Asymptotic Class Numbers of Lattices

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Timo Rosnau

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Asymptotic Class Numbers of Lattices

Fakultät für Mathematik
Technische Universität Dortmund

Erstgutachter: Prof. Dr. Rudolf Scharlau

Zweitgutachter: Prof. Dr. Peter Zeiner

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Introduction

In 1905, Minkowski has shown that most of the $\mathbb{Z}$-lattices possess a trivial automorphism group (see [Min05]). That is, if we fix the dimension and let an upper bound D on the determinant $\det L$ of the lattices L tend towards infinity, the number of isomorphism classes of lattices with trivial automorphism group grows with a higher order of magnitude of D than the class number of lattices with non-trivial automorphism groups. These orders of magnitude depend on the dimension n. Biermann established an analogous result for any given genus of lattices of determinant at most D in his 1981 dissertation (see [Bie81]).

Roughly speaking, the number of isomorphism classes of lattices grows with the determinant and the dimension. This growth, however, is not a strict one since the class number heavily depends on the prime factorisation of the determinant. Thus, useful statements can only be given for somehow averaged class numbers. The most natural approach is considering the number of all isomorphism classes of lattices with a determinant smaller than or equal to a given bound.

In this work we will determine the asymptotic growth of class numbers for two- and three-dimensional lattices with a prescribed automorphism group. Hopefully, this may be a step towards more general results.

The first chapter gives basic definitions. In particular, we describe the kind of classes of lattices that we want to count. A connection to classes of integral (Gram) matrices which are easier to enumerate than the lattices themselves is established.

In the second chapter we introduce several reduction theories and investigate their interrelations. If a reduction theory ensures that exactly one lattice of each isomorphism class is reduced, it can be used as a tool for the determination of class numbers.

The third chapter contains the first principal part of this thesis: We develop formulas for the asymptotic growth of class numbers of two-dimensional integral lattices. We count reduced Gram matrices of determinant smaller than or equal to D that belong to a given geometric type (rectangular, centered rectangular, rhombic, square, hexagonal). The biggest order of magnitude with which the class number of one of these types grows is given by $c \cdot D \log D$ where c is some constant. A theorem by Minkowski (Theorem 3.5) tells us that the number of all two-dimensional lattices grows approximately like $\frac{\pi}{9} D^{\frac{3}{2}}$. Thus, this must also be the first order term of growth of classes of oblique lattices. Nevertheless, we spend the rest of the chapter with calculating their asymptotic growth in more detail, since we can use these results for the calculations in the three-dimensional case.

In the second main part (chapter four) we consider three-dimensional lattices. In the cases of higher symmetry, that is for automorphism groups of order 48 down to 8, we use a theorem of Delange (Theorem 4.2) to determine the terms of highest order of the asymptotic growth. The application of this method becomes increasingly difficult for smaller automorphism groups. We switch back to a more direct approach (counting Gram matrices) and calculate class numbers of complete Bravais classes instead of single geometric types to avoid these difficulties. For triclinic lattices (that is lattices with trivial automorphism groups) the Theorem of Minkowski yields $\frac{\zeta(3)}{24} D^2$ as growth rate. Larger automorphism groups lead to a smaller growth rate (the biggest being $c \cdot D^{\frac{3}{2}}$ in the monoclinic case of groups of order four). For details see Table 4 which compiles the first order term of the asymptotic growth of each of the 14 Bravais types.

In the appendix we collect some information about the three-dimensional geometric types for which the automorphism group has at least eight elements: graphical sketch of the lattice, shortest vectors, reduced basis, Gram matrix, generators of the automorphism group.

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1 Class Numbers of Lattices with Symmetries

Let K be an algebraic number field and V be a vector space over K with $\dim V = n$.

Definition 1.1. An **integral lattice with bilinear form** is a pair (L, b) of a symmetric bilinear form $b : V \times V \to K$ and a finitely generated submodule L of $(V, +)$ whose elements $v, w \in L$ fulfil the integrity condition $b(v, w) \in \mathcal{O}_K$.

Abbreviatorily, we speak of **(integral) lattices**, if it is clear from the context which bilinear form should be used.

Below, we only consider lattices of full rank, equivalently, L generates the vector space V.

Definition 1.2. A **lattice basis** consists of n vectors $v_1, \ldots, v_n \in V$ that generate the lattice (L, b), that is, $L = \mathcal{O}_K v_1 + \ldots + \mathcal{O}_K v_n$.

Given such a lattice basis we define the **Gram matrix** of the lattice with respect to this basis by

$$B := (b(v_i, v_j))_{i,j=1}^n \in {\mathcal{O}_K}^{n \times n}.$$

Remarks 1.3. If $\mathcal{O}_K$ is a principal ideal domain, then the class number of K is 1, and thus every lattice has a basis.

For $v, w \in V$ with coordinate vectors $x, y \in K^n$ with respect to the lattice basis, we have $b(v, w) = x^t B y$.

Definition 1.4. The square class $[\det B]$ of $\det B$ in $\mathcal{O}_K / (\mathcal{O}_K^\times)^2$ is called **determinant** $\det(L, b)$ of the lattice.

Remark 1.5. If we fix the lattice, different lattice bases yield Gram matrices, whose determinants belong to the same square class. To be specific, let S be the matrix of the base change from $(v_1, \ldots, v_n)$ to another basis $(w_1, \ldots, w_n)$ and let B' be the corresponding Gram matrix, then

$$x^t B y = b(v, w) = (Sx)^t B' (Sy) = x^t (S^t B' S) y.$$

Therefore, $B = S^t B' S$, which implies $\det B = \det B' \cdot (\det S)^2$. Since S describes the base change between *lattice* bases, its entries are elements of $\mathcal{O}_K$. So $\det S \in \mathcal{O}_K^\times$ which implies $[\det B] = [\det B']$, and hence the determinant of a lattice is well-defined.

Definition 1.6. For $i = 1, 2$ let (L_i, b_i) be lattices with bilinear forms. A linear transformation $F \in \mathrm{GL}(V)$ is called an **isometry** from (L_1, b_1) to (L_2, b_2) if $F(L_1) = L_2$ and $b_1 = b_2 \circ (F, F)$, that is, F is a bijective mapping between the lattices with an orthogonality property. Here, $(F, F) : V \times V \to V \times V$ for $v, w \in V$ is defined by $(F, F)(v, w) := (F(v), F(w))$.

The lattices are **isometric**, in short $(L_1, b_1) \cong (L_2, b_2)$, if there is an isometry from L_1 to L_2 (which is the case, if and only if there is a reverse isometry).

By $[L, b]$ we denote the equivalence class of (L, b) with respect to this relation.

The group $\mathcal{O}(L, b) := \{F \in \mathrm{GL}(V) \mid F(L) = L,\ b = b \circ (F, F)\}$ is called **orthogonal group** or **isometry group** of the lattice.

Lemma 1.7. *Isometric lattices possess the same determinant.*

Proof. Let F be given as in Definition 1.6 and let $(v_1, \ldots, v_n)$ be a lattice basis of L_1. We claim that $(F(v_1), \ldots, F(v_n))$ is a lattice basis of L_2. For $x \in L_2$ we have $F^{-1}(x) \in L_1$, hence there are $\alpha_i \in \mathcal{O}_K,\ i = 1, \ldots, n$ with $F^{-1}(x) = \sum_{i=1}^{n} \alpha_i v_i$. The transformation F is linear, therefore $x = \sum_{i=1}^{n} \alpha_i F(v_i)$. Thus, the vectors $F(v_i),\ i = 1, \ldots, n$ generate L_2.

Applying the orthogonality property of F on the basis vectors of L_1 yields equality of the Gram matrices of L_1 and L_2 with respect to the given bases. In particular, the determinants are equal. □

Lemma 1.8. *If two lattices $(L_i, b_i),\ i = 1, 2$ are isometric, then their orthogonal groups are conjugate in* $\mathrm{GL}(V)$*. To be specific:*

$$F : (L_1, b_1) \xrightarrow{\cong} (L_2, b_2) \text{ is an isometry. } \Rightarrow F\mathcal{O}(L_1, b_1)F^{-1} = \mathcal{O}(L_2, b_2).$$

Proof. Let F be an isometry from L_1 to L_2 and let $g \in \mathcal{O}(L_1, b_1)$. From the definitions it follows that

$$F(L_1) = L_2, \quad b_1 = b_2 \circ (F, F)$$

and

$$g(L_1) = L_1, \quad b_1 = b_1 \circ (g, g).$$

Since F is bijective, the first part implies $b_2 = b_1 \circ (F^{-1}, F^{-1})$. The composition FgF^{-1} is an element of the isometry group of the second lattice because of

$$FgF^{-1}(L_2) = Fg(L_1) = F(L_1) = L_2$$

and

$$b_2 \circ (FgF^{-1}, FgF^{-1}) = \underbrace{\underbrace{b_2 \circ (F, F)}_{=b_1} \circ (g, g)}_{=b_1} \circ (F^{-1}, F^{-1}) = b_2.$$

It follows that $F\mathcal{O}(L_1, b_1)F^{-1} \subset \mathcal{O}(L_2, b_2)$. By symmetry reasons (use F^{-1} instead of F as an isometry between the lattices) we even have equality. □

We want to classify the lattices in V according to their symmetries.

Definition 1.9. We define two different class numbers for groups. Recall that V is a vector space over an algebraic number field K with ring of integers $\mathcal{O}_K$.

1. Let G be a subgroup of $\mathrm{GL}(V)$ and let $d \in \mathcal{O}_K$. We set

   $$\widetilde{M_G}(d) := \{[L, b] \mid (L, b) \text{ is a lattice in } V \text{ with } \det(L, b) = [d] \text{ and } G = \mathcal{O}(L, b)\}.$$

   The cardinality $\widetilde{h}_G(d) := |\widetilde{M_G}(d)|$ of this set of isometry classes of lattices is called **class number of G with respect to d.**

2. Let $L \subseteq V$ be a finitely generated, free $\mathcal{O}_K$-module and let G be a subgroup of $\mathrm{GL}(L) := \{F \in \mathrm{GL}(V) \mid F(L) = L\}$. We define

   $$M_G^L(d) := \{[L, b] \mid (L, b) \text{ is a lattice in } V \text{ with } \det(L, b) = [d] \text{ and } G = \mathcal{O}(L, b)\}.$$

   The **Bravais class number of G with respect to d** is the cardinality $h_G^L(d) := |M_G^L(d)|$ of this second set. It depends on a given L.

We finally define the **aggregated (Bravais) class numbers of the group** G as functions depending on an upper bound x for the determinant:

$$\widetilde{H}_G(x) := \sum_{1\leq d\leq x} \widetilde{h}_G(d) \text{ and } H_G^L(x) := \sum_{1\leq d\leq x} h_G^L(d).$$

Remark 1.10. For L fixed, we have $M_G^L(d) \subseteq \widetilde{M}_G(d)$ and hence $h_G^L(d) \leq \widetilde{h}_G(d)$.

Lemma 1.11. *Let V be a K-space and let $L, L_1, L_2 \subseteq V$ be finitely generated free modules over $\mathcal{O}_K$.*

a) If G_1 and G_2 are conjugate subgroups of $\mathrm{GL}(V)$, then their class numbers coincide for every $d \in \mathcal{O}_K$, that is, $\widetilde{h}_{G_1}(d) = \widetilde{h}_{G_2}(d)$.

b) If there is $S \in \mathrm{GL}(V)$ with $G_2 = SG_1S^{-1}$ and $L_2 = SL_1$, then $h_{G_1}^{L_1}(d) = h_{G_2}^{L_2}(d)$ for every $d \in \mathcal{O}_K$.

c) If G_1 and G_2 are conjugate subgroups of $\mathrm{GL}(L)$, then their Bravais class numbers coincide for every $d \in \mathcal{O}_K$, that is, $h_{G_1}^L(d) = h_{G_2}^L(d)$.

d) If G is a subgroup of $\mathrm{GL}(V)$ and there is $S \in \mathrm{GL}(V)$ with $G = SGS^{-1}$ and $L_2 = SL_1$, then $h_G^{L_1}(d) = h_G^{L_2}(d)$.

Proof. a) We prove the equality of the class numbers by showing that the underlying sets $\widetilde{M}_i := \widetilde{M}_{G_i}(d),\ i = 1, 2$ are identical.

Since G_1 and G_2 are conjugate, there is $S \in \mathrm{GL}(V)$ with $G_2 = SG_1S^{-1}$. Let $A \in \widetilde{M}_1$ and $(L, b) \in A$ with $G_1 = \mathcal{O}(L, b)$. Apparently, the mapping

$$S : (L, b) \longrightarrow (\underbrace{SL}_{L':=}, \underbrace{b \circ (S^{-1}, S^{-1})}_{b':=})$$

is an isometry. In particular, we have $A = [L, b] = [L', b']$. Moreover, by Lemma 1.8 we have $S\mathcal{O}(L, b)S^{-1} = \mathcal{O}(L', b')$ and thus

$$G_2 = SG_1S^{-1} = S\mathcal{O}(L, b)S^{-1} = \mathcal{O}(L', b').$$

The determinants of both lattices L and L' are equal to $[d]$. (For L', this follows from Lemma 1.7.) Therefore, A is also an element of $\widetilde{M}_2$. As in the proof of Lemma 1.8 the converse inclusion $\widetilde{M}_2 \subset \widetilde{M}_1$ follows by symmetry reasons.

b) The proof of a) can be adapted. We choose $A \in M_{G_1}^{L_1}(d)$ and $(L_1, b_1) \in A$ with $G_1 = \mathcal{O}(L_1, b_1)$. If we define $b_2 := b_1 \circ (S^{-1}, S^{-1})$, then $S : (L_1, b_1) \longrightarrow (L_2, b_2)$ is an isometry.

c) This is a special case of b).

d) This is another, even more obvious special case of b).

□

Example 1.12. Let $K := \mathbb{Q}, V := \mathbb{Q}^2, L := \mathbb{Z}^2$ and $d = 8$. Moreover, let

$$G_1 := \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \text{ and } G_2 := \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Then

$$M_{G_1}^L(d) = \left\{ [\mathbb{Z}^2, b_B] \,\middle|\, B \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \right\} \right\},$$
$$M_{G_2}^L(d) = \left\{ [\mathbb{Z}^2, b_B] \,\middle|\, B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \right\}$$

and

$$\widetilde{M}_{G_1}(d) = \widetilde{M}_{G_2}(d) = M_{G_1}^L(d) \cup M_{G_2}^L(d).$$

The first equality holds, since $SG_1S^{-1} = G_2$ for $S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. For the second equality, the definitions of the involved sets clearly imply that the right-hand side is included in the left-hand side. We use the geometric classification of two-dimensional lattices (see Table on page 12) for the inverse inclusion: Only lattices that are rectangular, centered rectangular, or rhombic possess automorphism groups of order 4. We assume B to be reduced.

- If $(\mathbb{Z}^2, b_B)$ is rectangular, then $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a < b$. For $d = 8$, this only admits the two matrices given in $M_{G_1}^L(d)$.
- If $(\mathbb{Z}^2, b_B)$ is centered rectangular, then $B = \begin{pmatrix} 2b & b \\ b & c \end{pmatrix}$ with $2b < c$. Furthermore, we have $8 = \det B = b(2c - b)$. We are looking for possible values for $b \in \{1, 2, 4, 8\}$. If $b = 1$ or $b = 8$, then $c = \frac{9}{2}$, which is not possible. In the remaining cases we get $c = 3$. But this would require $b < \frac{3}{2}$. So, there are no centered rectangular lattices with determinant 8.
- If $(\mathbb{Z}^2, b_B)$ is rhombic, then $B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $2b < a$. We have $8 = a^2 - b^2 = (a + b)(a - b)$ and $a + b > a - b > 0$. Hence, $(a + b, a - b) \in \{(8, 1), (4, 2)\}$. The first case cannot occur since it would imply $a, b \notin \mathbb{Z}$. The second case is given in $M_{G_2}^L(d)$.

Therefore, the class numbers are $h_{G_1}^L(d) = 2, h_{G_2}^L(d) = 1$ and $\widetilde{h}_{G_1}(d) = \widetilde{h}_{G_2}(d) = 3$.

If we consider two lattices (L_1, b_1) and (L_2, b_2) that are Bravais equivalent (see Definition A.1) and have the same automorphism group $G = \mathcal{O}(L_1, b_1) = \mathcal{O}(L_2, b_2)$, then we have $h_G^{L_1}(d) = h_G^{L_2}(d)$ thanks to part d) of the above Lemma. This shows that the name *Bravais class number of* G for the quantity $h_G^L(d)$ is justified.

