^{n} (a_k, b_k) \subset \mathbb{R}^n$$

for $a_k, b_k \in \mathbb{R}$ and $a_k < b_k$, $1 \leq k \leq n$. Furthermore, for $m \in \mathbb{N}_0$ let

$$H^m_{\text{per}} := \overline{C^\infty_{\text{per}}(U)}$$

be the space of periodic H^m-functions in U where the closure is taken with respect to the H^m-norm.
We also assume $\varepsilon > 0$ to be a fixed parameter and for simplicity, we suppress the usual indices of functions with ε dependency. We also allow constants depending on ε in this section.

We will prove the existence and uniqueness of weak periodic solutions for the evolution equation

$$\begin{cases} \partial_t u = -\nabla_{L^2}\mathcal{W}_\varepsilon(u) = \Big(\Delta - \dfrac{1}{\varepsilon^2}F''(u)\Big)v \text{ in } (0,\infty)\times U & (5.1a) \\ \quad v = -\varepsilon\Delta u + \dfrac{1}{\varepsilon}F'(u) \qquad \text{in } (0,\infty)\times U & (5.1b) \\ u(0) = u_0 \qquad \text{in } U & (5.1c) \end{cases}$$

with initial data $u_0 \in H^2_{\text{per}}(U)$. Precisely, we are interested in periodic functions which satisfy the following

Definition 5.1. We call $u : [0,\infty) \times U \to \mathbb{R}$ a weak periodic solution of (5.1) if

$$u \in H^1(0,T; L^2_{\text{per}}(U)) \cap L^\infty\big(0,T; H^2_{\text{per}}(U)\big) \tag{5.2}$$

$$v := -\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u) \in L^2\big(0,T; H^2_{\text{per}}(U)\big) \tag{5.3}$$

for all $T > 0$ and if (5.1a)-(5.1c) hold as identities in $L^2_{\text{per}}(U)$ for almost all $t > 0$. Due to the continuous embedding

$$H^1(0,T; L^2_{\text{per}}(U)) \cap L^\infty\big(0,T; H^2_{\text{per}}(U)\big) \hookrightarrow C\big([0,T]; L^2_{\text{per}}(U)\big),$$

(5.1c) is well defined.

Remark. Obviously, every classical periodic solution of (5.1) especially is a weak periodic solution.

The existence proof follows a classical time discretization scheme for gradient flows. The idea is to solve a minimization problem for a modified functional in each time step such that its first variation corresponds to a backward Euler scheme (see Proposition 5.2). Afterwards, the convergence of the resulting step functions in time is proven in Theorem 5.6 using uniform bounds on the step functions and a compactness argument. In Proposition 5.7 the uniqueness of solutions to (5.1) is established by a contradiction argument. Finally, we conclude the chapter with an argument which shows that due to the smoothness of F, every weak solution is in fact a function in $C^\infty((0,\infty), C^\infty_{\text{per}}(U))$ (see Proposition 5.8).
At this point we want to comment shortly on the time discretization technique described above. This method is well known and a common approach to show existence of solutions to evolution problems which have a gradient flow structure. It goes back to Luckhaus and Sturzenhecker and their contributions to the mean curvature flow equation [LuSt95] and was later generalized by De Giorgi to the notion of minimal movement for even non differentiable energies [DeG93].

The structure of this section and many details of the proof follow [CoLa11] where the authors consider the same phase field model with a volume constraint instead of periodicity. While our setting simplifies the calculations in some steps, additional arguments concerning the periodicity of the solutions are necessary.

We begin with the definition of the modified energy functionals: For $\tau > 0$ and $f \in L^2(U)$ we define

$$J_{\tau,f}(w) := \frac{1}{2}\int_U (w-f)^2 \, dx + \tau\mathcal{W}_\varepsilon(w), \quad w \in H^2_{\text{per}}(U)$$

and observe that a minimizer $u \in H^2_{\text{per}}(U)$ of $J_{\tau,f}$ satisfies the Euler-Lagrange equation

$$\frac{1}{\tau}(u-f) = -\nabla_{L^2}\mathcal{W}_\varepsilon(u) \quad \text{in } U,$$

which reminds of the gradient flow equation (5.1a) with a discretized time derivative on the left-hand side.
The existence of these minimizers is shown now.

Proposition 5.2. *For every $\tau > 0$ and every $f \in L^2(U)$ there exists a miminizer u of $J_{\tau,f}$ in $H^2_{per}(U)$.*

Proof. We apply the direct method of the calculus of variations. Let $\tau > 0$ and $f \in L^2(U)$. Since $J_{\tau,f}$ is obviously bounded from below by 0, there exists a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in $H^2_{\text{per}}(U)$ such that

$$J_{\tau,f}(u_k) \longrightarrow m := \inf_{v\in H^2_{\text{per}}(U)} J_{\tau,f}(v) \geq 0$$

as $k \to \infty$. Particularly, $\mathcal{W}_\varepsilon(u_k)$ is bounded uniformly in k and due to Lemma 3.11, we obtain

$$\|u_k\|^2_{H^2(U)} \leq C(\varepsilon)\big(1 + \mathcal{W}_\varepsilon(u_k)\big) \leq C(\varepsilon).$$

As in the proof of Theorem 3.12 we use the weak precompactness of $H^2_{\text{per}}(U)$ to find a subsequence of $(u_k)_{k\in\mathbb{N}}$ which converges weakly in $H^2_{\text{per}}(U)$ and uniformly in $C^0(\overline{U})$ by the general Sobolev inequality ([AdFo03], Theorem 4.12) towards a function $u \in H^2(U)$. Since $H^2_{\text{per}}(U) \subset H^2(U)$ is closed and convex and thus weakly closed, we deduce $u \in H^2_{\text{per}}(U)$.
Again as in Theorem 3.12, we obtain

$$m \leq J_{\tau,f}(u) \leq \liminf_{k\to\infty} J_{\tau,f}(u_k) = m$$

and u is a minimizer of $J_{\tau,f}$ in $H^2_{\text{per}}(U)$. □

Before we formulate the time discretization scheme, we collect some properties of the *diffuse curvature*

$$v := -\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u) \tag{5.4}$$

related to a minimizer u of $J_{\tau,f}$.

Lemma 5.3. *Let $u \in H^2_{per}(U)$ be a minimizer of $J_{\tau,f}$ and let v be defined as in* (5.4)*. Then the following statements hold.*

(i) v is in $H^2_{per}(U)$.

(ii) For all $\psi \in L^2_{per}(U)$ the Euler-Lagrange equation

$$\int_U \left[\frac{1}{\tau}(u - f) - \Delta v + \frac{1}{\varepsilon^2}F''(u)v\right]\psi\,dx = 0 \tag{5.5}$$

holds.

(iii) We have the energy estimate

$$\left\| -\Delta v + \frac{1}{\varepsilon^2} F''(u)v \right\|_{L^2(U)} \leq \frac{1}{\tau} \|u - f\|_{L^2(U)} . \tag{5.6}$$

Proof. For every $\psi \in H^2_{\text{per}}(U)$ and $s > 0$ the function $u + s\psi$ still belongs to $H^2_{\text{per}}(U)$ und since u minimizes $J_{\tau,f}$ we obtain,

$$0 = \frac{1}{\tau}\frac{d}{ds} J_{\tau,f}(u + s\psi)\Big|_{s=0} = \int_U \left[\frac{1}{\tau}(u - f)\psi + v \left(-\Delta\psi + \frac{1}{\varepsilon^2} F''(u)\psi \right) \right] dx. \tag{5.7}$$

To prove (5.5) it remains to show (i) and integrate by parts twice in the equation above where the boundary integrals vanish due to the periodicity of ψ and u. Then, (5.5) obviously holds even for $\psi \in L^2_{\text{per}}(U)$ by approximation.
For the regularity of v we note that the elliptic PDE (in w)

$$\Delta w = \frac{1}{\tau}(u - f) + \frac{1}{\varepsilon^2} F''(u)v \in L^2(U)$$

with $⨍_U w\,dx = 0$ has a unique solution w in $H^1_{\text{per}}(U)$ by the Lax-Milgram theorem. Now, standard regularity theory yields that w belongs to $H^2(U)$ and thus, $w \in H^2_{\text{per}}(U)$. Consequently, we know from (5.7) that $w = v - ⨍_U v\,dx$ which means (i) and (5.5) follows by two integrations by parts as mentioned above.

For the energy estimate (5.6) let $\eta \in (0, 1)$ and let $\varphi_\eta \in H^2_{\text{per}}(U)$ denote the unique solution of the auxiliary problem

$$\varphi_\eta - \eta\Delta\varphi_\eta = -\Delta v + \frac{1}{\varepsilon^2} F''(u)v \in L^2(U). \tag{5.8}$$

Again, the existence and uniqueness of φ_η is an immediate consequence of the Lax-Milgram theorem and standard regularity theory. We choose $\psi = \varphi_\eta$ in (5.5) and use (5.8) to obtain

$$\int_U \left[\frac{1}{\tau}(u - f) + \varphi_\eta - \eta\Delta\varphi_\eta \right] \varphi_\eta\,dx = 0.$$

Rearranging terms and integrating by parts in the third term on the left-hand side yields

$$\begin{aligned} \|\varphi_\eta\|^2_{L^2(U)} \leq \int_U \varphi_\eta^2 + \eta\,|\nabla\varphi_\eta|^2\,dx = -\int_U \frac{1}{\tau}(u - f)\varphi_\eta\,dx \\ \leq \frac{1}{\tau}\|u - f\|_{L^2(U)} \|\varphi_\eta\|_{L^2(U)} \end{aligned}$$

and thus,

$$\|\varphi_\eta\|_{L^2(U)} \leq \frac{1}{\tau} \|u - f\|_{L^2(U)} . \tag{5.9}$$

In order to prove (5.6) it is sufficient to pass to the limit $\eta \to 0$ in the inequality above. To see this, we test (5.8) with φ_η to obtain

$$\|\varphi_\eta\|^2_{L^2(U)} + \|\sqrt{\eta}\nabla\varphi_\eta\|^2_{L^2(U)} \leq \left\|-\Delta v + \frac{1}{\varepsilon^2}F''(u)v\right\|_{L^2(U)} \|\varphi_\eta\|_{L^2(U)}$$

and hence,

$$\|\varphi_\eta\|_{L^2(U)} \leq \left\|-\Delta v + \frac{1}{\varepsilon^2}F''(u)v\right\|_{L^2(U)}$$

and then

$$\|\sqrt{\eta}\nabla\varphi_\eta\|_{L^2(U)} \leq \left\|-\Delta v + \frac{1}{\varepsilon^2}F''(u)v\right\|_{L^2(U)}.$$

Due to the reflexivity of $L^2(U)$, there exists a non relabeled subsequence $\eta \to 0$ such that

$$\begin{aligned} \varphi_\eta &\rightharpoonup \varphi \quad \text{weakly in } L^2(U), \\ \sqrt{\eta}\nabla\varphi_\eta &\rightharpoonup \rho \quad \text{weakly in } L^2(U) \end{aligned}$$

with $\varphi, \rho \in L^2(U)$ as $\eta \to 0$. We can now consider the limit in the weak formulation of (5.8) which implies for arbitrary $w \in H^1_{\text{per}}(U)$

$$\int_U \left(-\Delta v + \frac{1}{\varepsilon^2}F''(u)v\right) w\,dx = \int_U \varphi_\eta w + \eta\nabla\varphi_\eta\nabla w\,dx \longrightarrow \int \varphi w\,dx$$

as $\eta \to 0$ and hence

$$\varphi = -\Delta v + \frac{1}{\varepsilon^2}F''(u)v.$$

Thus, using the weak lower semicontinuity of the L^2-norm we obtain (5.6) from (5.9) by

$$\left\|-\Delta v + \frac{1}{\varepsilon^2}F''(u)v\right\|_{L^2(U)} = \|\varphi\|_{L^2(U)} \leq \liminf_{\eta\to 0} \|\varphi_\eta\|_{L^2(U)} \leq \frac{1}{\tau}\|u - f\|_{L^2(U)}. \qquad \square$$

We continue by formulating the time discretization scheme for the diffuse Willmore flow. Let $u_0 \in H^2_{\text{per}}(U)$ be an initial value and fix a time step width $\tau > 0$. For $m \geq 0$ and with J_{τ,u^τ_m} from above we then define a sequence $(u^\tau_k)_{k\in\mathbb{N}}$ in $H^2_{\text{per}}(U)$ inductively by

$$\begin{aligned} &u^\tau_0 := u_0, \\ &u^\tau_{m+1} \text{ minimizes } J_{\tau,u^\tau_m} \text{ in } H^2_{\text{per}}(U), \end{aligned}$$

and also set

$$v^\tau_m := -\varepsilon\Delta u^\tau_m + \frac{1}{\varepsilon}F'(u^\tau_m).$$

With these sequences in hand, we define ($H^2_{\text{per}}(U)$-valued) step functions u^τ, v^τ in time by

$$u^\tau(t) := u^\tau_m \quad \text{and} \quad v^\tau(t) := v^\tau_m$$

for $t \in [m\tau, (m+1)\tau)$ and $m \geq 0$. Our plan is to prove the convergence of the above defined step functions towards a solution of the gradient flow in a suitable sense. We prepare the main theorem by deriving an H^2-bound uniform in time for u_τ as well as uniform bounds for v^τ in appropriate Bochner Spaces (see below).

Lemma 5.4. *Let $\tau > 0$ be an arbitrary time step width and $t_2 > t_1 \geq 0$. Then the estimates*

$$\mathcal{W}_\varepsilon\big(u^\tau(t_2)\big) \leq \mathcal{W}_\varepsilon\big(u^\tau(t_1)\big) \leq \mathcal{W}_\varepsilon(u_0), \tag{5.10}$$

$$\sup_{t\geq 0} \|v^\tau(t)\|^2_{L^2(U)} \leq 2\varepsilon \mathcal{W}_\varepsilon(u_0), \tag{5.11}$$

$$\|u^\tau(t_2) - u^\tau(t_1)\|^2_{L^2(U)} \leq 2\mathcal{W}_\varepsilon(u_0)(\tau + t_2 - t_1), \tag{5.12}$$

$$\int_\tau^\infty \left\| -\Delta v^\tau(t) + \frac{1}{\varepsilon^2} F''\big(u^\tau(t)\big) v^\tau(t) \right\|^2_{L^2(U)} dt \leq 2\mathcal{W}_\varepsilon(u_0), \tag{5.13}$$

hold.

Proof. We have $u^\tau_m \in H^2_{\text{per}}(U)$ for every $m \geq 0$ and since u^τ_{m+1} is a minimizer of J_{τ, u^τ_m} by definition, we obtain

$$\begin{aligned} &\frac{1}{2\tau} \left\| u^\tau_{m+1} - u^\tau_m \right\|^2_{L^2(U)} + \mathcal{W}_\varepsilon(u^\tau_{m+1}) \\ =&\frac{1}{\tau} J_{\tau, u^\tau_m}(u^\tau_{m+1}) \leq \frac{1}{\tau} J_{\tau, u^\tau_m}(u^\tau_m) = \mathcal{W}_\varepsilon(u^\tau_m). \end{aligned} \tag{5.14}$$

Now let $t_2 > t_1 \geq 0$ and set $m_1 = \lfloor \frac{t_1}{\tau} \rfloor$ and $m_2 = \lfloor \frac{t_2}{\tau} \rfloor$. Then, $m_2 \geq m_1$ and (5.14) together with an induction argument yields

$$\mathcal{W}_\varepsilon\big(u^\tau(t_2)\big) = \mathcal{W}_\varepsilon(u^\tau_{m_2}) \leq \mathcal{W}_\varepsilon(u^\tau_{m_1}) = \mathcal{W}_\varepsilon\big(u^\tau(t_1)\big)$$

which is (5.10). We then deduce (5.11) by

$$\frac{1}{2\varepsilon} \sup_{t\geq 0} \|v^\tau(t)\|^2_{L^2(U)} = \sup_{t\geq 0} \mathcal{W}_\varepsilon\big(u^\tau(t)\big) = \sup_{m\geq 0} \mathcal{W}_\varepsilon(u^\tau_m) \leq \mathcal{W}_\varepsilon(u^\tau_0) = \mathcal{W}_\varepsilon(u_0).$$

Summing up (5.14) over all $m \geq 0$ yields

$$\begin{aligned} &\frac{1}{2\tau} \sum_{m=0}^\infty \left\| u^\tau_{m+1} - u^\tau_m \right\|^2_{L^2(U)} \leq \sum_{m=0}^\infty \big(\mathcal{W}_\varepsilon(u^\tau_m) - \mathcal{W}_\varepsilon(u^\tau_{m+1})\big) \\ \leq& W_\varepsilon(u^\tau_0) = \mathcal{W}_\varepsilon(u_0) \end{aligned} \tag{5.15}$$

and due to the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|u^\tau(t_2) - u^\tau(t_1)\|_{L^2(U)} = \left\|u^\tau_{m_2} - u^\tau_{m_1}\right\|_{L^2(U)} &\leq \sum_{m=m_1}^{m_2-1} \left\|u^\tau_{m+1} - u^\tau_m\right\|_{L^2(U)} \\ &\leq \sqrt{m_2 - m_1}\sqrt{\sum_{m=m_1}^{m_2-1} \left\|u^\tau_{m+1} - u^\tau_m\right\|^2_{L^2(U)}} \\ &\leq \sqrt{1 + \frac{t_2 - t_1}{\tau}}\sqrt{2\tau W_\varepsilon(u_0)} \\ &= \sqrt{2\mathcal{W}_\varepsilon(u_0)}\sqrt{\tau + (t_2 - t_1)},\end{aligned}$$

which gives (5.12) after squaring both sides.
To prove (5.13) we again make use the fact that u^τ_{m+1} minimizes J_{τ,u^τ_m}. (5.6) yields

$$\left\|-\Delta v^\tau_{m+1} + \frac{1}{\varepsilon^2}F''(u^\tau_{m+1})v^\tau_{m+1}\right\|_{L^2(U)} \leq \frac{1}{\tau}\left\|u^\tau_{m+1} - u^\tau_m\right\|_{L^2(U)}$$

and we apply this inequality together with (5.15) to estimate the left hand side of (5.13). This gives

$$\begin{aligned}&\int_\tau^\infty \left\|-\Delta v^\tau(t) + \frac{1}{\varepsilon^2}F''\big(u^\tau(t)\big)v^\tau(t)\right\|^2_{L^2(U)} dt \\ =&\sum_{m=0}^\infty \int_{(m+1)\tau}^{(m+2)\tau} \left\|-\Delta v^\tau_{m+1} + \frac{1}{\varepsilon^2}F''(u^\tau_{m+1})v^\tau_{m+1}\right\|^2_{L^2(U)} dt \\ \leq&\frac{1}{\tau}\sum_{m=0}^\infty \left\|u^\tau_{m+1} - u^\tau_m\right\|^2_{L^2(U)} \leq 2\mathcal{W}_\varepsilon(u_0)\end{aligned}$$

and completes the proof. □

In order to show convergence of u^τ and v^τ as $\tau \to 0$, we depend on uniform bounds of these functions in $L^\infty(0,T;H^2(U))$ and $L^2(\tau,T;H^2(U))$, respectively.

Lemma 5.5. *For every $T > 0$ there exists a constant $C = C(U,T,u_0,F,\varepsilon)$ such that for every $\tau \in (0,1) \cap (0,T)$ the estimates*

$$\sup_{t\in[0,T]} \|u^\tau(t)\|_{H^2(U)} \leq C \tag{5.16}$$

and

$$\int_\tau^T \|v^\tau(t)\|^2_{H^2(U)}\, dt \leq C \tag{5.17}$$

hold.

Proof. Let $T > 0$ be arbitrary. For every $t \in [0,T]$ the spacial H^2-bound for $u^\tau(t)$ from

Lemma 3.11 combined with (5.11) reads

$$\begin{aligned}\|u^\tau(t)\|_{H^2(U)} \leq & C\left(1+\sqrt{\mathcal{W}_\varepsilon(u^\tau(t))}\right) = C\left(1+\frac{1}{\sqrt{2\varepsilon}}\|v^\tau(t)\|_{L^2(U)}\right)\\ \leq & C\left(1+\sqrt{\mathcal{W}_\varepsilon(u_0)}\right)\end{aligned}$$

which is (5.16) as the right hand side does not depend on t.
For the second inequality we immediately see that $F''\big(u^\tau(t)\big)$ is uniformly bounded in $L^\infty((0,T)\times U)$ by (5.16) and the fact that $H^2(U)$ embeds continuously into $L^\infty(U)$ for space dimensions $1\leq n\leq 3$. Together with (5.11) this yields a bound in $L^\infty(0,T;L^2(U))$ for the product $F''(u^\tau)v^\tau$ independent of τ.
Keeping this result in mind we obtain

$$\begin{aligned}&\int_\tau^T \|\Delta v^\tau(t)\|^2_{L^2(U)}\,dt\\ \leq & \int_\tau^T C\left\|-\Delta v^\tau(t)+\frac{1}{\varepsilon^2}F''\big(u^\tau(t)\big)v^\tau(t)\right\|^2_{L^2(U)}+\frac{C}{\varepsilon^4}\left\|F''\big(u^\tau(t)\big)v^\tau(t)\right\|^2_{L^2(U)}\,dt\\ \leq & C\left(\mathcal{W}_\varepsilon(u_0)\right)\end{aligned}$$

where we have used inequality (5.13) to estimate the first summand in the last step together with $\tau<1$. The control of $\Delta v^\tau(t)$ immediately provides a bound for the complete second derivative D^2v^τ in $L^2(\tau,T;L^2(U))$ and the $L^\infty(0,\infty;L^2(U))$-bound for v^τ from (5.11) also yields a uniform bound for v^τ in $L^2(\tau,T;L^2(U))$. Now, by a standard interpolation argument we also have

$$\begin{aligned}\int_\tau^T\int_U |\nabla v^\tau(t)|^2\,dx\,dt =& -\int_\tau^T\int_U v^\tau(t)\Delta v^\tau(t)\,dx\,dt\\ \leq & \frac{1}{2}\int_\tau^T \|v^\tau(t)\|^2_{L^2(U)}+\|\Delta v^\tau(t)\|^2_{L^2(U)}\,dt\\ \leq & \frac{1}{2}(T-\tau)\sup_{t\geq 0}\|v^\tau(t)\|^2_{L^2(U)}+\frac{1}{2}\int_\tau^T\|\Delta v^\tau(t)\|^2_{L^2(U)}\,dt\\ \leq & C\left(\mathcal{W}_\varepsilon(u_0)\right)\end{aligned}$$

which yields the uniform boundedness of v^τ in $L^2(\tau,T;H^2(U))$ and therefore, (5.17). □

We now use the inequalities from above for an Arzelà-Ascoli argument to find converging subsequences $(u^{\tau_k})_{k\in\mathbb{N}}$ and $(v^{\tau_k})_{k\in\mathbb{N}}$. Afterwards, the limit function is shown to be a strong solution of the gradient flow (5.1).

Theorem 5.6. *For initial data $u_0\in H^2_{per}(U)$ there exists a weak periodic solution of (5.1) u in the sense of Definition 5.1 which moreover satisfies*

$$u\in C\big([0,\infty)\times\overline{U}\big). \tag{5.18}$$

Proof. Let $T > 0$. By (5.16) the set

$$K := \{u^\tau(t) \,:\, t \in [0,T], \tau \in (0,1) \cap (0,T)\} \subset H^2_{\text{per}}(U)$$

is bounded and since the embedding $H^2(U) \hookrightarrow C^0(\overline{U})$ is compact for the considered space dimensions, $\overline{K} \subset C^0(\overline{U})$ is compact where the closure was taken with respect to the H^2-norm.
Furthermore, (5.12) yields for $0 \leq t_1 \leq t_2 \leq T$

$$\limsup_{\tau \to 0} \|u^\tau(t_2) - u^\tau(t_1)\|_{L^2(U)} \leq \sqrt{2\mathcal{W}_\varepsilon(u_0)(t_2 - t_1)} \longrightarrow 0$$

as $t_1 \to t_2$. Therefore, we are able to apply a refined version of the Arzelà-Ascoli theorem (see [AmGiSa08], Prop 3.3.1) to find a subsequence $(\tau_k)_{k\in\mathbb{N}}$ and a function $u \,:\, [0,T] \to C(\overline{U})$ such that

$$u^{\tau_k}(t) \longrightarrow u(t) \quad \text{uniformly for all } t \in [0,T]$$

and thus,

$$u^{\tau_k} \longrightarrow u \quad \text{in } C^0\big([0,T] \times \overline{U}\big) \tag{5.19}$$

as $k \to \infty$ which yields (5.18).
On the other hand, (5.16) and the fact that $L^\infty(0,T;H^2(U)) = \big(L^1(0,T;H^2(U))\big)^*$ imply the existence of a further (not relabeled) subsequence and $\tilde{u} \in L^\infty(0,T;H^2(U))$ such that

$$u^{\tau_k} \overset{*}{\rightharpoonup} \tilde{u} \quad \text{in } L^\infty\big(0,T;H^2(U)\big). \tag{5.20}$$

The two limit functions u and $\tilde{u}$ coincide almost everywhere since for every testfunction $\varphi \in C_c^\infty([0,T] \times \overline{U})$ we have

$$\begin{aligned}
&\left| \int_0^T \int_U (u - \tilde{u})\, \varphi \, dx\, dt \right| \\
&\leq \left| \int_0^T \int_U (u - u^{\tau_k})\, \varphi \, dx\, dt \right| + \left| \int_0^T \int_U (u^{\tau_k} - \tilde{u})\, \varphi \, dx\, dt \right| \\
&\leq \|u - u^{\tau_k}\|_{C^0([0,T]\times\overline{U})} \, \|\varphi\|_{L^1((0,T)\times U)} + \|u^{\tau_k} - \tilde{u}\|_{L^\infty(0,T;H^2(U))} \, \|\varphi\|_{L^1(0,T;H^2(U))} \\
&\quad \longrightarrow 0
\end{aligned}$$

as $k \to \infty$. Furthermore, $u(t)$ is a periodic function for every $t \in [0,T]$. Indeed, by (5.16) we can find for every fixed $t \in [0,T]$ and every subsequence $(u^{\tau_{k_j}}(t))_{j\in\mathbb{N}}$ of $(u^{\tau_k}(t))_{k\in\mathbb{N}}$ a further subsequence which converges in $H^2(U)$ and with the same argument as above, we conclude

$$u^{\tau_k}(t) \rightharpoonup u(t) \quad \text{in } H^2(U) \text{ for } t \in [0,T]$$

as $k \to \infty$. Finally, since $H^2_{\text{per}}(U)$ is a closed and convex and therefore weakly closed subset of $H^2(U)$, $u(t)$ is in $H^2_{\text{per}}(U)$ for every $t \in [0,T]$ and the second part of (5.2) is

shown.
To prove (5.3) we observe that (5.11) yields the existence of another (again not relabeled) subsequence of $(\tau_k)_{k\in\mathbb{N}}$ such that

$$v^{\tau_k} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty\big(0,T;L^2(U)\big)$$

and (5.19),(5.20) let us pass immediately to the limit in the definition of v^{τ_k} to deduce

$$v = -\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u).$$

We set

$$\tilde{v}^{\tau_k} := v^{\tau_k}\chi_{[\tau_k,T]}$$

where $\chi_{[\tau_k,T]}$ is the time dependent cutoff function of the interval $[\tau_k,T]$ and obtain

$$\|\tilde{v}^{\tau_k}\|_{L^2(0,T;H^2(U))} = \|v^{\tau_k}\|_{L^2(\tau_k,T;H^2(U))} \le C$$

by (5.17). Therefore,

$$\tilde{v}^{\tau_k} \rightharpoonup \tilde{v} \quad \text{in } L^2\big(0,T;H^2(U)\big) \tag{5.21}$$

holds with

$$\|\tilde{v}\|_{L^2(0,T;H^2(U))} \le C$$

as the norm is weak lower semicontinuous. As above, it remains to show that the limit functions v and $\tilde{v}$ coincide. We see for arbitrary $\varphi \in C_c^\infty([0,T]\times\overline{U})$ that

$$\begin{aligned}
&\left|\int_0^T\int_U (v-\tilde{v})\varphi\,dx\,dt\right| \\
&\le \left|\int_0^T\int_U (v-v^{\tau_k})\varphi\,dx\,dt\right| + \left|\int_0^T\int_U (v^{\tau_k}-\tilde{v})\varphi\,dx\,dt\right|
\end{aligned}$$

where the first summand becomes small due to the weak*-convergence of $(v^{\tau_k})_{k\in\mathbb{N}}$ in $L^\infty(0,T;L^2(U))$. Now, (5.3) follows since the second term can be estimated by

$$\begin{aligned}
&\left|\int_0^T\int_U (v^{\tau_k}-\tilde{v}^{\tau_k})\varphi\,dx\,dt\right| + \left|\int_0^T\int_U (\tilde{v}^{\tau_k}-\tilde{v})\varphi\,dx\,dt\right| \\
\le &\left|\int_0^{\tau_k}\int_U v^{\tau_k}\varphi\,dx\,dt\right| + \left|\int_0^T\int_U (\tilde{v}^{\tau_k}-\tilde{v})\varphi\,dx\,dt\right| \\
\le &\tau_k\,\|v^{\tau_k}\|_{L^\infty(0,T;L^2(U))}\,\|\varphi\|_{L^\infty(0,T;L^2(U))} + \left|\int_0^T\int_U (\tilde{v}^{\tau_k}-\tilde{v})\varphi\,dx\,dt\right|
\end{aligned}$$

and therefore also converges to 0 as $k\to\infty$ by (5.11) and (5.21).