Lemma 1.13. *Two lattices are isometric if and only if the corresponding Gram matrices are congruent by a matrix in* $\mathrm{GL}_n(\mathcal{O}_K)$. *(In this case, we speak of* **integral congruence***.)*

Proof. If (L_1, b_1) and (L_2, b_2) are isometric, then the proof of Lemma 1.7 tells us that they have the same Gram matrix with respect to suitable bases. By Remark 1.5 Gram matrices of the same lattice with respect to different bases belong to the same congruence class in

$(\mathcal{O}_K)^{n\times n}$. The base change is described by an integral, integrally invertible matrix, which proves the first part.

For the converse, let (L_1, b_1) and (L_2, b_2) be lattices with Gram matrices B_1 and B_2 with respect to the lattice bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$, respectively. Furthermore, let $S \in \mathrm{GL}_n(\mathcal{O}_K)$ with $B_1 = S^t B_2 S$. We define $F_S(v_j) := \sum_{i=1}^n s_{ij} w_i,\ j = 1, \dots, n$. We claim, that this is an isometry from L_1 to L_2:
Since S is regular and has entries belonging to $\mathcal{O}_K$, the images $F_S(v_j)$ of the basis vectors form a lattice basis of L_2. This implies $F_S(L_1) = L_2$. The orthogonality property of F_S can be seen in the following way:

$$\begin{aligned} b_1(v_i, v_j) = (B_1)_{ij} = (S^t B_2 S)_{ij} &= \left(\sum_{k=1}^n \sum_{l=1}^n s_{ki} s_{lj} b_2(w_k, w_l) \right)_{ij} \\ &= b_2 \left(\sum_{k=1}^n s_{ki} w_k, \sum_{l=1}^n s_{lj} w_l \right) = b_2(F_S(v_i), F_S(v_j)). \end{aligned}$$

□

General Assumption. *From now on, let $K = \mathbb{Q}$ and $V = K^n = \mathbb{Q}^n$.*

Remark 1.14. Since $\mathcal{O}_K^{\times 2} = \mathbb{Z}^{\times 2} = \{1\}$, we have $[d] = d$ for all $d \in \mathbb{Z}$ and may thus omit the brackets from expressions as $[\det B]$, $[d]$, etc..

Lemma 1.15. *Let (L, b) be a lattice with bilinear form. There exists $B \in \mathbb{Z}_{\mathrm{sym}}^{n\times n}$, such that $(L, b) \cong (\mathbb{Z}^n, b_B)$ holds, where $b_B(x, y) := x^t B y$.*

Proof. Let $(v_1, \dots, v_n)$ be a lattice basis of L. We define a linear transformation $F \in \mathrm{GL}(V)$ by $F(v_i) := e_i,\ i = 1, \dots, n$. Here, $\mathcal{E} = (e_1, \dots, e_n)$ denotes the standard basis of $\mathbb{Q}^n$. We choose B to be the Gram matrix of L with respect to the basis $v_1, \dots, v_n \in V$. This way, F becomes an isometry between the two lattices. □

Proposition 1.16. *There is a bijection φ between the isometry classes of n-dimensional lattices with bilinear forms and integral congruency classes of elements of $\mathbb{Z}_{\mathrm{sym}}^{n\times n}$.*

Proof. An isometry class $[L, b]$ with $(L, b) \cong (\mathbb{Z}^n, b_B)$ is assigned to the congruence class of B. This map is well-defined and injective by Lemma 1.13. Because of the existence of $(\mathbb{Z}^n, b_B)$, it is surjective as well. □

Remark 1.17. We consider a right group action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathbb{Z}_{\mathrm{sym}}^{n\times n}$, which is induced by the integral congruence of matrices:

$$\begin{aligned} . : \mathbb{Z}_{\mathrm{sym}}^{n\times n} \times \mathrm{GL}_n(\mathbb{Z}) &\longrightarrow \mathbb{Z}_{\mathrm{sym}}^{n\times n} \\ (B, S) &\longmapsto B.S := S^t B S. \end{aligned}$$

An orbit of this group action can be described with the help of the bijection φ from Proposition 1.16 as follows: $B.\mathrm{GL}_n(\mathbb{Z}) = \varphi([\mathbb{Z}^n, b_B])$. By Proposition 1.16 the orbits correspond to the isometry classes of lattices. Furthermore, the isometry groups of lattices can also be characterised using this group action.

By choosing the canonical basis $\mathcal{E}$ we can identify $\mathrm{GL}(V) = \mathrm{GL}(\mathbb{Q}^n)$ with $\mathrm{GL}_n(\mathbb{Q})$.

Lemma 1.18. *Let $B \in \mathbb{Z}_{\mathrm{sym}}^{n\times n}$ and consider the lattice $(\mathbb{Z}^n, b_B)$. Its orthogonal group is given by the stabilizer ${}_B\mathrm{GL}_n(\mathbb{Z})$ of the matrix B.*

Proof. We have the following chain of equalities:

$$\begin{aligned}\mathcal{O}(\mathbb{Z}^n, b_B) &= \{F \in \mathrm{GL}(V) \mid F(\mathbb{Z}^n) = \mathbb{Z}^n,\ b_B = b_B \circ (F, F)\}\\ &= \{A \in \mathrm{GL}_n(\mathbb{Q}) \mid A\mathbb{Z}^n = \mathbb{Z}^n,\ B = A^tBA\}\\ &= \{A \in \mathrm{GL}_n(\mathbb{Z}) \mid B.A = B\}\\ &= {}_B\mathrm{GL}_n(\mathbb{Z}).\end{aligned}$$

Lemma 1.19. *Each two elements of an orbit of a group action possess conjugate stabilizers. More specifically, let $* : X \times G \to X$ be a right action of a group G on a set X and let $x \in X, g \in G$. Then, ${}_{x*g}G = g^{-1}{}_xGg$.*

Proof. We have

$$\begin{aligned}{}_{x*g}G &= \{h \in G \mid (x * g) * h = x * g\} = \{h \in G \mid x * (ghg^{-1}) = x\}\\ &= \{g^{-1}hg \in G \mid x * h = x\} = g^{-1}{}_xGg.\end{aligned}$$

Proposition 1.20. *Let $d \in \mathbb{Z}$ and let G be a subgroup of $\mathrm{GL}_n(\mathbb{Q})$. The class number $\widetilde{h}_G(d)$ equals the number of congruence classes $B.\mathrm{GL}_n(\mathbb{Z})$ of elements $B \in \mathbb{Z}^{n\times n}_{\mathrm{sym}}$ with determinant $\det B = d$ whose stabilizer ${}_B\mathrm{GL}_n(\mathbb{Z})$ is also a representative of the conjugacy class of G in $\mathrm{GL}_n(\mathbb{Q})$, that is, $\widetilde{h}_G(d) = |N_G(d)|$, where*

$$N_G(d) := \{B.\mathrm{GL}_n(\mathbb{Z}) \mid B \in \mathbb{Z}^{n\times n}_{\mathrm{sym}}, \det B = d, \exists S \in \mathrm{GL}_n(\mathbb{Q}) : SGS^{-1} = {}_B\mathrm{GL}_n(\mathbb{Z})\}.$$

Proof. By Definition 1.9 we have

$$\widetilde{h}_G(d) = |\{[L, b] \mid (L, b) \text{ is a lattice in } V \text{ with } \det(L, b) = d \text{ and } G = \mathcal{O}(L, b)\}|.$$

If $B.\mathrm{GL}_n(\mathbb{Z}) \in N_G(d)$, then there is $S \in \mathrm{GL}_n(\mathbb{Q})$ with $SGS^{-1} = {}_B\mathrm{GL}_n(\mathbb{Z})$. We define $G' := SGS^{-1}$. Now, Lemma 1.18 implies $G' = \mathcal{O}(\mathbb{Z}^n, b_B)$, hence $[\mathbb{Z}^n, b_B] \in M_{G'}(d)$. Since G and G' are conjugate, we know from the proof of Lemma 1.11 that $M_G(d) = M_{G'}(d)$. Thus $[\mathbb{Z}^n, b_B] \in M_G(d)$ and with the help of the bijection φ from Proposition 1.16 we can conclude $\widetilde{h}_G(d) \geq |N_G(d)|$.

Conversely, let $G = \mathcal{O}(L, b)$ for a lattice (L, b) with determinant d, that is, $[L, b] \in M_G(d)$. By Lemma 1.15 there is a matrix $B \in \mathbb{Z}^{n\times n}_{\mathrm{sym}}$ with $(L, b) \cong (\mathbb{Z}^n, b_B)$. Lemma 1.8 ensures the existence of $S \in \mathrm{GL}_n(\mathbb{Q})$ with $\mathcal{O}(L, b) = S^{-1}\mathcal{O}(\mathbb{Z}^n, b_B)S$. By Lemma 1.18 this implies

$$SGS^{-1} = \mathcal{O}(\mathbb{Z}^n, b_B) = {}_B\mathrm{GL}_n(\mathbb{Z}).$$

Now we can see that $B.\mathrm{GL}_n(\mathbb{Z}) \in N_G(d)$ and hence $\widetilde{h}_G(d) \leq |N_G(d)|$. □

2 Reduction Theories

Typically, reduction theories are established for quadratic forms and not for lattices. This is why we first introduce some notions concerning quadratic forms. Later on, thanks to the correspondence between lattices and quadratic forms, most of the results can easily be translated.

Definition 2.1. Let $n \in \mathbb{N}$. A **quadratic form** f is a homogeneous polynomial of degree two in n indeterminates with coefficients in a commutative ring R, that is,

$$f = \sum_{1 \leq i \leq j \leq n} f_{ij} X_i X_j, \ \ f_{ij} \in R.$$

We assume $\operatorname{char} R \neq 2$. For $n = 2, 3, 4, \ldots$ the form f is referred to as being **binary, ternary, quaternary** etc..

Definition 2.2. For a given form f we define its **Gram matrix** $G_f := (a_{ij})_{ij} \in R^{n \times n}$ through $a_i := a_{ii} := f_{ii}$ and $a_{ij} := \frac{1}{2} f_{ij}$ for $i \neq j$.

For the following let $R = \mathbb{Q}$. The form f is called **positive definite**, if its Gram matrix is positive definite.

Two forms f and g are called **equivalent**, if their Gram matrices are integrally congruent.

Remark 2.3. According to Lemma 1.13 the notion of equivalence of quadratic forms corresponds to the notion of isometry of lattices with bilinear forms.

Definition 2.4. A quadratic form f with Gram matrix $G_f = (a_{ij})_{ij}$ is called **Minkowski semi-reduced**, if, for all $k = 1, \ldots, n$, we have

$$a_{kk} \leq f(s) \text{ for all } s \in \mathbb{Z}^n \text{ with } \operatorname{ggT}(s_k, s_{k+1}, \ldots, s_n) = 1.$$

A Minkowski semi-reduced form is called **Minkowski reduced**, if $a_{k,k+1} \geq 0$ holds for all $k = 1, \ldots, n-1$.

It is well known and easy to see that a binary quadratic form is Minkowski reduced if and only if $0 \leq 2a_{12} \leq a_{11} \leq a_{22}$ holds.

Definition 2.5. From now on, let f be a ternary form with Gram matrix

$$G_f = \begin{pmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{pmatrix}; \text{ in Gauß' notation this reads } f = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_{23} & a_{13} & a_{12} \end{pmatrix}.$$

The **adjoint** F of f is given by

$$F = \begin{pmatrix} a_{23}^2 - a_2 a_3 & a_{13}^2 - a_1 a_3 & a_{12}^2 - a_1 a_2 \\ a_1 a_{23} - a_{12} a_{13} & a_2 a_{13} - a_{12} a_{23} & a_3 a_{12} - a_{13} a_{23} \end{pmatrix}.$$

Let

$$G_F = \begin{pmatrix} \bar{a}_1 & \bar{a}_{12} & \bar{a}_{13} \\ \bar{a}_{12} & \bar{a}_2 & \bar{a}_{23} \\ \bar{a}_{13} & \bar{a}_{23} & \bar{a}_3 \end{pmatrix} \text{ be the Gram matrix of } F.$$

1. A positive definite form f is called **Brandt reduced**, if it is Minkowski semi-reduced and fulfils the following conditions:

- If we arrange the set $\{|a_{23}|, |a_{13}|, |a_{12}|\}$ and form an ascending triple (m_1, m_2, m_3) of its elements, this triple is lexicographically minimal among all triples of equivalent Minkowski semi-reduced forms.
- If there are equivalent Minkowski semi-reduced forms with the same triple, then the given one is lexicographically closest to the canonical arrangement $(|a_{23}|, |a_{13}|, |a_{12}|)$.
- The inequalities $a_{12} \geq 0$ and $a_{13} \geq 0$ hold.

2. A positive definite form f is called **Eisenstein reduced**, if it meets the following conditions:

(1) $$(a_{12}, a_{13}, a_{23} > 0) \text{ or } (a_{12}, a_{13}, a_{23} \leq 0),$$

(2) $$2|a_{12}| \leq a_1, \quad 2|a_{13}| \leq a_1, \quad 2|a_{23}| \leq a_2,$$

(3) $$a_1 \leq a_2 \leq a_3, \quad -a_{12} - a_{13} - a_{23} \leq \frac{1}{2}(a_1 + a_2),$$

(4) $$\begin{array}{lcl} (a_1 = a_2) & \Rightarrow & |a_{23}| \leq |a_{13}|, \\ (a_2 = a_3) & \Rightarrow & |a_{13}| \leq |a_{12}|, \end{array}$$

(5) $$\left(-a_{12} - a_{13} - a_{23} = \frac{1}{2}(a_1 + a_2)\right) \Rightarrow a_1 + 2a_{13} + a_{12} \leq 0,$$

(6) $$\begin{array}{lcl} (a_1 = 2a_{12}) & \Rightarrow & a_{13} \leq 2a_{23}, \\ (a_1 = 2a_{13}) & \Rightarrow & a_{12} \leq 2a_{23}, \\ (a_2 = 2a_{23}) & \Rightarrow & a_{12} \leq 2a_{13}, \end{array}$$

(7) $$\begin{array}{lcl} (a_1 = -2a_{12}) & \Rightarrow & a_{13} = 0, \\ (a_1 = -2a_{13}) & \Rightarrow & a_{12} = 0, \\ (a_2 = -2a_{23}) & \Rightarrow & a_{12} = 0. \end{array}$$

3. A positive definite form f is called **Seeber reduced**, if (1)–(4) and the following additional conditions hold:

(8) $$\left(-a_{12} - a_{13} - a_{23} = \frac{1}{2}(a_1 + a_2)\right) \Rightarrow 2|\bar{a}_{23}| \leq \bar{a}_3,$$

(9) $$\begin{array}{lcl} (a_1 = \pm 2a_{12}) & \Rightarrow & 2|\bar{a}_{12}| \leq |\bar{a}_2|, \\ (a_1 = \pm 2a_{13}) & \Rightarrow & 2|\bar{a}_{13}| \leq |\bar{a}_3|, \\ (a_2 = \pm 2a_{23}) & \Rightarrow & 2|\bar{a}_{23}| \leq |\bar{a}_3|. \end{array}$$

4. A positive definite form f is called **Schiemann reduced**, if it is Minkowski semi-reduced, fulfils condition (6) and the following conditions as well:

(10) $$a_{12} \geq 0, a_{13} \geq 0 \text{ and } (a_{12} = 0 \vee a_{13} = 0) \Rightarrow a_{23} \geq 0,$$

(11) $$\begin{array}{lcl} (a_1 = a_2) & \Rightarrow & |a_{23}| \leq a_{13}, \\ (a_2 = a_3) & \Rightarrow & a_{13} \leq a_{12}, \end{array}$$

$$(12) \qquad \left(a_{12} + a_{13} - a_{23} = \frac{1}{2}(a_1 + a_2)\right) \Rightarrow a_1 - 2a_{13} - a_{12} \leq 0,$$

$$(13) \qquad 2a_{23} > -a_2.$$

Lemma 2.6. *Let f be a positive definite ternary quadratic form. It is Minkowski semi-reduced if and only if one of the following sets of conditions is fulfilled by f:*

(A) $0 < a_1 \leq a_2 \leq a_3$ *and* $|a_{12}| \leq \frac{1}{2}a_1,\ |a_{13}| \leq \frac{1}{2}a_1,\ |a_{23}| \leq \frac{1}{2}a_2$ *as well as*

$$\begin{aligned} -a_{12} - a_{13} - a_{23} &\leq \frac{1}{2}(a_1 + a_2), \\ -a_{12} + a_{13} + a_{23} &\leq \frac{1}{2}(a_1 + a_2), \\ a_{12} - a_{13} + a_{23} &\leq \frac{1}{2}(a_1 + a_2), \\ a_{12} + a_{13} - a_{23} &\leq \frac{1}{2}(a_1 + a_2). \end{aligned}$$

(B) $0 < a_1 \leq a_2 \leq a_3$ *and* $|a_{12}| \leq \frac{1}{2}a_1,\ |a_{13}| \leq \frac{1}{2}a_1,\ |a_{23}| \leq \frac{1}{2}a_2$ *as well as*

$$a_{12}a_{13}a_{23} < 0 \Rightarrow |a_{12}| + |a_{13}| + |a_{23}| \leq \frac{1}{2}(a_1 + a_2).$$

Proof. Since f is positive definite by assumption, we have $0 < a_1$. For the fact that the remaining inequalities of **(A)** hold if and only if f is Minkowski semi-reduced, see [vdW56], page 290. For this result a proposition of Minkowski is essential, claiming that for $n \leq 4$ in the situation of Definition 2.4 we can assume $s_j \in \{0, \pm 1\}$.

It remains to show, that the conditions of **(A)** and **(B)** are equivalent. First, we assume **(A)** to be true and let $a_{12}a_{13}a_{23} < 0$. The number of negative elements in $\{a_{12}, a_{13}, a_{23}\}$ is either one or three. So $|a_{12}| + |a_{13}| + |a_{23}|$ is just the left-hand side of one of the last four conditions of **(A)**. Now, let **(B)** be true. If we have $a_{12}a_{13}a_{23} < 0$ in addition, then

$$\pm a_{12} \pm a_{13} \pm a_{23} \leq |a_{12}| + |a_{13}| + |a_{23}| \leq \frac{1}{2}(a_1 + a_2).$$

For $a_{12}a_{13}a_{23} = 0$, at least one of the factors of the left-hand side equals zero. Then each of the four inequalities of **(A)** that we have to show is a special case of

$$(14) \qquad \pm a_{ij} \pm a_{kl} \leq |a_{ij}| + |a_{kl}| \leq \frac{1}{2}a_i + \frac{1}{2}a_k \leq \frac{1}{2}(a_1 + a_2).$$

But this is true for

$$(i, j), (k, l) \in \{(1, 2), (1, 3), (2, 3)\},\ (i, j) \neq (k, l)$$

since

$$|a_{ij}| \leq \frac{1}{2}a_i \text{ and } a_1 \leq a_2$$

hold. Finally, if $a_{12}a_{13}a_{23} > 0$, then one or three of the factors are positive. Hence, at least one summand of each left-hand side of **(A)** is negative. Therefore, (14) implies each of the four inequalities that we have to prove.

□

Lemma 2.7. *A positive definite form f is Eisenstein reduced if and only if it is Seeber reduced. In particular, in this case it is Minkowski semi-reduced.*

Proof. The first claim is stated by Eisenstein on page 143 of [Eis51]; see also [Dic23], chapter IX, page 210. I could not find a detailed proof in the literature, however.

Let f be Eisenstein reduced. We wish to show that f is Minkowski semi-reduced. Since f is positive definite, we have $a_1 > 0$. Condition (2) and the first part of (3) complete the first line of **(B)** in Lemma 2.6. Suppose $a_{12}a_{13}a_{23} < 0$ then (1) implies $a_{12}, a_{13}, a_{23} < 0$. It follows that

$$|a_{12}| + |a_{13}| + |a_{23}| = -a_{12} - a_{13} - a_{23} \leq \frac{1}{2}(a_1 + a_2)$$

thanks to the second part of (3). Thus, **(B)** is true.
We have seen that a ternary positive definite quadratic form is already Minkowski semi-reduced if it fulfils conditions (1)–(3).

□

Lemma 2.8. *A positive definite form f is Schiemann reduced if and only if it fulfils the following conditions.*
Boundary conditions (decide which forms on the boundary of the cone of Schiemann reduced forms are included):

$$\begin{aligned} (a_1 = 2a_{12}) &\Rightarrow a_{13} \leq 2a_{23}, \\ (a_1 = 2a_{13}) &\Rightarrow a_{12} \leq 2a_{23}, \\ (a_2 = 2a_{23}) &\Rightarrow a_{12} \leq 2a_{13}, \end{aligned} \tag{6}$$

$$\begin{aligned} (a_1 = a_2) &\Rightarrow |a_{23}| \leq a_{13}, \\ (a_2 = a_3) &\Rightarrow a_{13} \leq a_{12}, \end{aligned} \tag{11}$$

$$\left(a_{12} + a_{13} - a_{23} = \frac{1}{2}(a_1 + a_2)\right) \Rightarrow a_1 - 2a_{13} - a_{12} \leq 0, \tag{12}$$

$$(a_{12} = 0 \vee a_{13} = 0) \Rightarrow a_{23} \geq 0 \tag{15}$$

Essential inequalities:

$$0 < a_1 \leq a_2 \leq a_3, \tag{16}$$

$$\begin{aligned} 0 \leq 2a_{12} &\leq a_1, \\ 0 \leq 2a_{13} &\leq a_1, \\ -a_2 < 2a_{23} &\leq a_2, \end{aligned} \tag{17}$$

$$a_{12} + a_{13} - a_{23} \leq \frac{1}{2}(a_1 + a_2). \tag{18}$$

Proof. Let f be a positive definite quadratic form that complies with the boundary conditions. We have to show that f fulfils conditions (16)–(18) if and only if f is Minkowski semi-reduced and $a_{12}, a_{13} \geq 0,\ 2a_{23} > -a_2$.

If the latter is true, (16) and (18) follow as a part of **(A)** by Lemma 2.6. Each inequality of condition (17) is either contained in **(A)** as well or already assumed to be true.

Now we suppose that (16)–(18) are fulfilled. The inequalities $a_{12}, a_{13} \geq 0,\ 2a_{23} > -a_2$ are given as part of (17). We still have to prove that f is Minkowski semi-reduced. We want to accomplish this by showing **(B)**. The first line of **(B)** follows from (16) and (17). Suppose $a_{12}a_{13}a_{23} < 0$, then $a_{23} < 0$ since a_{12} and a_{13} are non-negative. With the help of (18) we get

$$|a_{12}| + |a_{13}| + |a_{23}| = a_{12} + a_{13} - a_{23} \leq \frac{1}{2}(a_1 + a_2).$$

Hence f fulfils **(B)** and is therefore Minkowski semi-reduced. □

For the sake of brevity of language we extend the notion of reducedness to matrices and lattice bases.

Definition 2.9. A matrix is called **reduced** if it arises as a Gram matrix of a reduced quadratic form. A lattice basis is called **reduced** if the corresponding Gram matrix is reduced.

3 Asymptotics of Planar Lattices

We begin our asymptotic examination with two-dimensional lattices (L, b_B) over $\mathbb{Z}$. Let the lattice be standard, that is, $L = \mathbb{Z}^2$. Furthermore, let its Gram matrix be given by

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ with } a, b, c \in \mathbb{Z} \ .$$

We assume that B is Minkowski reduced; see Definition 2.4. It is known and easy to see that every lattice admits a unique reduced Gram matrix. These matrices can be classified according to the following table:

Case	**Name**	**Index**	**Conditions**
I	oblique	0	$0 < 2b < a < c$
II	rectangular	re	$0 = 2b < a < c$
III	centered rectangular	cr	$0 < 2b = a < c$
III'	rhombic	rh	$0 < 2b < a = c$
IV	square	sq	$0 = 2b < a = c$
V	hexagonal	hex	$0 < 2b = a = c$

Definition 3.1. We will refer to the six different cases of the above table as the **geometric types** of two-dimensional lattices. The number of isometry classes of lattices of a given geometric type with determinant D is denoted by $h_i(D)$, where $i \in \{0, \mathrm{re}, \mathrm{cr}, \mathrm{rh}, \mathrm{sq}, \mathrm{hex}\}$ is the corresponding index. The sum of all class numbers $h_i(d)$ of lattices with determinant $d \leq D$ is denoted by $H_i(D)$.

Remark 3.2. Four of the six geometric types coincide with a Bravais class, that is, their class number equals the Bravais class number $h_G^L(D)$ from Definition 1.9 for $L = \mathbb{Z}^2$ and a certain group G. The centered rectangular type and the rhombic type belong to the same Bravais class. More precisely, we have

$$h_i(D) = h^{\mathbb{Z}^2}_{\mathcal{O}\left(\mathbb{Z}^2, b_{B_i}\right)}(D), \ \ i \in \{0, \mathrm{re}, \mathrm{sq}, \mathrm{hex}\}$$

with

$$B_0 := \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \ B_{\mathrm{re}} := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \ B_{\mathrm{sq}} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B_{\mathrm{hex}} := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We define $G_i := \mathcal{O}\left(\mathbb{Z}^2, b_{B_i}\right)$ We list the corresponding groups

$$\begin{aligned}
G_0 &= \{\pm \mathrm{id}\}, \\
G_{\mathrm{re}} &= \left\{\pm \mathrm{id}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \\
G_{\mathrm{sq}} &= \left\{\pm \mathrm{id}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \\
G_{\mathrm{hex}} &= \left\{\pm \mathrm{id}, \pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\right\},
\end{aligned}$$

and note that no two of them are conjugated since they all have different orders. If we set

$$B_{\text{cr}} := \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } B_{\text{rh}} := \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

however, then

$$G_{\text{cr}} = \left\{ \pm \text{id}, \ \pm \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\} \text{ and } G_{\text{rh}} = \left\{ \pm \text{id}, \ \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

are two groups that are conjugated in $\mathrm{GL}(2,\mathbb{Z})$ by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ (but not integrally conjugated to G_{re}). Hence, according to part c) of Lemma 1.11, we have

$$h^{\mathbb{Z}^2}_{\mathcal{O}(\mathbb{Z}^2, b_{B_{\text{cr}}})}(D) = h^{\mathbb{Z}^2}_{\mathcal{O}\left(\mathbb{Z}^2, b_{B_{\text{rh}}}\right)}(D).$$

In the remainder of this chapter we shall investigate the behaviour of $H_i(D)$ for $D \to \infty$.

Definition 3.3. We fix some notation.

1. For $x \in \mathbb{R}$ let $\lfloor x \rfloor, \lceil x \rceil$ and $\{x\}$ denote the floor function, ceiling function and fractional part of x, respectively. That is,

$$\begin{aligned} \lfloor x \rfloor &:= \max\{k \in \mathbb{Z} \mid k \leq x\}, \\ \lceil x \rceil &:= \min\{k \in \mathbb{Z} \mid k \geq x\}, \\ \{x\} &:= x - \lfloor x \rfloor. \end{aligned}$$

2. Let $n \in \mathbb{N}$. We define $\sigma_\alpha(n) := \sum_{q|n} q^\alpha$, the so-called sum-of-divisors function.

3. Let $Q(n) := |\{k \in \mathbb{N} \mid k^2 \leq n\}|$ be the number of squares less than or equal to n.

The general case **I** of lattices with trivial automorphism group will be dealt with later on, applying a result of Minkowski.

II In the case of a rectangular lattice the determinant is $D_{\text{re}} = ac$. The number of (natural) divisors of a natural number n is odd if and only if n is a square. In this case the divisor $\sqrt{n}$ will not be counted, since it corresponds to a square lattice. Therefore, the class number arises as $h_{\text{re}}(D) = \lfloor \frac{1}{2}\sigma_0(D) \rfloor$, $\sigma_0(D)$ being the number of divisors of D. For the aggregated class numbers it follows that

$$\begin{aligned} H_{\text{re}}(D) &= \sum_{k=1}^{D} h_{\text{re}}(k) = \sum_{k=1}^{D} \left\lfloor \frac{1}{2}\sigma_0(k) \right\rfloor = \sum_{k=1}^{D} \left(\frac{1}{2}\sigma_0(k) - \left\{ \frac{1}{2}\sigma_0(k) \right\} \right) \\ &= \frac{1}{2} \sum_{k=1}^{D} \sigma_0(k) - \sum_{k=1}^{D} \left\{ \frac{1}{2}\sigma_0(k) \right\}. \end{aligned}$$

The asymptotic behaviour of the sum-of-divisors function is well known (see [Apo76], Theorem 3.3, for example) and the absolute values of the summands of the second sum are smaller than 1, which gives us

$$H_{\text{re}}(D) = \frac{1}{2}\left(D \log D + (2C-1)D + \mathcal{O}(\sqrt{D}) \right) + \mathcal{O}(D) = \frac{D}{2} \log D + \mathcal{O}(D),$$

where C is the Euler-Mascheroni constant.

So, for large bounds D, the aggregated class number $H_{\mathrm{re}}(D)$ can be approximated by $\frac{D}{2}\log D$.

We notice that the estimate of the fractional parts is very rough. We obtain an error term of smaller order by calculating the sum of the class numbers avoiding the floor function:

$$H_{\mathrm{re}}(D) = \frac{1}{2}\left(\sum_{k=1}^{D}\sigma_0(k) - Q(D)\right).$$

For a given $n \in \mathbb{N}$ let $k := Q(n)$. Then the definition of Q yields $k^2 \leq n < (k+1)^2$, and taking the square root we get

$$Q(n) = k \leq \sqrt{n} < k+1 = Q(n)+1.$$

Thus, $Q(D) = \sqrt{D} + \mathcal{O}(1)$, and finally

$$\begin{aligned} H_{\mathrm{re}}(D) &= \frac{1}{2}\left(\sum_{k=1}^{D}\sigma_0(k) - \left(\sqrt{D} + \mathcal{O}(1)\right)\right) \\ &= \frac{1}{2}\left(D\log D + (2C-1)D + \mathcal{O}\left(\sqrt{D}\right)\right) - \frac{1}{2}\sqrt{D} + \mathcal{O}(1) \\ &= \frac{D}{2}\log D + \left(C - \frac{1}{2}\right)D + \mathcal{O}\left(\sqrt{D}\right). \end{aligned}$$

III If the lattice is centered rectangular, its Gram matrix is specified by b and c, resulting in the determinant $D_{\mathrm{cr}} = 2bc - b^2 = b(2c-b)$. We define $m_1 := b$ and $m_2 := 2c - b$, which makes (m_1, m_2) a pair of divisors of D_{cr}, i.e. $D_{\mathrm{cr}} = m_1 m_2$. This pair fulfils the condition $3m_1 < m_2$, since

$$2b < c \;\Leftrightarrow\; b < \frac{1}{2}c \;\Leftrightarrow\; m_1 < \frac{1}{4}(m_1+m_2) \;\Leftrightarrow\; \frac{3}{4}m_1 < \frac{1}{4}m_2.$$

Moreover, the congruence $m_1 \equiv m_2 \mod 2$ holds because of $m_2 - m_1 = 2(c-b)$.

Conversely, each pair of divisors with these properties determines a centered rectangular lattice with determinant $m_1 m_2$ via the definitions $b := m_1$ and $c := \frac{1}{2}(m_1+m_2)$ as well as $a := 2b$. Thus, the class number $h_{\mathrm{cr}}(D)$ equals the number of such pairs. We calculate the sum $H_{\mathrm{cr}}(D)$ by adaption of the proof of the asymptotical formula for the sum-of-divisors function. We note the following auxiliary statements.

Lemma 3.4. *Let $x \geq 1$ be a real number. Then*

(a) $$\sum_{n\leq x}\frac{1}{n} = \log x + C + \mathcal{O}\left(\frac{1}{x}\right),$$

(b) $$\sum_{n\leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + \mathcal{O}(x^\alpha), \text{ if } \alpha \geq 0.$$

Proof. See [Apo76], Theorem 3.2. □

Remark. In case of $x \in \mathbb{N}$ the order of magnitude of the error term in both formulas corresponds to the last summand on the left-hand side, respectively. Hence, the summation condition $n \leq x$ can be replaced by $n < x$, if required.