Finally, we show that the limit functions solve (5.1a). Let $\psi \in C_c^\infty([0,T);C^\infty_{\text{per}}(U))$ and

$\tau > 0$ sufficiently small. We define $m = \lfloor \frac{T}{\tau} \rfloor$ and set

$$\psi_j^\tau := \fint_{j\tau}^{(j+1)\tau} \psi(s)\, ds \in C^\infty_{\text{per}}(U), \quad j = 0, \dots, m-1$$

as well as

$$\psi^\tau(t) := \psi_j^\tau \quad \text{for } t \in \big[j\tau, (j+1)\tau\big).$$

Moreover, we extend ψ^τ by $\psi^\tau(t) := \psi(0)$ for $t < 0$.

Due to the definition of u^τ_{j+1}, $j \in \{0, \dots, m-1\}$, we consider (5.5) with $\psi = \psi^\tau_j$ and therefore obtain by rearranging

$$\begin{aligned} \int_U (u^\tau_{j+1} - u^\tau_j)\psi^\tau_j \, dx &= \tau \int_U \left(\Delta v^\tau_{j+1} - \frac{1}{\varepsilon^2} F''(u^\tau_{j+1}) v^\tau_{j+1} \right) \psi^\tau_j \, dx \\ &= \int_{(j+1)\tau}^{(j+2)\tau} \int_U \left(\Delta v^\tau - \frac{1}{\varepsilon^2} F''(u^\tau) v^\tau \right) \psi^\tau(\cdot - \tau)\, dx\, ds. \end{aligned} \tag{5.22}$$

The left-hand side can be written as

$$\begin{aligned} &\int_U (u^\tau_{j+1} - u^\tau_j)\psi^\tau_j \, dx = \int_{j\tau}^{(j+1)\tau} \int_U \partial^\tau_t u^\tau \psi^\tau \, dx\, ds \\ &= - \int_{j\tau}^{(j+1)\tau} \int_U u^\tau \partial^{-\tau}_t \psi^\tau \, dx\, ds + \fint_{(j+1)\tau}^{(j+2)\tau} \int_U u^\tau \psi^\tau(\cdot - \tau)\, dx\, ds \\ &\quad - \fint_{j\tau}^{(j+1)\tau} \int_U u^\tau \psi^\tau(\cdot - \tau)\, dx\, ds \end{aligned}$$

where ∂^τ_t denotes the discrete (time) derivative and where we have integrated by parts discretely in the last step. Hence, by summing (5.22) over $j = 0, \dots m-1$, canceling the twice appearing terms on the left hand side, and using $\psi^\tau \equiv 0$ in $[(m-1)\tau, T]$ for sufficiently small τ, this yields

$$\begin{aligned} &- \int_0^{m\tau} \int_U u^\tau \partial^{-\tau}_t \psi^\tau \, dx\, ds - \fint_0^\tau \int_U u^\tau \psi^\tau(\cdot - \tau)\, dx\, ds \\ &= \int_\tau^T \int_U \left(\Delta v^\tau - \frac{1}{\varepsilon^2} F''(u^\tau) v^\tau \right) \psi^\tau(\cdot - \tau)\, dx\, ds. \end{aligned} \tag{5.23}$$

Now, we pass to the limit $\tau \to 0$ and recognize that

$$\partial^{-\tau}_t \psi^\tau \longrightarrow \partial_t \psi \quad \text{uniformly in } (0, T] \times \overline{U} \tag{5.24}$$

as $\tau \to 0$. Indeed, for $t > 0$ we can assume $\tau > 0$ sufficiently small such that $t \in [j\tau, (j+1)\tau)$, $j \in \{1, \dots, m-1\}$. As above, $\psi^\tau(t) = \psi(t) \equiv 0$ even holds for $t > (m-2)\tau$ by definition and therefore, $\partial^{-\tau}_t \psi^\tau(t)$ and $\partial_t \psi(t)$ both vanish for

$t > (m-1)\tau$. We obtain by the definition of ψ^τ (independent of $x \in \overline{U}$)

$$\begin{aligned}
&\left|\partial_t^{-\tau}\psi^\tau(t) - \partial_t\psi(t)\right| = \left|\frac{1}{\tau}\big(\psi^\tau(t) - \psi^\tau(t-\tau)\big) - \partial_t\psi(t)\right| \\
=&\left|\frac{1}{\tau}\fint_{j\tau}^{(j+1)\tau} \psi(s) - \psi(s-\tau) - \tau\partial_t\psi(t)\,ds\right| = \left|\fint_{j\tau}^{(j+1)\tau} \partial_t\psi(\rho) - \partial_t\psi(t)\,ds\right| \\
\leq&\fint_{j\tau}^{(j+1)\tau} \|\psi\|_{C^2([0,T]\times\overline{U})}\,|\rho - t|\;ds \leq 2\,\|\psi\|_{C^2([0,T]\times\overline{U})}\,\tau \\
&\longrightarrow 0
\end{aligned}$$

with $\rho \in (s-\tau, s)$ due to the mean value theorem.

We choose $\tau = \tau_k$ from above such that (5.19), (5.20) and (5.21) hold and take the limit $k \to \infty$ in (5.23). Due to the uniform convergence of $(u^{\tau_k})_{k\in\mathbb{N}}$ in $[0,T]\times\overline{U}$ from (5.19) and (5.24), we obtain for the left hand side of (5.23)

$$\begin{aligned}
&\int_0^{m\tau_k}\int_U u^{\tau_k}\partial_t^{-\tau_k}\psi^{\tau_k}\,dx\,ds - \fint_0^{\tau_k}\int_U u^{\tau_k}\psi^{\tau_k}(\cdot-\tau_k)\,dx\,ds \\
=&\int_0^{m\tau_k}\int_U u^{\tau_k}\partial_t^{-\tau_k}\psi^{\tau_k}\,dx\,ds - \int_U u_0\psi(0)\,dx \\
&\longrightarrow -\int_0^T\int_U u\partial_t\psi\,dx\,ds - \int_U u_0\psi(0)\,dx
\end{aligned}$$

as $k \to \infty$. For the right hand side we observe that

$$\psi^\tau(\cdot-\tau) \longrightarrow \psi \quad \text{uniformly in } (0,T]\times\overline{U}$$

since for $t \in [j\tau, (j+1)\tau)$, $j \in 1, \cdots, m-1$ we have similar to the argument for (5.24)

$$\begin{aligned}
&|\psi^\tau(t-\tau) - \psi(t)| = \left|\psi^\tau_{j-1} - \psi(t)\right| = \left|\fint_{(j-1)\tau}^{j\tau}\psi(s)\,ds - \psi(t)\right| \\
=&\,|\psi(\rho) - \psi(t)| \leq 2\,\|\psi\|_{C^1([0,T]\times U)}\,\tau \longrightarrow 0
\end{aligned}$$

as $\tau \to 0$, where we have used the mean value theorem of integral calculus with $\rho \in ((j-1)\tau, j\tau)$. Therefore, the convergence of

$$\begin{aligned}
&\int_\tau^T\int_U\left(\Delta v^\tau - \frac{1}{\varepsilon^2}F''(u^\tau)v^\tau\right)\psi^\tau(\cdot-\tau)\,dx\,ds \\
=&\int_0^T\int_U\left(\Delta\tilde{v}^\tau - \frac{1}{\varepsilon^2}F''(u^\tau)\tilde{v}^\tau\right)\psi^\tau(\cdot-\tau)\,dx\,ds
\end{aligned}$$

follows with the weak convergence of $(\tilde{v}^{\tau_k})_{k\in\mathbb{N}}$ in $L^2(0,T;H^2(U))$ from (5.21) and the uniform convergence of $(u^{\tau_k})_{k\in\mathbb{N}}$ from (5.19).

Summarized, we deduce

$$-\int_0^T\int_U u\partial_t\psi\,dx\,ds-\int_U u_0\psi(0)\,dx=\int_0^T\int_U\left(\Delta v-\frac{1}{\varepsilon^2}F''(u)v\right)\psi\,dx\,ds \tag{5.25}$$

from (5.22) in the limit for all $\psi\in C_c^\infty([0,T);C^\infty_{\text{per}}(U))$. For $\psi\in C_c^\infty((0,T);C^\infty_{\text{per}}(U))$ in (5.25) we obtain

$$-\int_0^T\int_U u\partial_t\psi\,dx\,ds=\int_0^T\int_U\left(\Delta v-\frac{1}{\varepsilon^2}F''(u)v\right)\psi\,dx\,ds$$

and thus, $u\in H^1(0,T;L^2_{\text{per}}(U))$, as the expression in parentheses on the right hand side is in $L^2(0,T;L^2_{\text{per}}(U))$, which proves (5.2).
As a last step, a partial integration in (5.25) yields (5.1) and this completes the proof of the theorem. □

Proposition 5.7. *Weak periodic solutions of* (5.1) *for given initial data* $u_0\in H^2_{per}(U)$ *are unique.*

Proof. Fix $T>0$ and let u^1 and u^2 be two weak periodic solutions of (5.1) with same initial data u_0 and let v^1 and v^2 denote the corresponding diffuse curvatures as in (5.4). Due to the regularity of u and v from (5.2) and (5.3), we can find a constant $K>0$ such that

$$\left\|u^i\right\|_{L^\infty([0,T]\times\bar U)}+\left\|v^i\right\|_{L^\infty(0,T;L^2(U))}+\left\|v^i\right\|_{L^2(0,T;L^\infty(U))}\le K \tag{5.26}$$

for $i=1,2$. Here, the first and last inequality both follow by the continuous embedding of $H^2(U)$ into $L^\infty(U)$.
We first observe that

$$\begin{aligned}
&\left|F''(u^1)v^1-F''(u^2)v^2\right|\\
\le&\left|F''(u^1)-F''(u^2)\right|\left|v^1\right|+\left|F''(u^2)\right|\left|v^1-v^2\right|\\
\le&\left\|F'''\right\|_{L^\infty(-K,K)}\left|u^1-u^2\right|\left|v^1\right|+\left\|F''\right\|_{L^\infty(-K,K)}\left|v^1-v^2\right|\\
\le&C\left(\left|v^1\right|\left|u^1-u^2\right|+\left|v^1-v^2\right|\right)
\end{aligned} \tag{5.27}$$

and that the difference of u^1 and u^2 satisfies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|u^1-u^2\right\|^2_{L^2(U)}\\
=&\int_U\partial_t(u^1-u^2)(u^1-u^2)\\
=&\int_U(v^1-v^2)\Delta(u^1-u^2)-\frac{1}{\varepsilon^2}\big(F''(u^1)v^1-F''(u^2)v^2\big)(u^1-u^2)\,dx
\end{aligned}$$

since both functions solve (5.1). Now, (5.27) and the definition of v from (5.4) yield

$$\frac{1}{2}\frac{d}{dt}\left\|u^1-u^2\right\|^2_{L^2(U)}$$

$$
\begin{aligned}
=&\frac{1}{\varepsilon}\int_U (v^1-v^2)\left(\frac{1}{\varepsilon}F'(u^1)-\frac{1}{\varepsilon}F'(u^2)-(v^1-v^2)\right)dx\\
&\quad -\frac{1}{\varepsilon^2}\int_U \left(F''(u^1)v^1-F''(u^2)v^2\right)(u^1-u^2)\,dx\\
\leq&\frac{1}{\varepsilon^2}\left\|F''\right\|_{L^\infty(-K,K)}\left\|v^1-v^2\right\|_{L^2(U)}\left\|u^1-u^2\right\|_{L^2(U)}-\frac{1}{\varepsilon}\left\|v^1-v^2\right\|^2_{L^2(U)}\\
&\quad +\frac{C}{\varepsilon^2}\int_U \left|v^1\right|\left|u^1-u^2\right|^2+\left|v^1-v^2\right|\left|u^1-u^2\right|\,dx\\
\leq&\ \frac{C}{\varepsilon^2}\left\|v^1-v^2\right\|_{L^2(U)}\left\|u^1-u^2\right\|_{L^2(U)}+\frac{C}{\varepsilon^2}\left\|v^1\right\|_{L^\infty(U)}\left\|u^1-u^2\right\|^2_{L^2(U)}\\
&\quad -\frac{1}{\varepsilon}\left\|v^1-v^2\right\|^2_{L^2(U)}\\
\leq&\frac{1}{\varepsilon}\left\|v^1-v^2\right\|^2_{L^2(U)}+\frac{C}{\varepsilon^3}\left\|u^1-u^2\right\|_{L^2(U)}+\frac{C}{\varepsilon^2}\left\|v^1\right\|_{L^\infty(U)}\left\|u^1-u^2\right\|^2_{L^2(U)}\\
&\quad -\frac{1}{\varepsilon}\left\|v^1-v^2\right\|^2_{L^2(U)}\\
\leq& C\left(1+\left\|v^1\right\|_{L^\infty(U)}\right)\left\|u^1-u^2\right\|^2_{L^2(U)}
\end{aligned}
$$

where we have applied Young's inequality in the fourth step to absorb the first term in the sum. Now, Gronwall's inequality (e.g., [Eva10], Appendix B.2.) immediately implies for every $t \in [0,T]$

$$
\begin{aligned}
\left\|(u^1-u^2)(t)\right\|^2_{L^2(U)} &\leq \left\|(u^1-u^2)(0)\right\|^2_{L^2(U)}\exp\left(C\int_0^t 1+\left\|v^1(s)\right\|_{L^\infty(U)}\,ds\right)\\
&\leq C\left\|(u^1-u^2)(0)\right\|^2_{L^2(U)}
\end{aligned}
$$

where the integral in the exponential term exists due to (5.26). Since $u^1(0)=u^2(0)=u_0$, this proves $u^1(t)=u^2(t)$ for all $t\in[0,T]$ in $L^2(U)$. □

Proposition 5.8. *Every weak periodic solution of* (5.1) *is a smooth classical solution in* $C^\infty((0,\infty);C^\infty_{per}(U))\cap C^\infty([0,\infty);H^2_{per}(U))$.

Proof. We first observe that a weak periodic solution u of (5.1) solves the biharmonic heat flow equation

$$
\partial_t u+\varepsilon\Delta^2 u=G(u), \tag{5.28}
$$

where

$$
G(u):=-\frac{2}{\varepsilon}F''(u)\Delta u-\frac{1}{\varepsilon}F'''(u)\left|\nabla u\right|^2+\frac{1}{\varepsilon^3}F'(u)F''(u)
$$

contains all remaining terms with less derivatives of u in (5.1a). The smoothness of u can now be shown by combining spacial regularity results from [La02] (or [MaMa12]) for the biharmonic heat flow with a bootstrap argument and applying ODE theory afterwards in order to prove the smoothness in time. Unfortunately, it is not possible to apply the parabolic regularity results from [La02] or [MaMa12] directly to establish higher space and time regularity simultaneously, as $G(u)$ contains derivatives of u (and

their products) and therefore does not inherit the complete regularity of u. This makes the more careful treatment necessary.
We sketch the argument here using formal differentiation and omit some technical details which for example can be found (in a similar context) in [La02, MaMa12, Eva10]. The proof can easily be made rigorous by using discrete difference quotients.

As $H^2_{\text{per}}(U)$ embeds continuously into $L^\infty(U)$ for the considered space dimensions, we have $u \in L^\infty([0,T] \times \overline{U})$ by (5.2) and therefore, all occurring derivatives of F are bounded uniformly in $[0,T] \times \overline{U}$. The regularity of v from (5.3) now implies

$$\Delta u = \frac{1}{\varepsilon^2} F'(u) - \frac{1}{\varepsilon} v \in L^2\big(0,T; H^2_{\text{per}}(U)\big)$$

and thus, $u \in L^2(0,T; H^4_{\text{per}}(U))$, which already yields $G(u) \in L^2(0,T; H^1_{\text{per}}(U))$ since for every $i = 1, \dots, n$, $\partial_i G(u)$ is bounded in $L^2(0,T; L^2_{\text{per}}(U))$. Indeed, we have

$$\begin{aligned} |\partial_i G(u)| =& \Big| - 2F'''(u)\partial_i u \Delta u - 2F''(u)\Delta\partial_i u - F^{(4)}(u)\left|\nabla u\right|^2 \partial_i u \\ & - 2F'''(u)\nabla u \cdot \nabla \partial_i u + \frac{1}{\varepsilon^2}\big(F''(u)^2 + F'(u)F'''(u)\big)\partial_i u\Big| \\ \leq & C\Big(\left|\partial_i u \Delta u\right| + \left|\Delta \partial_i u\right| + \left|\nabla u\right|^2 \left|\partial_i u\right| + \left|\nabla u\right| \left|\nabla \partial_i u\right| + \left|\partial_i u\right| \Big) \end{aligned}$$

and the boundedness of the second and last term follows directly as $u \in L^\infty([0,T] \times \overline{U})$. For the the first and fourth summand we remark that

$$\begin{aligned} D^2 u &\in L^2(0,T; H^2_{\text{per}}(U)) \hookrightarrow L^2(0,T; L^\infty(U)) \\ \partial_i u &\in L^\infty(0,T; H^1_{\text{per}}(U)) \hookrightarrow L^\infty(0,T; L^2_{\text{per}}(U)) \end{aligned}$$

and therefore,

$$\int_0^T \int_U (\partial_i u)^2 (\Delta u)^2 \, dx \, dt \leq \left\| \int_U (\partial_i u)^2 \, dx \right\|_{L^\infty(0,T)} \int_0^T \left\|\Delta u\right\|^2_{L^\infty(U)} \, dt < \infty.$$

At last, for space dimensions $n = 1, 2, 3$,

$$u \in L^\infty(0,T; H^2_{\text{per}}(U)) \hookrightarrow L^\infty(0,T; W^{1,6}_{\text{per}}(U))$$

which yields that the third summand is bounded in $L^2(0,T; L^2_{\text{per}}(U))$.
Hence, we formally derive

$$\partial_t(\partial_i u) + \varepsilon \Delta^2(\partial_i u) = \partial_i G(u) \in L^2(0,T; L^2_{\text{per}}(U))$$

and using the regularity results from [La02] (in their localized versions), we obtain

$$\partial_i u \in H^1\big(0,T; L^2_{\text{per}}(U)\big) \cap L^\infty\big(0,T; H^2_{\text{per}}(U)\big) \cap L^2\big(0,T; H^4_{\text{per}}(U)\big)$$

and then

$$u \in H^1\big(0,T; H^1_{\text{per}}(U)\big) \cap L^\infty\big(0,T; H^3_{\text{per}}(U)\big) \cap L^2\big(0,T; H^5_{\text{per}}(U)\big).$$

Now, the enhanced regularity of u can be used to show $G(u) \in L^2(0,T;H^2_{\text{per}}(U))$ with analogous calculations as above and repeating the argument gives

$$u \in H^1\big(0,T;H^2_{\text{per}}(U)\big) \cap L^\infty\big(0,T;H^4_{\text{per}}(U)\big) \cap L^2\big(0,T;H^6_{\text{per}}(U)\big)$$

and so on. By this bootstrap argument we finally obtain

$$u \in H^1\big(0,T;H^k_{\text{per}}(U)\big) \cap L^\infty\big(0,T;H^k_{\text{per}}(U)\big) \cap L^2\big(0,T;H^k_{\text{per}}(U)\big)$$

for all $k \geq 0$ and consequently, u is a smooth function in the space variable. For the differentiability in time we argue similarly. We see that $G(u) \in H^1(0,T;H^k_{\text{per}}(U))$ for all $k \geq 0$ and (formally) differentiating (5.28) yields

$$\partial_t(\partial_t u) = -\varepsilon\Delta^2(\partial_t u) + \partial_t G(u) \in L^2(0,T;H^k_{\text{per}}(U))$$

which especially implies $u \in H^2(0,T;H^k_{\text{per}}(U))$ for all k. This provides better time regularity for $G(u)$ and with another bootstrap argument we successively obtain $u \in H^j(0,T;H^k_{\text{per}}(U))$ for all $j,k \geq 0$ which proves the proposition.
To show the time regularity of u in $t=0$ we fix $j \geq 0$ and $0 \leq k \leq 2$. For $t \leq 1$ we obtain

$$\begin{aligned} &\left\|\partial_t^j u^{(k)}(t) - \partial_t^j u^{(k)}(0)\right\|^2_{L^2(U)} \leq \int_U \left(\int_0^t \left|\partial_t^{j+1} u^{(k)}\right| \, ds\right)^2 \, dx \\ \leq t &\left\|\partial_t^{j+1} u^{(k)}\right\|^2_{L^2(0,1;L^2_{\text{per}}(U))} \leq t\,\|u\|^2_{H^{j+1}(0,1;H^2_{\text{per}}(U))} \\ &\longrightarrow 0 \end{aligned}$$

as $t \to 0$. This holds true for all $j \geq 0$ and hence implies $u \in C^\infty([0,\infty);H^2_{\text{per}}(U))$. □

5.2 Slow motion: Heuristic justification and analytical results

We deal with the question how the gradient flow of $\mathcal{W}_\varepsilon$ qualitatively behaves in one dimension and whether transition layers show up dynamic metastability. This question has not yet been addressed for the diffuse Willmore flow but is well studied for gradient flows of the diffuse area functional $\mathcal{E}_\varepsilon$. There, this phenomenon is distinguished by several stages of the evolution. In a (fast) first energy relaxation stage the initial value forms almost constant regions and steep transition layers between them. Afterwards, the configuration appears to be stationary but in fact evolves exponentially slow on a long timescale. Both stages are followed by a possible third stage during which the two closest layers collide and annihilate. This process repeats until a global minimizer of the underlying energy functional is attained.
Our interest results from the observed dynamics in [EsRäRö14] (which appear numerically nearly stable after an initial energy relaxation) and is also motivated by the already mentioned metastable behavior of other gradient flows appearing in theory of phase transitions as the Allen-Cahn or Cahn-Hilliard equation.
A full analysis of the slow motion stage for the Allen-Cahn equation can be found in

[CaPe89] and [FuHa89] where a geometrical approach is used to characterize the exponentially slow layer movement. Using weaker conditions on the inital values Bronsard and Kohn [BrKo90] could prove slightly weaker results on this stage by energy-based techniques.
[Ch04] contains a detailed description of all stages and especially characterizes the initial energy relaxation. We also refer to [OtRe07] where the authors develop general abstract conditions on an energy functional such that its gradient flow shows up metastability. Afterwards, they exploit the gradient flow structure of the Allen-Cahn equation to derive similar results as in [Ch04] by energy based methods.
For a complete analysis of the Cahn-Hilliard equation we refer to [ScWe18] which is based on the (modified) scheme of [OtRe07] and the citations therein. We also point out that [ScWe18] heavily relies on the relaxation framework introduced in [OtWe14] which makes extensive use of the mass conservation of solutions to the Cahn-Hilliard equation.

The Allen-Cahn and Cahn-Hilliard equation are gradient flows of the diffuse surface energy functional $\mathcal{E}_\varepsilon$ (with respect either to the L^2-norm or the H^{-1}-norm). Here, we consider the gradient flow of the diffuse Willmore functional $\mathcal{W}_\varepsilon$ instead, which makes it difficult to adapt most of the listed approaches above. As a major challenge, solutions of its gradient flow (5.1) lack both of a parabolic maximum principle (as in the case of the Allen-Cahn equation) and of the mass conservation property (as in the Cahn-Hilliard equation).
In the following, we will restrict on the slow motion stage in which the dynamics are mainly driven by the layer locations and especially on the question how the competitor phase fields u^* constructed in Theorem 3.10 (as a prototype for configurations with formed transition layers and exponentially small energy) on an interval $[0, L]$, $L > 0$ evolve in time. Our aim is to describe the motion speed of u and especially its zero positions, see Propositions 5.10 and 5.14 below. Unfortunately, it turns out extremely difficult to determine the direction in which the zeros move. This specific problem still remains open and we comment on it at the end of the section via an heuristic argument.

As the motion speed of a solution u of (5.1) is directly related to its energy decrease we expect an exponentially small motion in the L^2-sense. This correspondence is true for every L^2-gradient flow equation although we formulate the inequality specifically for our problem in the next Proposition. In this section space derivatives of u will be denoted by u' as before while we write $\partial_t u$ for its time derivative.

Proposition 5.9. *Let u be a weak solution of* (5.1) *in the sense of Definition 5.1 for $\varepsilon > 0$ with initial condition $u_0 \in H^2_{per}\big((0, L)\big)$. Then for all $T > 0$ we have the inequality*

$$\int_0^T \int_0^L (\partial_t u)^2 \, dx \, dt = \mathcal{W}_\varepsilon(u_0) - \mathcal{W}_\varepsilon\big(u(\cdot, T)\big) \leq \mathcal{W}_\varepsilon(u_0). \tag{5.29}$$

Proof. From Proposition 5.8 we obtain $u \in C^\infty((0, \infty); C^\infty_{\text{per}}(U)) \cap C^\infty([0, \infty); H^2_{\text{per}}(U))$. We multiply (5.1a) by $\partial_t u$ and integrate over $(0, T) \times (0, L)$. After an integration by parts on the right hand side in space this yields

$$\begin{aligned}&\int_0^T\int_0^L (\partial_t u)^2\,dx\,dt\\ =&-\frac{1}{\varepsilon}\int_0^T\int_0^L\Big(-\varepsilon u''+\frac{1}{\varepsilon}F'(u)\Big)\Big(-\varepsilon(\partial_t u)''+\frac{1}{\varepsilon}F''(u)\partial_t u\Big)\,dx\,dt\\ =&-\frac{1}{2\varepsilon}\int_0^T\frac{d}{dt}\int_0^L\Big(-\varepsilon u''+\frac{1}{\varepsilon}F'(u)\Big)^2\,dx\,dt\\ =&\mathcal{W}_\varepsilon(u_0)-\mathcal{W}_\varepsilon\big(u(\cdot,T)\big)\end{aligned}$$

where we have used the continuity in time of the inner integral down to $t=0$. The statement then follows as $W_\varepsilon(u(\cdot,T))\geq 0$ for all $T\geq 0$. □

Proposition 5.9 gives a first estimate on the motion speed of u and only yields a bound on $\partial_t u$ in $L^2((0,T)\times(0,L))$. Nevertheless, we can use the gradient inequality (5.29) to obtain better insight into the dynamic behavior. Propositions 5.10 and 5.14 are in the spirit of [BrKo90] where in contrast to other approaches to the slow motion stage of the Allen-Cahn equation the results are merely due to the equation's gradient flow structure and energy based methods. This makes it suitable to adapt the results in our case, as we have a good understanding of the energy orders of configurations with a prescribed number of transition layers from Section 3.2.