We calculate $H_{\mathrm{cr}}(D)$ by counting the possible values of m_2 for each admissible m_1. The given restrictions $3m_1 < m_2$ and $D \geq D_{\mathrm{cr}} = m_1 m_2$ imply an upper bound $m_1 < \sqrt{\frac{D}{3}}$. Having preassigned m_1, it follows from the determinant condition $D_{\mathrm{cr}} \leq D$ that $1 \leq m_2 \leq \lfloor \frac{D}{m_1} \rfloor$. By $3m_1 < m_2$ this sharpens to $3m_1 < m_2 \leq \lfloor \frac{D}{m_1} \rfloor$. Of these $\lfloor \frac{D}{m_1} \rfloor - 3m_1$ possibilities for m_2 every second one has the same parity as m_1, which gives us

$$H_{\mathrm{cr}}(D) = \sum_{m_1 < \sqrt{\frac{D}{3}}} \left(\frac{1}{2} \left(\left\lfloor \frac{D}{m_1} \right\rfloor - 3m_1 \right) + \mathcal{O}(1) \right)$$

as an approximation for the desired class number. With the help of Lemma 3.4 we can simplify this expression in the following way:

$$\begin{aligned}
H_{\mathrm{cr}}(D) &= \sum_{n < \sqrt{\frac{D}{3}}} \left(\frac{1}{2} \left(\frac{D}{n} - 3n + \mathcal{O}(1) \right) + \mathcal{O}(1) \right) \\
&= \frac{D}{2} \sum_{n < \sqrt{\frac{D}{3}}} \frac{1}{n} - \frac{3}{2} \sum_{n < \sqrt{\frac{D}{3}}} n + \mathcal{O}\left(\sqrt{\frac{D}{3}} \right) \\
&= \frac{D}{2} \left(\log \sqrt{\frac{D}{3}} + C + \mathcal{O}\left(\sqrt{\frac{3}{D}} \right) \right) - \frac{3}{2} \left(\frac{\frac{D}{3}}{2} + \mathcal{O}\left(\sqrt{\frac{D}{3}} \right) \right) + \mathcal{O}\left(\sqrt{\frac{D}{3}} \right) \\
&= \frac{D}{4}(\log D - \log 3) + \frac{C}{2} D - \frac{1}{4} D + \mathcal{O}\left(\sqrt{D} \right) \\
&= \frac{D}{4} \log D + \frac{2C - \log 3 - 1}{4} D + \mathcal{O}\left(\sqrt{D} \right).
\end{aligned}$$

III' Let a and b be the given entries of the Gram matrix. We set $m_1 := a - b$ and $m_2 := a + b$. Because of $D_{\mathrm{rh}} = a^2 - b^2 = (a-b)(a+b) = m_1 m_2$, we have to count those pairs of divisors (m_1, m_2) of D with $m_1 < m_2$ and $m_1 \equiv m_2 \mod 2$, whose difference $2b$ is smaller than their average a, to determine $h_{\mathrm{rh}}(D)$. Again, such a pair (m_1, m_2) uniquely determines a rhombic lattice by $a := c := \frac{1}{2}(m_1 + m_2)$ and $b := \frac{1}{2}(m_2 - m_1)$. Considering

$$2b < a \Leftrightarrow m_2 - m_1 < \frac{1}{2}(m_1 + m_2) \Leftrightarrow 2m_2 < 2m_1 + m_1 + m_2 ,$$

we conclude $m_2 < 3m_1$, which in turn yields $m_2 < \sqrt{3D_{\mathrm{rh}}} \leq \sqrt{3D}$. As opposed to the centered rectangular case, the upper bound on the determinant here leads to a restriction on the greater factor m_2. This is why we first choose m_2 allowing for $1 \leq m_2 < \sqrt{3D}$. Now, m_1 has to satisfy $\frac{1}{3} m_2 < m_1 \leq \min\{m_2 - 1, \frac{D}{m_2}\}$ and the parity condition $m_1 \equiv m_2 \mod 2$. The rhombic case is more complicated than the centered rectangular case, since we get two competing upper bounds on the last chosen factor, whereas before, the supposed ordering $m_1 < m_2$ of the factors was already contained in the stricter inequality $3m_1 < m_2$, which resulted from the specific form of the lattice in question. So far, we have found

$$H_{\mathrm{rh}}(D) = \sum_{\substack{m_2\in\mathbb{N}\\ 1\le m_2<\sqrt{3D}}} \quad \sum_{\substack{m_1\in\mathbb{N}\\ \frac{1}{3}m_2<m_1\le\min\left\{m_2-1,\frac{D}{m_2}\right\}\\ m_1\equiv m_2 \mod 2}} 1\ .$$

To determine $\min\{m_2-1, \frac{D}{m_2}\}$ we consider

$$m_2 - 1 \le \frac{D}{m_2} \ \Leftrightarrow\ (m_2-1)m_2 \le D \ \Leftrightarrow\ m_2^2 - m_2 - D \le 0.$$

The parabola function $f(x) = x^2 - x - D$ has the zeros $x_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4}+D}$ with $x_1 < 0 < 1 < x_2$ for positive D. Therefore, since $m_2 \in \mathbb{N}$,

$$m_2 - 1 \le \frac{D}{m_2} \ \Leftrightarrow\ m_2 \le \frac{1}{2} + \sqrt{\frac{1}{4}+D}.$$

Thus, the first bound on m_1 applies if $m_2 \le x_2$, and the second one if $m_2 \ge x_2$. That is,

$$\min\left\{m_2-1, \frac{D}{m_2}\right\} = \begin{cases} m_2-1, & m_2\le x_2 \\ \frac{D}{m_2}, & m_2 \ge x_2 \end{cases}\ .$$

We split the exterior sum accordingly:

$$H_{\mathrm{rh}}(D) = \sum_{\substack{m_2\in\mathbb{N}\\ m_2\le\frac{1}{2}+\sqrt{\frac{1}{4}+D}}} \ \sum_{\substack{m_1\in\mathbb{N}\\ \frac{1}{3}m_2<m_1\le m_2-1\\ m_1\equiv m_2 \mod 2}} 1 \ + \sum_{\substack{m_2\in\mathbb{N}\\ \frac{1}{2}+\sqrt{\frac{1}{4}+D}<m_2<\sqrt{3D}}} \ \sum_{\substack{m_1\in\mathbb{N}\\ \frac{1}{3}m_2<m_1\le\frac{D}{m_2}\\ m_1\equiv m_2 \mod 2}} 1\ . \tag{19}$$

Since the summands of the interior sums are all equal (to 1), we can get rid of the parity condition and multiply by $\frac{1}{2}$ instead. The first interior sum can be approximated in the following way:

$$\begin{aligned}\sum_{\substack{m_1\in\mathbb{N}\\ \frac{1}{3}m_2<m_1\le m_2-1\\ m_1\equiv m_2 \mod 2}} 1 &= \frac{1}{2}\sum_{\substack{m_1\in\mathbb{N}\\ \frac{1}{3}m_2<m_1\le m_2-1}} 1 + \mathcal{O}(1) = \frac{1}{2}\left(m_2 - 1 - \left\lfloor\frac{1}{3}m_2\right\rfloor\right) + \mathcal{O}(1)\\ &= \frac{1}{2}\left(m_2 - \frac{1}{3}m_2 + \mathcal{O}(1)\right) + \mathcal{O}(1) = \frac{m_2}{3} + \mathcal{O}(1)\ .\end{aligned}$$

Similar steps yield

$$\sum_{\substack{m_1\in\mathbb{N}\\ \frac{1}{3}m_2<m_1\le\frac{D}{m_2}\\ m_1\equiv m_2 \mod 2}} 1 = \frac{1}{2}\left(\left\lfloor\frac{D}{m_2}\right\rfloor - \left\lfloor\frac{1}{3}m_2\right\rfloor\right) + \mathcal{O}(1) = \frac{D}{2m_2} - \frac{m_2}{6} + \mathcal{O}(1)$$

for the second interior sum. We include these results in (19) and get

$$\begin{aligned}
H_{\mathrm{rh}}(D) &= \sum_{\substack{m_2\in\mathbb{N}\\ m_2\le\frac12+\sqrt{\frac14+D}}} \left(\frac{m_2}{3}+\mathcal{O}(1)\right) + \sum_{\substack{m_2\in\mathbb{N}\\ \frac12+\sqrt{\frac14+D}<m_2<\sqrt{3D}}} \left(\frac{D}{2m_2}-\frac{m_2}{6}+\mathcal{O}(1)\right)\\
&= \sum_{\substack{m_2\in\mathbb{N}\\ m_2\le\frac12+\sqrt{\frac14+D}}} \frac{m_2}{3} + \sum_{\substack{m_2\in\mathbb{N}\\ \frac12+\sqrt{\frac14+D}<m_2<\sqrt{3D}}} \left(\frac{D}{2m_2}-\frac{m_2}{6}\right)+\mathcal{O}\left(\sqrt{D}\right)\\
&= \sum_{\substack{m_2\in\mathbb{N}\\ m_2\le\sqrt{D}}} \frac{m_2}{3} + \sum_{\substack{m_2\in\mathbb{N}\\ \sqrt{D}<m_2\le\sqrt{3D}}} \left(\frac{D}{2m_2}-\frac{m_2}{6}\right)+\mathcal{O}\left(\sqrt{D}\right).
\end{aligned}$$

Now, we rearrange the sums such that both of them start at 1. Then we can apply Lemma 3.4 and end up with

$$\begin{aligned}
H_{\mathrm{rh}}(D) &= \sum_{\substack{m_2\in\mathbb{N}\\ m_2\le\sqrt{D}}} \left(\frac{m_2}{2}-\frac{D}{2m_2}\right) + \sum_{\substack{m_2\in\mathbb{N}\\ m_2\le\sqrt{3D}}} \left(\frac{D}{2m_2}-\frac{m_2}{6}\right)+\mathcal{O}\left(\sqrt{D}\right)\\
&= \frac12\left(\frac{D}{2}+\mathcal{O}\left(\sqrt{D}\right)\right)-\frac{D}{2}\left(\log\sqrt{D}+C+\mathcal{O}\left(\frac{1}{\sqrt{D}}\right)\right)\\
&\quad+\frac{D}{2}\left(\log\sqrt{3D}+C+\mathcal{O}\left(\frac{1}{\sqrt{3D}}\right)\right)-\frac16\left(\frac{3D}{2}+\mathcal{O}\left(\sqrt{3D}\right)\right)+\mathcal{O}\left(\sqrt{D}\right)\\
&= \frac{D}{2}\left(\log\sqrt{3D}-\log\sqrt{D}\right)+\mathcal{O}\left(\sqrt{D}\right)=\frac{D}{4}\log 3+\mathcal{O}\left(\sqrt{D}\right)
\end{aligned}$$

for the aggregated class number.

IV Thanks to $D_{\mathrm{sq}} = a^2$, we obviously have $h_{\mathrm{sq}}(D) \in \{0,1\}$. Furthermore, $h_{\mathrm{sq}} = 1$ if and only if D is a square. The aggregated class number $H_{\mathrm{sq}}(D)$ simply equals the number of squares $Q(D)$ less than or equal to D. Therefore, asymptotically, it grows like $\sqrt{D}$, since

$$H_{\mathrm{sq}}(D) = Q(D) = \sqrt{D}+\mathcal{O}(1).$$

V Again, we look at the conditions on the Gram matrix's entries. In the hexagonal case those lead to $D_{\mathrm{hex}} = 3b^2$. Analogously to the last case, we conclude $h_{\mathrm{hex}}(D) \in \{0,1\}$, and $h_{\mathrm{hex}}(D) = 1$ if and only if $\frac{D}{3}$ is a square. In adding up, we thus count squares less than or equal to $\frac{D}{3}$. This gives rise to $H_{\mathrm{hex}}(3D) = H_{\mathrm{sq}}(D)$. The resulting order of magnitude $\sqrt{\frac{D}{3}}$ is similar to the square lattice case:

$$H_{\mathrm{hex}}(D) = Q\left(\frac{D}{3}\right) = \sqrt{\frac{D}{3}}+\mathcal{O}(1).$$

I The class number in the generic case $H_0(D)$ can now be determined as the difference of the overall class number $H(D)$ and the sum of all special class numbers (cases II-V). For the overall class number we have the following result of Minkowski.

Theorem 3.5. *The aggregated class number $H(D)$ of all isometry classes of lattices with determinant less than or equal to D is given by*

$$H(D) = v_2 D^{\frac32}+\mathcal{O}(D\log D)$$

in the two-dimensional case, whereas for $n > 2$

$$H(D) = v_n D^{\frac{n+1}{2}} + \mathcal{O}\left(D^{\frac{n+1}{2}-\frac{1}{n}}\right).$$

The constant v_n can be viewed as the volume of the domain of rational, Minkowski reduced quadratic forms with determinant less than or equal to 1. It can be explicitly computed by

$$v_n = \frac{2}{(n+1)\Gamma(\frac{1}{2})^{\frac{n^2+n-2}{2}}} \prod_{k=2}^{n} \Gamma\left(\frac{k}{2}\right)\zeta(k),$$

where Γ and ζ denote the gamma and Riemann zeta function, respectively.

Proof. See [Min05], §16. □

Corollary. *The aggregated class number of lattices in the generic case is given by*

$$H_0(D) = \frac{\pi}{9} D^{\frac{3}{2}} + \mathcal{O}(D \log D).$$

Proof. From $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$ and $\zeta(2) = \frac{\pi^2}{6}$ it follows that $v_2 = \frac{\pi}{9}$. So, the right-hand side of the asserted equation describes the overall class number $H(D)$. Furthermore, we have

$$\sum_{i \in \{\mathrm{re},\mathrm{cr},\mathrm{rh},\mathrm{sq},\mathrm{hex}\}} H_i(D) = \frac{3}{4} D \log D + \frac{6C-3}{4} D + \mathcal{O}(\sqrt{D}).$$

Hence, the order of magnitude $D \log D$ of the expressions which we have to subtract, does not exceed those of the error term in Minkowski's statement. □

The results obtained so far are contained in Table 1 at the end of the chapter.

Although we have already determined the main terms of the growth rate of the class numbers for $D \to \infty$ in each of the cases given by the classification at the beginning of the chapter, we will study once again the asymptotic behaviour of the generic class number. Using a direct approach, we shall decrease the order of magnitude of the error term. Some of the calculations will be useful for the investigation of three-dimensional lattices in a later chapter. There, we will benefit from having established two-dimensional results as accurately as possible.
For the formulation of our results in the following proposition we use a slightly modified fractional part of a real number. For $x \in \mathbb{R}$ let

$$\langle x \rangle := \begin{cases} \{x\}, & x \notin \mathbb{Z} \\ 1, & x \in \mathbb{Z} \end{cases},$$

such that for all $x \in \mathbb{R}$ the equation $\lceil x \rceil - 1 = x - \langle x \rangle$ holds.

We state two formulas for $H_0(D)$. The first one allows a smaller bound on the error term but contains a double sum of possibly greater (i.e. linear) order of magnitude, which we are so far unable to convert to an explicit expression.

Proposition 3.6. *For the aggregated class number of two-dimensional lattices with trivial automorphism group the following equations hold:*

(a)
$$H_0(D) = \frac{\pi}{9}D^{\frac{3}{2}} - \frac{3}{8}D\log D + \left(\frac{1}{8} - \frac{3C}{4} - \frac{\log 2}{2} + \frac{\log 3}{4}\right)D + \sum_{\substack{a,c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left\langle\sqrt{ac-D}\right\rangle + \mathcal{O}\left(D^{\frac{3}{4}}\right),$$

(b)
$$H_0(D) = \frac{\pi}{9}D^{\frac{3}{2}} - \frac{3}{8}D\log D + \mathcal{O}\left(D\right).$$

To deduce the second claim from the first one, it suffices to show that the remaining double sum grows at most linearly with D. More precisely, we will see that

Remark 3.7.

$$0 \le \sum_{\substack{a,c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left\langle\sqrt{ac-D}\right\rangle \le D\left(\log 2 - \frac{\log 3}{2}\right) + \mathcal{O}\left(\sqrt{D}\right).$$

This estimate is based on the inequalities $0 \le \langle x\rangle \le 1$, which hold for every $x \in \mathbb{R}$. Numerical calculations done by Peter Zeiner with the help of Wolfram Research's Mathematica suggest that on average the summand $\left\langle\sqrt{ac-D}\right\rangle$ can be approximated by $\frac{1}{2}$. The difference between the double sum and $D\left(\frac{\log 2}{2} - \frac{\log 3}{4}\right)$ seems to have smaller order of magnitude than $D^{\frac{3}{4}}$. If true, this would give us

Conjecture 3.8.

$$H_0(D) = \frac{\pi}{9}D^{\frac{3}{2}} - \frac{3}{8}D\log D + D\left(\frac{1}{8} - \frac{3C}{4}\right) + \mathcal{O}\left(D^{\frac{3}{4}}\right).$$

These observations may convince the reader that the first formula is of some value despite its rather technical shape.
For the proof of Proposition 3.6 and other calculations as well we need the following general result.

Lemma 3.9 (Euler-Mclaurin formula)**.** *Let $x, y \in \mathbb{R}$ with $0 < x < y$ and let f be a continuously differentiable function with domain $[x, y]$, then*

$$\sum_{\substack{n\in\mathbb{N}\\ x<n\le y}} f(n) = \int_x^y f(t)\mathrm{d}t + \int_x^y (t - \lfloor t\rfloor)f'(t)\mathrm{d}t + f(y)(\lfloor y\rfloor - y) - f(x)(\lfloor x\rfloor - x).$$

Proof. See Theorem 3.1 in [Apo76]. □

Proof of Proposition 3.6. We count the admissible Gram matrices by summation over their entries:

(20)
$$H_0(D) = \sum_{\substack{a,b,c\in\mathbb{N}\\ 2b<a<c\\ ac-b^2\le D}} 1\,.$$

In a first step, we determine the number of possible values for b. This enables us to carry out the summation over b which yields

$$H_0(D) = \sum_{\substack{a,c\in\mathbb{N}\\ a<c\\ ac-\frac{a^2}{4}<D}} \left(\left\lfloor \frac{a-1}{2} \right\rfloor - \max\left\{0, \left\lceil \operatorname{Re}\sqrt{ac-D} \right\rceil - 1\right\} \right).$$

Since the remaining summand vanishes for $c = \frac{D}{a} + \frac{a}{4}$ in virtue of $\left\lfloor \frac{a-1}{2} \right\rfloor - \left(\left\lceil \frac{a}{2} \right\rceil - 1\right) = 0$, we can weaken the second inequality under the summation symbol. This is helpful for the subsequent application of the Euler-Maclaurin formula. We get

$$\begin{aligned} H_0(D) &= \sum_{\substack{a,c\in\mathbb{N}\\ a<c\\ ac-\frac{a^2}{4}\leq D}} \left(\left\lfloor \frac{a-1}{2} \right\rfloor - \max\left\{0, \left\lceil \operatorname{Re}\sqrt{ac-D} \right\rceil - 1\right\} \right) \\ &= \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq \frac{D}{a}+\frac{a}{4}}} \left(\left\lfloor \frac{a-1}{2} \right\rfloor - \max\left\{0, \left\lceil \operatorname{Re}\sqrt{ac-D} \right\rceil - 1\right\} \right). \end{aligned}$$

Now, we sum separately and omit taking the floor function of $\frac{a-1}{2}$. The arising error is corrected by the second sum. The maximum can only be greater than zero if $D < ac$. So, we have

$$H_0(D) = \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq \frac{D}{a}+\frac{a}{4}}} \frac{a-1}{2} - \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq \frac{D}{a}+\frac{a}{4}\\ a\equiv_2 0}} \frac{1}{2} - \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq \frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left(\left\lceil \sqrt{ac-D} \right\rceil - 1\right).$$

Regarding the first expression, we can evaluate the sum over c, since the summands do not depend on it. We have to take the floor function of the upper bound on c, because the bound is not necessarily an integer. Afterwards, we replace that term by the difference between the exact upper bound and its fractional part:

$$\begin{aligned} \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq \frac{D}{a}+\frac{a}{4}}} \frac{a-1}{2} &= \frac{1}{2} \sum_{\substack{a\in\mathbb{N}\\ a^2<\frac{4D}{3}}} (a-1) \left(\left\lfloor \frac{D}{a} + \frac{a}{4} \right\rfloor - a \right) \\ &= \frac{1}{2} \sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} (a-1) \left(\frac{D}{a} - \frac{3a}{4} - \left\{ \frac{D}{a} + \frac{a}{4} \right\} \right) \\ &= \frac{1}{2} \sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \left(D - \frac{3}{4}a^2 + \frac{3}{4}a - \frac{D}{a} \right) - \sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \frac{a}{2} \left\{ \frac{D}{a} + \frac{a}{4} \right\} + \mathcal{O}\left(\sqrt{D}\right). \end{aligned} \tag{21}$$

The second sum is treated in the same way, however, we have to keep in mind the additional condition $a \equiv_2 0$:

$$\begin{aligned} \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq \frac{D}{a}+\frac{a}{4}\\ a\equiv_2 0}} \frac{1}{2} &= \frac{1}{2} \sum_{\substack{a\in 2\mathbb{N}\\ a^2<\frac{4D}{3}}} \left(\left\lfloor \frac{D}{a} + \frac{a}{4} \right\rfloor - a \right) = \frac{1}{2} \sum_{\substack{a\in\mathbb{N}\\ a^2<\frac{D}{3}}} \left(\frac{D}{2a} - \frac{3a}{2} - \left\{ \frac{D}{2a} + \frac{a}{2} \right\} \right) \\ &= \frac{1}{2} \sum_{a=1}^{\left\lceil \sqrt{\frac{D}{3}} \right\rceil - 1} \left(\frac{D}{2a} - \frac{3a}{2} \right) + \mathcal{O}\left(\sqrt{D}\right). \end{aligned} \tag{22}$$

If we start summing up with regard to c first in the last double sum, we get two lower bounds on c depending on a, namely a and $\frac{D}{a}$. We add two mutually exclusive conditions on a, each guaranteeing that one of these bounds is greater than the other and partition the exterior sum accordingly:

$$\sum_{\substack{a,c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left(\left\lceil\sqrt{ac-D}\right\rceil-1\right)$$

$$=\sum_{\substack{a\in\mathbb{N}\\ a^2\le D}}\ \sum_{\substack{c\in\mathbb{N}\\ \frac{D}{a}<c\le\frac{D}{a}+\frac{a}{4}}} \left(\left\lceil\sqrt{ac-D}\right\rceil-1\right)+\sum_{\substack{a\in\mathbb{N}\\ D<a^2<\frac{4D}{3}}}\ \sum_{\substack{c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}}} \left(\left\lceil\sqrt{ac-D}\right\rceil-1\right).$$

Then, we use the modified fractional part $\left\langle\sqrt{ac-D}\right\rangle$ to evaluate the ceiling function, which gives us

$$=\sum_{\substack{a\in\mathbb{N}\\ a^2\le D}}\ \sum_{\substack{c\in\mathbb{N}\\ \frac{D}{a}<c\le\frac{D}{a}+\frac{a}{4}}} \sqrt{ac-D}+\sum_{\substack{a\in\mathbb{N}\\ D<a^2<\frac{4D}{3}}}\ \sum_{\substack{c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}}} \sqrt{ac-D}-\sum_{\substack{a,c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left\langle\sqrt{ac-D}\right\rangle.$$

We apply the Euler-Maclaurin formula (Lemma 3.9) on the first two interior sums, which contain the explicit expression $\sqrt{ac-D}$:

$$\sum_{\substack{a,c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left(\left\lceil\sqrt{ac-D}\right\rceil-1\right)$$

$$=\sum_{a=1}^{\lfloor\sqrt{D}\rfloor}\left(\int_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}\sqrt{at-D}\,\mathrm{d}t+\int_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t+\frac{a}{2}\left(\frac{1}{2}-\left\{\frac{D}{a}+\frac{a}{4}\right\}\right)\right) \tag{23}$$

$$+\sum_{\substack{a\in\mathbb{N}\\ D<a^2<\frac{4D}{3}}}\left(\int_{a}^{\frac{D}{a}+\frac{a}{4}}\sqrt{at-D}\,\mathrm{d}t+\int_{a}^{\frac{D}{a}+\frac{a}{4}}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t+\frac{a}{2}\left(\frac{1}{2}-\left\{\frac{D}{a}+\frac{a}{4}\right\}\right)\right. \tag{24}$$

$$\left.+\sqrt{a^2-D}\left(\{a\}-\frac{1}{2}\right)\right)-\sum_{\substack{a,c\in\mathbb{N}\\ a<c\le\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left\langle\sqrt{ac-D}\right\rangle. \tag{25}$$

The two main integrals of the Euler-Maclaurin formula are evaluated separately. We begin with the second one:

$$\int_{a}^{\frac{D}{a}+\frac{a}{4}}\sqrt{at-D}\,\mathrm{d}t=\left[\frac{2(at-D)^{\frac{3}{2}}}{3a}\right]_{a}^{\frac{D}{a}+\frac{a}{4}}=\frac{a^2}{12}-\frac{2\left(a^2-D\right)^{\frac{3}{2}}}{3a}\ .$$

The term corresponding to the lower limit of the first integral vanishes, hence

$$\int_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}\sqrt{at-D}\,\mathrm{d}t=\left[\frac{2(at-D)^{\frac{3}{2}}}{3a}\right]_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}=\frac{a^2}{12}\ .$$

Each third term from lines (23) and (24) can be allocated with line (21), such that in particular all fractional parts of $\frac{D}{a}+\frac{a}{4}$ cancel out. Similarly, we include the term $\frac{a^2}{12}$, arising at the computation of the main integrals, in line (21). The second sum carries over unchanged from line (22). The first term in line (25) simplifies since $\{a\} = 0$ for $a \in \mathbb{N}$. Altogether, we get

$$
\begin{aligned}
&(26) \quad H_0(D) = \frac{1}{2} \sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \left(D - \frac{11}{12}a^2 + \frac{a}{4} - \frac{D}{a} \right) - \frac{1}{2} \sum_{a=1}^{\left\lceil \sqrt{\frac{D}{3}} \right\rceil - 1} \left(\frac{D}{2a} - \frac{3a}{2} \right) + \mathcal{O}\left(\sqrt{D}\right) \\
&(27) \quad + \frac{2}{3} \sum_{a=\left\lfloor \sqrt{D} \right\rfloor + 1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \frac{(a^2 - D)^{\frac{3}{2}}}{a} + \frac{1}{2} \sum_{a=\left\lfloor \sqrt{D} \right\rfloor + 1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \sqrt{a^2 - D} \\
&(28) \quad - \sum_{a=1}^{\left\lfloor \sqrt{D} \right\rfloor} \int_{\frac{D}{a}}^{\frac{D}{a} + \frac{a}{4}} \frac{a\left(\{t\} - \frac{1}{2}\right)}{2\sqrt{at - D}} \,\mathrm{d}t - \sum_{a=\left\lfloor \sqrt{D} \right\rfloor + 1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \int_{a}^{\frac{D}{a} + \frac{a}{4}} \frac{a\left(\{t\} - \frac{1}{2}\right)}{2\sqrt{at - D}} \,\mathrm{d}t \\
&(29) \quad + \sum_{\substack{a, c \in \mathbb{N} \\ a < c \leq \frac{D}{a} + \frac{a}{4} \\ \frac{D}{a} < c}} \left\langle \sqrt{ac - D} \right\rangle .
\end{aligned}
$$

The sums over the remaining (error) integrals in line (28) do not grow faster than $D^{\frac{3}{4}}$, the sum in line (29) probably grows linearly with D (see Remark 3.7). For now, we ignore the latter. The other expressions will be dealt with separately.

For the first term in line (26) we use the well known asymptotic formulas for the sum of the first n natural numbers, their squares, and their reciprocals (cf. Lemma 3.4). To improve the readability we define $m := 2\sqrt{\frac{D}{3}}$ and $M := \lceil m \rceil - 1$. We have

$$
\begin{aligned}
&\frac{1}{2} \sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}} \right\rceil - 1} \left(D - \frac{11}{12}a^2 + \frac{a}{4} - \frac{D}{a} \right) \\
&= \frac{D}{2} \cdot M - \frac{11}{24} \cdot \frac{M(M+1)(2M+1)}{6} + \frac{1}{8} \cdot \frac{M(M+1)}{2} - \frac{D}{2} \left(\log M + C + \mathcal{O}\left(\frac{1}{M}\right) \right) \\
&= \frac{D}{2} \cdot M - \frac{11}{72} \cdot M^3 - \frac{1}{6} \cdot M^2 - \frac{1}{72} \cdot M - \frac{D}{2} \cdot \log M - \frac{DC}{2} + \mathcal{O}\left(\sqrt{D}\right).
\end{aligned}
$$

We determine appropriate expressions for the occurring powers of M:

$$
\begin{aligned}
M &= m - \langle m \rangle = m + \mathcal{O}(1) = \mathcal{O}\left(\sqrt{D}\right), \\
M^2 &= m^2 + \mathcal{O}\left(\sqrt{D}\right) = \frac{4}{3} \cdot D + \mathcal{O}\left(\sqrt{D}\right), \\
M^3 &= m^3 - 3m^2 \langle m \rangle + \mathcal{O}\left(\sqrt{D}\right) = \frac{8}{3\sqrt{3}} \cdot D^{\frac{3}{2}} - 4D \left\langle 2\sqrt{\frac{D}{3}} \right\rangle + \mathcal{O}\left(\sqrt{D}\right).
\end{aligned}
$$

This gives us

$$
\begin{aligned}
&\frac{1}{2}\sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\left(D-\frac{11}{12}\cdot a^2+\frac{a}{4}-\frac{D}{a}\right)\\
&=\frac{D}{2}\left(m-\langle m\rangle\right)-\frac{11}{72}\left(m^3-3m^2\langle m\rangle\right)-\frac{1}{6}m^2-\frac{D}{2}\log\left(m-\langle m\rangle\right)-\frac{DC}{2}+\mathcal{O}\left(\sqrt{D}\right)\\
&=\left(\frac{1}{\sqrt{3}}-\frac{11}{27\sqrt{3}}\right)D^{\frac{3}{2}}\\
&\quad-\left(\langle m\rangle-\frac{11\langle m\rangle}{9}+\frac{4}{9}+\log\sqrt{D}+\log\frac{2}{\sqrt{3}}+\log\left(1-\frac{\langle m\rangle}{m}\right)+C\right)\frac{D}{2}+\mathcal{O}\left(\sqrt{D}\right)\\
&=\frac{16}{27\sqrt{3}}D^{\frac{3}{2}}-\frac{D}{4}\log D+\left(\frac{1}{9}\left\langle 2\sqrt{\frac{D}{3}}\right\rangle-\frac{2}{9}-\frac{\log 2}{2}+\frac{\log 3}{4}-\frac{C}{2}\right)D+\mathcal{O}\left(\sqrt{D}\right),
\end{aligned}
\tag{30}
$$

because of

$$
\log\left(1-\frac{\langle m\rangle}{m}\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(-\frac{\langle m\rangle}{m}\right)^k=-\sum_{k=1}^{\infty}\frac{\langle m\rangle^k}{km^k}=\mathcal{O}\left(\frac{1}{m}\right)=\mathcal{O}\left(\frac{1}{\sqrt{D}}\right).
$$

The calculation for the second expression in line (26) is done analogously. We define $n:=\sqrt{\frac{D}{3}}$ and $N:=\lceil n\rceil-1$ and get

$$
\begin{aligned}
\frac{1}{2}\sum_{a=1}^{\left\lceil\sqrt{\frac{D}{3}}\right\rceil-1}\left(\frac{3a}{2}-\frac{D}{2a}\right)&=\frac{3}{2}\cdot\frac{N(N+1)}{2}-\frac{D}{4}\left(\log N+C+\mathcal{O}\left(\frac{1}{N}\right)\right)\\
&=\frac{3}{8}\cdot N^2+\frac{3}{8}\cdot N-\frac{D}{4}\log N-\frac{DC}{4}+\mathcal{O}\left(\sqrt{D}\right).
\end{aligned}
$$

With the help of

$$
\begin{aligned}
N&=n-\langle n\rangle=n+\mathcal{O}(1)=\mathcal{O}\left(\sqrt{D}\right)\quad\text{and}\\
N^2&=n^2+\mathcal{O}\left(\sqrt{D}\right)=\frac{D}{3}+\mathcal{O}\left(\sqrt{D}\right)
\end{aligned}
$$

we end up with

$$
\begin{aligned}
\frac{1}{2}\sum_{a=1}^{\left\lceil\sqrt{\frac{D}{3}}\right\rceil-1}\left(\frac{3a}{2}-\frac{D}{2a}\right)&=\frac{3}{8}n^2-\frac{D}{4}\log\left(n-\langle n\rangle\right)-\frac{DC}{4}+\mathcal{O}\left(\sqrt{D}\right)\\
&=\frac{D}{8}-\frac{D}{4}\left(\log\sqrt{D}+\log\frac{1}{\sqrt{3}}+\log\left(1-\frac{\langle n\rangle}{n}\right)\right)-\frac{DC}{4}+\mathcal{O}\left(\sqrt{D}\right)\\
&=-\frac{D}{8}\log D+\left(\frac{1}{8}+\frac{\log 3}{8}-\frac{C}{4}\right)D+\mathcal{O}\left(\sqrt{D}\right).
\end{aligned}
\tag{31}
$$

The following term from line (27) is calculated by using the Euler-Maclaurin formula. To do that, the upper bound on the index of summation should involve a weak inequality rather than a strict one, necessitating a correction term.