The first result describes the time evolution of initial configurations with small energy. As we will see such phase fields stay almost constant on a large time scale.

Proposition 5.10. *For $\ell\in(0,L)$ let $(u_0^\varepsilon)_{\varepsilon>0}$ be a sequence of initial values in $H^2_{per}\big((0,L)\big)$ with $u(0)=u(\ell)=u(L)=0$ and*

$$u_0^\varepsilon\longrightarrow v:=2\chi_{(0,\ell)-1}\quad \text{in } L^1\big((0,L)\big),\tag{5.30}$$

$$\mathcal{W}_\varepsilon(u_0^\varepsilon)\longrightarrow 0\tag{5.31}$$

as $\varepsilon\to 0$. For the solution $u^\varepsilon\in C^\infty((0,\infty);C^\infty_{per}((0,L)))\cap C^\infty([0,\infty);H^2_{per}(U))$ of (5.1) with initial value u_0^ε, $\varepsilon>0$, we then obtain for all $T>0$

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\int_0^L|u^\varepsilon(\cdot,t)-v|\,dx=0.\tag{5.32}$$

Proof. For $T>0$ we observe

$$\sup_{0\leq t\leq T}\int_0^L|u^\varepsilon(\cdot,t)-v|\,dx\leq\sup_{0\leq t\leq T}\int_0^L|u^\varepsilon(\cdot,t)-u_0^\varepsilon|\,dx+\int_0^L|u_0^\varepsilon-v|\,dx$$

where the second summand tends to zero by (5.30). For the first term we apply the fundamental theorem of calculus for the estimate

$$\begin{aligned}
&\sup_{0\le t\le T}\int_0^L |u^\varepsilon(\cdot,t)-u_0^\varepsilon|\,dx \le \sup_{0\le t\le T}\int_0^L\int_0^t |\partial_t u^\varepsilon|\,ds\,dx\\
&\le \int_0^L\int_0^T |\partial_t u^\varepsilon|\,ds\,dx \le \sqrt{LT}\left(\int_0^L\int_0^T |\partial_t u^\varepsilon|^2\,ds\,dx\right)^{\frac12}\\
&\le \sqrt{LT}\sqrt{\mathcal{W}_\varepsilon(u_0^\varepsilon)}\\
&\quad\longrightarrow 0
\end{aligned}$$

as $\varepsilon \to 0$ where the last inequality follows from (5.29). □

As we have information about the least energy order of $\mathcal{W}_\varepsilon$ from the scaling law, we can improve the result above for a special choice of initial values.

Proposition 5.11. *In the situation of Proposition 5.10 choose $(u_0^\varepsilon)_{\varepsilon>0}$ such that*

$$\mathcal{W}_\varepsilon(u_0^\varepsilon) \le \frac{C}{\varepsilon^2}\left|e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{L-\ell}{\varepsilon}}\right|^2 \le \frac{2C}{\varepsilon^2}e^{-2\alpha\frac{\min\{\ell,L-\ell\}}{\varepsilon}} \tag{5.33}$$

holds for small $\varepsilon>0$. Then, we have for all constants $m>0$, $D<2\alpha$

$$\lim_{\varepsilon\to 0}\sup\left\{\int_0^L |u^\varepsilon(\cdot,t)-v|\,dx \,:\, 0\le t\le m\varepsilon^2 e^{D\frac{\min\{\ell,L-\ell\}}{\varepsilon}}\right\} = 0.$$

Proof. This follows directly by repeating the proof of Proposition 5.10. Note that the ε dependence of T only becomes relevant in the last inequality, where it is now compensated by the smallness of $\sqrt{\mathcal{W}_\varepsilon(u_0^\varepsilon)}$. □

Remark. For $\varepsilon>0$ sufficiently small, $\frac{\ell}{\varepsilon}, \frac{L-\ell}{\varepsilon} > \ell_4$ is satisfied and we can choose u_0^ε to be the constructed competitor function u^* from the proof of Theorem 3.10 which satisfies (5.33).

We have proven that initial configurations with exponentially small energy hardly move for exponentially long times which explains why such phase fields appear numerically stable. We continue by analyzing the movement of transition layers explicitly.
Before we formulate our result and further considerations we prove a preparatory lemma which basically says that functions in M_ℓ^ε with small diffuse Willmore energy are almost step functions with values in $\{-1,1\}$.

Lemma 5.12. *Let $\varepsilon,\ell>0$ with $\frac{\ell}{\varepsilon}, \frac{L-\ell}{\varepsilon} > \ell_4$ and consider $u\in M_\ell^\varepsilon$ with*

$$\mathcal{W}_\varepsilon(u) \le \frac{C}{\varepsilon^2}\left|e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{L-\ell}{\varepsilon}}\right|^2.$$

For sufficiently small $\delta>0$ there exists $\varepsilon_0=\varepsilon_0(\delta)$ such that for all $\varepsilon<\varepsilon_0$ we have

$$\left|u^2(x)-1\right| \le \delta \quad \textit{for all } x\in[\delta,\ell-\delta]\cup[\ell+\delta,L-\delta]. \tag{5.34}$$

Proof. By Corollary 3.4 (5.34) holds for $q_{\ell,\varepsilon}$ in $[\delta,\ell-\delta]$ (for $-q_{r,\varepsilon}$ in $[\ell+\delta,L-\delta]$, respectively). If $\varepsilon>0$ is sufficiently small u inherits the property due to the estimate

(3.14).
Precisely, we choose $\delta > 0$ small and observe for $x \in [\delta, \ell - \delta]$

$$|u(x) - q_{\ell,\varepsilon}(x)| \leq C \, \|u - q_{\ell,\varepsilon}\|_{H^1\left((0,\ell)\right)} \leq C\sqrt{\varepsilon}\sqrt{\mathcal{W}_\varepsilon(u)} \leq \frac{C}{\sqrt{\varepsilon}} \left| e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{L-\ell}{\varepsilon}} \right|$$

by estimate (3.14) which becomes smaller as $\frac{\delta}{6}$ for sufficiently small $\varepsilon > 0$. Moreover, we directly deduce that $|u(x)| < 2$ as u is close to $q_{\ell,\varepsilon}$ and $0 \leq q_{\ell,\varepsilon} < 1$.
Together with Corollary 3.4 we can also choose ε sufficiently small such that

$$|q_{\ell,\varepsilon} - 1| \leq \frac{\delta}{6} \quad \text{in } [\delta, \ell - \delta]$$

to obtain

$$\left|u^2(x) - 1\right| = |u(x) + 1| \, |u(x) - 1| \leq 3\Big(|u(x) - q_{\ell,\varepsilon}| + |q_{\ell,\varepsilon} - 1| \Big) = \delta.$$

For $x \in [\ell + \delta, L - \delta]$ the same argument applies and this proves (5.34). □

For the rest of this section we consider $\varepsilon, \ell_0 > 0$ satisfying $\frac{\ell_0}{\varepsilon}, \frac{L-\ell_0}{\varepsilon} > \ell_4$ (with ℓ_4 from Theorem 3.10) and choose $\tilde{u}_0 \in M^\varepsilon_{\ell_0}$ to be the competitor function u^* constructed in the proof of the scaling law. We now define $u_0 \in H^2_{\text{per}}\big((0, L)\big)$ by

$$u_0(\cdot) := \tilde{u}_0\left(\cdot - \frac{\ell}{2}\right)$$

taking the periodicity of $\tilde{u}_0$ into account. The shift by $\frac{\ell}{2}$ ensures that no zero lies on the ends of the interval at time $t = 0$. As we are interested in the movement of the zero positions, this will simplify the formulation of the results below. We denote the zero positions of u_0 by $x_0^1, x_0^2 \in (0, L)$, $x_0^1 < x_0^2$, satisfying (after possible change of indices)

$$x_0^2 - x_0^1 = \ell_0.$$

and hence, $u_0 > 0$ in (x_0^1, x_0^2). Further, let $u \in C^\infty((0, \infty); C^\infty_{\text{per}}((0, L)))$ be the solution of (5.1) satisfying $u(\cdot, 0) = u_0(\cdot)$.
We can assume that the number of zeros of $u(\cdot, t)$ does not change and that they stay transversal for large times as by Theorem 3.9 and the small energy size of u_0 increasing the number of zeros is not energetically preferable (see the proof of Theorem 3.12 for a similar argument). At a time $t \geq 0$ we will denote the zeros of $u(\cdot, t)$ by $x^1(t)$ and $x^2(t)$ with $x^1(0) = x_0^1$ and $x^2(0) = x_0^2$ and remark that by the implicit function theorem the mappings

$$x^i \,:\, t \mapsto x^i(t), \quad t \geq 0, \, i = 1, 2$$

are differentiable. For convenience, we also define the function

$$\ell(t) := x^2(t) - x^1(t)$$

which satisfies $\ell(0) = \ell_0$. We restrict ourselves to the analysis of two transition layers here but remark that the results below can easily be generalized for configurations with more layers, provided that neighboring zeros are far enough away from each other.

We are going to apply Lemma 5.12 for different points in time and therefore have to ensure that the requirements of Theorem 3.9, on which we relied in the proof, are satisfied for large times $T > 0$. In particular we have to check condition (3.9) for positive times T. As we consider the gradient flow of $\mathcal{W}_\varepsilon$ it is not a priori clear how the diffuse surface energy of u evolves in time.

Lemma 5.13. *For $T > 0$ and small $\varepsilon > 0$ we have*

$$\mathcal{E}_\varepsilon\big(u(\cdot,T)\big) \le \mathcal{E}_\varepsilon(u_0) + \frac{\sqrt{2T}C_2}{\varepsilon^{\frac{3}{2}}}\left|e^{-\alpha\frac{\ell_0}{\varepsilon}} - e^{-\alpha\frac{L-\ell_0}{\varepsilon}}\right|^2$$

and hence, condition (3.9) *is still satisfied for exponentially long times.*

Proof. For $T > 0$ we apply Proposition 5.9 to obtain

$$\begin{aligned}
&\int_0^T \frac{d}{dt}\mathcal{E}_\varepsilon(u)\,ds = \int_0^T \frac{d}{dt}\int_0^L \frac{\varepsilon}{2}(u')^2 + \frac{1}{\varepsilon}F(u)\,dx\,ds\\
=&\int_0^T\int_0^L \Big(-\varepsilon u'' + \frac{1}{\varepsilon}F'(u)\Big)\,\partial_t u\,dx\,ds\\
\le&\sqrt{2\varepsilon}\left(\int_0^T \mathcal{W}_\varepsilon(u)\,ds\right)^{\frac{1}{2}}\left(\int_0^T\int_0^L(\partial_t u)^2\,dx\,ds\right)^{\frac{1}{2}}\\
\le&\sqrt{2\varepsilon T}\mathcal{W}_\varepsilon(u_0)
\end{aligned}$$

and hence,

$$\begin{aligned}
\mathcal{E}_\varepsilon\big(u(\cdot,T)\big) \le&\mathcal{E}_\varepsilon(u_0) + \sqrt{2\varepsilon T}\mathcal{W}_\varepsilon(u_0)\\
=&\mathcal{E}_\varepsilon(u_0) + \frac{\sqrt{2T}C_2}{\varepsilon^{\frac{3}{2}}}\left|e^{-\alpha\frac{\ell_0}{\varepsilon}} - e^{-\alpha\frac{L-\ell_0}{\varepsilon}}\right|^2.
\end{aligned}$$ □

Now we can characterize the movement speed of the layer positions of u.

Proposition 5.14. *For $\delta_1 > 0$ we define*

$$T_\varepsilon^{\delta_1} := \inf\Big\{t \ge 0 \,:\, \big|x^i(t) - x_0^i\big| > \delta_1 \text{ for some } i \in \{1,2\}\Big\}.$$

If δ_1 is sufficiently small there exists $\varepsilon_0 = \varepsilon_0(\delta_1) > 0$ such that

$$T_\varepsilon^{\delta_1} \ge C\delta_1^2\Big(\mathcal{W}_\varepsilon(u_0)\Big)^{-1} \tag{5.35}$$

holds for every $\varepsilon < \varepsilon_0$. This means that the transition layers require at least exponential long time (in terms of $\frac{\ell}{\varepsilon}$ and $\frac{L-\ell_0}{\varepsilon}$) to travel the prescribed distance δ_1.

Proof. Let $\delta_1 > 0$ be small. We notice that there is nothing to show if $T_\varepsilon^{\delta_1} = \infty$ and hence, we can assume that $T_\varepsilon^{\delta_1} < \infty$. Since the zeros of u move continuously in time, there exists $i \in \{1,2\}$ with

$$\left|x^i\big(T_\varepsilon^{\delta_1}\big) - x_0^i\right| = \delta_1$$

which implies

$$\int_0^L \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - u_0\right| \, dx \geq C\delta_1. \tag{5.36}$$

Indeed, this follows from the fact that both $u\big(\cdot, T_\varepsilon^{\delta_1}\big)$ and u_0 are basically step functions by Lemma 5.12 with one of their discontinuities δ_1 away from each other. We have

$$\begin{aligned}
&\int_0^L \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - u_0\right| \, dx \\
\geq &\int_0^L \left|\big(2\chi_{\left(x^1(T_\varepsilon^{\delta_1}) - x^2(T_\varepsilon^{\delta_1})\right)} - 1\big) - \big(2\chi_{(x_0^1, x_0^2)} - 1\big)\right| \, dx \\
&\quad - \int_0^L \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - \big(2\chi_{\left(x^1(T_\varepsilon^{\delta_1}) - x^2(T_\varepsilon^{\delta_1})\right)} - 1\big)\right| \, dx \\
&\quad - \int_0^L \left|u_0 - \big(2\chi_{(x_0^1, x_0^2)} - 1\big)\right| \, dx
\end{aligned} \tag{5.37}$$

where the first integral is greater or equal to $2\delta_1$. The remaining integrals can be treated analogously and we therefore concentrate on one of them. As $u(\cdot, T_\varepsilon^{\delta_1})$ is periodic we can perform a shift of $x^1(T_\varepsilon^{\delta_1})$ in space (and a transformation of variables afterwards) to obtain

$$\begin{aligned}
&\int_0^L \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - \big(2\chi_{\left(x^1(T_\varepsilon^{\delta_1}), x^2(T_\varepsilon^{\delta_1})\right)} - 1\big)\right| \, dx \\
= &\int_0^L \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - \big(2\chi_{\left(0, \ell(T_\varepsilon^{\delta_1})\right)} - 1\big)\right| \, dx
\end{aligned}$$

as well as $u(0, T_\varepsilon^{\delta_1}) = u(\ell(T_\varepsilon^{\delta_1}), T_\varepsilon^{\delta_1}) = 0$. Since

$$\mathcal{W}_\varepsilon\Big(u\big(\cdot, T_\varepsilon^{\delta_1}\big)\Big) \leq \mathcal{W}_\varepsilon(u_0) = \frac{C_2}{\varepsilon^2}\left|e^{-\alpha\frac{\ell_0}{\varepsilon}} - e^{-\alpha\frac{L-\ell_0}{\varepsilon}}\right|^2$$

and due to Lemma 5.13 we have $u(\cdot, T_\varepsilon^{\delta_1}) \in M^\varepsilon_{\ell(T_\varepsilon^{\delta_1})}$ and we can apply Lemma 5.12 for $\delta \ll \delta_1$. This yields for $I_\delta := (\delta, \ell - \delta) \cup (\ell + \delta, L - \delta) \subset [0, L]$

$$\begin{aligned}
&\int_0^L \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - \big(2\chi_{\left(0, \ell(T_\varepsilon^{\delta_1})\right)} - 1\big)\right| \, dx \\
= &\int_{I_\delta} \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - \big(2\chi_{\left(0, \ell(T_\varepsilon^{\delta_1})\right)} - 1\big)\right| \, dx \\
&\quad + \int_{[0,L]\setminus I_\delta} \left|u\big(\cdot, T_\varepsilon^{\delta_1}\big) - \big(2\chi_{\left(0, \ell(T_\varepsilon^{\delta_1})\right)} - 1\big)\right| \, dx \\
\leq &\delta \left|I_\delta\right| + C\left|[0, L] \setminus I_\delta\right| \leq C\delta
\end{aligned}$$

and for sufficiently small $\delta > 0$ we combine the estimates of the single integrals in (5.37) to conclude (5.36).
Now, by the fundamental theorem of calculus and Proposition 5.9 we have

$$C\delta_1 \leq \int_0^{T_\varepsilon^{\delta_1}} \int_0^L |\partial_t u| \, dx \, ds \leq \sqrt{LT_\varepsilon^{\delta_1}} \left(\int_0^{T_\varepsilon^{\delta_1}} \int_0^L (\partial_t u)^2 \right)^{\frac{1}{2}} \leq \sqrt{LT_\varepsilon^{\delta_1}} \sqrt{\mathcal{W}_\varepsilon(u_0)}$$

and (5.35) follows by rearranging. □

The proven results above give a profound understanding of the slow time evolution of configurations with small energy under the diffuse Willmore flow and the velocity of moving phase transitions. However, they do not explain in which direction the layer locations move in time. This problem remains unsolved in general although we expect that the zeros of u will distribute equally on $(0, \ell)$ and maximize their mutual distance taking the periodicity into account. We present a partial result on the movement direction as an immediate consequence of the scaling law from Proposition 3.16. We will prove that whenever $\ell_0 > L - \ell_0$ (the case "$<$" is absolutely analogous) the scaling law directly implies that $\ell(t)$ cannot exceed ℓ_0 by more than a constant times ε in the "wrong" direction for all times $t > 0$.

Proposition 5.15. *Assume that $\ell_0 > L - \ell_0$ and define $\tilde{C} := \frac{C_2}{C_1} \geq 1$ with C_1, C_2 from Theorem 3.10. Then we have for all $t > 0$*

$$\ell(t) \leq \ell_0 + \left(\sqrt{\tilde{C}} - 1\right) \frac{\varepsilon}{\alpha}. \tag{5.38}$$

Proof. For a fixed $t > 0$ we omit the argument of $\ell(t)$ and write $\ell := \ell(t)$ in the following. We can assume that $\ell > L - \ell$ and $\ell > \ell_0$ as there is nothing to show otherwise. From (3.16) we obtain

$$\frac{C_1}{\varepsilon^2} \left| e^{-\alpha \frac{\ell}{\varepsilon}} - e^{-\alpha \frac{L-\ell}{\varepsilon}} \right|^2 \leq \mathcal{W}_\varepsilon(u) \leq W_\varepsilon(u_0) \leq \frac{C_2}{\varepsilon^2} \left| e^{-\alpha \frac{\ell_0}{\varepsilon}} - e^{-\alpha \frac{L-\ell_0}{\varepsilon}} \right|^2$$

and taking the square root after rearranging gives

$$e^{-\alpha \frac{L-\ell}{\varepsilon}} - e^{-\alpha \frac{\ell}{\varepsilon}} \leq \sqrt{\tilde{C}} \left(e^{-\alpha \frac{L-\ell_0}{\varepsilon}} - e^{-\alpha \frac{\ell_0}{\varepsilon}} \right).$$

A multiplication by $e^{\alpha \frac{L-\ell_0}{\varepsilon}}$ yields

$$e^{\alpha \frac{\ell-\ell_0}{\varepsilon}} - e^{\alpha \frac{L-(\ell_0+\varepsilon)}{\varepsilon}} \leq \sqrt{\tilde{C}} \left(1 - e^{\alpha \frac{L-2\ell_0}{\varepsilon}} \right)$$

and hence since $\tilde{C} \geq 1$

$$\begin{aligned} 1 + \frac{\alpha}{\varepsilon}(\ell - \ell_0) &\leq e^{\alpha \frac{\ell-\ell_0}{\varepsilon}} \leq \sqrt{\tilde{C}} - \sqrt{\tilde{C}} e^{\alpha \frac{L-2\ell_0}{\varepsilon}} + e^{\alpha \frac{L-(\ell_0+\varepsilon)}{\varepsilon}} \\ \leq& \sqrt{\tilde{C}} + e^{\alpha \frac{L}{\varepsilon}} \left(e^{-\alpha \frac{\ell_0+\ell}{\varepsilon}} - e^{-\alpha \frac{2\ell_0}{\varepsilon}} \right) = \sqrt{\tilde{C}} - \frac{\alpha}{\varepsilon} e^{\alpha \frac{L}{\varepsilon}} e^{-\alpha \frac{\ell_0 - \tilde{\ell}}{\varepsilon}} (\ell - \ell_0) \end{aligned}$$

due to the basic inequality $1 + s \leq e^s$ for $s \geq 0$ and with $\tilde{\ell}$ between ℓ_0 and ℓ determined by the mean value theorem. The second summand in the last step above is negative and we neglect it in the inequality which immediately yields (5.38) after rearranging. □

We can adapt the argument from the proof of Proposition 5.15 for an heuristic derivation of an ODE model which describes the evolution of the zeros of u.

Heuristic ODE-model for the zero position. The scaling law from Theorem (3.12) lets us formally derive an ODE system describing the motion of zero positions of a configuration u under the diffuse Willmore flow. Although the constants C_1, C_2 in the scaling law (3.20) are different in general, we now assume that there exists a constant $C_0 = C_0(L) > 0$ such that

$$\min_{M_\ell^\varepsilon} \mathcal{W}_\varepsilon \approx \frac{C_0}{\varepsilon^2} \left| e^{-\alpha\frac{\ell}{\varepsilon}} - e^{-\alpha\frac{L-\ell}{\varepsilon}} \right|^2$$

holds for all $\ell > 0$ with $\frac{\ell}{\varepsilon}, \frac{L-\ell}{\varepsilon} > \ell^*$ (with ℓ^* from Theorem 3.12) and where we have neglected possible lower order terms.
As the functions $x^1(t), x^2(t)$ are differentiable it is reasonable to describe their behavior by an ODE system.
We define the functional

$$H_\varepsilon(u) := \tilde{H}_\varepsilon(x^1, x^2) := \frac{C_0}{\varepsilon^2} \left| e^{-\alpha\frac{x^2-x^1}{\varepsilon}} - e^{-\alpha\frac{L-(x^2-x^1)}{\varepsilon}} \right|^2$$

and calculate the corresponding gradient flow for x^1, x^2

$$\partial_t \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = -\nabla \tilde{H}_\varepsilon(x^1, x^2).$$

This yields the system

$$\begin{aligned} \partial_t x^1 &= \frac{2\alpha C_0}{\varepsilon^3} \left(e^{-2\alpha\frac{L-(x^2-x^1)}{\varepsilon}} - e^{-2\alpha\frac{(x^2-x^1)}{\varepsilon}} \right), \\ \partial_t x^2 &= \frac{2\alpha C_0}{\varepsilon^3} \left(e^{-2\alpha\frac{(x^2-x^1)}{\varepsilon}} - e^{-2\alpha\frac{L-(x^2-x^1)}{\varepsilon}} \right). \end{aligned} \tag{5.39}$$

which describes the time evolution of both zero positions of u completely. Notice that for $\ell = x^2 - x^1 > \frac{L}{2}\,(< \frac{L}{2})$ we have $\partial_t x^1 > 0\,(< 0)$ and $\partial_t x^2 < 0\,(> 0)$ and $\partial_t x^1 = \partial_t x^2 = 0$ for $x^2 - x^1 = \frac{L}{2}$. This means that the zeros always maximize their distance to each other (regarding periodicity) and distribute equally on the interval which is in accordance to our observation that only in these cases the energy can vanish completely. However, the motion of zeros is exponentially slow and (5.39) is an intelligent guess for their velocity.

The derivation of (5.39) was done under the assumption that there exists an optimal constant C_0 in the scaling law. Moreover, it is not clear that the gradient flow for x^1 and x^2 really displays the actual behavior of the zeros. While the first point is probably difficult to solve (as the constant in the crucial estimate (3.14) is only given implicitly) this method yields the correct dynamics in similar contexts. In [BeNaNo15] the authors prove a scaling law for the Ginzburg-Landau energy $\mathcal{E}_\varepsilon$ in a similar setting with an explicitly determined sharp constant. As the energy scaling is solely dependent on the zero positions (as in our case) they formally derive an ODE system for their motion under the gradient flow of $\mathcal{E}_\varepsilon$ (the Allen-Cahn equation) which coincides (up to an multiplicative constant) with the rigorously proven results from [CaPe89].

We end this section with a few thoughts about the zero movement and give a small outlook how it could be proved.
Even if it would be possible to match the constants C_1 and C_2 in the scaling law (which does not even have to be possible in general) this does not immediately justify the ODE system (5.39).
The in our view most promising approach is to follow the ideas from [CaPe89] for the slow motion of the Allen-Cahn equation. There, the authors identify a potential and prove that the time evolution is driven by potential gaps between the transition layers. We remark that the gradient flow of $\mathcal{W}_\varepsilon$ still gives rise to a potential in the sense that a solution of (5.1a) satisfies

$$(\partial_t u)u' = \partial_x \Big[-\varepsilon^2 u''' u' + \frac{\varepsilon^2}{2}(u'')^2 + F''(u)(u')^2 - \frac{1}{2\varepsilon^2}F'(u)^2 \Big] =: \partial_x P(u).$$

For this strategy it is crucial to have a precise understanding of the shape of quasi stationary states which still lacks in our case and therefore, this could be a first point for later research.

6 The diffuse approximation of $\mathcal{W}$ for nonsmooth configurations

6.1 Γ-convergence of $\mathcal{W}_\varepsilon$ for intersecting boundary curves

In connection with the Van der Waals-Cahn-Hilliard theory of phase transitions De Giorgi stated the following conjecture regarding the Γ-limit of certain diffuse energy functionals approximating the sum of the area and Willmore energy of the sharp phase boundary. We state the precise conjecture from [DeG91], Conjecture 4, in a slightly corrected form with a factor 2 in front of the Laplacian and slightly enhanced regularity for the function u (see also [RöSc06]).

Conjecture (De Giorgi): For $\Omega \subset \mathbb{R}^n$ and $\lambda > 0$ define the functionals $G_p : L^1(\Omega) \to \mathbb{R}$, $p > 0$ by

$$G_p(u) := \int_\Omega \left[\left(\frac{2\Delta u}{p} - p \sin u \right)^2 + \lambda \right] \left[\frac{|\nabla u|^2}{p} + p(1 - \cos u) \right] dx, \tag{6.1}$$

if $u \in H^2(\Omega)$, and $G_p(u) := \infty$ if $u \in L^1(\Omega) \setminus H^2(\Omega)$. Then there exists $k \in \mathbb{R}$, depending only on n, such that for any $E \subset \Omega$ with $\partial E \cap \Omega \in C^2$ and $u = 2\pi \chi_E$

$$\Gamma(L^1(\Omega)) - \lim_{p \to \infty} G_p(u) = 8\sqrt{2}\lambda \mathcal{H}^{n-1}(\partial E \cap \Omega) + k \int_{\partial E \cap \Omega} |\mathbf{H}_{\partial E}|^2 \, d\mathcal{H}^{n-1}.$$

Due to the large mathematical relevance of phase transition theory, De Giorgi's conjecture has been studied intensely in the following years. In this work we consider a modified problem for closely related functionals which originally has been formulated in [BePa93].