$$\frac{2}{3}\sum_{a=\lfloor\sqrt{D}\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\frac{(a^2-D)^{\frac{3}{2}}}{a}=\frac{2}{3}\sum_{\sqrt{D}<a<2\sqrt{\frac{D}{3}}}\frac{(a^2-D)^{\frac{3}{2}}}{a}$$

$$(32)\qquad =\frac{2}{3}\left(\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}\frac{(t^2-D)^{\frac{3}{2}}}{t}\,\mathrm{d}t+\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right)\left(\{t\}-\frac{1}{2}\right)\mathrm{d}t\right.$$
$$\left.+\frac{D}{6}\left(\frac{1}{2}-\left\{2\sqrt{\frac{D}{3}}\right\}\right)-\left\{\begin{array}{ll}\frac{D}{6}, & 2\sqrt{\frac{D}{3}}\in\mathbb{N}\\ 0, & \text{otherwise}\end{array}\right\}\right)$$

We will show later on that the asymptotic growth of the second integral does not exceed that of $\sqrt{D}$. The corresponding parts of the last two terms cancel out with the modified fractional part of $2\sqrt{\frac{D}{3}}$ in line (30) of the calculation of the first sum. It remains to determine the main integral:

$$\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}\frac{(t^2-D)^{\frac{3}{2}}}{t}\,\mathrm{d}t=\lim_{\varepsilon\searrow 0}\left[\frac{t^2-4D}{3}\sqrt{t^2-D}-D^{\frac{3}{2}}\arctan\left(\frac{\sqrt{D}}{\sqrt{t^2-D}}\right)\right]_{\sqrt{D+\varepsilon}}^{2\sqrt{\frac{D}{3}}}$$
$$=\frac{-8}{9\sqrt{3}}D^{\frac{3}{2}}-\arctan\left(\sqrt{3}\right)D^{\frac{3}{2}}+\lim_{\varepsilon\searrow 0}\arctan\left(\sqrt{\frac{D}{\varepsilon}}\right)D^{\frac{3}{2}}$$
$$=\left(\frac{-8}{9\sqrt{3}}-\frac{\pi}{3}+\frac{\pi}{2}\right)D^{\frac{3}{2}}=\left(\frac{\pi}{6}-\frac{8}{9\sqrt{3}}\right)D^{\frac{3}{2}}\ .$$

This implies

$$(33)\qquad \frac{2}{3}\sum_{a=\lfloor\sqrt{D}\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\frac{(a^2-D)^{\frac{3}{2}}}{a}=\left(\frac{\pi}{9}-\frac{16}{27\sqrt{3}}\right)D^{\frac{3}{2}}+\left(\frac{1}{18}-\frac{1}{9}\left\langle 2\sqrt{\frac{D}{3}}\right\rangle\right)D+\mathcal{O}\left(\sqrt{D}\right)\ .$$

The Euler-Maclaurin formula can also be applied to the second expression from line (27):

$$\frac{1}{2}\sum_{a=\lfloor\sqrt{D}\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\sqrt{a^2-D}=\frac{1}{2}\sum_{\sqrt{D}<a<2\sqrt{\frac{D}{3}}}\sqrt{a^2-D}$$

$$(34)\qquad =\frac{1}{2}\left(\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}\sqrt{t^2-D}\,\mathrm{d}t+\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}\frac{t}{\sqrt{t^2-D}}\{t\}\,\mathrm{d}t\right.$$
$$\left.-\sqrt{\frac{D}{3}}\left\{2\sqrt{\frac{D}{3}}\right\}-\left\{\begin{array}{ll}\sqrt{\frac{D}{3}}, & 2\sqrt{\frac{D}{3}}\in\mathbb{N}\\ 0, & \text{otherwise}\end{array}\right\}\right).$$

All terms except the main integral grow at most as fast as $\sqrt{D}$ for $D\to\infty$. We will prove this claim for the second integral further down the line, for the correction term it is obviously true. With

$$\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}\sqrt{t^2-D}\,\mathrm{d}t=\left[\frac{t\sqrt{t^2-D}}{2}-\frac{D\log\left(t+\sqrt{t^2-D}\right)}{2}\right]_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}}$$
$$=\left(\frac{D}{3}-\frac{D}{4}\log\left(3D\right)+\frac{D}{4}\log(D)\right)=\left(\frac{1}{3}-\frac{\log 3}{4}\right)D$$

it follows that

$$(35)\qquad \frac{1}{2}\sum_{a=\left\lfloor\sqrt{D}\right\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\sqrt{a^2-D}=\left(\frac{1}{6}-\frac{\log 3}{8}\right)D+\mathcal{O}\left(\sqrt{D}\right).$$

By adding up the results from (30), (31), (33) and (35), we obtain

$$H_0(D)=\frac{\pi}{9}D^{\frac{3}{2}}-\frac{3}{8}D\log D+\left(\frac{1}{8}-\frac{3C}{4}-\frac{\log 2}{2}+\frac{\log 3}{4}\right)D+\sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}}\left\langle\sqrt{ac-D}\right\rangle+\mathcal{O}\left(D^{\frac{3}{4}}\right),$$

the first assertion of the proposition.

We still have to show the estimates which we claimed to be true without proof so far. We start with the error integral from line (32). Its domain of integration will be partitioned into three intervals:

$$\left[\sqrt{D},2\sqrt{\frac{D}{3}}\right]=\left[\sqrt{D},\left\lceil\sqrt{D}\right\rceil\right]\cup\left[\left\lceil\sqrt{D}\right\rceil,\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor\right]\cup\left[\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor,2\sqrt{\frac{D}{3}}\right].$$

This can be done since for $D\geq 42$ (even for $D>\frac{3}{7-4\sqrt{3}}$) we obtain $\left\lceil\sqrt{D}\right\rceil\leq\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor$. Having established these additional boundaries, we estimate each part integral separately. For the first section

$$\left|\int_{\sqrt{D}}^{\left\lceil\sqrt{D}\right\rceil}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right)\left(\{t\}-\frac{1}{2}\right)\mathrm{d}t\right|$$

$$\leq\frac{1}{2}\int_{\sqrt{D}}^{\left\lceil\sqrt{D}\right\rceil}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right|\mathrm{d}t=\frac{1}{2}\left[\frac{(t^2-D)^{\frac{3}{2}}}{t}\right]_{\sqrt{D}}^{\left\lceil\sqrt{D}\right\rceil}=\frac{\left(\left\lceil\sqrt{D}\right\rceil^2-D\right)^{\frac{3}{2}}}{2\left\lceil\sqrt{D}\right\rceil}$$

$$\leq\frac{\left(\left(\sqrt{D}+1\right)^2-D\right)^{\frac{3}{2}}}{2\sqrt{D}}=\frac{\left(2\sqrt{D}+1\right)^{\frac{3}{2}}}{2\sqrt{D}}=\mathcal{O}\left(D^{\frac{1}{4}}\right)$$

holds. A similar calculation for the rightmost integral yields D as the maximal order of magnitude at first glance. So, we apply a different approach, using that the length of the integral is at most one, to get a stricter bound:

$$\left|\int_{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor}^{2\sqrt{\frac{D}{3}}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right)\left(\{t\}-\frac{1}{2}\right)\mathrm{d}t\right|$$

$$\leq\frac{1}{2}\int_{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor}^{2\sqrt{\frac{D}{3}}}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right|\mathrm{d}t\leq\frac{1}{2}\max_{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor\leq t\leq 2\sqrt{\frac{D}{3}}}\left|3\sqrt{t^2-D}-\frac{(t^2-D)^{\frac{3}{2}}}{t^2}\right|$$

$$\leq\frac{3}{2}\max_{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor\leq t\leq 2\sqrt{\frac{D}{3}}}\sqrt{t^2-D}+\frac{1}{2}\max_{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor\leq t\leq 2\sqrt{\frac{D}{3}}}\left(t^2-D\right)^{\frac{3}{2}}\left(\min_{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor\leq t\leq 2\sqrt{\frac{D}{3}}}t^2\right)^{-1}$$

$$\leq\frac{3}{2}\sqrt{\frac{D}{3}}+\frac{\left(\frac{D}{3}\right)^{\frac{3}{2}}}{2\left(2\sqrt{\frac{D}{3}}-1\right)^2}\leq\frac{\sqrt{3D}}{2}+\frac{\left(\frac{D}{3}\right)^{\frac{3}{2}}}{2\left(\sqrt{\frac{D}{3}}\right)^2}=2\sqrt{\frac{D}{3}}=\mathcal{O}\left(\sqrt{D}\right).$$

The middle section is bounded by integers. For each interval of the form $[n, n+1], n \in \mathbb{Z}$, the function $\frac{\{t\}^2-\{t\}}{2}$ is an antiderivative for $\{t\} - \frac{1}{2}$. Evaluated at integer numbers, it yields zero, and its absolute value is less than or equal to $\frac{1}{8}$. We exploit this in applying partial integration.

$$\left|\int_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right)\left(\{t\}-\frac{1}{2}\right)\mathrm{d}t\right|$$

$$= \left|\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right)\frac{\{t\}^2-\{t\}}{2}\right]_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor} - \int_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right)\frac{\{t\}^2-\{t\}}{2}\,\mathrm{d}t\right|$$

$$\leq \frac{1}{8}\int_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor}\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right|\mathrm{d}t = \frac{1}{8}\int_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor}\left|\frac{t^2\left(2t^2-D\right)+2D^2}{t^3\sqrt{t^2-D}}\right|\mathrm{d}t$$

$$= \frac{1}{8}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(t^2-D)^{\frac{3}{2}}}{t}\right)\right]_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor} = \frac{1}{8}\left[3\sqrt{t^2-D} - \frac{(t^2-D)^{\frac{3}{2}}}{t^2}\right]_{\lceil\sqrt{D}\rceil}^{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor}$$

$$\leq \frac{1}{8}\left(3\sqrt{\left\lfloor 2\sqrt{\frac{D}{3}}\right\rfloor^2 - D} + \frac{\left(\left\lceil\sqrt{D}\right\rceil^2 - D\right)^{\frac{3}{2}}}{\left\lceil\sqrt{D}\right\rceil^2}\right)$$

$$\leq \frac{1}{8}\left(3\sqrt{\frac{4D}{3} - D} + \frac{\left(\left(\sqrt{D}+1\right)^2 - D\right)^{\frac{3}{2}}}{D}\right) = \frac{1}{8}\left(\sqrt{3D} + \frac{\left(2\sqrt{D}+1\right)^{\frac{3}{2}}}{D}\right) = \mathcal{O}\left(\sqrt{D}\right)$$

Therefore, none of the three integrals has order of magnitude greater than $\sqrt{D}$. Thanks to the triangle inequality, this also holds for their sum.
Next, we treat the error integral from line (34):

$$\left|\int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}} \frac{t}{\sqrt{t^2-D}}\{t\}\,\mathrm{d}t\right| \leq \int_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}} \frac{t}{\sqrt{t^2-D}}\,\mathrm{d}t = \left[\sqrt{t^2-D}\right]_{\sqrt{D}}^{2\sqrt{\frac{D}{3}}} = \sqrt{\frac{D}{3}} = \mathcal{O}\left(\sqrt{D}\right).$$

Last, we have to deal with the sums of line (28). We begin with the integrals of the first sum, and suppose $\left\lceil\frac{D}{a}\right\rceil + 1 \leq \left\lfloor\frac{D}{a} + \frac{a}{4}\right\rfloor$ to be true as an additional condition. Thereby, we can divide the domain of integration in three sections again:

$$\left[\frac{D}{a}, \frac{D}{a} + \frac{a}{4}\right] = \left[\frac{D}{a}, \left\lceil\frac{D}{a}\right\rceil + 1\right] \cup \left[\left\lceil\frac{D}{a}\right\rceil + 1, \left\lfloor\frac{D}{a} + \frac{a}{4}\right\rfloor\right] \cup \left[\left\lfloor\frac{D}{a} + \frac{a}{4}\right\rfloor, \frac{D}{a} + \frac{a}{4}\right].$$

For the leftmost part we get

$$\text{(36)} \quad \left|\int_{\frac{D}{a}}^{\left\lceil\frac{D}{a}\right\rceil+1} \frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right| \leq \int_{\frac{D}{a}}^{\frac{D}{a}+2} \frac{a}{4\sqrt{at-D}}\,\mathrm{d}t = \int_0^{2a}\frac{1}{4\sqrt{x}}\,\mathrm{d}x = \left[\frac{\sqrt{x}}{2}\right]_0^{2a} = \sqrt{\frac{a}{2}}$$

by using the substitution $x = at - D$. As above, the middle section is calculated using the integrity of its boundaries and a primitive function for $\{t\}-\frac{1}{2}$. This yields

$$
\begin{aligned}
&\left|\int_{\lceil\frac{D}{a}\rceil+1}^{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right| \\
&=\left|\left[\frac{a}{2\sqrt{at-D}}\cdot\frac{\{t\}^2-\{t\}}{2}\right]_{\lceil\frac{D}{a}\rceil+1}^{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor}-\int_{\lceil\frac{D}{a}\rceil+1}^{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{2\sqrt{at-D}}\right)\right)\frac{\{t\}^2-\{t\}}{2}\,\mathrm{d}t\right| \\
&\le\frac{1}{2}\int_{\lceil\frac{D}{a}\rceil+1}^{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor}\left|\frac{-a^2}{4\left(at-D\right)^{\frac{3}{2}}}\right|\,\mathrm{d}t=-\frac{1}{2}\left[\frac{a}{2\sqrt{at-D}}\right]_{\lceil\frac{D}{a}\rceil+1}^{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor} \\
&\le\frac{a}{4\sqrt{a\left\lceil\frac{D}{a}\right\rceil+a-D}}-\frac{a}{4\sqrt{a\left\lfloor\frac{D}{a}+\frac{a}{4}\right\rfloor-D}}\le\frac{\sqrt{a}}{4}-\frac{a}{4\sqrt{a\left\lfloor\frac{D}{a}+\frac{a}{4}\right\rfloor-D}}\,. \qquad (37)
\end{aligned}
$$

The third integral can be bounded in the following way:

$$
\begin{aligned}
&\left|\int_{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor}^{\frac{D}{a}+\frac{a}{4}}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right|\le\frac{1}{2}\int_{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor}^{\frac{D}{a}+\frac{a}{4}}\left|\frac{a}{2\sqrt{at-D}}\right|\,\mathrm{d}t \\
&\le\frac{1}{2}\left\{\frac{D}{a}+\frac{a}{4}\right\}\max_{\lfloor\frac{D}{a}+\frac{a}{4}\rfloor\le t\le\frac{D}{a}+\frac{a}{4}}\frac{a}{2\sqrt{at-D}}\le\frac{a}{4\sqrt{a\left\lfloor\frac{D}{a}+\frac{a}{4}\right\rfloor-D}}\,. \qquad (38)
\end{aligned}
$$

We see that $\left(\frac{1}{\sqrt{2}}+\frac{1}{4}\right)\sqrt{a}$ is an upper bound on the absolute value of the overall integral by adding up the results from lines (36)–(38).

It remains to look at the two cases, in which the additional condition $\left\lceil\frac{D}{a}\right\rceil+1\le\left\lfloor\frac{D}{a}+\frac{a}{4}\right\rfloor$ is not true. Supposing $\left\lfloor\frac{D}{a}+\frac{a}{4}\right\rfloor\in\left\{\left\lceil\frac{D}{a}\right\rceil,\left\lceil\frac{D}{a}\right\rceil-1\right\}$ yields $\frac{D}{a}+\frac{a}{4}<\left\lceil\frac{D}{a}\right\rceil+1<\frac{D}{a}+2$. As in (36) we conclude

$$
\left|\int_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right|\le\sqrt{\frac{a}{2}}\,.
$$

All in all, we have shown that

$$
\left|\int_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right|\le\left(\frac{1}{\sqrt{2}}+\frac{1}{4}\right)\sqrt{a}<\sqrt{a}\,,
$$

and we get the following bound for the sum:

$$
\left|\sum_{a=1}^{\lfloor\sqrt{D}\rfloor}\int_{\frac{D}{a}}^{\frac{D}{a}+\frac{a}{4}}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right|\le\sum_{a=1}^{\lfloor\sqrt{D}\rfloor}\sqrt{a}=\frac{\left\lfloor\sqrt{D}\right\rfloor^{\frac{3}{2}}}{\frac{3}{2}}+\mathcal{O}\left(\sqrt{\left\lfloor\sqrt{D}\right\rfloor}\right)=\mathcal{O}\left(D^{\frac{3}{4}}\right).
$$

Now, we look at the second sum from line (28). Our modus operandi will be quite similar. We first suppose $a+1\le\left\lfloor\frac{D}{a}+\frac{a}{4}\right\rfloor$ and employ these two numbers as division points for the domain of integration from a to $\frac{D}{a}+\frac{a}{4}$. Recall that $a>\sqrt{D}$.