Conjecture (De Giorgi, modified): For $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ define the functionals $\mathcal{F}_\varepsilon : L^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{F}_\varepsilon(u) &:= \mathcal{E}_\varepsilon(u) + \mathcal{W}_\varepsilon(u) \\ &:= \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \right) dx + \int_\Omega \frac{1}{2\varepsilon} \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} F'(u) \right)^2 dx, \end{aligned} \tag{6.2}$$

if $u \in L^1(\Omega) \cap H^2(\Omega)$ and $\mathcal{F}_\varepsilon(u) := \infty$ if $u \in L^1(\Omega) \setminus H^2(\Omega)$. Moreover, let $E \subset \Omega$

with $\partial E \cap \Omega \in C^2$ and indicator function $\chi := 2\chi_E - 1$ and set $\sigma := \int_{-1}^{1} \sqrt{2F}\, ds$. Then $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ is $\Gamma(L^1(\Omega))$-convergent in χ and

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\chi) = \sigma \mathcal{H}^{n-1}(\partial E \cap \Omega) + \sigma \int_{\partial E\cap\Omega} \frac{1}{2} \left|\mathbf{H}_{\partial E}\right|^2 \, d\mathcal{H}^{n-1}. \tag{6.3}$$

Remark. In comparison to the original conjecture, Bellettini and Paolini changed the double well potential from $(1 - \cos u)$ to F (as before) and replaced the *energy density* $\varepsilon \left|\nabla u\right|^2 + \frac{1}{\varepsilon}(1 - \cos u)$ in front of the diffuse curvature term by a factor $\frac{1}{\varepsilon}$ where we set $\varepsilon := \frac{1}{p}$ in (6.1). The last mentioned modification simplifies the structure of the considered functionals and can roughly be motivated by the fact that the phase transition happens on a layer of size ε which means that the energy density above is of order $\frac{1}{\varepsilon}$ in the transition layers. For further details of the problem we refer to [BePa93, LoMa00, RöSc06] without any claim on completeness.
As another advantage of the modified formulation, the constant factors in front of both summands of the limit functional now coincide and σ can be interpreted as the "cost" of an optimal one-dimensional transition from -1 to 1 in terms of the diffuse surface area functional (see [MoMo77]).
In comparison to [BePa93] and [RöSc06], we added a factor $\frac{1}{2}$ in front of the Willmore functional and its diffuse approximation to be consistent with the notation of this thesis.

The first summand of $\mathcal{F}_\varepsilon$ is (up to the constant σ) a diffuse approximation of the surface area of $\partial E \cap \Omega$ and its Γ-convergence has already been proven in [MoMo77] (see also [Mo87]) for more general sets E with finite perimeter (see, e.g., Section 2.3 for a definition).
The Γ-convergence of the second term has turned out to be more complicated to prove and several authors have contributed in the investigation of the problem. The lim sup-part of (6.3) was shown in 1993 by Bellettini and Paolini who managed to construct a proper recovery sequence for the problem (see [BePa93, BeMu05] and also Section 2.6). The lim inf-estimate was accomplished by Bellettini and Mugnai in [BeMu05] for rotationally symmetric configurations in two dimensions while Moser could prove the estimate in $\mathbb{R}^3$ for data which are monotone in one space direction [Mo05]. In 2006 Röger and Schätzle managed to prove the whole lim inf estimate for dimensions $n = 2, 3$ in smooth limit points and thereby could prove the modified conjecture of De Giorgi in these dimensions [RöSc06]. Their approach uses the framework of geometric measure theory and the concept of varifolds as suggested in [HuTo00] and [To02].

In this chapter we are going to generalize the result in two dimensions to a wider class of sets E where our main focus lies on (self-)intersecting boundary curves in which case the conjecture has not been proven yet. In particular, we will show that these configurations have a limit energy given by the sum of the energies for every boundary curve without any "penalty" for the intersections and independent of the intersection angles (see (6.7) below). These configurations for example appear in the long time behavior of numerical simulations for the diffuse Willmore flow as shown in Figure 1.1. We refer to [EsRäRö14] for the precise numerical results.

We give a short outline of this section. In a first step we introduce a class $\mathcal{S}$ of indicator functions (and thereby the class of sets E) for which we will prove Γ-convergence of $\mathcal{F}_\varepsilon$ (see the following Section 6.2 for several examples of indicator functions in $\mathcal{S}$). Afterwards in Theorem 6.3, we formulate our main result whose proof is basically divided into two propositions.

We begin by defining an auxiliary class of characteristic functions $u = 2\chi_E - 1$ where we allow ∂E to be the union of finitely many closed C^2 curves with finitely many transversal (self) intersections such that ∂E locally looks like a line or actually is the intersection of two straight line segments. Observe, that we do not allow more than two boundary portions to intersect in one point in particular.
Precisely, we consider $E \subset \Omega$ such that:

i) There exist $N \in \mathbb{N}$ closed C^2-curves $\varphi_1, \dots, \varphi_N : S^1 \to \Omega$ in Ω such that

$$\partial E = \bigcup_{i=1}^{N} \varphi_i(S^1)$$

and

$$\Lambda(x) := \#\Big\{(k, s) \,:\, \varphi_k(s) = x\Big\} \leq 2 \tag{6.4}$$

for all $x \in \partial E$.

ii) The set $A := \{x \in \partial E \colon \Lambda(x) = 2\} \subset \partial E$ (which consists of the intersection points of ∂E) is finite and for every $x \in A$ there exist one-dimensional subspaces $P_1, P_2 \in G(2,1)$, $P_1 \neq P_2$ and an open neighborhood $U \subset \mathbb{R}^2$ containing the origin such that

$$\partial E \cap \Big(x + U\Big) = x + \Big((P_1 \cup P_2) \cap U\Big).$$

For every point $x \in \partial E \setminus A$ the classical tangent $T_x(\partial E)$ on ∂E is well defined.

The class of the considered indicator functions is then given by

$$\mathcal{S}_0 := \big\{u \in L^1(\Omega) \,:\, u = 2\chi_E - 1 \text{ with } E \subset\subset \Omega \text{ and conditions i)-ii) above}\big\}$$

and we define the Willmore energy of $u \in \mathcal{S}_0$ to be

$$\mathcal{W}(u) := \mathcal{W}(\partial E) := \sum_{i=1}^{N} \mathcal{W}(\varphi_i)$$

where the closed curves φ_i, $1 \leq i \leq N$ describe the positions of the phase transitions of u as in i) and with $\mathcal{W}(\varphi_i) := \mathcal{W}(\varphi_i(S^1))$ for all i. Finally, we set

$$\mathcal{F}(u) := \sigma\big(\mathcal{H}^1(\partial E \cap \Omega) + \mathcal{W}(\partial E)\big) = \sigma \sum_{i=1}^{N} \big(L(\varphi_i) + \mathcal{W}(\varphi_i)\big)$$

with $\sigma = \int_{-1}^{1} \sqrt{2F}\, ds$ as above. Based on $\mathcal{S}_0$, the class of functions $\mathcal{S}$ can now be constructed by an approximation argument. We consider indicator functions of sets $E \subset \Omega$ such that

iii) There exists an integral 1-varifold $V \in \mathbb{V}_1(\Omega)$ with weight measure $\|V\| := \theta_V \mathcal{H}^1 \lfloor M$ on Ω for a 1-rectifiable subset $M \subset \mathbb{R}^2$ and $\theta_V \in \mathbb{N}$ $\mathcal{H}^1$-a.e. on M such that

$$|\nabla \chi_E| \leq \|V\| . \tag{6.5}$$

This condition especially implies that $\partial^* E \subset \operatorname{supp} \|V\| = M$. Moreover, V shall have generalized mean curvature $\mathbf{H}_V \in L^2(\|V\|)$.

iv) There exists a sequence $(u^{(k)})_{k \in \mathbb{N}}$ in $\mathcal{S}_0$ such that

$$u^{(k)} \to u \quad \text{in } L^1(\Omega)$$

and

$$\mathcal{F}(u^{(k)}) \to \mathcal{F}^*(u) < \infty \tag{6.6}$$

as $k \to \infty$ where we define

$$\mathcal{F}^*(u) := \sigma \inf_{V \text{ as above}} \int_\Omega 1 + \frac{1}{2} |\mathbf{H}_V|^2 \, d\|V\| . \tag{6.7}$$

and set

$$\mathcal{S} := \left\{ u \in L^1(\Omega) \, : \, u = 2\chi_E - 1 \text{ with } E \subset\subset \Omega \text{ and conditions iii)-iv) above} \right\}.$$

Remark. The definition of $\mathcal{F}^*$ extends $\mathcal{F}$ in the sense that we have $\mathcal{F}^* = \mathcal{F}$ on $\mathcal{S}_0$. To see this consider $u = 2\chi_E - 1 \in \mathcal{S}_0$. Let $V \in \mathbb{V}_1(\Omega)$ be the integral 1-varifold with weight measure $\|V\| = \mathcal{H}^1 \lfloor \partial E$. Then (6.5) is obviously satisfied and $\mathbf{H}_V = \mathbf{H}_{\mathrm{Im}(\varphi_i)}$ holds $\mathcal{H}^1$-a.e. on $\mathrm{Im}(\varphi_i)$, $1 \leq i \leq N$. Hence, we especially have $\mathbf{H}_V \in L^2(\|V\|)$ as ∂E is C^2-regular away from the intersection points. This yields

$$\mathcal{F}^*(u) \leq \sigma \int_{\partial E} 1 + \frac{1}{2} |\mathbf{H}_{\partial E}|^2 \, d\mathcal{H}^1 = \sigma \sum_{i=1}^N \int_{\mathrm{Im}(\varphi_i)} 1 + \frac{1}{2} |\mathbf{H}_{\partial E}|^2 \, d\mathcal{H}^1 = \mathcal{F}(u).$$

Now let V as in iii) with (6.5) and $\|V\| = \theta_V \mathcal{H}^1 \lfloor M$. We obtain

$$\begin{aligned} &\int_M \left(1 + \frac{1}{2} |\mathbf{H}_V|^2 \right) \theta_V \, d\mathcal{H}^1 \geq \int_{\partial E} 1 + \frac{1}{2} |\mathbf{H}_V|^2 \, d\mathcal{H}^1 \\ = &\int_{\partial E} 1 + \frac{1}{2} |\mathbf{H}_{\partial E}| \, d\mathcal{H}^1 = \sigma^{-1} \mathcal{F}(u) \end{aligned}$$

where the equality in the second step follows from [Sc09], Corollary 4.3. By taking the infimum over all V as in iii) the statement is shown.

Remark. Notice that the conditions on $u \in \mathcal{S}$ are fairly natural regarding the generalization of De Giorgi's conjecture. Indeed, we can show that whenever a sequence of sharp interfaces $(u^{(k)})_{k \in \mathbb{N}}$ in $\mathcal{S}_0$ has uniformly bounded energy $\mathcal{F}(u^{(k)})$ there exists a

subsequence and a set $E \subset\subset \mathbb{R}^2$ such that the associated phase boundaries converge to a varifold as in iii):
Let $(u^{(k)})_{k\in\mathbb{N}}$ be a sequence in $\mathcal{S}_0$ with $u^{(k)} = 2\chi_{E_k} - 1$ for $k \geq 1$ and assume

$$\liminf_{k\to\infty} \mathcal{F}(u^{(k)}) = \liminf_{k\to\infty} \sigma \left(\mathcal{H}^1(\partial E_k) + \frac{1}{2} \|\mathbf{H}_{\partial E_k}\|^2_{L^2(\mathcal{H}^1 \lfloor \partial E_k)} \right) < \infty. \tag{6.8}$$

As the sequence is obviously bounded in $L^1(\Omega)$ the first part of the inequality implies that (at least for a subsequence) $\left|\nabla u^{(k)}\right|(\Omega)$ is bounded and by the compactness theorem for BV functions (see, e.g., [EvGa92], 5.2.3) we can extract a further (non relabeled) subsequence such that

$$u^{(k)} \longrightarrow u \quad \text{in } L^1(\Omega) \tag{6.9}$$

as $k \to \infty$ and $u = 2\chi_E - 1$ for a set $E \subset\subset \mathbb{R}^2$ of finite perimeter.

We further denote the induced unit density varifolds by $V_k \in \mathbb{V}_1(\Omega)$ which means $\|V_k\| = \mathcal{H}^1 \lfloor \partial E_k$ for $k \geq 1$. Particularly, (6.8) implies for all subsets $U \subset\subset \Omega$

$$\liminf_{k\to\infty} \|V_k\|(U) = \liminf_{k\to\infty} \mathcal{H}^1(\partial E_k \cap U) < \infty$$

as well as

$$\liminf_{k\to\infty} |\delta V_k|(U) \leq \liminf_{k\to\infty} \|\mathbf{H}_{\partial E_k}\|_{L^1(\mathcal{H}^1 \lfloor \partial E_k)} \leq C \liminf_{k\to\infty} \|\mathbf{H}_{\partial E_k}\|_{L^2(\mathcal{H}^1 \lfloor \partial E_k)} < \infty.$$

By Allard's integral compactness theorem (see Theorem 2.14) there exists a subsequence again denoted by $(V_k)_{k\in\mathbb{N}}$ which converges weakly to an integral 1-varifold $V \in V_1(\Omega)$. Moreover, this immediately induces $\delta V_k \overset{*}{\rightharpoonup} \delta V$ as $k \to \infty$ by definition. It remains to show that V has a generalized mean curvature $\mathbf{H}_V \in L^2(\|V\|)$. The lower semi-continuity of weak measure limits (see Proposition A.7) yields

$$\begin{aligned} |\delta V(\eta)| = \lim_{k\to\infty} |\delta V_k(\eta)| &= \lim_{k\to\infty} \left| \int_\Omega \mathbf{H}_{\partial E_k} \cdot \eta \, d\|V_k\| \right| \\ &\leq \liminf_{k\to\infty} \|\mathbf{H}_{\partial E_k}\|_{L^2(\|V_k\|)} \|\eta\|_{L^2(\|V_k\|)} \\ &\leq \liminf_{k\to\infty} \|\mathbf{H}_{\partial E_k}\|_{L^2(\|V_k\|)} \|\eta\|_{L^2(\|V\|)} \\ &\leq C \|\eta\|_{L^2(\|V\|)} \end{aligned}$$

for all vector fields $\eta \in C^0_c(\Omega; \mathbb{R}^2)$ and thus, δV is a bounded linear form on the reflexive space $L^2(\|V\|)$ for which exists a function $\mathbf{H}_V \in L^2(\|V\|)$, i.e. the generalized mean curvature of V, such that

$$\delta V(\eta) = -\int_\Omega \mathbf{H}_V \cdot \eta \, d\|V\|$$

for all $\eta \in C^0_c(\Omega; \mathbb{R}^2)$.
It remains to show that V satisfies (6.5). Let M denote the support of $\|V\|$ and for an

open set $U \subset \Omega$ consider $\varphi \in C_c^0(U;\mathbb{R}^2)$ with $|\varphi| \leq 1$. We set $K := \operatorname{supp}\varphi$ and obtain

$$\begin{aligned}\int_E \nabla\cdot\varphi\,dx &= \lim_{k\to\infty}\int_{E_k}\nabla\cdot\varphi\,dx = -\lim_{k\to\infty}\int_{\partial E_k}\varphi\cdot\nu_{\partial E_k}\,d\mathcal{H}^1\\ &\leq \limsup_{k\to\infty}\mathcal{H}^1(\partial E_k\cap K) \leq \mathcal{H}^1(M\cap K) \leq \mathcal{H}^1\lfloor M(U) \leq \|V\|(U)\end{aligned}$$

by (6.9) and the weak convergence of $\|V_k\| = \mathcal{H}^1\lfloor\partial E_k = |\nabla\chi_{E_k}|$. By definition we now have

$$|\nabla\chi_E|(U) = \sup\left\{\int_E \nabla\cdot\varphi\,dx\ :\ \varphi\in C_c^0(U;\mathbb{R}^2),\ |\varphi|\leq 1\right\} \leq \|V\|(U)$$

which holds for all open sets $U \subset \Omega$. This proves (6.5).

The observation above shows that we can assume without loss of generality that every sequence $(u^{(k)})_{k\in\mathbb{N}}$ of indicator functions in $\mathcal{S}_0$ with uniformly bounded energy $\mathcal{F}$ induces a limit integer 1-varifold as above.
Nevertheless, the energies $\mathcal{F}(u^{(k)})$ do not have to converge towards $\mathcal{F}^*(u)$ as in general we only have

$$\mathcal{F}^*(u) \leq \lim_{k\to\infty}\mathcal{F}(u^{(k)}).$$

This makes (6.6) the crucial condition to functions in $\mathcal{S}$.
Finally, the definition of the generalized energy functional $\mathcal{F}^*$ is rather canonical as the infimum ensures that ∂^*E (or rather the induced boundary measure $\mathcal{H}^1\lfloor\partial^*E$) is extended to an integer 1-varifold with generalized mean curvature in an energetic optimal way. In fact, we will show in Proposition 6.2 that the infimum in (6.7) is always attained.

The set $\mathcal{S}$ has been defined by an abstract approximation argument. We will give several examples of included elements in Section 6.2, which will provide a better insight into its structure.

Lemma 6.1. *The generalized energy functional $\mathcal{F}^*$ is L^1-lower semicontinuous on $\mathcal{S}$, i.e.,*

$$\mathcal{F}^*(u) \leq \liminf_{k\to\infty}\mathcal{F}^*(u_k)$$

for all $u\in\mathcal{S}$ and sequences $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{S}$ with $u_k\to u$ in $L^1(\Omega)$ as $k\to\infty$.

Proof. Let $u = 2\chi_E - 1 \in \mathcal{S}$ and $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{S}$ with $u_k \to u$ in $L^1(\Omega)$. We can assume that the limes inferior is finite as there is nothing to show otherwise. For $k\in\mathbb{N}$ we choose $V_k \in \mathbb{V}_1(\Omega)$ as in iii) with

$$\sigma\int_\Omega 1 + \frac{1}{2}|\mathbf{H}_{V_k}|^2\,d\|V_k\| \leq \mathcal{F}^*(u_k) + \frac{1}{k}$$

and thus obtain

$$\sigma \liminf_{k\to\infty} \int_\Omega 1 + \frac{1}{2} |\mathbf{H}_{V_k}|^2 \, d\|V_k\| \le \liminf_{k\to\infty} \mathcal{F}^*(u_k) < \infty.$$

As in the remark above, Allard's compactness theorem yields $V_k \overset{*}{\rightharpoonup} V$ for a (non relabeled) subsequence and again, we deduce

$$\|\mathbf{H}_V\|_{L^2(\|V\|)} = \sup_{\substack{g\in C_c^0(\Omega;\mathbb{R}^2)\\ \|g\|_{L^2(\|V\|)}\le 1}} \int_\Omega \mathbf{H}_V \cdot g \, d\|V\| \le \liminf_{k\to\infty} \|\mathbf{H}_{V_k}\|_{L^2(\|V_k\|)}$$

as well as

$$|\nabla \chi_E| \le \|V\|.$$

Therefore, V satisfies the requirements in iii) for u and we have

$$\begin{aligned}
\mathcal{F}^*(u) \le & \sigma \int_\Omega 1 + \frac{1}{2} |\mathbf{H}_V|^2 \, d\|V\| \\
\le & \sigma \liminf_{k\to\infty} \|V_k\|(\Omega) + \frac{1}{2}\sigma \liminf_{k\to\infty} \|H_{V_k}\|^2_{L^2(\|V_k\|)} \\
\le & \sigma \liminf_{k\to\infty} \int_\Omega 1 + \frac{1}{2} |\mathbf{H}_{V_k}|^2 \, d\|V_k\| \\
\le & \liminf_{k\to\infty} \mathcal{F}^*(u_k)
\end{aligned}$$

since $\|V_k\| \overset{*}{\rightharpoonup} \|V\|$ for $k \to \infty$ and this completes the proof. □

Proposition 6.2. *For every $u = 2\chi_E - 1 \in \mathcal{S}$ with $E \subset\subset \Omega$ and finite energy $\mathcal{F}^*(u) < \infty$ there always exists a minimizing varifold in* (6.7) *and therefore, we have*

$$\mathcal{F}^*(u) = \sigma \min_{V \text{ as in iii)}} \int_\Omega 1 + \frac{1}{2} |\mathbf{H}_V|^2 \, d\|V\|. \tag{6.10}$$

Proof. This follows immediately by choosing $u_k = u$ for all $k \in \mathbb{N}$ in the proof of Lemma 6.1. □

The main result of this chapter is contained in the theorem below.

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $E \subset\subset \Omega$ such that its indicator function $u := 2\chi_E - 1$ belongs to $\mathcal{S}$. Then*

$$\Gamma(L^1) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = \mathcal{F}^*(u)$$

holds with $\mathcal{F}^*$, $\mathcal{F}_\varepsilon$ defined as in (6.7), (6.2).

Following the definition and characterization of Γ-convergence in Section 2.1 the proof reduces to the following statements.

i) *(Lower bound inequality)* Let $(u_\varepsilon)_{\varepsilon>0}$ be an arbitrary sequence in $L^1(\Omega)$ with $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Then

$$\mathcal{F}^*(u) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon)$$

holds.

ii) *(Recovery sequence)* There exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega)$ with $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ such that

$$\mathcal{F}^*(u) = \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

We prove statements i) and ii) separately in the Propositions 6.4 and 6.8. The lower bound inequality follows with the results from [RöSc06] and we therefore restrict to a sketch of the proof.

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $E \subset\subset \Omega$ such that $u := 2\chi_E - 1$ lies in $\mathcal{S}$. Further, let $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega)$ with $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Then*

$$\mathcal{F}^*(u) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon). \tag{6.11}$$

Proof. We follow the proof of Röger and Schätzle from [RöSc06], Theorem 2.1. By smoothening and standard approximation we can assume without loss of generality that $u_\varepsilon \in C^2(\Omega)$ for all $\varepsilon > 0$. Now, we define the *diffuse curvatures* $v_\varepsilon \in C^0(\Omega)$ by

$$v_\varepsilon := -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)$$

together with the Radon measures

$$\mu_\varepsilon := \left(\frac{\varepsilon}{2}\left|\nabla u_\varepsilon\right|^2 + \frac{1}{\varepsilon}F(u_\varepsilon)\right)\mathcal{L}^2,$$
$$\alpha_\varepsilon := \frac{1}{2\varepsilon}v_\varepsilon^2\mathcal{L}^2,$$

which localize the diffuse approximations of the surface and Willmore functional in the definition of $\mathcal{F}_\varepsilon$ in (6.2). In particular, we have

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \mu_\varepsilon(\Omega) + \alpha_\varepsilon(\Omega).$$

We can assume that $\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon) < \infty$, since otherwise (6.11) clearly holds. In the following, we restrict ourselves to a subsequence (for the sake of notation still indicated by ε) which realizes the limes inferior and therefore obtain

$$\mu_\varepsilon(\Omega) + \alpha_\varepsilon(\Omega) \leq C. \tag{6.12}$$

Since the functions u_ε approximate the indicator function u, the measures μ_ε roughly display the position of the phase transitions of u_ε. To give this interpretation a more geometric meaning, an approximate "normal direction" is assigned to μ_ε on the interface:

For $\varepsilon > 0$ we define the general varifold $V_\varepsilon \in \mathbb{V}_1(\Omega)$ induced by μ_ε via

$$V_\varepsilon(\phi) := \int_\Omega \phi\big(x, P_{\nabla u_\varepsilon}(x)\big)\, d\mu_\varepsilon(x), \quad \phi \in C_c^0(\Omega \times G(2,1)),$$

where

$$P_{\nabla u_\varepsilon}(x) := \begin{cases} \mathrm{Id} - \left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \otimes \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right)(x), & \text{if } \nabla u_\varepsilon(x) \neq 0 \\ 0, & \text{else} \end{cases}$$

describes the orthogonal projection onto the level line of u_ε in x. We note that by definition we have $\|V_\varepsilon\| = \mu_\varepsilon$ for the weight measure of V_ε and inequality (6.12) therefore induces

$$V_\varepsilon(\Omega \times G(2,1)) = \|V_\varepsilon\|(\Omega) = \mu_\varepsilon(\Omega) \leq C.$$

Thus, the weak compactness theorem for Radon measures (see Proposition A.8) yields the existence of another subsequence (again indicated by ε) such that

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu, \quad \alpha_\varepsilon \overset{*}{\rightharpoonup} \alpha \quad \text{weakly* in } C_0^0(\Omega)^* \tag{6.13}$$

as well as

$$V_\varepsilon \overset{*}{\rightharpoonup} V \quad \text{weakly* in } C_0^0\big(\Omega \times G(2,1)\big)^*$$

as $\varepsilon \to 0$.
The classical result of Modica and Mortola from [MoMo77] (see also [Mo87]) for the Γ-convergence of the diffuse surface functionals already includes the estimate

$$\sigma \mathcal{H}^1 \lfloor \partial^* E(U) \leq \mu(U) \leq \liminf_{\varepsilon \to 0} \mu_\varepsilon(U) \tag{6.14}$$

for all $U \subset \Omega$ and we can concentrate on the second summand of $\mathcal{F}_\varepsilon$. We remark that the inequality above is independent of the boundary restrictions on E we made in the formulation of the proposition. Essentially, the statement still holds true for every set with finite perimeter.
Since

$$\mu \overset{*}{\leftharpoonup} \mu_\varepsilon = \|V_\varepsilon\| \overset{*}{\rightharpoonup} \|V\|$$

as $\varepsilon \to 0$, we obtain $\mu = \|V\|$. By Theorem 4.1 from [RöSc06], V is rectifiable and has generalized mean curvature $\mathbf{H}_V \in L^2(\Omega)$. Furthermore, we have

$$\frac{1}{2} \int_\Omega |\mathbf{H}_V|^2 \, d\mu \leq \alpha(\Omega) \leq \liminf_{\varepsilon \to 0} \alpha_\varepsilon(\Omega) = \liminf_{\varepsilon \to 0} \mathcal{W}_\varepsilon(u_\varepsilon) < \infty. \tag{6.15}$$

Theorem 5.1 in [RöSc06] proves that $\sigma^{-1}\mu$ has an integral density and (6.14) ensures $|\nabla \chi_E| \leq \sigma^{-1}\mu$ which is (6.5). Hence, by the definition of the generalized energy $\mathcal{F}^*$ in

(6.7) and the fact that $\mathbf{H}_V = \mathbf{H}_{\sigma^{-1}V}$ this yields

$$\mathcal{F}^*(u) \leq \sigma \int_\Omega 1 + \frac{1}{2} \left|\mathbf{H}_{\sigma^{-1}V}\right|^2 \, d(\sigma^{-1} \|V\|) = \int_\Omega 1 + \frac{1}{2} \left|\mathbf{H}_V\right| \, d\mu \tag{6.16}$$

and due to (6.14), (6.15), (6.16) we finally obtain

$$\begin{aligned}\mathcal{F}^*(u) &\leq \int_\Omega 1 + \frac{1}{2} \left|\mathbf{H}_V\right|^2 \, d\mu \\ &= \mu(\Omega) + \frac{1}{2} \int_\Omega \left|\mathbf{H}_V\right|^2 \, d\mu \\ &\leq \liminf_{\varepsilon\to 0} \mu_\varepsilon(\Omega) + \liminf_{\varepsilon\to 0} \alpha_\varepsilon(\Omega) \\ &\leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon),\end{aligned}$$

which is the lower bound inequality in u. □

Remark. In the formulation of Proposition 6.4 we assume that $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$. In fact, this requirement can be guaranteed (at least for a subsequence) for every sequence $(u_\varepsilon)_{\varepsilon>0}$ with uniformly bounded energy $\mathcal{F}_\varepsilon(u_\varepsilon)$. This compactness property has already been shown for a sequence in $L^1(\Omega)$ with bounded diffuse surface energy $\mathcal{E}_\varepsilon$ in [Mo87], Proposition 3. As $\mathcal{F}_\varepsilon$ contains $\mathcal{E}_\varepsilon$, the statement follows instantly.

The primary effort of this section is the construction of recovery sequences and therefore the suitable approximation of $u \in \mathcal{S}$ in $L^1(\Omega)$ such that the generalized Willmore energy of u is recovered in the limit. For indicator functions of sets with C^2-boundary this was already established in [BePa93] and the main difficulty in this work is the handling of intersecting boundary curves. In the following, we will prove a local approximation result for these intersections from which we deduce the existence of recovery sequences for every $u \in \mathcal{S}_0$. Afterwards, the main result for general $u \in \mathcal{S}$ follows directly by an diagonal argument in Proposition 6.8.

Concerning the local approximation of boundary intersections we restrict ourselves to the following simple case: For $R > 0$ consider an open ball $B := B(0, 4R) \subset \mathbb{R}^2$ around the origin and let $u = 2\chi_E - 1$ with $E \subset B$ such that

$$\partial E \cap B = \operatorname{Im}(\varphi_1) \cup \operatorname{Im}(\varphi_2)$$

where φ_1, φ_2 are C^2-curves in B with $\operatorname{Im}(\varphi_1) \cap \operatorname{Im}(\varphi_2) = \{0\}$ and which intersect ∂B transversally in pairwise distinct points. Moreover we assume that

$$\operatorname{Im}(\varphi_i) = g_i \text{ in } B(0, 3R), \quad i = 1, 2$$

for two distinct lines g_1, g_2 through the origin (see Figure 6.1). After a possible rotation we can assume that E is symmetric with respect to both coordinate axes in $B(0, 3R)$, i.e., there exists a vector $v_0 \in S^1$ pointing into the first quadrant such that

$$g_1 = \{tv_0 \,:\, t \in \mathbb{R}\} \quad \text{and} \quad g_2 = \left\{ t \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} v_0 \,:\, t \in \mathbb{R} \right\}.$$

Moreover, we assume that $E \cap \{x_1 = 0\} \neq \emptyset$ as in Figure 6.1.
In the following lemma we construct a sequence of H^2-functions which approximates u in B in the desired sense.

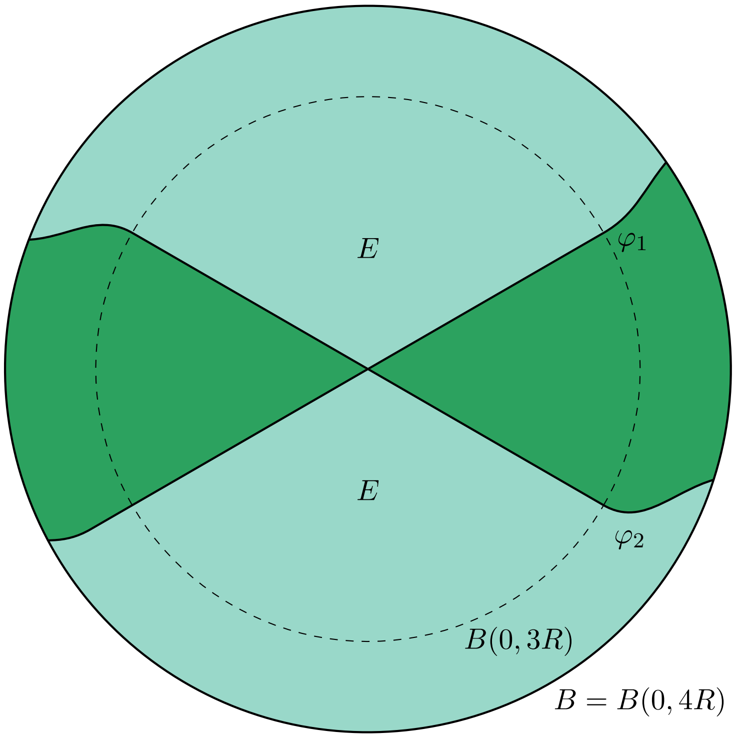


Figure 6.1: A localized cross shaped intersection of ∂E inside B.

Lemma 6.5. *Let $u = 2\chi_E - 1$ as described above. Then there exists a sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ in $H^2(B)$ such that*

$$u_\varepsilon \to u \quad \text{in } L^1(B) \tag{6.17}$$

and

$$\mathcal{F}_\varepsilon(u_\varepsilon) \to \mathcal{F}(u) = \sigma\big(\mathcal{H}^1(\partial E \cap B) + \mathcal{W}(\varphi_1) + \mathcal{W}(\varphi_2)\big) \tag{6.18}$$

as $\varepsilon \to 0$.

The basic idea is to approximate the intersection of φ_1 and φ_2 inside a ball around the origin which shrinks with $\varepsilon \searrow 0$ by a 4-ended solution of the stationary Allen-Cahn

equation (see Section 2.5) corresponding to v_0. Outside of this ball we use a slightly modified version of the standard interface approximation from [BePa93] (see also Section 2.6). This approach approximates the sharp phase transition from -1 to 1 with ε-scaled versions of the one-dimensional optimal profile γ. Thereby, the contribution of the diffuse transition to the total energy $\mathcal{F}$ vanishes in the limit which produces the desired convergence of the approximation to the sharp interface. For technical reasons it will be necessary to perform a small shift of the diffuse interfaces which vanishes with $\varepsilon \searrow 0$ to connect the ends of the inner approximation with the diffuse interfaces.
Before we precise the construction of $(u_\varepsilon)_{\varepsilon>0}$ and show the convergence (6.17) and (6.18), we need two auxiliary results the first describing the qualitative behavior of entire solutions to the Allen-Cahn equation.

Lemma 6.6. *For $\varepsilon > 0$ let $u_\varepsilon \in C^2(\mathbb{R}^2)$ be a solution of the ε-dependent Allen-Cahn equation*

$$-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} F'(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^2 \tag{6.19}$$

with $|u_\varepsilon| < 1$. Then

$$\left|u_\varepsilon(x)^2 - 1\right| + \varepsilon \left|\nabla u_\varepsilon(x)\right| + \varepsilon^2 \left|D^2 u_\varepsilon(x)\right| \leq C e^{-\frac{\alpha}{\varepsilon} \operatorname{dist}(x, \{u_\varepsilon = 0\})}$$

holds for every $x \in \mathbb{R}^2$ with positive constants $\alpha, C > 0$ independent of u_ε.

Proof. The result is a scaled version of Lemma 4.2 in [KoLiPa12] for entire solutions of the Allen-Cahn equation in the case $\varepsilon = 1$. With the usual rescaling $u = u_\varepsilon(\varepsilon \cdot)$ the estimate therein reads

$$\left|u(y)^2 - 1\right| + \left|\nabla u(y)\right| + \left|D^2 u(y)\right| \leq C e^{-\alpha \operatorname{dist}(y, \{u=0\}}$$

for all $y \in \mathbb{R}^2$ and hence with $x = \varepsilon y$

$$\left|u_\varepsilon(x)^2 - 1\right| + \varepsilon \left|\nabla u_\varepsilon(x)\right| + \varepsilon^2 \left|D^2 u_\varepsilon(x)\right| \leq C e^{-\alpha \operatorname{dist}\left(\frac{x}{\varepsilon}, \{u_\varepsilon(\varepsilon \cdot)=0\}\right)}.$$

Now the statement follows from the fact that

$$\operatorname{dist}\left(\frac{x}{\varepsilon}, \{u_\varepsilon(\varepsilon \cdot) = 0\}\right) = \operatorname{dist}\left(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\{u_\varepsilon = 0\}\right) = \frac{1}{\varepsilon} \operatorname{dist}\left(x, \{u_\varepsilon = 0\}\right). \qquad \square$$

Remark. Lemma 6.6 immediately implies a similar statement for solutions of the one-dimensional equation: Let $v_\varepsilon \in C^2(\mathbb{R})$ be a solution of

$$-\varepsilon v_\varepsilon'' + \frac{1}{\varepsilon} F'(v_\varepsilon) = 0 \quad \text{in } \mathbb{R}$$

with $|v_\varepsilon| < 1$. Then we can define a solution of (6.19) by

$$u_\varepsilon(x_1, x_2) \coloneqq v_\varepsilon(x_1).$$

This yields

$$\left|v_\varepsilon(x)^2 - 1\right| + \varepsilon \left|v_\varepsilon'(x)\right| + \varepsilon^2 \left|v_\varepsilon''(x)\right| \le Ce^{-\frac{\alpha}{\varepsilon}\operatorname{dist}(x,\{v_\varepsilon=0\})}, \quad x \in \mathbb{R}.$$

Lemma 6.7. *For $h \in C^2(\mathbb{R})$ and a constant $0 \le \lambda \le 1$ we define $H_\lambda \in C^2(\mathbb{R}^2)$ by*

$$H_\lambda(a,b) := h\big(\lambda a + (1-\lambda)b\big) - \lambda h(a) - (1-\lambda)h(b), \quad a,b \in \mathbb{R}.$$

Then there exists a function $r_\lambda \in C^0(\mathbb{R}^2)$ (also depending continuously on λ) such that

$$H_\lambda(a,b) = (a-b)^2 r_\lambda(a,b), \quad a,b \in \mathbb{R}. \tag{6.20}$$

Proof. We define new variables $x := a-b$ and $y := b$ and obtain $\tilde{H} \in C^2(\mathbb{R}^2)$ by

$$\tilde{H}(x,y) := h(\lambda x + y) - \lambda h(x+y) - (1-\lambda)h(y) = H_\lambda(a,b).$$

We compute

$$\partial_x \tilde{H}(x,y) = \lambda h'(\lambda x + y) - \lambda h'(x+y)$$

and directly see that $\tilde{H}(0,y) = \partial_x \tilde{H}(0,y) = 0$ for all $y \in \mathbb{R}$. By using the Taylor formula of $\tilde{H}$ we find a function $\tilde{r}_\lambda \in C^0(\mathbb{R}^2)$ such that

$$\tilde{H}(x,y) = x^2 \tilde{r}_\lambda(x,y), \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Hence, the statement follows by resetting variables as

$$r_\lambda(a,b) := \tilde{r}_\lambda(a-b,b), \quad a,b \in \mathbb{R}.$$ □

With the foregoing auxiliary results we are ready to prove Lemma 6.5.

Proof of Lemma 6.5. For the whole proof let $d : B \to \mathbb{R}$ denote the signed distance function of the set $E = \{u = 1\}$ which for $x \in B$ is given by

$$d(x) := \operatorname{dist}(x, B \setminus E) - \operatorname{dist}(x, B).$$

Further, we divide g_1 and g_2 into four distinct half-lines which are given by

$$\tilde{G}_i := \{tv_i \,:\, t \ge 0\}$$

with

$$v_1 = v_0, \quad v_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} v_0, \quad v_3 = -v_1, \quad v_4 = -v_2 \tag{6.21}$$

and set $\tilde{G} := \bigcup_{j=1}^4 \tilde{G}_j$.
Finally, we fix $\delta > 0$ with

$$\frac{1}{3} < \delta < \frac{1}{2}. \tag{6.22}$$

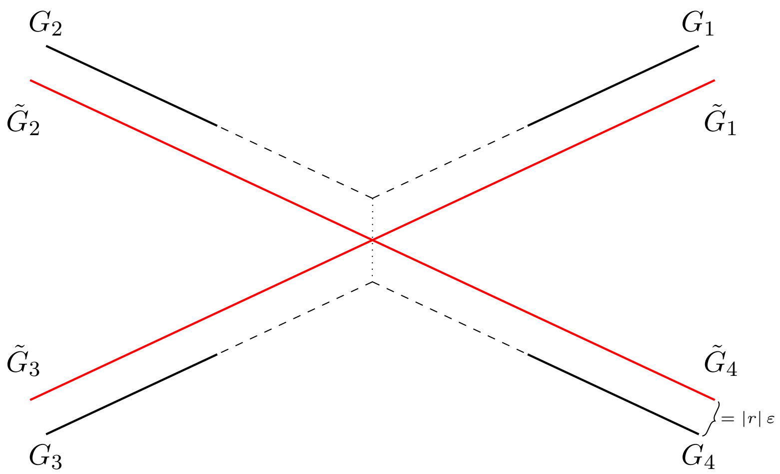


Figure 6.2: The half lines $\tilde{G}_i$ and the corresponding end G_i of $u_\varepsilon^{\mathrm{in}}$.

We follow the idea of construction which already was given above:
Inner construction: According to Lemma 2.21, there exists a 4-ended solution of (6.19) with values in $(-1,1)$ corresponding to v_0 and some $r \in \mathbb{R}$ such that (after a possible rotation) its ends G_i (as in (2.10)) are parallel to the half-lines $\tilde{G}_i$ with mutual distance $r\varepsilon$ (see Figure 6.2). We refer to this solution as $u_\varepsilon^{\mathrm{in}}$ in the following and choose a corresponding approximate solution (according to Definition 2.20) which we denote by u_ε^G.
Outer construction: For $\zeta \in C_c^\infty(\mathbb{R})$ with $0 \le \zeta \le 1$, $\operatorname{supp}\zeta = [-2,2]$, and $\zeta \equiv 1$ on $[-1,1]$ we set

$$\zeta_\varepsilon(s) := \zeta\left(\frac{s}{\varepsilon^{\delta+\frac{1}{2}}}\right), \quad \text{for } s \in \mathbb{R}$$

and assuming ε sufficiently small we define a truncated version of the optimal profile γ_ε by

$$w_\varepsilon(s) := \zeta_\varepsilon(s)\gamma_\varepsilon(s) + \big(1-\zeta_\varepsilon(s)\big)\operatorname{sgn}(s).$$

Thus, we have

$$w_\varepsilon(s) = \gamma_\varepsilon(s) \text{ for } |s| \le \varepsilon^{\delta+\frac{1}{2}} \quad \text{and} \quad w_\varepsilon(s) = \operatorname{sgn}(s) \text{ for } |s| \ge 2\varepsilon^{\delta+\frac{1}{2}}$$

as well as $w_\varepsilon \in C^\infty(\mathbb{R})$. Notice that $\delta < \delta + \frac{1}{2} < 1$ for δ as chosen above and since γ_ε makes its transition from -1 to 1 on a domain of size ε around zero, w_ε only differs significantly from γ_ε in a region where γ_ε is almost constant.
Precisely, Lemma 6.6 and the remark thereafter rise up an estimate for the distance between w_ε and γ_ε in $C^2(\mathbb{R})$ as for $|s| \ge \varepsilon^{\delta+\frac{1}{2}}$ we have

$$\begin{aligned}|w_\varepsilon(s)-\gamma_\varepsilon(s)| &= \big|\big(1-\zeta_\varepsilon(s)\big)\big(\operatorname{sgn}(s)-\gamma_\varepsilon(s)\big)\big| \le |\operatorname{sgn}(s)-\gamma_\varepsilon(s)| \\ &\le |\operatorname{sgn}(s)-\gamma_\varepsilon(s)|\,|\operatorname{sgn}(s)+\gamma_\varepsilon(s)| = \big|1-\gamma_\varepsilon(s)^2\big|\end{aligned}$$

$$\leq C e^{-\frac{C}{\varepsilon}|s|} \leq C e^{-C\varepsilon^{\delta-\frac{1}{2}}} \tag{6.23}$$

and analogously,

$$\begin{aligned}
\left|w_\varepsilon'(s) - \gamma_\varepsilon'(s)\right| &= \left|\zeta_\varepsilon'(s)\big(\gamma_\varepsilon(s) - \operatorname{sgn}(s)\big) - \big(1-\zeta_\varepsilon(s)\big)\gamma_\varepsilon'(s)\right| \\
&\leq \left\|\zeta_\varepsilon'\right\|_{L^\infty(\mathbb{R})} \left|1-\gamma_\varepsilon^2(s)\right| + \left|\gamma_\varepsilon'(s)\right| \\
&\leq \frac{C}{\varepsilon^{\delta+\frac{1}{2}}} \left\|\zeta'\right\|_{L^\infty(\mathbb{R})} e^{-\frac{C}{\varepsilon}|s|} + \frac{C}{\varepsilon} e^{-\frac{C}{\varepsilon}|s|} \leq \frac{C}{\varepsilon} e^{-C\varepsilon^{\delta-\frac{1}{2}}}
\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}
\left|w_\varepsilon''(s) - \gamma_\varepsilon''(s)\right| &\leq \left|\big(1-\zeta_\varepsilon(s)\big)\gamma_\varepsilon''(s)\right| + 2\left|\zeta_\varepsilon'(s)\gamma_\varepsilon'(s)\right| + \left|\big(\operatorname{sgn}(s) - \gamma_\varepsilon(s)\big)\zeta_\varepsilon''(s)\right| \\
&\leq \left|\gamma_\varepsilon''(s)\right| + \frac{2}{\varepsilon^{\delta+\frac{1}{2}}} \left\|\zeta'\right\|_{L^\infty(\mathbb{R})} \left|\gamma_\varepsilon'(s)\right| + \frac{1}{\varepsilon^{2\delta+1}} \left\|\zeta''\right\|_{L^\infty(\mathbb{R})} \left|1-\gamma_\varepsilon^2(s)\right| \\
&\leq \left(\frac{C}{\varepsilon^2} + \frac{C}{\varepsilon^{\delta+\frac{3}{2}}} + \frac{C}{\varepsilon^{2\delta+1}}\right) e^{-\frac{C}{\varepsilon}|s|} \\
&\leq \frac{C}{\varepsilon^2} e^{-C\varepsilon^{\delta-\frac{1}{2}}}.
\end{aligned} \tag{6.25}$$

Our plan to define the outer approximation $u_\varepsilon^{\text{out}}$ is to evaluate $u_\varepsilon^{\text{out}}$ in the signed distance of E (as it has been done in [BePa93]), however this construction does not properly go with the chosen inner approximation of the boundary intersection $u_\varepsilon^{\text{in}}$. Since the ends of $u_\varepsilon^{\text{in}}$ are not exactly given by the half-lines $\tilde{G}_i$, $i = 1, \dots, 4$ but by their shifted versions G_i, it is not possible to glue both constructions together with vanishing cost in terms of energy. We solve this problem by performing the necessary shift of the G_i on the annulus $B(0, 2R) \setminus B(0, R)$ and adding a small (in terms of ε) correction term on d. Precisely, we define a function $v : B \setminus \{0\} \to \{v_1, v_2, v_3, v_4\}$ by

$$v(x) := \begin{cases} v_1, & \text{if } x_1 > 0 \text{ and } x_2 \geq 0 \\ v_2, & \text{if } x_1 \leq 0 \text{ and } x_2 > 0 \\ v_3, & \text{if } x_1 < 0 \text{ and } x_2 \leq 0 \\ v_4, & \text{if } x_1 \geq 0 \text{ and } x_2 < 0 \end{cases}$$

which for $x = (x_1, x_2) \in B \setminus \{0\}$ yields the direction of the half-line which is closest to x. Moreover, we choose $\theta \in C^\infty([0, \infty))$ with $0 \leq \theta \leq 1$ such that

$$\theta(s) = \begin{cases} 1, & \text{if } x \leq R \\ 0, & \text{if } x > 2R \end{cases}$$

and define for $x \in B \setminus \{0\}$

$$u_\varepsilon^{\text{out}}(x) := w_\varepsilon\big(d_\theta^\varepsilon(x)\big) := w_\varepsilon\Big(d(x) - \varepsilon r \theta\big(x \cdot v(x)\big)\Big). \tag{6.26}$$

Particularly, we always place the optimal profile γ_ε in perpendicular direction to ∂E over the interface ignoring the small shift $r\varepsilon\theta$ during this choice. We refer to Figure 6.3 where we describe the structure of $u_\varepsilon^{\text{out}}$ in a neighborhood of one $\tilde{G}_i$.

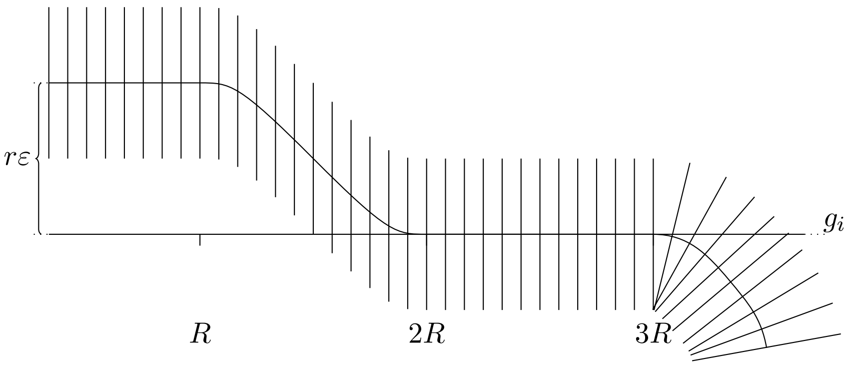


Figure 6.3: The directions in which $u_\varepsilon^{\text{out}}$ places w_ε on the sharp interface.

Although v is not even continuous on the coordinate axis, $u_\varepsilon^{\text{out}}$ still is in $C^\infty\big(B\setminus B(0,\varepsilon^\delta)\big)$ for ε sufficiently small since it is constant to 1 (or -1, respectively) on the axes in $B\setminus B(0,\varepsilon^\delta)\big)$. We further point out, that $x\cdot v(x)$ is always positive as both factors lie in the same quadrant.
Finally, with a smooth radial symmetric cutoff function $\eta\in C^\infty(B)$ with $0\le\eta\le 1$ given by

$$\eta(x) := \begin{cases} 1, & \text{in } B(0,1),\\ 0, & \text{in } \mathbb{R}^2\setminus B(0,2)\end{cases}$$

we set

$$\eta_\varepsilon(x) := \eta\left(\frac{x}{\varepsilon^\delta}\right)$$

and are now able to define $u_\varepsilon : B\to\mathbb{R}$ by interpolating the inner and outer construction due to

$$u_\varepsilon(x) := \eta_\varepsilon(x)u_\varepsilon^{\text{in}}(x) + \big(1-\eta_\varepsilon(x)\big)u_\varepsilon^{\text{out}}(x), \quad x\in B.$$

Note, that η_ε satisfies

$$|\nabla\eta_\varepsilon(x)|\le C\varepsilon^{-\delta} \quad \text{and} \quad |\Delta\eta_\varepsilon(x)|\le C\varepsilon^{-2\delta} \tag{6.27}$$

for all $x\in B$ and that clearly, u_ε is in $H^2(B)$ by definition.
It is also easy to see that u_ε approximates u in $L^1(\Omega)$ as $\varepsilon\to 0$ and therefore, (6.17) holds. Indeed, the constructed function satisfies $|u_\varepsilon|\le 1$ and we have

$$\int_{B(0,2\varepsilon^\delta)} |u_\varepsilon - u|\,dx \le 2\left|B(0,2\varepsilon^\delta)\right| = 8\pi\varepsilon^{2\delta} \longrightarrow 0$$

as well as

$$\begin{aligned}\int_{B\setminus B(0,2\varepsilon^\delta)} |u_\varepsilon - u|\,dx &= \int_{B\setminus B(0,2\varepsilon^\delta)} \left|u_\varepsilon^{\text{out}} - u\right|\,dx\\ &\le 2\left|\{x\in B\,:\,|x|\ge 2\varepsilon^\delta \text{ and } |d(x)| < 2\varepsilon^{\delta+\frac12} + r\varepsilon\}\right|\end{aligned}$$

$$
\begin{aligned}
&\leq 4\left(2\varepsilon^{\delta+\frac{1}{2}} + r\varepsilon\right)\big(L(\varphi_1)+L(\varphi_2)\big)\\
&\longrightarrow 0
\end{aligned}
$$

as $\varepsilon \to 0$.
In order to prove (6.18), we consider the convergence of both summands in F_ε separately and start with the second term $\mathcal{W}_\varepsilon$. Since B divides into

$$B = B(0,\varepsilon^\delta) \cup \Big(B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)\Big) \cup \Big(B(0,3R)\setminus B(0,2\varepsilon^\delta)\Big) \cup \big(B\setminus B(0,3R)\big)$$

we can split $\mathcal{W}_\varepsilon(u_\varepsilon)$ into four integrals and consider the convergence of the single terms one by one. For the first summand we have

$$\frac{1}{2\varepsilon}\int_{B(0,\varepsilon^\delta)}\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)\Big)^2\,dx = \frac{1}{2\varepsilon}\int_{B(0,\varepsilon^\delta)}\Big(-\varepsilon\Delta u_\varepsilon^{\mathrm{in}} + \frac{1}{\varepsilon}F'(u_\varepsilon^{\mathrm{in}})\Big)^2\,dx = 0$$

as $u_\varepsilon^{\mathrm{in}}$ solves (6.19).
In $B\setminus B(0,3R)$ where ∂E has non vanishing curvature and where $u = u_\varepsilon^{\mathrm{out}} = w_\varepsilon(d(x))$ by definition, we obtain

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\int_{B\setminus B(0,3R)}\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)\Big)^2\,dx\\
=&\frac{1}{2\varepsilon}\int_{B\setminus B(0,3R)}\Big(-\varepsilon\Delta u_\varepsilon^{\mathrm{out}} + \frac{1}{\varepsilon}F'(u_\varepsilon^{\mathrm{out}})\Big)^2\,dx\\
\longrightarrow\,& \sigma\big(\mathcal{W}(\varphi_1)+\mathcal{W}(\varphi_2)\big)
\end{aligned}
\tag{6.28}
$$

as $\varepsilon \to 0$ by [BePa93]. Indeed, the argument stays true although we have chosen a different width of the stripes around φ_1 and φ_2 but since $\delta + \frac{1}{2} < 1$ is is still larger than order ε (see the remark at the end of Section 2.6).
In order to show the convergence of the integral over the third region we first note that for sufficiently small $\varepsilon > 0$ and $x \in B(0,3R)\setminus B(0,2\varepsilon^\delta)$ we have $u(x) = u_\varepsilon^{\mathrm{out}}(x) = \pm 1$ if $|d_\theta^\varepsilon(x)| \geq 2\varepsilon^{\delta+\frac{1}{2}}$. Consequently, the integral over $B(0,3R)\setminus B(0,2\varepsilon^\delta)$ reduces to the integral over a neighborhood U_ε of $\tilde{G}\cap\big(B(0,3R)\setminus B(0,2\varepsilon^\delta)\big)$ given by

$$U_\varepsilon \coloneqq \{x \in B(0,3R)\setminus B(0,2\varepsilon^\delta) \,:\, |d_\theta^\varepsilon(x)| < 2\varepsilon^{\delta+\frac{1}{2}}\} \tag{6.29}$$

and due to $|d| \leq |d_\theta^\varepsilon| + C\varepsilon$ for small $\varepsilon > 0$ we have

$$|U_\varepsilon| \leq C\varepsilon^{\delta+\frac{1}{2}}. \tag{6.30}$$

Since this region consists of four same sized components around the four half-lines $\tilde{G}_i$, $i = 1,\dots,4$, we restrict to one of them and assume (after a possible rotation) that the regarded half-line is the positive x_1-axis. For the sake of notation we still refer to this region as U_ε.
For $x = (x_1,x_2) \in U_\varepsilon$ we then obtain with $d_\theta^\varepsilon(x) = x_2 - r\varepsilon\theta(x_1)$

$$u_\varepsilon^{\mathrm{out}}(x) = w_\varepsilon\big(x_2 - r\varepsilon\theta(x_1)\big) \tag{6.31}$$

and

$$\nabla u_\varepsilon^{\text{out}}(x) = \begin{pmatrix} -r\varepsilon\theta'(x_1) \\ 1 \end{pmatrix} w_\varepsilon'\big(x_2 - r\varepsilon\theta(x_1)\big) \tag{6.32}$$

as well as

$$\begin{aligned}\Delta u_\varepsilon^{\text{out}}(x) =& w_\varepsilon''(x_2 - r\varepsilon\theta(x_1))\Big(1 + r^2\varepsilon^2\big(\theta'(x_1)\big)^2\Big) \\ &- w_\varepsilon'\big(x_2 - r\varepsilon\theta(x_1)\big) r\varepsilon\theta''(x_1).\end{aligned} \tag{6.33}$$