We apply the substitution from (36) to the first section:

$$
\begin{aligned}
\left|\int_{a}^{a+1}\frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right|&\le\left[\frac{\sqrt{x}}{2}\right]_{a^2-D}^{a^2+a-D}=\frac{1}{2}\left(\sqrt{a^2+a-D}-\sqrt{a^2-D}\right) \\
&=\frac{a}{2\left(\sqrt{a^2+a-D}+\sqrt{a^2-D}\right)}\le\frac{a}{2\sqrt{a}}=\frac{\sqrt{a}}{2}\,.
\end{aligned}
$$

As in (37) we cover the middle section:

$$\left|\int_{a+1}^{\left\lfloor \frac{D}{a}+\frac{a}{4}\right\rfloor} \frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right| \leq \frac{a}{4\sqrt{a^2+a-D}} - \frac{a}{4\sqrt{a\left\lfloor \frac{D}{a}+\frac{a}{4}\right\rfloor - D}}$$
$$\leq \frac{\sqrt{a}}{4} - \frac{a}{4\sqrt{a\left\lfloor \frac{D}{a}+\frac{a}{4}\right\rfloor - D}}\ .$$

The third integral has already been bounded by the above subtrahend (see line (38)). Hence we find $\frac{3}{4}\sqrt{a}$ as an upper bound for the whole integral.

We finally look at $a = \left\lfloor \frac{D}{a}+\frac{a}{4}\right\rfloor$ and use the substitution $x = at - D$ again.

$$\left|\int_{a}^{\frac{D}{a}+\frac{a}{4}} \frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right| \leq \left[\frac{\sqrt{x}}{2}\right]_{a^2-D}^{\frac{a^2}{4}} = \frac{1}{2}\left(\frac{a}{2} - \sqrt{a^2-D}\right) = \frac{\frac{a^2}{4}-(a^2-D)}{a+2\sqrt{a^2-D}}$$
$$\leq \frac{D}{a} - \frac{3}{4}a = \left(\frac{D}{a}+\frac{a}{4}\right) - a < 1 \leq \sqrt{a}$$

We see now, that the sum fulfils

$$\left|\sum_{a=\left\lfloor\sqrt{D}\right\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1} \int_{a}^{\frac{D}{a}+\frac{a}{4}} \frac{a\left(\{t\}-\frac{1}{2}\right)}{2\sqrt{at-D}}\,\mathrm{d}t\right| \leq \sum_{a=\left\lfloor\sqrt{D}\right\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1} \sqrt{a} = \mathcal{O}\left(D^{\frac{3}{4}}\right)\ .$$

This completes the proof of the first statement of Proposition 3.6.

To prove part (b) we show that the remaining double sum has at most linear growth. We observe, that the fractional part $\left\langle\sqrt{ac-D}\right\rangle$ is trivially bounded by one, which will do the job:

$$\sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} \left\langle\sqrt{ac-D}\right\rangle \leq \sum_{\substack{a,c\in\mathbb{N}\\ a<c\leq\frac{D}{a}+\frac{a}{4}\\ \frac{D}{a}<c}} 1 = \sum_{a=1}^{\left\lfloor\sqrt{D}\right\rfloor} \sum_{\substack{c\in\mathbb{N}\\ \frac{D}{a}<c\leq\frac{D}{a}+\frac{a}{4}}} 1 + \sum_{a=\left\lfloor\sqrt{D}\right\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1} \sum_{\substack{c\in\mathbb{N}\\ a<c\leq\frac{D}{a}+\frac{a}{4}}} 1$$
$$= \sum_{a=1}^{\left\lfloor\sqrt{D}\right\rfloor}\left(\frac{a}{4}+\mathcal{O}(1)\right) + \sum_{a=\left\lfloor\sqrt{D}\right\rfloor+1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\left(\frac{D}{a}-\frac{3}{4}a+\mathcal{O}(1)\right)$$
$$= \frac{\sqrt{D}\left(\sqrt{D}+1\right)}{8} + \sum_{a=1}^{\left\lceil 2\sqrt{\frac{D}{3}}\right\rceil-1}\left(\frac{D}{a}-\frac{3}{4}a\right) + \sum_{a=1}^{\left\lfloor\sqrt{D}\right\rfloor}\left(\frac{3}{4}a-\frac{D}{a}\right)+\mathcal{O}\left(\sqrt{D}\right)$$
$$= \frac{D}{8}+D\log\left(2\sqrt{\frac{D}{3}}\right)+DC-\frac{3\left(2\sqrt{\frac{D}{3}}-1\right)2\sqrt{\frac{D}{3}}}{8} + \frac{3\sqrt{D}\left(\sqrt{D}+1\right)}{8}$$
$$-D\log\left(\sqrt{D}\right) - DC + \mathcal{O}\left(\sqrt{D}\right)$$
$$= D\left(\frac{1}{8}+\log 2+\frac{\log D-\log 3}{2}-\frac{1}{2}+\frac{3}{8}-\frac{\log D}{2}\right)+\mathcal{O}\left(\sqrt{D}\right)$$
$$= D\left(\log 2-\frac{\log 3}{2}\right)+\mathcal{O}\left(\sqrt{D}\right) = \mathcal{O}(D)\ .$$

□

We summarize our results in the following table.

Table 1: Aggregated Class Numbers of Planar Lattices

Case	Name	Aggregated Bravais Class Number
I	oblique	$H_0(D) = \frac{\pi}{9} D^{\frac{3}{2}} - \frac{3}{8} D \log D + \mathcal{O}(D)$
II	rectangular	$H_{\mathrm{re}}(D) = \frac{D}{2} \log D + \left(C - \frac{1}{2}\right) D + \mathcal{O}\big(\sqrt{D}\big)$
III	centered rectangular	$H_{\mathrm{cr}}(D) = \frac{D}{4} \log D + \frac{2C - \log 3 - 1}{4} D + \mathcal{O}\big(\sqrt{D}\big)$
III'	rhombic	$H_{\mathrm{rh}}(D) = \frac{\log 3}{4} D + \mathcal{O}\big(\sqrt{D}\big)$
IV	square	$H_{\mathrm{sq}}(D) = \sqrt{D} + \mathcal{O}(1)$
V	hexagonal	$H_{\mathrm{hex}}(D) = \sqrt{\frac{D}{3}} + \mathcal{O}(1)$

Remarks 3.10. 1. We observe that

$$\frac{1}{2} H_{\mathrm{re}}(D) = H_{\mathrm{cr}}(D) + H_{\mathrm{rh}}(D) + \mathcal{O}\big(\sqrt{D}\big).$$

We recall from Remark 3.2 that $H_{\mathrm{cr}}(D) + H_{\mathrm{rh}}(D)$ is an aggregated Bravais class number, whereas the individual summands are not.

2. We employed a C-program to count all lattices with determinant up to a given bound in order to check the plausibility of the results of Table 1. We denote the output for all lattices with determinant less than or equal to 10 million:

Table 2: Numerical Results for Planar Lattices

Case	Name	Exact Cl. Nr.	Predicted Cl. Nr.	Relative Error
	all lattices	11.096.955.716	11.038.431.406,440111	-0,005302
I	oblique	10.974.911.701	10.974.909.430,262256	-0,000000
II	rectangular	81.361.101	81.362.634,903807	0,000019
III	cent. rect.	37.933.877	37.934.786,730233	0,000024
III'	rhombic	2.744.050	2.746.530,721670	0,000903
IV	square	3.162	3.162,277660	0,000088
V	hexagonal	1.825	1.825,741858	0,000406

The exact class number is determined by counting the possible entries of a Gram matrix corresponding to the lattice.

For the cases II - V, the predicted class number is the evaluation of the explicit terms from Table 1 for $D = 10.000.000$. As the predicted class number for oblique lattices we take

$$\frac{\pi}{9} D^{\frac{3}{2}} - \frac{3}{8} D \log D + D \left(\frac{1}{8} - \frac{3C}{4} \right)$$

from Conjecture 3.8. The number of all lattices is compared to $\frac{\pi}{9}D^{\frac{3}{2}}$, the result of Minkowski (see Theorem 3.5).

The relative error is given by

$$\frac{\text{predicted class number} - \text{exact class number}}{\text{predicted class number}}\ .$$

We observe the exceptionally small relative error for lattices with trivial automorphism group. This further supports Peter Zeiner's Conjecture 3.8.

4 Asymptotics of some Three-Dimensional Lattices

In this chapter we want to determine aggregated class numbers of three-dimensional lattices. The Schiemann reduced lattices form a set of representatives for the isometry classes of all three-dimensional lattices (see [Sch94]). In Appendix A we enumerate all groups $G \subseteq \mathrm{GL}_3(\mathbb{Z})$ that occur as isometry groups of Schiemann reduced lattices $(\mathbb{Z}^3, b_B)$. For each group we shall investigate the asymptotic behaviour of the aggregated class number

$$H_G(x) := \sum_{1 \le d \le x} h_G(d) \tag{39}$$

for $x \to \infty$, where

$$h_G(d) := h_G^{\mathbb{Z}^3}(d)\ ; \tag{40}$$

see Definition 1.9. Furthermore, the list of these (rational conjugacy classes of) groups G is subdivided into the list of Bravais classes of lattices $(\mathbb{Z}^3, b_B)$, where $\mathcal{O}(\mathbb{Z}^3, b_B) = G = {}_B\mathrm{GL}_3(\mathbb{Z})$. These Bravais classes are in one-to-one correspondence with the $\mathrm{GL}_3(\mathbb{Z})$-conjugacy classes of the occurring groups G. Technically, most of the Bravais classes are subdivided further by a geometrically motivated case distinction. Table 5 gives a Gram matrix B in each of these cases. For each index i of Table 5 we write

$$h_i(d) := h_{G_i}(d).$$

Definition 4.1. Let G be a subgroup of $\mathrm{GL}_3(\mathbb{Z})$. We define the **corresponding Dirichlet series** by

$$F_G(s) := \sum_{d=1}^{\infty} h_G(d) d^{-s},\ \ s \in \mathbb{C}.$$

In the first part of this chapter we will use this series for a given isometry group G. Under some additional requirements it will help us to find a real function f_G with $f_G(x) \sim H_G(x)$ for $x \to \infty$. (Here, the symbol $\sim$ denotes the asymptotic equivalence of functions, that is $\lim_{x\to\infty} \frac{f_G(x)}{H_G(x)} = 1$.)
The results obtained by this method will be summarized in Table 3.

We use a special form of Delange's Theorem from [BSZ14, Appendix A].

Proposition 4.2. *Let $F(s) = \sum_{d=1}^{\infty} h(d) d^{-s}$ be a Dirichlet series with non-negative coefficients $h(d)$. Suppose that there is $\alpha > 0$ such that $F(s)$ converges in the half-plane $\mathbb{H}_\alpha := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$. Assume in addition that $F(s)$ is holomorphic on the punctured line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = \alpha\} \setminus \{\alpha\}$.*
Let $n \in \mathbb{N}_0$ and let $g(s)$ be holomorphic at $s = \alpha$ with $g(\alpha) \neq 0$. Suppose that $F(s)$ has a singularity of the form $F(s) = g(s)/(s-\alpha)^{n+1}$ in α when approaching from $\mathbb{H}_\alpha$.
Then, for $x \to \infty$ we have the asymptotic equivalence

$$\sum_{d \le x} h(d) \sim \frac{g(\alpha)}{\alpha \cdot n!}\ x^\alpha (\log x)^n\,.$$

Proof. A more general version of this result can be found as Theorem 7.28 in [Ten15, Chapter II.7]. It is proven in [Del54]. □

Our first aim is to determine the Dirichlet series $F_G(s)$. We start with the following auxiliary result which is a special case of the Euler-Maclaurin formula (see Lemma 3.9).

Corollary 4.3. *Let $x, y \in \mathbb{N}$ with $x < y$, let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ and $\alpha t + \beta > 0$ for all $t \in [x, y]$, and for $s \in \mathbb{C}$ let $\sigma := \operatorname{Re} s$. Define $f : [x, y] \longrightarrow \mathbb{C}$ by $f(t) := (\alpha t + \beta)^{-s}$. Then*

$$\begin{aligned}\sum_{n=x+1}^{y} f(n) &= \frac{(\alpha x + \beta)^{1-s} - (\alpha y + \beta)^{1-s}}{\alpha(s-1)} - \varphi_{x,y}^{\alpha,\beta}(s) \\ &= \frac{(\alpha x + \beta)^{1-s} - (\alpha y + \beta)^{1-s}}{\alpha(s-1)} + \frac{1}{2(\alpha y + \beta)^s} - \frac{1}{2(\alpha x + \beta)^s} - \chi_{x,y}^{\alpha,\beta}(s) \ ,\end{aligned}$$

where we can estimate the error terms as follows:

$$\begin{aligned}\left|\varphi_{x,y}^{\alpha,\beta}(s)\right| &\leq \frac{|s|}{|\sigma|} \cdot \left|\frac{1}{(\alpha x + \beta)^\sigma} - \frac{1}{(\alpha y + \beta)^\sigma}\right| \\ \text{and } \left|\chi_{x,y}^{\alpha,\beta}(s)\right| &\leq \frac{|s| \cdot |s+1| \cdot \alpha}{8(\sigma+1)} \left(\frac{1}{(\alpha x + \beta)^{\sigma+1}} - \frac{1}{(\alpha y + \beta)^{\sigma+1}}\right) .\end{aligned}$$

Proof. Since the function f is continuously differentiable on $[x, y]$, we can apply Lemma 3.9 and get

$$\begin{aligned}\sum_{n=x+1}^{y} f(n) &= \int_x^y f(t)\mathrm{d}t + \int_x^y (t - \lfloor t \rfloor) f'(t)\mathrm{d}t \\ &= \left[\frac{(\alpha t + \beta)^{1-s}}{\alpha(1-s)}\right]_x^y + \int_x^y \frac{(t - \lfloor t \rfloor)(-s)\alpha}{(\alpha t + \beta)^{s+1}}\mathrm{d}t \\ &= \frac{(\alpha x + \beta)^{1-s} - (\alpha y + \beta)^{1-s}}{\alpha(s-1)} - \underbrace{s\alpha \int_x^y \frac{(t - \lfloor t \rfloor)}{(\alpha t + \beta)^{s+1}}\mathrm{d}t}_{\varphi_{x,y}^{\alpha,\beta}(s) :=}\end{aligned}$$

with

$$\begin{aligned}\left|\varphi_{x,y}^{\alpha,\beta}(s)\right| &= |s\alpha| \cdot \left|\int_x^y \frac{(t - \lfloor t \rfloor)}{(\alpha t + \beta)^{s+1}}\mathrm{d}t\right| \leq |s\alpha| \int_x^y \frac{1}{(\alpha t + \beta)^{\sigma+1}}\mathrm{d}t \\ &= |s\alpha| \left[\frac{(\alpha t + \beta)^{-\sigma}}{-\sigma\alpha}\right]_x^y = \frac{|s|}{\sigma} \cdot \frac{|\alpha|}{\alpha} \left(\frac{1}{(\alpha x + \beta)^\sigma} - \frac{1}{(\alpha y + \beta)^\sigma}\right) \\ &= \frac{|s|}{|\sigma|} \cdot \left|\frac{1}{(\alpha x + \beta)^\sigma} - \frac{1}{(\alpha y + \beta)^\sigma}\right| .\end{aligned}$$

This proves the first assertion. For the second one we calculate $\varphi_{x,y}^{\alpha,\beta}(s)$ more precisely using that $\{t\} = t - \lfloor t \rfloor$ equals $\frac{1}{2}$ on average. We have

$$\varphi_{x,y}^{\alpha,\beta}(s) = s\alpha \int_x^y \frac{(t - \lfloor t \rfloor)}{(\alpha t + \beta)^{s+1}}\mathrm{d}t = \frac{s\alpha}{2} \int_x^y \frac{1}{(\alpha t + \beta)^{s+1}}\mathrm{d}t + s\alpha \int_x^y \frac{\{t\} - \frac{1}{2}}{(\alpha t + \beta)^{s+1}}\mathrm{d}t \ .$$

As on page 26, in the proof of Proposition 3.6 (class number of two-dimensional lattices with trivial automorphism groups), we have $\frac{\{t\}^2 - \{t\}}{2}$ as an antiderivative for $\{t\} - \frac{1}{2}$ on each interval of the form $[n, n+1], n \in \mathbb{Z}$. We use the integrality of the integration bounds and apply partial integration to the second integral:

$$\begin{aligned}\varphi_{x,y}^{\alpha,\beta}(s) &= \frac{s\alpha}{2}\left[\frac{(\alpha t+\beta)^{-s}}{-s\alpha}\right]_x^y + \left[\frac{s\alpha}{(\alpha t+\beta)^{s+1}}\cdot\frac{\{t\}^2-\{t\}}{2}\right]_x^y \\ &\quad - \int_x^y \frac{-s(s+1)\alpha^2}{(\alpha t+\beta)^{s+2}}\cdot\frac{\{t\}^2-\{t\}}{2}\mathrm{d}t \\ &= \frac{1}{2(\alpha x+\beta)^s} - \frac{1}{2(\alpha y+\beta)^s} + \underbrace{\frac{s(s+1)\alpha^2}{2}\int_x^y \frac{\{t\}^2-\{t\}}{(\alpha t+\beta)^{s+2}}\mathrm{d}t}_{\chi_{x,y}^{\alpha,\beta}(s):=}\,.\end{aligned}$$

It remains to show the estimate on $\chi_{x,y}^{\alpha,\beta}(s)$. We get

$$\begin{aligned}\left|\chi_{x,y}^{\alpha,\beta}(s)\right| &\leq \frac{|s|\cdot|s+1|\cdot\alpha^2}{2}\int_x^y\left|\frac{\{t\}^2-\{t\}}{(\alpha t+\beta)^{s+2}}\right|\mathrm{d}t \leq \frac{|s|\cdot|s+1|\cdot\alpha^2}{8}\int_x^y\frac{1}{(\alpha t+\beta)^{\sigma+2}}\mathrm{d}t \\ &= \frac{|s|\cdot|s+1|\cdot\alpha^2}{8}\left[\frac{(\alpha t+\beta)^{-(\sigma+1)}}{-\alpha(\sigma+1)}\right]_x^y \\ &= \frac{|s|\cdot|s+1|\cdot\alpha}{8(\sigma+1)}\left(\frac{1}{(\alpha x+\beta)^{\sigma+1}} - \frac{1}{(\alpha y+\beta)^{\sigma+1}}\right).\end{aligned}$$

□

Now, we will use this result to approximate the Dirichlet series for some isometry groups of three-dimensional lattices.