The integral over U_ε reads

$$\begin{aligned}&\frac{1}{2\varepsilon}\int_{U_\varepsilon}\left(-\varepsilon\Delta u_\varepsilon^{\text{out}} + \frac{1}{\varepsilon}F'(u_\varepsilon^{\text{out}})\right)^2 dx \\ =&\frac{1}{2\varepsilon}\int_{U_\varepsilon}\left(-\varepsilon\Delta(w_\varepsilon\circ d_\theta^\varepsilon) + \frac{1}{\varepsilon}F'(w_\varepsilon\circ d_\theta^\varepsilon)\right)^2 dx \\ \le&\frac{C}{\varepsilon}\int_{U_\varepsilon}\big(-\varepsilon\Delta(w_\varepsilon\circ d_\theta^\varepsilon) + \varepsilon\Delta(\gamma_\varepsilon\circ d_\theta^\varepsilon)\big)^2 dx \\ &+\frac{C}{\varepsilon}\int_{U_\varepsilon}\left(\frac{1}{\varepsilon}F'(w_\varepsilon\circ d_\theta^\varepsilon) - \frac{1}{\varepsilon}F'(\gamma_\varepsilon\circ d_\theta^\varepsilon)\right)^2 dx \\ &+\frac{C}{\varepsilon}\int_{U_\varepsilon}\left(-\varepsilon\Delta(\gamma_\varepsilon\circ d_\theta^\varepsilon) + \frac{1}{\varepsilon}F'(\gamma_\varepsilon\circ d_\theta^\varepsilon)\right)^2 dx \\ &=: H_1 + H_2 + H_3\end{aligned} \tag{6.34}$$

and using (6.31), (6.32), (6.33) (and the analogous results for $\gamma_\varepsilon\circ d_\theta^\varepsilon$) as well as the estimates (6.23), (6.24), (6.25) on the C^2-distance between w_ε and γ_ε we treat the three terms separately. For H_1 this gives

$$\begin{aligned}H_1 =& C\varepsilon\int_{U_\varepsilon}\Bigg(\Big(w_\varepsilon''\big(d_\theta^\varepsilon(x)\big) - \gamma_\varepsilon''\big(d_\theta^\varepsilon(x)\big)\Big)\Big(1 + r^2\varepsilon^2\big(\theta'(x_1)\big)^2\Big) \\ &- \Big(w_\varepsilon'\big(d_\theta^\varepsilon(x)\big) - \gamma_\varepsilon'\big(d_\theta^\varepsilon(x)\big)\Big) r\varepsilon\theta''(x_1)\Bigg)^2 dx \\ \le& C\varepsilon\int_{U_\varepsilon}\big|w_\varepsilon''\big(d_\theta^\varepsilon(x)\big) - \gamma_\varepsilon''\big(d_\theta^\varepsilon(x)\big)\big|^2 + \big|w_\varepsilon'\big(d_\theta^\varepsilon(x)\big) - \gamma_\varepsilon'\big(d_\theta^\varepsilon(x)\big)\big|^2 dx \\ \le& \frac{C}{\varepsilon^3}\,|U_\varepsilon|\, e^{-C\varepsilon^{\delta-\frac{1}{2}}} \\ \le& C\varepsilon^{\delta-\frac{5}{2}} e^{-C\varepsilon^{\delta-\frac{1}{2}}}\end{aligned}$$

and by the same arguments we obtain

$$\begin{aligned}H_2 =& \frac{C}{\varepsilon}\int_{U_\varepsilon}\left(\frac{1}{\varepsilon}F'\Big(w_\varepsilon\big(d_\theta^\varepsilon(x)\big)\Big) - \frac{1}{\varepsilon}F'\Big(\gamma_\varepsilon\big(d_\theta^\varepsilon(x)\big)\Big)\right)^2 dx \\ =& \frac{C}{\varepsilon^3}\,\|F''\|^2_{L^\infty([-1,1])}\int_{U_\varepsilon}\big|w_\varepsilon\big(d_\theta^\varepsilon(x)\big) - \gamma_\varepsilon\big(d_\theta^\varepsilon(x)\big)\big|^2 dx\end{aligned}$$

$$\leq C\varepsilon^{\delta-\frac{5}{2}}e^{-C\varepsilon^{\delta-\frac{1}{2}}}$$

as well as

$$\begin{aligned}
H_3 &= \frac{C}{\varepsilon}\int_{U_\varepsilon}\left(-\varepsilon\Delta\gamma_\varepsilon\big(d_\theta^\varepsilon(x)\big)+\frac{1}{\varepsilon}F'\Big(\gamma_\varepsilon\big(d_\theta^\varepsilon(x)\big)\Big)\right)^2 dx \\
&= \frac{C}{\varepsilon}\int_{U_\varepsilon}\Bigg(-\varepsilon\gamma_\varepsilon''\big(d_\theta^\varepsilon(x)\big)\Big(1+r^2\varepsilon^2\big(\theta'(x_1)\big)^2\Big) \\
&\qquad +r\varepsilon^2\theta''(x_1)\gamma_\varepsilon'\big(d_\theta^\varepsilon(x)\big)+\frac{1}{\varepsilon}F'\Big(\gamma_\varepsilon\big(d_\theta^\varepsilon(x)\big)\Big)\Bigg)^2 dx \\
&= \frac{C}{\varepsilon}\int_{U_\varepsilon}\left(-r^2\varepsilon^3\big(\theta'(x_1)\big)^2\gamma_\varepsilon''\big(d_\theta^\varepsilon(x)\big)+r\varepsilon^2\theta''(x_1)\gamma_\varepsilon'\big(d_\theta^\varepsilon(x)\big)\right)^2 dx \\
&\leq C\varepsilon^{-1}\varepsilon^{\delta+\frac{1}{2}}\varepsilon^2 \\
&= C\varepsilon^{\delta+\frac{3}{2}}.
\end{aligned}$$

Hence, the integral over U_ε and therefore over $B(0,3R)\setminus B(0,2\varepsilon^\delta)$ vanishes as $\varepsilon\to 0$ and it remains to show that the integral over the annulus $B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)$ also tends to 0.
Since $|x\cdot v(x)|\leq 2\varepsilon^\delta < R$ we have $d_\theta^\varepsilon(x)=d(x)-r\varepsilon$ for $x\in B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)$ and

$$\{d_\theta^\varepsilon=0\}\cap\big(B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)\big)=\left(\bigcup_{i=1}^4 G_i\right)\cap\big(B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)\big)$$

consists of four straight line segments. We distinguish between two disjoint regions A_ε^1 and A_ε^2 on the annulus (see Figure 6.4). In

$$A_\varepsilon^1 := \{|d_\theta^\varepsilon|\leq\varepsilon^{2\delta}\}\cap\Big(B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)\Big)$$

$u_\varepsilon^{\text{out}}=w_\varepsilon\circ d_\theta^\varepsilon$ does its complete transition from -1 to 1 or vice versa (for sufficiently small $\varepsilon>0$) since $2\delta<\delta+\frac{1}{2}$. This implies that $u_\varepsilon^{\text{out}}$ is constant ± 1 on each connected component of the remaining part

$$A_\varepsilon^2 := \Big(B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)\Big)\setminus A_\varepsilon^1$$

and we particularly observe

$$\big|A_\varepsilon^1\big|\leq C\varepsilon^{3\delta}\quad\text{and}\quad\big|A_\varepsilon^2\big|\leq C\varepsilon^{2\delta}. \tag{6.35}$$

Due to Corollary 2.22 the zero set of $u_\varepsilon^{\text{in}}$ stays exponentially close to G (in terms of ε). Indeed, (2.13) reads for all $x\in\{u_\varepsilon^{\text{in}}=0\}$ with $|x|\geq\varepsilon^\delta$

$$\operatorname{dist}\big(x,\{d_\theta^\varepsilon=0\}\cap\{|x|\geq\varepsilon^\delta\}\big)\leq\frac{C}{\varepsilon}e^{-C\varepsilon^{2\delta-2}}$$

and thus, there exists a constant $C > 0$ such that

$$\operatorname{dist}(A_\varepsilon^2, \{u_\varepsilon^{\text{in}} = 0\}) \geq C\varepsilon^{2\delta}$$

is satisfied. Combined with Lemma 6.6 this yields the estimate for $x \in A_\varepsilon^2$

$$\begin{aligned} &\left|\left(u_\varepsilon^{\text{in}}(x)^2\right) - 1\right| + \varepsilon\left|\nabla u_\varepsilon^{\text{in}}(x)\right| + \varepsilon^2\left|D^2 u_\varepsilon^{\text{in}}(x)\right| \\ \leq & C e^{-\frac{C}{\varepsilon}\operatorname{dist}(x,\{u_\varepsilon^{\text{in}}=0\})} \leq C e^{-C\varepsilon^{2\delta-1}} \end{aligned} \tag{6.36}$$

which tends to 0 exponentially fast as $\varepsilon \to 0$ since $2\delta - 1 < 0$.

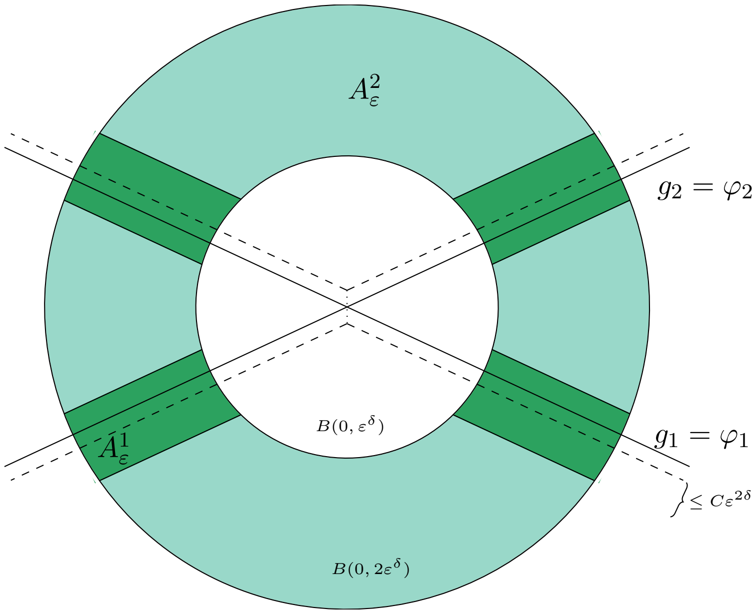


Figure 6.4: Sketch of the construction inside $B(0, 2\varepsilon^\delta)$

With these considerations in mind we have

$$\begin{aligned} &\frac{1}{2\varepsilon}\int_{B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)} \Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)\Big)^2\,dx \\ =&\frac{1}{2\varepsilon}\int_{A_\varepsilon^1\cup A_\varepsilon^2} \Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}F'(u_\varepsilon)\Big)^2\,dx \\ =&\frac{1}{2\varepsilon}\int_{A_\varepsilon^1\cup A_\varepsilon^2} \Big(-\varepsilon\Delta\big(\eta_\varepsilon u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon)u_\varepsilon^{\text{out}}\big) + \frac{1}{\varepsilon}F'\big(\eta_\varepsilon u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon)u_\varepsilon^{\text{out}}\big)\Big)^2\,dx \\ =&\frac{1}{2\varepsilon}\int_{A_\varepsilon^1\cup A_\varepsilon^2} \Big(-\varepsilon\Delta\eta_\varepsilon(u_\varepsilon^{\text{in}} - u_\varepsilon^{\text{out}}) - 2\varepsilon\nabla\eta_\varepsilon\cdot(\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}}) - \varepsilon\eta_\varepsilon\Delta u_\varepsilon^{\text{in}} \\ &-\varepsilon(1-\eta_\varepsilon)\Delta u_\varepsilon^{\text{out}} + \frac{1}{\varepsilon}F'\big(\eta_\varepsilon u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon)u_\varepsilon^{\text{out}}\big)\Big)^2\,dx \\ \leq& C\varepsilon\int_{A_\varepsilon^1\cup A_\varepsilon^2} \left(\Delta\eta_\varepsilon(u_\varepsilon^{\text{in}} - u_\varepsilon^{\text{out}})\right)^2\,dx + C\varepsilon\int_{A_\varepsilon^1\cup A_\varepsilon^2} \left(\nabla\eta_\varepsilon\cdot(\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}})\right)^2\,dx \\ &+ C\frac{1}{\varepsilon}\int_{A_\varepsilon^1\cup A_\varepsilon^2} \left(-\varepsilon\eta_\varepsilon\Delta u_\varepsilon^{\text{in}} - \varepsilon(1-\eta_\varepsilon)\Delta u_\varepsilon^{\text{out}} + \frac{1}{\varepsilon}F'(\eta_\varepsilon u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon)u_\varepsilon^{\text{out}})\right)^2\,dx \end{aligned}$$

$$:= I_1 + I_2 + I_3.$$

We show that all integrals converge to 0 as $\varepsilon \to 0$. By (6.27) and (6.35) we obtain

$$\begin{aligned} I_1 &\le C\varepsilon \left|A_\varepsilon^1 \cup A_\varepsilon^2\right| \|\Delta \eta_\varepsilon\|^2_{L^\infty(\mathbb{R}^2)} \left\|u_\varepsilon^{\text{in}} - u_\varepsilon^{\text{out}}\right\|^2_{L^\infty(A_\varepsilon^1 \cup A_\varepsilon^2)} \\ &\le C\varepsilon\varepsilon^{2\delta}\varepsilon^{-4\delta} = C\varepsilon^{1-2\delta} \\ &\longrightarrow 0 \end{aligned}$$

as $\varepsilon \to 0$ since $\delta < \frac{1}{2}$ and $-1 \le u_\varepsilon^{\text{in}}, u_\varepsilon^{\text{out}} < 1$.
For the second term we have

$$\begin{aligned} I_2 &\le C\varepsilon \int_{A_\varepsilon^1 \cup A_\varepsilon^2} |\nabla \eta_\varepsilon|^2 \left|\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}}\right|^2 \, dx \\ &\le C\varepsilon^{1-2\delta} \int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}}\right|^2 \, dx + C\varepsilon^{1-2\delta} \int_{A_\varepsilon^2} \left|\nabla u_\varepsilon^{\text{in}}\right|^2 \, dx \\ &\le C\varepsilon^{1-2\delta} \int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}}\right|^2 \, dx + C\varepsilon^{1-2\delta} \left|A_\varepsilon^2\right| \left\|\nabla u_\varepsilon^{\text{in}}\right\|^2_{L^\infty(A_\varepsilon^2)}. \end{aligned} \tag{6.37}$$

Inequality (6.36) yields an estimate for the second summand as

$$\begin{aligned} C\varepsilon^{1-2\delta} \left|A_\varepsilon^2\right| \left\|\nabla u_\varepsilon^{\text{in}}\right\|^2_{L^\infty(A_\varepsilon^2)} &\le C\varepsilon^{1-2\delta}\varepsilon^{2\delta}\varepsilon^{-2}e^{-C\varepsilon^{2\delta-1}} \\ &= C\varepsilon^{-1}e^{-C\varepsilon^{2\delta-1}} \end{aligned}$$

which tends to 0 exponentially fast since $2\delta - 1 < 0$ and it remains to show that the first term in (6.37) vanishes as $\varepsilon \to 0$.
We use the fact that the phase transition profiles of $u_\varepsilon^{\text{in}}$ and $u_\varepsilon^{\text{out}}$ from -1 to 1 or vice versa, respectively, are both comparable to the optimal profile γ_ε for small ε and sufficiently close to the interface. Since A_ε^1 consists of four components with size of the same order, we restrict ourselves to one of them and skip the index i for the vector v_i in the following. Notice that by construction the approximate solution u_ε^G of (6.19) then can be written as $\gamma_\varepsilon(\cdot \cdot v^\perp - r\varepsilon) = \gamma_\varepsilon \circ d_\theta^\varepsilon$ in A_ε^1 for sufficiently small $\varepsilon > 0$ (after a possible multiplication of all occurring terms by -1). Due to Proposition 2.21 and (6.24), this implies

$$\begin{aligned} &\int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}}\right|^2 \, dx \\ \le& C \int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^G\right|^2 \, dx + C \int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^G - \nabla u_\varepsilon^{\text{out}}\right|^2 \, dx \\ \le& C \left\|\varepsilon^{-2}e^{-C\frac{|\cdot|^2}{\varepsilon^2}}\right\|_{L^\infty(A_\varepsilon^1)} \int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^{\text{in}}(x) - \nabla u_\varepsilon^G(x)\right|^2 \varepsilon^2 e^{C\frac{|x|^2}{\varepsilon^2}} \, dx \\ &\quad + C \int_{A_\varepsilon^1} |\nabla(\gamma_\varepsilon \circ d_\theta^\varepsilon) - \nabla(w_\varepsilon \circ d_\theta^\varepsilon)|^2 \, dx \\ \le& C\varepsilon^{-2}e^{-C\varepsilon^{2\delta-2}} + C\left|A_\varepsilon^1\right| \left\|\gamma_\varepsilon' - w_\varepsilon'\right\|^2_{L^\infty(\mathbb{R})} \|\nabla d_\theta^\varepsilon\|^2_{L^\infty(A_\varepsilon^1)} \\ \le& C\varepsilon^{-2}e^{-C\varepsilon^{2\delta-2}} + C\varepsilon^{3\delta-2}e^{-C\varepsilon^{\delta-\frac{1}{2}}} \end{aligned}$$

as $|\nabla d^\varepsilon_\theta| = |\nabla d| \equiv 1$. Therefore, I_2 tends to 0 exponentially fast as $\varepsilon \to 0$.
To understand the behavior of I_3 we first consider the remaining integral over A^2_ε and again, equation (6.36) yields its exponential smallness. In fact, since $u^{\text{out}}_\varepsilon$ is constant to ± 1 on each component of A^2_ε we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{A^2_\varepsilon}\left(-\varepsilon\eta_\varepsilon\Delta u^{\text{in}}_\varepsilon - \varepsilon(1-\eta_\varepsilon)\Delta u^{\text{out}}_\varepsilon + \frac{1}{\varepsilon}F'\big(\eta_\varepsilon u^{\text{in}}_\varepsilon + (1-\eta_\varepsilon)u^{\text{out}}_\varepsilon\big)\right)^2 dx\\
=&\frac{1}{\varepsilon}\int_{A^2_\varepsilon}\left(-\varepsilon\eta_\varepsilon\Delta u^{\text{in}}_\varepsilon + \frac{1}{\varepsilon}F'\big(\eta_\varepsilon u^{\text{in}}_\varepsilon \pm (1-\eta_\varepsilon)\big)\right)^2 dx\\
\le& C\varepsilon\int_{A^2_\varepsilon}|\eta_\varepsilon|^2\left|\Delta u^{\text{in}}_\varepsilon\right|^2 dx + \frac{C}{\varepsilon^3}\int_{A^2_\varepsilon}\Big(F'\big(\eta_\varepsilon u^{\text{in}}_\varepsilon \pm (1-\eta_\varepsilon)\big) - F'(\pm 1)\Big)^2 dx\\
\le& C\varepsilon^{2\delta+1}\left\|\Delta u^{\text{in}}_\varepsilon\right\|^2_{L^\infty(A^2_\varepsilon)} + C\varepsilon^{2\delta-3}\left\|F''\right\|^2_{L^\infty([-1,1])}\left\|\eta_\varepsilon\right\|^2_{L^\infty(B)}\left\|u^{\text{in}}_\varepsilon \mp 1\right\|^2_{L^\infty(A^2_\varepsilon)}\\
\le& C\varepsilon^{2\delta-3}e^{-C\varepsilon^{2\delta-1}} \longrightarrow 0
\end{aligned}
$$

as $\varepsilon \to 0$ by the mean value theorem.
For the integral over A^1_ε we use that $u^{\text{in}}_\varepsilon$ is a solution of (6.19) and that $u^{\text{out}}_\varepsilon$ almost solves this equation. By Lemma 6.7 applied to F' we obtain with $r_{\eta_\varepsilon}(x)$ as in (6.20)

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{A^1_\varepsilon}\left(-\varepsilon\eta_\varepsilon\Delta u^{\text{in}}_\varepsilon - \varepsilon(1-\eta_\varepsilon)\Delta u^{\text{out}}_\varepsilon + \frac{1}{\varepsilon}F'\big(\eta_\varepsilon u^{\text{in}}_\varepsilon + (1-\eta_\varepsilon)u^{\text{out}}_\varepsilon\big)\right)^2 dx\\
\le&\frac{C}{\varepsilon}\int_{A^1_\varepsilon}\left(-\frac{1}{\varepsilon}\eta_\varepsilon F'(u^{\text{in}}_\varepsilon) - \frac{1}{\varepsilon}(1-\eta_\varepsilon)F'(u^{\text{out}}_\varepsilon) + \frac{1}{\varepsilon}F'\big(\eta_\varepsilon u^{\text{in}}_\varepsilon + (1-\eta_\varepsilon)u^{\text{out}}_\varepsilon\big)\right)^2 dx\\
&\quad + \frac{C}{\varepsilon}\int_{A^1_\varepsilon}\left(-\varepsilon(1-\eta_\varepsilon)\Delta u^{\text{out}}_\varepsilon + \frac{1}{\varepsilon}(1-\eta_\varepsilon)F'(u^{\text{out}}_\varepsilon)\right)^2 dx\\
\le&\frac{C}{\varepsilon^3}\left\|r_{\eta_\varepsilon}(u^{\text{in}}_\varepsilon, u^{\text{out}}_\varepsilon)\right\|^2_{L^\infty(A^1_\varepsilon)}\int_{A^1_\varepsilon}(u^{\text{in}}_\varepsilon - u^{\text{out}}_\varepsilon)^4\,dx\\
&\quad + \frac{C}{\varepsilon}\int_{A^1_\varepsilon}\left(-\varepsilon\Delta u^{\text{out}}_\varepsilon + \frac{1}{\varepsilon}F'(u^{\text{out}}_\varepsilon)\right)^2 dx\\
\le&\frac{C}{\varepsilon^3}\int_{A^1_\varepsilon}(u^{\text{in}}_\varepsilon - u^{\text{out}}_\varepsilon)^2\,dx + \frac{C}{\varepsilon}\int_{A^1_\varepsilon}\left(-\varepsilon\Delta u^{\text{out}}_\varepsilon + \frac{1}{\varepsilon}F'(u^{\text{out}}_\varepsilon)\right)^2 dx. \qquad (6.38)
\end{aligned}
$$

Similar to the estimate for I_2 above we see that

$$
\begin{aligned}
&\int_{A^1_\varepsilon}(u^{\text{in}}_\varepsilon - u^{\text{out}}_\varepsilon)^2\,dx\\
\le& C\int_{A^1_\varepsilon}(u^{\text{in}}_\varepsilon - u^G_\varepsilon)^2\,dx + C\int_{A^1_\varepsilon}(u^G_\varepsilon - u^{\text{out}}_\varepsilon)^2\,dx\\
\le& C\varepsilon^{-2}e^{-C\varepsilon^{2\delta-2}} + C\varepsilon^{3\delta}e^{-C\varepsilon^{\delta-\frac{1}{2}}}
\end{aligned}
$$

by Proposition 2.21 and (6.23)
The second summand in (6.38) approximates the Willmore energy of the shifted half-lines G_i, $i = 1, \ldots, 4$ in A^1_ε and vanishes therefore as ε tends to 0. Precisely, we make

use of the fact that $d_\theta^\varepsilon = d - r\varepsilon$ in $B(0, 2\varepsilon^\delta)$ and hence,

$$\Delta u_\varepsilon^{\mathrm{out}} = w_\varepsilon''(d_\theta^\varepsilon)\,|\nabla d|^2 + w_\varepsilon'(d_\theta^\varepsilon)\Delta d = w_\varepsilon''(d_\theta^\varepsilon) \tag{6.39}$$

as $|\nabla d| \equiv 1$ and $\Delta d = 0$ in this region. Then the assumption follows exactly as in (6.34) above with the inequalities (6.23)-(6.25) since

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{A_\varepsilon^1}\left(-\varepsilon\Delta u_\varepsilon^{\mathrm{out}} + \frac{1}{\varepsilon}F'(u_\varepsilon^{\mathrm{out}})\right)^2 dx \\
\leq &\frac{1}{\varepsilon}\int_{B(0,R)}\left(-\varepsilon\Delta(w_\varepsilon \circ d_\theta^\varepsilon) + \frac{1}{\varepsilon}F'(w_\varepsilon \circ d_\theta^\varepsilon)\right)^2 dx \\
\leq &C\varepsilon\int_{B(0,R)}\Big(-w_\varepsilon''\big(d(x) - r\varepsilon\big) + \gamma_\varepsilon''\big(d(x) - r\varepsilon\big)\Big)^2 dx \\
&+ \frac{C}{\varepsilon^3}\int_{B(0,R)}\Big(F'\Big(w_\varepsilon\big(d(x) - r\varepsilon\big)\Big) - F'\Big(\gamma_\varepsilon\big(d(x) - r\varepsilon\big)\Big)\Big)^2 dx \\
\leq &\frac{C}{\varepsilon^3}e^{-C\varepsilon^{\delta - \frac{1}{2}}}.
\end{aligned}$$

This finally yields

$$I_3 \longrightarrow 0, \quad \varepsilon \to 0$$

and completes the proof of convergence for $\mathcal{W}_\varepsilon(u_\varepsilon)$.
The handling of the remaining diffuse surface part $\mathcal{E}_\varepsilon(u_\varepsilon)$ of $\mathcal{F}_\varepsilon(u_\varepsilon)$ is less complicated to show due to its lower order (in terms of involved derivatives of u_ε). However, in contrast to the calculations above, the contribution of $u_\varepsilon^{\mathrm{in}}$ inside $B(0, \varepsilon^\delta)$ to the total surface area does not vanish and we have to analyze this part more carefully.
Remark, that there exists a compact set $K \subset \mathbb{R}^2$ around the origin such that outside of εK the zero set of $u_\varepsilon^{\mathrm{in}}$ consists of four disjoint pieces which are asymptotic to the four half-lines G_i, $i = 1, \ldots, 4$. Since εK completely lies in the ball $B(0, \varepsilon^{\frac{3}{4}})$ for sufficiently small ε we can argue as above by dividing the ball $B(0, \varepsilon^\delta)$. For the inner part we obtain by Lemma 6.6

$$\int_{B(0,\varepsilon^{\frac{3}{4}})} \frac{\varepsilon}{2}\left|\nabla u_\varepsilon^{\mathrm{in}}\right|^2 + \frac{1}{\varepsilon}F(u_\varepsilon^{\mathrm{in}})\,dx \leq C\varepsilon^{-1}\left|B(0, \varepsilon^{\frac{3}{4}})\right| = C\varepsilon^{\frac{1}{2}}$$

which vanishes as $\varepsilon \to 0$. The remaining part of $B(0, \varepsilon^\delta)$ itself divides into two disjoint sets. We define

$$B_\varepsilon^1 := \left\{x \in B(0, \varepsilon^\delta) \setminus B(0, \varepsilon^{\frac{3}{4}}) \,:\, \operatorname{dist}(x, \{u_\varepsilon = 0\}) < \varepsilon^{\frac{3}{4}}\right\} \text{ with } \left|B_\varepsilon^1\right| \leq C\varepsilon^{\delta + \frac{3}{4}}$$

and

$$B_\varepsilon^2 := \left\{x \in B(0, \varepsilon^\delta) \setminus B(0, \varepsilon^{\frac{3}{4}}) \,:\, \operatorname{dist}(x, \{u_\varepsilon = 0\}) \geq \varepsilon^{\frac{3}{4}}\right\} \text{ with } \left|B_\varepsilon^2\right| \leq C\varepsilon^{2\delta}$$

and hence observe with another application of Lemma 6.6 and (6.22)

$$\int_{B_\varepsilon^1} \frac{\varepsilon}{2} \left|\nabla u_\varepsilon^{\text{in}}\right|^2 + \frac{1}{\varepsilon} F(u_\varepsilon^{\text{in}})\, dx \leq C\varepsilon^{-1} \left|B_\varepsilon^1\right| = C\varepsilon^{-1+\delta+\frac{3}{4}} \leq C\varepsilon^{\frac{1}{12}}$$

as well as

$$\int_{B_\varepsilon^2} \frac{\varepsilon}{2} \left|\nabla u_\varepsilon^{\text{in}}\right|^2 + \frac{1}{\varepsilon} F(u_\varepsilon^{\text{in}})\, dx \leq C\varepsilon^{-1+2\delta} e^{-\frac{C}{\varepsilon}\operatorname{dist}(x,\{u=0\})} \leq C\varepsilon^{-1+2\delta} e^{-C\varepsilon^{-\frac{1}{4}}}.$$

Therefore,

$$\int_{B(0,\varepsilon^\delta)} \frac{\varepsilon}{2} \left|\nabla u_\varepsilon^{\text{in}}\right|^2 + \frac{1}{\varepsilon} F(u_\varepsilon^{\text{in}})\, dx \longrightarrow 0 \tag{6.40}$$

as $\varepsilon \to 0$. For the annulus $B(0, 2\varepsilon^\delta) \setminus B(0, \varepsilon^\delta) = A_\varepsilon^1 \cup A_\varepsilon^2$ we calculate

$$\begin{aligned}
&\int_{A_\varepsilon^1 \cup A_\varepsilon^2} \frac{\varepsilon}{2} \left|\nabla \eta_\varepsilon (u_\varepsilon^{\text{in}} - u_\varepsilon^{\text{out}}) + \eta_\varepsilon \nabla u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon)\nabla u_\varepsilon^{\text{out}}\right|^2 \\
&\qquad + \frac{1}{\varepsilon} F\big(\eta_\varepsilon u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon) u_\varepsilon^{\text{out}}\big)\, dx \\
&\leq C\varepsilon \int_{A_\varepsilon^1 \cup A_\varepsilon^2} |\nabla \eta_\varepsilon|^2 \left|u_\varepsilon^{\text{in}} - u_\varepsilon^{\text{out}}\right|^2 dx + C\varepsilon \int_{A_\varepsilon^1 \cup A_\varepsilon^2} |\eta_\varepsilon|^2 \left|\nabla u_\varepsilon^{\text{in}} - \nabla u_\varepsilon^{\text{out}}\right|^2 dx \\
&\qquad + C\varepsilon \int_{A_\varepsilon^1 \cup A_\varepsilon^2} \left|\nabla u_\varepsilon^{\text{out}}\right|^2 dx + \frac{1}{\varepsilon} \int_{A_\varepsilon^1 \cup A_\varepsilon^2} F\big(\eta_\varepsilon u_\varepsilon^{\text{in}} + (1-\eta_\varepsilon) u_\varepsilon^{\text{out}}\big)\, dx \\
&:= J_1 + J_2 + J_3 + J_4
\end{aligned}$$

and treat the summands separately. Analogously to the estimates of I_1 and I_2 in the calculations for $\mathcal{W}_\varepsilon(u_\varepsilon)$ above we obtain for J_1 and J_2

$$J_1 \leq C\varepsilon \longrightarrow 0$$

and

$$J_2 \leq C\varepsilon^{3\delta-1} e^{-C\varepsilon^{2\delta-1}} \longrightarrow 0$$

as $\varepsilon \to 0$.
Since $u_\varepsilon^{\text{out}}$ is constant ± 1 on each component of A_ε^2 the integral in J_3 reduces to A_ε^1 which yields

$$\begin{aligned}
J_3 = C\varepsilon \int_{A_\varepsilon^1} \left|\nabla u_\varepsilon^{\text{out}}\right|^2 dx &\leq C\varepsilon\varepsilon^{3\delta} \left\|w_\varepsilon'\right\|_{L^\infty(\mathbb{R})}^2 \|\nabla d_\theta^\varepsilon\|_{L^\infty(A_\varepsilon^1)}^2 \\
&\leq C\varepsilon^{3\delta+1} \left\|\gamma_\varepsilon'\right\|_{L^\infty(\mathbb{R})}^2 \leq C \left\|\gamma'\right\|_{L^\infty(\mathbb{R})}^2 \varepsilon^{3\delta-1} \\
&\longrightarrow 0
\end{aligned}$$

where we have used the fact that $|\nabla d_\theta^\varepsilon| = |\nabla d| = 1$ in $B(0, 2\varepsilon^\delta)$.For the remaining term J_4 we argue similarly as for I_3 above. Due to the small size of A_ε^1 and the mean value theorem, we obtain

$$\begin{aligned}
J_4 =& \frac{1}{\varepsilon}\int_{A^1_\varepsilon} F\big(\eta_\varepsilon u^{\text{in}}_\varepsilon + (1-\eta_\varepsilon)u^{\text{out}}_\varepsilon\big)\,dx + \frac{1}{\varepsilon}\int_{A^2_\varepsilon} F\big(\eta_\varepsilon u^{\text{in}}_\varepsilon \pm (1-\eta_\varepsilon)\big) - F(\pm 1)\,dx \\
\le& C\varepsilon^{-1}\varepsilon^{3\delta}\,\|F\|_{L^\infty([-1,1])} + C\varepsilon^{-1}\varepsilon^{2\delta}\,\|F'\|_{L^\infty([-1,1])}\,\|\eta_\varepsilon\|_{L^\infty(B)}\,\|u^{\text{in}}_\varepsilon \mp 1\|_{L^\infty(A^2_\varepsilon)} \\
\le& C\varepsilon^{3\delta-1} + C\varepsilon^{2\delta-1}e^{-C\varepsilon^{2\delta-1}} \longrightarrow 0.
\end{aligned}$$

Hence, the contribution of the annulus $B(0,2\varepsilon^\delta)\setminus B(0,\varepsilon^\delta)$ and by (6.40) therefore of the whole ball $B(0,2\varepsilon^\delta)$ to the diffuse surface energy $\mathcal{E}_\varepsilon(u_\varepsilon)$ vanishes as $\varepsilon \to 0$.
On $B\setminus B(0,3R)$ we have $u_\varepsilon = u^{\text{out}}_\varepsilon = w_\varepsilon \circ d$ and thus, a direct application of the result of [MoMo77] (or [Mo87]) yields

$$\int_{B\setminus B(0,3R)} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}F(u_\varepsilon)\,dx \longrightarrow \sigma\mathcal{H}^1\Big(\partial E \cap \big(B\setminus B(0,3R)\big)\Big).$$

In order to show the convergence of the remaining integral over $B(0,3R)\setminus B(0,2\varepsilon^\delta)$, we have to adapt the proof of Modica and Mortola since our construction of $u^{\text{out}}_\varepsilon$ involves the disturbed signed distance function d^ε_θ instead of d. For that purpose it will turn out to be crucial that the small correction of d has an almost vanishing gradient. As in the corresponding considerations for the Willmore energy of u_ε, we can restrict to the set U_ε from (6.29) and consider only one of its connected components. Again we assume that the regarded half-line is the positive x_1-axis and that (6.31), (6.32) hold. We see, that

$$\begin{aligned}
&\int_{U_\varepsilon} \frac{\varepsilon}{2}\left|\nabla u^{\text{out}}_\varepsilon\right|^2 + \frac{1}{\varepsilon}F(u^{\text{out}}_\varepsilon)\,dx \\
=&\int_{U_\varepsilon} \frac{\varepsilon}{2}|\nabla(w_\varepsilon\circ d^\varepsilon_\theta)|^2 + \frac{1}{\varepsilon}F\big(w_\varepsilon\circ d^\varepsilon_\theta\big)\,dx \\
=&\int_{U_\varepsilon} \frac{\varepsilon}{2}|\nabla(\gamma_\varepsilon\circ d^\varepsilon_\theta)|^2 + \frac{1}{\varepsilon}F\big(\gamma_\varepsilon\circ d^\varepsilon_\theta\big)\,dx + \frac{\varepsilon}{2}\int_{U_\varepsilon}\left|\nabla\big(w_\varepsilon\circ d^\varepsilon_\theta\big)\right|^2 - \left|\nabla\big(\gamma_\varepsilon\circ d^\varepsilon_\theta\big)\right|^2 dx \\
&+\frac{1}{\varepsilon}\int_{U_\varepsilon} F\big(w_\varepsilon\circ d^\varepsilon_\theta\big) - F\big(\gamma_\varepsilon\circ d^\varepsilon_\theta\big)\,dx \\
:=&K_1 + K_2 + K_3.
\end{aligned}$$

and we will prove that the last two summands vanish as $\varepsilon \to 0$ while the first one approximates up to the constant σ the length of $\partial E \cap U_\varepsilon$ which converges to $3R\sigma$ in the limit.
Indeed, by

$$|\nabla d^\varepsilon_\theta(x)|^2 = 1 + r^2\varepsilon^2\big(\theta(x_1)\big)^2 \le C \quad \text{for } x = (x_1,x_2)\in U_\varepsilon$$

as well as (6.24), (6.26), and (6.30) we obtain for the second summand

$$\begin{aligned}
K_2 \le& \frac{\varepsilon}{2}\int_{U_\varepsilon}\left|\big(w'_\varepsilon + \gamma'_\varepsilon\big)\circ d^\varepsilon_\theta\right|\left|\big(w'_\varepsilon - \gamma'_\varepsilon\big)\circ d^\varepsilon_\theta\right| |\nabla d^\varepsilon_\theta|^2\,dx \\
\le& C\varepsilon\int_{U_\varepsilon}\left|\big(\zeta'_\varepsilon(\gamma_\varepsilon - \operatorname{sgn}) + \gamma'_\varepsilon(\zeta_\varepsilon+1)\big)\circ d^\varepsilon_\theta\right|\left|\big(w'_\varepsilon - \gamma'_\varepsilon\big)\circ d^\varepsilon_\theta\right|\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon^{\delta+\frac12}\varepsilon^{-1}e^{-C\varepsilon^{\delta-\frac12}}\\
&\le C\varepsilon^{\delta-\frac12}e^{-C\varepsilon^{\delta-\frac12}}.
\end{aligned}
$$

Using the mean value theorem the third term can be controlled by

$$
K_3 \le C\varepsilon^{\delta-\frac12}e^{-C\varepsilon^{\delta-\frac12}}
$$

analogously to the estimate of H_2 above. Hence, K_2 and K_3 vanish as expected for $\varepsilon \to 0$. For K_1 we use the fact that the optimal profile satisfies

$$
\gamma_\varepsilon' = \frac{1}{\varepsilon}\sqrt{2F(\gamma_\varepsilon)}
$$

and hence,

$$
\begin{aligned}
K_1 &= \int_{U_\varepsilon} \frac{\varepsilon}{2}\left|\gamma_\varepsilon'\circ d_\theta^\varepsilon\right|^2|\nabla d_\theta^\varepsilon|^2 + \frac{1}{\varepsilon}F(\gamma_\varepsilon\circ d_\theta^\varepsilon)\,dx\\
&= \int_{U_\varepsilon} \frac12\sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\,|\nabla d_\theta^\varepsilon|^2 + \frac12\sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\,dx\\
&= \int_{U_\varepsilon} \frac12\sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\left(1+|\nabla d_\theta^\varepsilon|^2\right)\,dx.
\end{aligned}
$$

Now, we observe for $x=(x_1,x_2)\in U_\varepsilon$ that

$$
\begin{aligned}
1+|\nabla d_\theta^\varepsilon(x)|^2 &= 2+r^2\varepsilon^2\big(\theta'(x_1)\big)^2 = 2\sqrt{1+r^2\varepsilon^2\big(\theta'(x_1)\big)^2}+O(\varepsilon^4)\\
&= 2\,|\nabla d_\theta^\varepsilon(x)| + O(\varepsilon^4)
\end{aligned}
$$

and consequently,

$$
K_1 = \int_{U_\varepsilon} \sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\,|\nabla d_\theta^\varepsilon|\;dx + O(\varepsilon^4)\int_{U_\varepsilon}\sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\,dx,
$$

where the second summand tends to 0 with $\varepsilon\to 0$ due to

$$
\int_{U_\varepsilon}\sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\,dx \le C\varepsilon^{\delta-1}.
$$

The coarea formula yields for the first integral

$$
\begin{aligned}
&\int_{U_\varepsilon}\sqrt{2F(\gamma_\varepsilon\circ d_\theta^\varepsilon)}(\gamma_\varepsilon'\circ d_\theta^\varepsilon)\,|\nabla d_\theta^\varepsilon|\;dx\\
&= \int_{-2\varepsilon^{\delta+\frac12}}^{2\varepsilon^{\delta+\frac12}} \sqrt{2F\big(\gamma_\varepsilon(t)\big)}\,\big|\gamma_\varepsilon'(t)\big|\,\mathcal{H}^1\big(\{z\in U_\varepsilon\;:\;d_\theta^\varepsilon(z)=t\}\big)\,dt.
\end{aligned}
$$

For $|t|\le 2\varepsilon^{\delta+\frac12}$ we can determine the measure inside the integral explicitly by

$$
\mathcal{H}^1\big(\{z\in U_\varepsilon\;:\;d_\theta^\varepsilon(z)=t\}\big) = \mathcal{H}^1\big(\{z\in U_\varepsilon\;:\;d_\theta^\varepsilon(z)=0\}\big)+h(t)
$$

with $h \in C^0(-2\varepsilon^{\delta+\frac{1}{2}}, 2\varepsilon^{\delta+\frac{1}{2}})$ given by

$$h(t) := -3R\left(1-\sqrt{1-\frac{t^2}{9R^2}}\right) + \operatorname{sgn}\big(t(t+2r\varepsilon)\big)\frac{r\varepsilon^{1-\delta}t + \frac{1}{2}\varepsilon^{-\delta}t^2}{\sqrt{1-\frac{r^2\varepsilon^2}{4\varepsilon^{2\delta}}} + \sqrt{1-\frac{(t+r\varepsilon)^2}{4\varepsilon^{2\delta}}}}$$

which converges to 0 as $t \to 0$. Now, by an application of the generalized mean value theorem for integrals there exists a null sequence of real numbers $(\xi_\varepsilon)_{\varepsilon>0}$ with $|\xi_\varepsilon| \le 2\varepsilon^{\delta+\frac{1}{2}}$ such that

$$\begin{aligned}&\int_{-2\varepsilon^{\delta+\frac{1}{2}}}^{2\varepsilon^{\delta+\frac{1}{2}}} \sqrt{2F\big(\gamma_\varepsilon(t)\big)}\,\big|\gamma_\varepsilon'(t)\big|\,\mathcal{H}^1\big(\{z\in U_\varepsilon \,:\, d_\theta^\varepsilon(z)=t\}\big)\,dt\\ =&\Big(\mathcal{H}^1\big(\{z\in U_\varepsilon \,:\, d_\theta^\varepsilon(z)=0\}\big)+\xi_\varepsilon\Big)\int_{-2\varepsilon^{\delta+\frac{1}{2}}}^{2\varepsilon^{\delta+\frac{1}{2}}} \sqrt{2F\big(\gamma_\varepsilon(t)\big)}\,\big|\gamma_\varepsilon'(t)\big|\,dt\end{aligned}$$

is satisfied. The expression on the right hand side converges with $\varepsilon \to 0$ as

$$\mathcal{H}^1\big(\{z\in U_\varepsilon \,:\, d_\theta^\varepsilon(z)=0\}\big) = L\big(\operatorname{graph}(\varepsilon r\theta)\cap U_\varepsilon\big) = \int_{2\varepsilon^\delta}^{3R}\sqrt{1+r\varepsilon\theta'}\,ds \longrightarrow 3R$$

follows immediately by Lebesgue's dominated convergence theorem and

$$\int_{-2\varepsilon^{\delta+\frac{1}{2}}}^{2\varepsilon^{\delta+\frac{1}{2}}} \sqrt{2F\big(\gamma_\varepsilon(t)\big)}\,\big|\gamma_\varepsilon'(t)\big|\,dt = \int_{-2\gamma\left(\varepsilon^{\delta-\frac{1}{2}}\right)}^{2\gamma\left(\varepsilon^{\delta-\frac{1}{2}}\right)}\sqrt{2F}\,ds \longrightarrow \int_{-1}^{1}\sqrt{2F}\,ds = \sigma.$$

We combine the calculations and finally obtain

$$K_1 \longrightarrow 3R\sigma$$

and thus,

$$\int_B \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}F(u_\varepsilon)\,dx \longrightarrow \sigma\mathcal{H}^1(\partial E\cap B)$$

as $\varepsilon \to 0$ which completes the proof of Lemma 6.5. □

Remark. In the definition of $\mathcal{S}_0$ we have restricted ourselves to boundary intersections of only two boundary portions which we could approximate by using a 4-ended solution as the inner part of the construction above. Unfortunately, it is not sufficient to replace $u_\varepsilon^{\text{in}}$ by a general $2k$-ended solution ($k \ge 2$) and directly approximate arbitrary intersections of more than two curves in a common point since these solutions only exist for a suitable set of intersection angles (satisfying a Toda-System, see [PiKoPa10], Theorem 1.1). Hence, another argument is required to handle those intersections and we revisit them in Section 6.2.

The construction in Lemma 6.5 yields a way how to handle occurring transversal intersections of transition layers in the approximation of functions $u \in \mathcal{S}_0$. With this method in hand we are able to construct for given $u \in \mathcal{S}_0$ an appropriate recovery sequence $(u_\varepsilon)_{\varepsilon>0}$ in $H^2(\Omega)$ which realizes the Γ-limit in $\mathcal{F}(u)$. By a simple diagonal argument the result stays true for $u \in \mathcal{S}$.

Proposition 6.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $E \subset \Omega$ such that $u := 2\chi - 1$ lies in $\mathcal{S}$. Then there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $H^2(\Omega)$ with $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ such that*

$$\mathcal{F}^*(u) = \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Proof. First, consider $u = 2\chi_E - 1 \in \mathcal{S}_0$. By definition, $\partial E \subset \Omega$ is the union of finitely many closed C^2-curves with at most finitely many self intersection points $x_1, \dots, x_m$ of the boundary. We define a radius $s > 0$ such that $B(x_j, s) \subset\subset \Omega$ for $1 \le j \le m$ and such that these balls are pairwise disjoint. Moreover, we choose s small enough such that for every j $\partial E \cap B(x_j, s)$ consists of two curve pieces which intersect $\partial B(x_j, s)$ transversally and which are equal to line segments in $B(x_j, \frac{3}{4}s)$.
Now, we apply Lemma 6.5 on every such ball and denote the resulting sequences by $(u_{j,\varepsilon})_{\varepsilon>0}$. For ε sufficiently small we then define

$$u_\varepsilon(x) := \begin{cases} u_{j,\varepsilon}(x), & \text{if } x \in B(x_j, s) \text{ for some } 1 \le j \le m \\ w_\varepsilon(d(x)), & \text{elsewhere in } \Omega \end{cases}$$

with w_ε from the outer construction in the proof of Lemma 6.5 (see Figure 6.5).

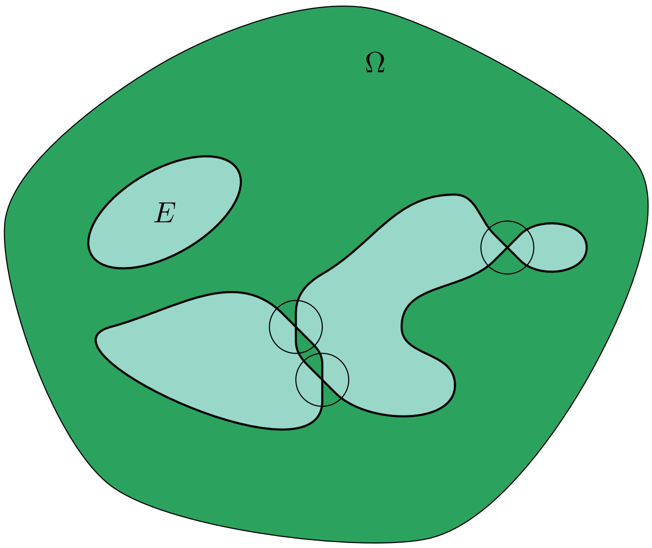


Figure 6.5: The whole approximation of $u = 2\chi_E - 1 \in \mathcal{S}_0$ in Ω.

Therefore, $u_\varepsilon \in H^2(\Omega)$ and Lemma 6.5 together with [BePa93] yield $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ and

$$\mathcal{F}(u) = \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Now, let $u \in \mathcal{S}$ and let $(u^{(k)})_{k\in\mathbb{N}}$ be sequence in $\mathcal{S}_0$ such that

$$u^{(k)} \longrightarrow u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \mathcal{F}(u^{(k)}) \longrightarrow \mathcal{F}^*(u)$$

as $k \to \infty$. By the consideration above we can find for each $k \in \mathbb{N}$ a sequence $(u_\varepsilon^{(k)})_{\varepsilon>0}$ with

$$u_\varepsilon^{(k)} \longrightarrow u^{(k)} \quad \text{in } L^1(\Omega) \quad \text{and} \quad \mathcal{F}_\varepsilon(u_\varepsilon^{(k)}) \longrightarrow \mathcal{F}(u^{(k)})$$

as $\varepsilon \to 0$. We will construct a diagonal sequence which will prove the proposition. For $k \in \mathbb{N}$ we choose $\varepsilon(k) > 0$ such that

$$\left\| u^{(k)} - u_\varepsilon^{(k)} \right\|_{L^1(\Omega)} < \frac{1}{k} \quad \text{and} \quad \left| \mathcal{F}(u^{(k)}) - \mathcal{F}_\varepsilon\big(u_\varepsilon^{(k)}\big) \right| < \frac{1}{k}$$

hold for every $\varepsilon < \varepsilon(k)$. Without loss of generality we assume that $\varepsilon(k) \searrow 0$ as $k \to \infty$ and hence, we can define for $0 < \varepsilon < 1$

$$\tilde{u}_\varepsilon := u_\varepsilon^{(k)} \quad \text{for } \varepsilon \in \big[\varepsilon(k+1), \varepsilon(k)\big).$$

This yields for $0 < \varepsilon < 1$

$$\begin{aligned} \|u - \tilde{u}_\varepsilon\|_{L^1(\Omega)} &\le \left\| u - u^{(k)} \right\|_{L^1(\Omega)} + \left\| u^{(k)} - \tilde{u}_\varepsilon \right\|_{L^1(\Omega)} \\ &\le \left\| u - u^{(k)} \right\|_{L^1(\Omega)} + \frac{1}{k} \end{aligned}$$

as well as

$$\begin{aligned} |\mathcal{F}^*(u) - \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon)| &\le \left| \mathcal{F}^*(u) - \mathcal{F}\big(u^{(k)}\big) \right| + \left| \mathcal{F}\big(u^{(k)}\big) - \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \right| \\ &\le \left| \mathcal{F}^*(u) - \mathcal{F}\big(u^{(k)}\big) \right| + \frac{1}{k} \end{aligned}$$

where we have chosen k such that $\varepsilon \in [\varepsilon(k+1), \varepsilon(k))$. With $\varepsilon \to 0$ we obtain $k \to \infty$ and hence

$$\tilde{u}_\varepsilon \longrightarrow u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \longrightarrow \mathcal{F}^*(u)$$

which completes the proof. □

6.2 Examples of configurations in $\mathcal{S}$

We conclude this chapter by giving several specific examples of configurations $u \in \mathcal{S}$ for which we have proven the Γ-convergence in Theorem 6.3. The definition of $\mathcal{S}$ is rather implicit and only given by an approximation argument. The examples below will provide not only a vivid interpretation of $\mathcal{S}$ but also yield an (incomplete) characterization of new Γ-limit points of $\mathcal{F}_\varepsilon$.

i) **Non vanishing curvature in intersection points.** Let $u = 2\chi_E - 1$ with $E \subset\subset \Omega$ as in the definition of $\mathcal{S}_0$ except that we now allow intersection points of curved boundary segments of ∂E. We show that $u \in \mathcal{S}$. For the proof we assume without loss of generality that there exists exactly one such intersection. The argument for the general case is identical.
Condition iii) in the definition of $\mathcal{S}$ is satisfied automatically as we can choose $V \in \mathbb{V}_1(\Omega)$ with $\|V\| = \mathcal{H}^1 \lfloor \partial E$. We also have

$$\mathcal{F}^*(u) = \sigma \sum_{i=1}^{N} \big(L(\varphi_i) + \mathcal{W}(\varphi_i)\big)$$

with the same argument as in the remark on page 104. It remains to show that there exists a sequence in $\mathcal{S}_0$ which approximates u in $L^1(\Omega)$ with the correct energy.

As in the proof of Lemma 6.5 we restrict to a (sufficiently small) ball $B = B(0,R) \subset \Omega$ around the considered intersection and assume $u = 2\chi_E - 1$ with

$$\partial E \cap B = \mathrm{Im}(\varphi_1) \cup \mathrm{Im}(\varphi_2)$$

for C^2-curves φ_1 and φ_2 which intersect transversally in the origin and are disjoint elsewhere in B. For a sufficiently small radius $R > 0$ we can assume that φ_i can be written as a graph of a C^2-function h_i over its tangent line segment g_i

$$\{rv_i \,:\, |r| \le R\}$$

for $i = 1, 2$ where we chose $v_i := \varphi_i'(0)$. We show that we can approximate each h_i by functions which are constant to zero in a small neighborhood of the origin such that both the length and the Willmore energy of its graph are preserved in the limit. Since we are able to construct a proper recovery sequence for each such approximating function a diagonal argument completes the proof of the claim.
To precise the argument sketched above we restrict to one φ_i and assume g_i to be the real line. Then we omit all indices i in the following. We have $h(0) = h'(0) = 0$ and for $|x| \le R$ we obtain the estimate

$$|h(x)| = \left| \int_0^x h'(t)\, dt \right| \le \left| \int_0^x \int_0^t h''(s)\, ds\, dt \right| \le |x^2| \, \|h''\|_{L^\infty([-R,R])} \tag{6.41}$$

and similarly,

$$|h'(x)| \le |x| \, \|h''\|_{L^\infty([-R,R])} . \tag{6.42}$$

With a smooth cutoff function $\xi : [-R, R] \to \mathbb{R}$ given by

$$\xi := \begin{cases} 0 & \text{in } |x| \leq \frac{1}{2} \\ 1 & \text{in } |x| \geq 1 \end{cases}$$

we define for $0 < \tau < \frac{1}{2}R$

$$h_\tau(x) := \xi\left(\frac{x}{\tau}\right) h(x), \quad x \in [-R, R],$$

see Figure 6.6.

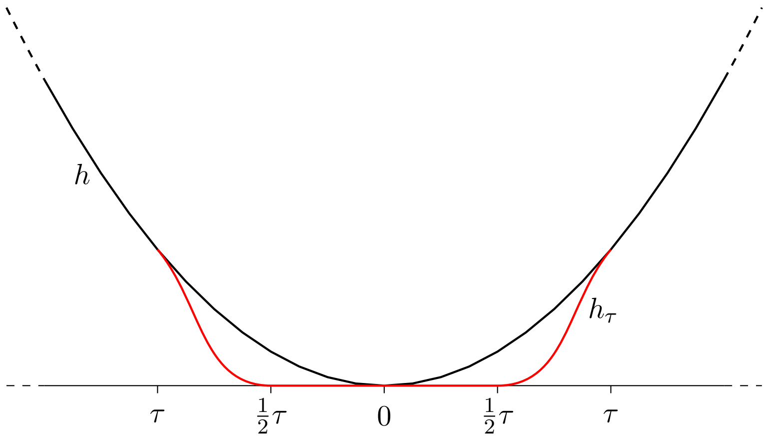


Figure 6.6: The graph of h and its approximation h_τ.

Inequalities (6.41) and (6.42) yield

$$\begin{aligned} &\int_{-R}^{R} (h_\tau(x) - h(x))^2 + (h'_\tau(x) - h'(x))^2 + (h''_\tau(x) - h''(x))^2 \, dx \\ =&\int_{-\tau}^{\tau} \left(\xi\left(\frac{x}{\tau}\right) h(x)\right)^2 + \left(\frac{1}{\tau}\xi'\left(\frac{x}{\tau}\right) h(x) + \left(\xi\left(\frac{x}{\tau}\right) - 1\right) h'(x)\right)^2 \\ &\quad + \left(\frac{1}{\tau^2}\xi''\left(\frac{x}{\tau}\right) h(x) + \frac{2}{\tau}\xi'\left(\frac{x}{\tau}\right) h'(x) + \left(\xi\left(\frac{x}{\tau}\right) - 1\right) h''(x)\right)^2 dx \\ \leq& C \int_{-\tau}^{\tau} \tau^4 + \tau^2 + 1 \, dx \\ \leq& C\tau \end{aligned}$$

for $0 < \tau < 1$ and therefore,

$$h_\tau \longrightarrow h \quad \text{in } H^2\big((-R, R)\big) \tag{6.43}$$

as $\tau \to 0$. Obviously, we also have

$$h_\tau^{(k)}(x) \longrightarrow h^{(k)}(x), \quad k = 0, 1, 2 \tag{6.44}$$

pointwise in $[-R, R] \setminus \{0\}$ as $\tau \to 0$ by the definition of h_τ.

Finally, we observe that

$$\mathcal{W}(\operatorname{graph} h_\tau) + L(\operatorname{graph} h_\tau) = \int_{-R}^{R} \frac{(h_\tau'')^2}{(1+(h_\tau')^2)^3} + \sqrt{1+(h_\tau')^2}\,dx$$
$$\longrightarrow \int_{-R}^{R} \frac{(h'')^2}{(1+(h')^2)^3} + \sqrt{1+(h')^2}\,dx = \mathcal{W}(\operatorname{graph} h) + L(\operatorname{graph} h)$$

as $\tau \to 0$ by the generalized Lebesgue dominated convergence theorem due to (6.43) and (6.44) and the fact that

$$\frac{(h_\tau'')^2}{(1+(h_\tau')^2)^3} + \sqrt{1+(h_\tau')^2} \le 1 + (h_\tau')^2 + (h_\tau'')^2$$

holds in $(-R,R)$ where we have used the general inequality $\sqrt{1+t} \le 1 + \frac{t}{2}$ for $t \ge 0$.
To construct a proper approximating sequence we can now argue as in the proof of Proposition 6.8. For $\tau > 0$ we denote the indicator function corresponding to the modified boundary curves from above by u_τ and due to the considerations above, the phase boundary is shaped like a cross inside of $B(0, \frac{\tau}{2})$. Hence, we can apply Lemma 6.5 on the ball $B_\tau := B\left(0, \frac{2\tau}{3}\right)$ and consequently, there exists a sequence $(u_\varepsilon^\tau)_{\varepsilon>0}$ which approximates u_τ in $L^1(B_\tau)$ and therefore in $L^1(B)$ such that

$$\mathcal{F}_\varepsilon(u_\varepsilon^\tau) \longrightarrow \mathcal{F}(u^\tau)$$

as $\varepsilon \to 0$. By choosing a sequence $(\tau_\varepsilon)_{\varepsilon>0}$ which tends to 0 slowly enough (e.g. choose $\tau_\varepsilon = \sqrt{\varepsilon}$) we can define the diagonal sequence as

$$u_\varepsilon := u_\varepsilon^{\tau_\varepsilon}.$$

ii) **H^2-curves.** Obviously, $\mathcal{S}$ includes indicator functions $u = 2\chi_E - 1$ of sets $E \subset\subset \Omega$ as in the definition of $\mathcal{S}_0$ but with ∂E given by closed H^2-regular curves instead of C^2-curves. In this case condition iii) in the definition of $\mathcal{S}$ is still satisfied by $V \in V_1(\Omega)$ with $\|V\| = \mathcal{H}^1 \lfloor \partial E$ and we especially have $\mathbf{H}_{\partial E} \in L^2(\mathcal{H}^1)$ and

$$\mathcal{F}^*(u) = \sigma \sum_{i=1}^{N} \big(L(\varphi_i) + \mathcal{W}(\varphi_i)\big).$$

We can approximate E with smoothly bounded sets E_k in $H^2(\Omega)$ and the convergence of energies follows as $\mathcal{F}$ is continuous with respect to the H^2-topology.

iii) **Countably many curves.** Let $u = 2\chi_E - 1$ with $E \subset\subset \Omega$ such that its (reduced) boundary is given by

$$\partial^* E = \bigcup_{i=1}^{\infty} \operatorname{Im}(\varphi_i)$$

for countably many closed C^2-curves φ_i, $i \ge 0$. Further, assume that every φ_k

has only a finite number of transversal (self) intersections with all curves and that there are no common points of more than two curves. Precisely, we assume

$$\Lambda(x) \leq 2 \quad \text{for all } x \in \partial E$$

with the multiplicity function Λ from (6.4) and additionally, that

$$\#\Big\{x \in \partial^*\Omega \,:\, \Lambda(x) = 2\Big\} < \infty.$$

We show that $u \in \mathcal{S}$ if

$$\sum_{i=1}^{\infty} \mathcal{F}(\varphi_i) = \sigma \sum_{i=1}^{\infty} \big(L(\varphi_i) + \mathcal{W}(\varphi_i)\big) < \infty.$$

Condition iii) in the definition of $\mathcal{S}$ is satisfied automatically since $\partial^* E$ is rectifiable and we can choose V with induced weight measure $\|V\| = \mathcal{H}^1\lfloor\partial^* E$. Especially, we have

$$\mathcal{F}^*(u) = \sum_{i=1}^{\infty} \mathcal{F}(\varphi_i) < \infty.$$

It remains to show that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{S}_0$ with $u_k = 2\chi_{E_k} - 1$ for all $k \geq 0$ which converges in $L^1(\Omega)$ towards u and approximates $\mathcal{F}(u)$. Therefore, we define $E_k \subset\subset \Omega$ for $k \geq 1$ by choosing its boundary as

$$\partial E_k = \bigcup_{i=1}^{k} \operatorname{Im}(\varphi_i)$$

and then set

$$E_k := \left\{x \in \Omega \,:\, n(\partial E_k; x) := \sum_{i=1}^{k} n(\varphi_i; x) \text{ is odd.}\right\}$$

where $n(\varphi; x)$ denotes the winding number of x with respect to φ_i. It is clear that $u_k = 2\chi_{E_k} - 1 \in \mathcal{S}_0$ for all $k \geq 1$ and by definition

$$\mathcal{F}(u_k) = \sigma \sum_{i=1}^{k} \Big(L(\varphi_i) + \mathcal{W}(\varphi_i)\Big) \longrightarrow \sigma \sum_{i=1}^{\infty} \Big(L(\varphi_i) + \mathcal{W}(\varphi_i)\Big) = \mathcal{F}^*(u) < \infty$$

as $k \to \infty$. We prove that $E_k \longrightarrow E$ in $L^1(\Omega)$ (i.e. $\chi_{E_k} \longrightarrow \chi_E$ in $L^1(\Omega)$) as $k \to \infty$. For all $k \geq 1$ we have

$$|\nabla \chi_{E_k}|(\Omega) = \sum_{i=1}^{k} L(\varphi_i) < \sum_{i=1}^{\infty} L(\varphi_i) < \sigma^{-1}\mathcal{F}(u) < \infty.$$

Hence, the sequence $(E_k)_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)$ and we can extract a subsequence

$(E_{k_j})_{j\in\mathbb{N}}$ with

$$E_{k_j} \longrightarrow \tilde{E}$$

for $E \subset \Omega$ ([EvGa92], 5.2.3, Theorem 4) and by the construction of E_k we have $\partial\tilde{E} = \partial E$ and hence

$$\tilde{E} = E \quad \text{or} \quad \tilde{E} = \Omega \setminus E.$$

To see that indeed $\tilde{E} = E$ holds we can argue as follows. Since $E \subset\subset \Omega$ there exists a ball $B \subset \Omega$ such that

$$n(\partial^* E; b) = \sum_{i=1}^{\infty} n(\varphi_i; b) = 0 \quad \text{for all } b \in B$$

which especially implies $B \cap E = \emptyset$. Hence, for every $k \geq 1$ we also have $n(\partial E_k; \cdot) = 0$ in B and thus $E_k \cap B = \emptyset$. The L^1-convergence of the subsequence above yields $\tilde{E} \cap B = \emptyset$ hence $\tilde{E}$ and E coincide on a set B with positive measure. This implies $\tilde{E} = E$ and since we can repeat the same argument for every subsequence of $(E_k)_{k\in\mathbb{N}}$ this finally proves

$$E_k \longrightarrow E$$

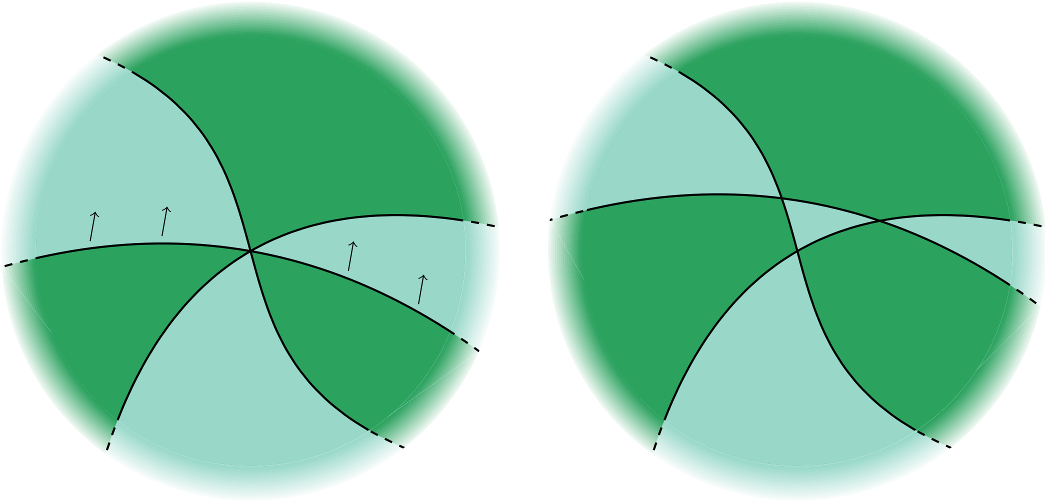
as $k \to \infty$.

iv) **Junction points of finitely many curve pieces.** $\mathcal{S}$ allows phase transitions with junction points of more than 2 curve pieces and arbitrary intersection angles. For example, a intersection of three curves in a common point (see Figure 6.7 (a)) can be approximated by slightly shifting one of the curves to the side which creates 3 intersection points of 2 curve pieces (Figure 6.7 (b)). By a diagonal argument, we find a sequence which simultaneously approximates each shifted configuration and lets the shift size shrink.
Remark that this construction gives rise to a new phase whose sign can be determined uniquely.
As mentioned in the remark after Lemma 6.5 the argument is crucial for junction points of higher order with arbitrary intersection angles.

v) **Non transversal intersections.** For $u \in \mathcal{S}$ the boundary intersections do not have to be transversal in general. For example, consider $u = 2\chi_E - 1$ with $\partial E \subset \Omega$ given by two round spheres S_1 and S_2 touching in one point $x_0 \in \Omega$ (see Figure 6.8). We can approximate the set by connecting both boundary components to a figure eight shaped curve which shows a transversal self intersection in x_0. Reducing the size of the intersection angle gives a sequence which converges to u in $L^1(\Omega)$ and which realizes (up to the constant σ) the Willmore energy and the surface area

$$\mathcal{W}(u) = \mathcal{W}(S_1) + \mathcal{W}(S_2), \qquad \mathcal{H}^1(E \cap \Omega) = \mathcal{H}^1(S_1) + \mathcal{H}^1(S_2).$$

Beyond the points above (and a combination of them) there are numerous other ways



(a) (b)

Figure 6.7: Approximation of a junction of three curves (a) by creating two additional intersection points and a new phase (b).

to construct elements in $\mathcal{S}$. For example, it is possible to approximate configurations with phase boundaries locally shaped as

$$\{x_1 = 0\} \cup \left\{ (x_1, x_2) \,:\, x_2 = x_1^4 \sin\left(\frac{1}{x_1}\right) \right\}.$$

$\partial^* E$ contains countable many self intersections which cumulate in one point. An approximation can be constructed with a similar argument as in i) by replacing the boundary in small neighborhoods of $x_1 = 0$ with a straight line segment.

At the end of this chapter, we point out that the limit functional $\mathcal{F}^*$ is not the lower semicontinuous envelope of $\mathcal{F}$ given by

$$\overline{\mathcal{F}}(u) := \inf \left\{ \liminf_{k\to\infty} \mathcal{F}(u_k) \,:\, u_k = 2\chi_{E_k} - 1,\ E_k\ C^2\text{-bounded},\ u_k \to u \text{ in } L^1(\Omega) \right\}$$

as for example a configuration $u = 2\chi_E - 1$ for a figure eight shaped set E satisfies $\mathcal{F}^*(u) < \infty$ but has infinite energy in terms of $\overline{\mathcal{F}}$.
It is also possible to construct configurations with an even number of cusps (see [BeDaPa93]) such that both energies are finite but $\mathcal{F}^*(u) < \overline{\mathcal{F}}(u) < \infty$.
In general, we have

$$\mathcal{F}^* \leq \overline{\mathcal{F}}. \tag{6.45}$$

This follows directly with the result of [BeMu07] where the authors fully characterize the lower semicontinuous envelope $\overline{\mathcal{F}}$. They prove that $\overline{\mathcal{F}}(u)$, $u = 2\chi_E - 1$, is given by the minimum of $\mathcal{F}$ evaluated in a class of varifolds which contain $\partial^* E$ and have a unique tangent in *every* point (see [BeMu07], Definition 4.1 for a precise description).

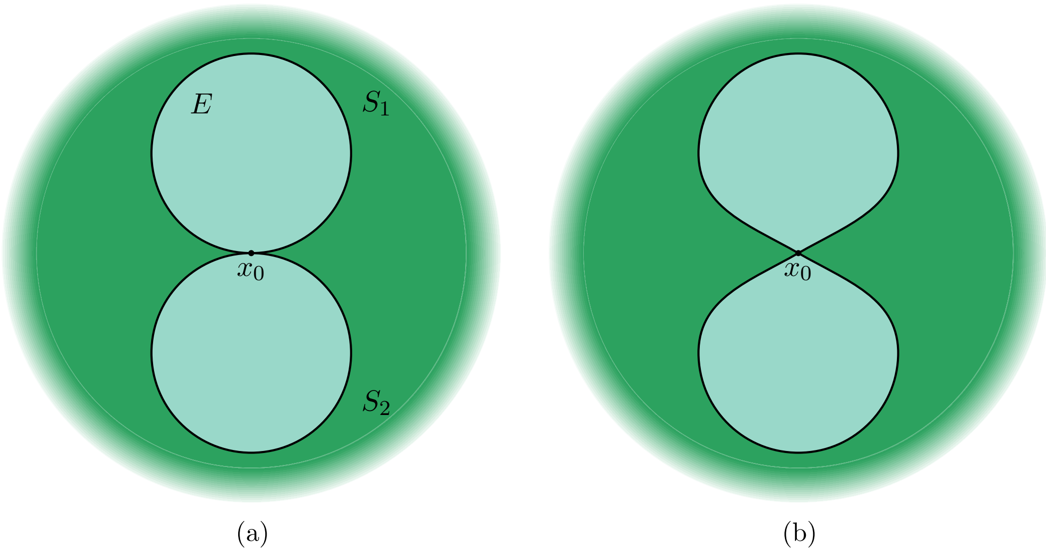


Figure 6.8: Approximation of two touching balls (a) by a sequence of figure eight shaped configurations (b).

In our case, the class of admissible varifolds in the definition of $\mathcal{S}$ contains the varifolds from Bellettini and Mugnai and hence, (6.45) holds.
However, a full and explicit characterization of $\mathcal{S}$ as in [BeMu05] is not available at the moment. We leave this as an open question for further research.

7 Summary and outlook

The main purpose of this thesis was the analysis of the diffuse Willmore functional and its L^2-gradient flow in the situation of interacting nonsmooth interfaces. This interest was especially motivated by configurations which occur in numerical simulations of the diffuse Willmore flow [EsRäRö14]. We will now interpret the appearing phenomena in the light of our proven results.
The simulations suggest that configurations with intersecting boundary curves are energetically preferable states although the sharp interface limit yields infinitely large values in these cases as corners have infinite curvature.
In Chapter 6 we explained this behavior by proving the Γ-convergence of the diffuse functionals $\mathcal{F}_\varepsilon$. The limit functional $\mathcal{F}^*$ extends the phase boundary to an integer varifold with generalized curvature and its energy value is determined by the support of this varifold and its geometry. In the case of the simulation snapshots of Figure 1.1 it is given by the sum of the single energies of the intersecting curves. Hence, the Willmore energy vanishes in the limit for configurations whose phase boundaries are given by straight lines without curvature.
This part of the thesis especially extends the Γ-convergence result from [RöSc06] to a larger class $\mathcal{S}$ of nonsmooth interfaces where we explicitly allow transversal intersections on the boundary. In Section 6.2 we have presented several examples of configurations $u \in \mathcal{S}$ which already yield a good idea how $\mathcal{S}$ can be described.
A question which directly arises from this work concerns the complete characterization of $\mathcal{S}$. A similar result by Bellettini and Mugnai in [BeMu07] for the lower semicontinuous envelope $\overline{\mathcal{F}}$ for H^2-bounded sets without intersections of the boundary suggests that this might be possible.

Another part of this work was dedicated to configurations which appear numerically stable in the diffuse Willmore flow. To analyze how planar interfaces interact with each other we considered a quasi one-dimensional situation of parallel stripes. We precisely determined the minimal energy order in terms of ε and the widths of neighboring stripes by a scaling law in Section 3.2. Particularly, we proved that quasi one-dimensional configurations always carry an exponentially small amount of energy in non symmetric situations. From these results we derived consequences for the diffuse Willmore flow with small initial energy in Chapter 5. We could show that the diffuse interfaces do evolve slowly in time. On an exponentially large time scale the layer locations stay almost constant and hence appear stable in numerical simulations. In view of slow motion phenomena occurring in gradient flows of the Ginzburg-Landau energy we expect the same behavior for the diffuse Willmore flow and the present thesis yields a first

rigorous result in this field.
Although we were not able to determine the movement direction of the zero positions we expect that they will maximize their mutual distance and distribute equally. We showed rigorously that the zeros cannot move asymptotically in the "wrong" direction. Additionally, we presented a heuristic argument which underlines our assumption that the gradient flow converges to the perfectly symmetric configuration. An exact description of the layer movement is of high mathematical interest. An idea for a possible approach was already presented at the end of chapter 5.2.

Concerning the analysis of the real two-dimensional case we considered situations of semi infinite rectangles which result from a slight modification of quasi one-dimensional stripes. We proved the existence of diffuse interfaces which approximate those configurations with energy constant to zero for rectangles with the same width. Our class of solutions can be seen as an analogue of the optimal arc profiles $q_{\ell,\varepsilon}$ in the higher dimensional case. The proof also showed that in the situation of differing rectangle widths all diffuse interfaces have to carry positive energy.
In comparison to the existence and characterization of the one-dimensional optimal arc profiles $q_{\ell,\varepsilon}$ in Section 3.1 the two-dimensional case in Chapter 4 turned out to be noticeably more difficult to prove. This already suggests that a full energy characterization (e.g., by another scaling law) will be complicated.

Finally, we can summarize that this thesis yields significant progress into the analysis of the diffuse Willmore functional and its L^2-gradient flow. Our results raise several further questions and interesting problems as a topic for future research.

Appendix

Maximum principles

Proposition A.1 (Weak maximum principle on narrow domains)**.** *Let* $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, *be a bounded domain which lies in a* narrow band *of sufficiently small width* $0 < \varepsilon < \varepsilon_0$, *i.e. there exist* $i \in \{1, \dots, n\}$ *and* $\alpha \in \mathbb{R}$ *such that*

$$\alpha < x_i < \alpha + \varepsilon \quad \textit{for all } x = (x_1, \dots, x_n) \in \Omega.$$

Consider a second order elliptic operator L *on* Ω *given by*

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{n} b_i(x)\partial_i + c(x)$$

with $a_{ij}, b_i, c \in L^\infty(\Omega)$ *for all* $1 \leq i, j \leq n$ *which is uniformly elliptic, i.e., there exist constants* $c_0, C_0 > 0$ *with*

$$c_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq C_0 |\xi|^2, \quad \textit{for all } \xi \in \mathbb{R}^n$$

and assume that

$$\sqrt{\sum_{i=1}^{n} b_i^2}\,,\ |c| \leq C_1$$

for another constant $C_1 > 0$.
Every function $w \in W^{2,n}_{loc}(\Omega)$ *with*

$$Lw \geq 0 \quad \textit{in } \Omega$$

and

$$\limsup_{x \to \partial\Omega} w(x) \leq 0$$

then satisfies $z \leq 0$ *in* Ω

Proof. [BeNi91], 1.2. □

Proposition A.2 (Weak maximum principle on small domains)**.** *For $d > 0$ let $\Omega \subset \mathbb{R}^n$ be a bounded domain with* $\operatorname{diam}\Omega \leq d$ *and assume L to be a second order elliptic operator as in Proposition A.1. There exists $\delta = \delta(n, d, c_0, C_1) > 0$ such that the following holds. If*

$$\mathcal{L}^n(\Omega) < \delta,$$

then every function $w \in W^{2,n}_{loc}(\Omega)$ with

$$Lw \geq 0 \quad in\ \Omega$$

and

$$\limsup_{x \to \partial\Omega} w(x) \leq 0$$

satisfies $z \leq 0$ in Ω

Proof. [BeNi91], Proposition 1.1. □

Basic facts on Radon measures

The results presented in this section are fairly standard and can be found e.g. in [EvGa92].
Let X be a locally compact and separable metric space.

Definition A.3. An outer measure μ on X is called *(positive) Radon measure* if

i) μ is Borel regular, i.e., μ is a Borel measure and for all $A \subset X$ there exists a Borel set B with $A \subset B$ and $\mu(A) = \mu(B)$.

ii) for all compact $K \subset X$ we have $\mu(K) < \infty$.

Remark. A Radon measure μ on X is regular in the sense that

$$\begin{aligned}
&\mu(A) = \inf \{\mu(U) \,:\, A \subset U,\, U \subset X \text{ open}\} && \text{for all } A \subset X\\
&\mu(A) = \sup \{\mu(U) \,:\, K \subset A,\, K \subset X \text{ compact}\} && \text{for all } \mu\text{-measurable } A \subset X.
\end{aligned}$$

Definition A.4. Let $M \subset X$ and μ a Radon measure on X. Then we define $\mu\lfloor M$ given by

$$(\mu\lfloor M)\,(A) := \mu(A \cap M) \quad \text{for all } A \subset X$$

as the *restriction of μ to M*.

The finiteness of a Radon measure μ on compact sets allows us to integrate functions $\phi \in C^0_c(X, \mathbb{R})$ and we write

$$\mu(\phi) := \int_X \phi \, d\mu.$$

Similarly, if H is a Hilbert space with inner product $(\cdot,\cdot)_H$ then each Radon measure μ on X and each μ-measurable function $\sigma : X \to H$ with $\|\sigma\|_H = 1$ μ-almost everywhere in X induce a linear functional $L : C_c^0(X, H) \to \mathbb{R}$ by

$$L(\phi) = \int_X (\phi, \sigma)_H \, d\mu \quad \text{for all } \phi \in C_c^0(X, H). \tag{A.1}$$

Vice versa, the next proposition states that under suitable assumptions, a linear functional $L : C_c^0(X, H) \to \mathbb{R}$ can be written in this way.

Proposition A.5 (Riesz representation theorem)**.** *Let $L : C_c^0(X, H) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\phi) \,:\, \phi \in C_c^0(X, H),\ \|\phi\|_H \leq 1,\ \operatorname{supp}\phi \subset K\} < \infty$$

for all compact $K \subset X$. Then there exists a Radon measure μ on X and a μ-measurable function $\sigma : X \to H$ with $\|\sigma\|_H = 1$ μ-almost everywhere in X such that (A.1) *holds. In this case, $\mu =: |L|$ is called the* variation measure of L *and we have*

$$|L|\,(U) := \sup\{L(\phi) \,:\, \phi \in C_c^0(X, H),\ \|\phi\|_H \leq 1,\ \operatorname{supp}\phi \subset U\}$$

for all open sets $U \subset X$.

Proof. [Si83], Theorem 4.1. □

Remark. By Proposition A.5 the Radon measures on X can be identified uniquely with the nonnegative linear functionals on $C_c^0(X, \mathbb{R})$ (see [Si83], Remark 4.3).

Definition A.6. A sequence of Radon measures $(\mu_k)_{k\in\mathbb{N}}$ on X converges weakly towards another Radon measure μ, in terms

$$\mu_k \overset{*}{\rightharpoonup} \mu$$

as $k \to \infty$ if

$$\lim_{k\to\infty} \mu_k(\phi) = \mu(\phi) \quad \text{for all } \phi \in C_c^0(X, \mathbb{R}).$$

Proposition A.7. *A sequence of Radon measures $(\mu_k)_{k\in\mathbb{N}}$ on X converging weakly towards μ has the lower semi-continuity property*

$$\mu(U) \leq \liminf_{j\to\infty} \mu_j(U)$$

for each open set $U \subset X$.

Proof. For $X = \mathbb{R}^n$ the proof can be found in [EvGa92], 1.9, Theorem 1. The general case is identical. □

With respect to the topology induced by Definition A.6 the space of Radon measures on X satisfies the following *compactness property*:

Proposition A.8. *Let* $(\mu_k)_{k\in\mathbb{N}}$ *be a sequence of Radon measures on* X *(as above) satisfying*

$$\sup_{k\geq 1}\mu_k(U) < \infty \quad \textit{for all open and relatively compact } U \subset X.$$

Then there exists a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ *of* $(\mu_k)_{k\in\mathbb{N}}$ *and a Radon measure* μ *on* X *with*

$$\mu_{k_j} \overset{*}{\rightharpoonup} \mu$$

as $j \to \infty$.

Proof. [Si83], Theorem 4.4. □

Next we want to define the density of Radon measures.

Definition A.9. For a Radon measure μ on X and $x \in X$ we define the *n-dimensional upper and lower densities* $\theta^{*n}(\mu, x)$, $\theta^n_*(\mu, x)$ of μ by

$$\theta^{*n}(\mu, x) := \limsup_{\rho\searrow 0} \frac{\mu(\overline{B(x,\rho)})}{\omega_n \rho^n}$$

and

$$\theta^n_*(\mu, x) := \liminf_{\rho\searrow 0} \frac{\mu(\overline{B(x,\rho)})}{\omega_n \rho^n}$$

where $\omega_n\rho^n$ is the n-dimensional Hausdorff- (or Lebesgue-) measure of a ball in $\mathbb{R}^n$ with radius $\rho > 0$.
Whenever $\theta^{*n}(\mu, x) = \theta^n_*(\mu, x)$ we just denote the value by $\theta^n(\mu, x)$ and call it *density* of μ.

Definition A.10. Let μ_1 and μ_2 be two measures on a space X. μ_1 is called *absolutely continuous with respect to* μ_2, in terms

$$\mu_1 \ll \mu_2$$

if $\mu_2(A) = 0$ implies $\mu_1(A) = 0$ for all $A \subset X$.

For the special case $X = \mathbb{R}^n$ we have the following differentiation theorem for Radon measures.

Proposition A.11 (Radon-Nikodym theorem)**.** *Let* μ_1 *and* μ_2 *be two Radon measures on* $\mathbb{R}^n$ *with*

$$\mu_1 \ll \mu_2.$$

Then there exists a μ_2*-measurable function* $\frac{d\mu_1}{d\mu_2}$ *called the* Radon-Nikodym derivative of μ_1 with respect to μ_2 *such that*

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2}\, d\mu_2 \quad \textit{for all } \mu_2\textit{-measurable subsets } A \subset \mathbb{R}^n.$$

For μ_2-almost every $x \in \mathbb{R}^n$ this function is given by

$$\frac{d\mu_1}{d\mu_2}(x) = \begin{cases} \lim\limits_{r\to 0} \frac{\mu_1(B(x,r))}{\mu_2(B(x,r))} & \textit{if } \mu_2(B(x,r)) > 0 \textit{ for all } r > 0 \\ \infty & \textit{if } \mu_2(B(x,r)) = 0 \textit{ for some } r > 0. \end{cases}$$

Proof. [EvGa92], 1.6.2, Theorem 2. □

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