Proposition 4.4. *For every index i Table 3 specifies a function F_i with $F_i(s) = F_{G_i}(s)$, where the summands $R_i(s)$ and $S_i(s)$ are holomorphic for* $\operatorname{Re} s > \frac{1}{3}$.

Proof. For every index i Table 5 in Appendix A contains a matrix B_i with $G_i = {}_{B_i}\mathrm{GL}_n(\mathbb{Z})$, that is, $(\mathbb{Z}^3, b_{B_i})$ is a lattice with automorphism group G_i. Since all matrices of Table 5 are supposed to be Schiemann reduced, different matrices correspond to non-isometric lattices. Thus, we can determine $h_i(d)$ by counting all reduced matrices of the given form with determinant d.

In the following calculations we will replace certain Dirichlet series by the Riemann zeta function $\zeta(s)$. In each case we suppose $\sigma = \operatorname{Re} s$ to be big enough for the equation to hold. In particular, all modifications must be admissible at the rightmost pole α of the resulting expression which is given in the third column of Table 3. We need this for the application of Proposition 4.2 later on (see Proposition 4.5).

$\boldsymbol{G_{48,p}}$ The reduced Gram matrix of a primitive cubic lattice is given by multiplying the identity matrix by $a \in \mathbb{N}$:

$$B_{48,p} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Its determinant is a^3. We have

$$h_{G_{48,p}}(d) = \sum_{\substack{a\in\mathbb{N} \\ a^3=d}} 1 \text{ and thus, } F_{G_{48,p}}(s) = \sum_{d=1}^{\infty}\frac{h_{G_{48,p}}(d)}{d^s} = \sum_{a=1}^{\infty}\frac{1}{a^{3s}} = \zeta(3s)$$

for the corresponding Dirichlet series. Since the Riemann zeta function $\zeta(s)$ is given by its Dirichlet series for $\operatorname{Re} s > 1$, the last equality holds for all complex numbers s with $\sigma > \frac{1}{3}$. The pole with the biggest real part of the resulting expression is $\alpha = \frac{1}{3}$.

Table 3: Dirichlet Series and Asymptotic Class Numbers

Index	$F_i(s)$ $\qquad (\operatorname{Re} s = \sigma \in \mathbb{R}^+)$	α	$f_i(x)$
$48, p$	$\zeta(3s)$	$\frac{1}{3}$	$x^{\frac{1}{3}}$
$48, i$	$\frac{1}{16^s}\zeta(3s)$	$\frac{1}{3}$	$2^{-\frac{4}{3}} \cdot x^{\frac{1}{3}}$
$48, f$	$\frac{1}{4^s}\zeta(3s)$	$\frac{1}{3}$	$2^{-\frac{2}{3}} \cdot x^{\frac{1}{3}}$
$24, p, 1$	$\frac{2}{6^s(s-1)}\zeta(3s-1) + \frac{1}{6^s}\zeta(3s) - R_{24,p,1}(s)$	1	$\frac{1}{3}\zeta(2)\, x$
$24, p, 2$	$\frac{2^{2s-1}}{3^s(2s-1)}\zeta(3s-1) + \frac{8^s-1}{2\cdot 6^s}\zeta(3s) - R_{24,p,2}(s)$	$\frac{2}{3}$	$\sqrt[3]{\frac{3}{4}} \cdot x^{\frac{2}{3}}$
$16, p, 1$	$\zeta(s)\zeta(2s) - \frac{1}{2s-1}\zeta(3s-1) - \zeta(3s) + R_{16,p}(s)$	1	$\zeta(2)\, x$
$16, p, 2$	$\frac{1}{2s-1}\zeta(3s-1) - R_{16,p}(s)$	$\frac{2}{3}$	$\frac{3}{2} \cdot x^{\frac{2}{3}}$
$16, i, 1$	$\frac{1}{4^s(s-1)}\zeta(3s-1) - R_{16,i,1}(s)$	1	$\frac{1}{4}\zeta(2)\, x$
$16, i, 2$	$\left(\frac{1}{4^s} - \frac{2}{16^s}\right)\frac{1}{2s-1}\zeta(3s-1) - \frac{1}{16^s}\zeta(3s) - R_{16,i,2}(s)$	$\frac{2}{3}$	$\frac{3}{8}\left(\sqrt[3]{4} - \sqrt[3]{2}\right) x^{\frac{2}{3}}$
$16, i, 3$	$\left(\frac{2}{16^s} - \frac{3}{36^s}\right)\frac{1}{2s-1}\zeta(3s-1) - \frac{1}{36^s}\zeta(3s) - R_{16,i,3}(s)$	$\frac{2}{3}$	$\frac{3}{8}\left(\sqrt[3]{2} - \sqrt[3]{\frac{4}{3}}\right) x^{\frac{2}{3}}$
$16, i, 4$	$\frac{3}{36^s(2s-1)}\zeta(3s-1) + \frac{1}{36^s}\zeta(3s) - R_{16,i,4}(s)$	$\frac{2}{3}$	$\frac{3}{8}\sqrt[3]{\frac{4}{3}} \cdot x^{\frac{2}{3}}$
$12, r, 1$	$\frac{4}{3\cdot 4^s(s-1)}\zeta(3s-1) - R_{12,r,1}(s)$	1	$\frac{1}{3}\zeta(2)\, x$
$12, r, 2$	$\left(\frac{1}{3} - \frac{4}{3\cdot 4^s}\right)\frac{1}{s-1}\zeta(3s-1) - \frac{1}{4^s}\zeta(3s) - R_{12,r,2}(s)$	$\frac{2}{3}$	$\frac{1}{2}\left(\sqrt[3]{4} - 1\right) x^{\frac{2}{3}}$
$12, r, 3$	$\left(\frac{1}{3} - \frac{4}{3\cdot 16^s}\right)\frac{1}{2s-1}\zeta(3s-1) - \frac{1}{16^s}\zeta(3s) - R_{12,r,3}(s)$	$\frac{2}{3}$	$\frac{1}{2}\left(1 - \frac{1}{\sqrt[3]{4}}\right) x^{\frac{2}{3}}$
$12, r, 4$	$\frac{4}{3\cdot 16^s(2s-1)}\zeta(3s-1) - R_{12,r,4}(s)$	$\frac{2}{3}$	$\frac{1}{2\sqrt[3]{4}} \cdot x^{\frac{2}{3}}$
$8, p$	$\frac{1}{6}\zeta(s)^3 - \frac{1}{2}\zeta(s)\zeta(2s) - \frac{1}{3}\zeta(3s)$	1	$\frac{1}{12} \cdot x(\log x)^2$
$8, f, 1$	$\frac{9\cdot\zeta(3s-2)}{2\cdot 36^s(s-1)^2} + \frac{3s\cdot\zeta(3s-1)}{2\cdot 36^s(s-1)(2s-1)} + \frac{R_{8,f,1}(s)}{s-1} + S_{8,f,1}(s)$	1	$\frac{1}{48} \cdot x(\log x)^2$
$8, f, 2$	$\frac{(9^s-9)\zeta(3s-2)}{2\cdot 36^s(s-1)^2} - \frac{(9^s(3s-2)+3s)\zeta(3s-1)}{2\cdot 36^s(s-1)(2s-1)} + \frac{R_{8,f,2}(s)}{s-1} + S_{8,f,2}(s)$	1	$\frac{\log 3}{12} \cdot x \log x$

For each index i the function $F_i(s)$ equals the Dirichlet series of the group G_i (see Proposition 4.4). Its abscissa of convergence is given by α, and $f_i(x)$ denotes the corresponding asymptotic class number (see Proposition 4.5). All the expressions $R_i(s)$ and $S_i(s)$ are holomorphic for $\operatorname{Re} s > \frac{1}{3}$.

$G_{48,i}$ The Gram matrix for the body-centered cubic lattice is given by

$$B_{48,i} = \begin{pmatrix} 3d & d & d \\ d & 3d & -d \\ d & -d & 3d \end{pmatrix} .$$

Since the Gram matrix is supposed to be reduced, we have $d \in \mathbb{N}$. The determinant of this Gram matrix is given by $16d^3$ and the corresponding Dirichlet series is

$$F_{G_{48,i}}(s) = \sum_{d=1}^{\infty} \frac{h_{G_{48,i}}(d)}{d^s} = \sum_{d=1}^{\infty} \frac{1}{(16d^3)^s} = \frac{\zeta(3s)}{16^s} .$$

Again, this holds for $\sigma > \frac{1}{3} = \alpha$.

$G_{48,f}$ From Table 5 we have the Gram matrix

$$B_{48,f} = \begin{pmatrix} 2d & d & d \\ d & 2d & d \\ d & d & 2d \end{pmatrix}$$

for the face-centered cubic case. As before, we have $d \in \mathbb{N}$. The determinant equals $4d^3$ and we get

$$F_{G_{48,f}}(s) = \sum_{d=1}^{\infty} \frac{h_{G_{48,f}}(d)}{d^s} = \sum_{d=1}^{\infty} \frac{1}{(4d^3)^s} = \frac{\zeta(3s)}{4^s} .$$

$G_{24,p,1}$ The Gram matrix

$$B_{24,p,1} = \begin{pmatrix} 2d & d & 0 \\ d & 2d & 0 \\ 0 & 0 & c \end{pmatrix}$$

for the first hexagonal case, where the shortest vectors are orthogonal to the sixfold axis, has two independent parameters c and d. The reduction conditions give $2d \le c$. The determinant is $3cd^2$. We get

$$F_{G_{24,p,1}}(s) = \sum_{d=1}^{\infty} \frac{h_{G_{24,p,1}}(d)}{d^s} = \sum_{\substack{c,d\in\mathbb{N} \\ 2d\le c}} \frac{1}{(3cd^2)^s} = \frac{1}{3^s} \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \sum_{c=2d}^{\infty} \frac{1}{c^s} .$$

We apply Corollary 4.3 for $(x, y, \alpha, \beta) = (2d, k, 1, 0)$ and suppose $\sigma > 1$. Then

$$\begin{aligned} \sum_{c=2d}^{\infty} \frac{1}{c^s} &= \lim_{k\to\infty} \left(\frac{1}{(2d)^s} + \frac{(2d)^{1-s} - k^{1-s}}{s-1} - \varphi_{2d,k}^{1,0}(s) \right) \\ &= \frac{1}{2^s d^s} + \frac{2}{2^s(s-1)d^{s-1}} - \lim_{k\to\infty} \varphi_{2d,k}^{1,0}(s) , \end{aligned}$$

and, using this, altogether we have

$$\begin{aligned} F_{G_{24,p,1}}(s) &= \frac{1}{6^s} \sum_{d=1}^{\infty} \frac{1}{d^{3s}} + \frac{2}{6^s(s-1)} \sum_{d=1}^{\infty} \frac{1}{d^{3s-1}} - \underbrace{\frac{1}{3^s} \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \lim_{k\to\infty} \varphi_{2d,k}^{1,0}(s)}_{R_{24,p,1}(s):=} \\ &= \frac{\zeta(3s)}{6^s} + \frac{2}{6^s(s-1)} \zeta(3s-1) - R_{24,p,1}(s) . \end{aligned}$$

Beginning with this case, we will not be able to express the Dirichlet series F_{G_i} by Riemann zeta functions in an exact equation but we will have to deal with certain remainder terms in most cases (the primitive orthorhombic case being the only exception). The approximation by zeta functions will nevertheless be useful if the error terms are given by functions that are analytic in a right half-plane of the complex numbers that contains the rightmost singularity of the explicit expressions.

The remainder term in this case can be estimated in the following way:

$$\begin{aligned}|R_{24,p,1}(s)| &\le \frac{1}{3^\sigma}\sum_{d=1}^{\infty}\frac{1}{d^{2\sigma}}\lim_{k\to\infty}\left|\varphi^{1,0}_{2d,k}(s)\right| \le \frac{1}{3^\sigma}\sum_{d=1}^{\infty}\frac{1}{d^{2\sigma}}\lim_{k\to\infty}\frac{|s|}{\sigma}\cdot\left|\frac{1}{(2d)^\sigma}-\frac{1}{k^\sigma}\right| \\ &= \frac{1}{3^\sigma}\cdot\frac{|s|}{\sigma}\sum_{d=1}^{\infty}\frac{1}{d^{2\sigma}(2d)^\sigma} = \frac{1}{6^\sigma}\cdot\frac{|s|}{\sigma}\zeta(3\sigma)\ .\end{aligned}$$

We can see now that the rest $R_{24,p,1}(s)$ is holomorphic for $\sigma > \frac{1}{3}$. Since the rightmost pole of the explicit expressions lies at $s = 1$, we will be able to apply Delange's Theorem (Proposition 4.2) to this situation.

$\boldsymbol{G_{24,p,2}}$ The second hexagonal case (shortest vector spans the sixfold axis) deals with Gram matrices of the form

$$B_{24,p,2} = \begin{pmatrix} a & 0 & 0 \\ 0 & 2f & f \\ 0 & f & 2f \end{pmatrix},$$

where $a \le 2f$. If we had $a = 2f$, reducedness would imply $|f| \le e = 0$, thus $f = 0$. But this is not possible. So, both inequalities have to be strict. The determinant of $B_{24,p,2}$ is given by $3af^2$ yielding

$$F_{G_{24,p,2}}(s) = \sum_{\substack{a,f\in\mathbb{N}\\ a<2f}}\frac{1}{(3af^2)^s} = \frac{1}{3^s}\sum_{a=1}^{\infty}\frac{1}{a^s}\sum_{\substack{f\in\mathbb{N}\\ a<2f}}\frac{1}{f^{2s}} = \frac{1}{3^s}\sum_{a=1}^{\infty}\frac{1}{a^s}\lim_{k\to\infty}\sum_{\substack{f\in\mathbb{N}\\ \frac{a}{2}<f\le k}}\frac{1}{f^{2s}}\ .$$

We want to apply the Euler-Maclaurin formula to the function $t \mapsto t^{-2s}$. For a odd we have $\frac{a}{2} \notin \mathbb{N}$, so we do not meet the premises of Corollary 4.3 and have to use Lemma 3.9 instead. The expressions of the right-hand side of the Euler-Maclaurin formula are: main integral, error integral and two error terms for non-integral summation bounds. Thanks to the special form of the considered function we can proceed as in the proof of Corollary 4.3 for the calculation of the two integrals. Because of $k \in \mathbb{Z}$ the second error term for the bounds vanishes. We get

$$\begin{aligned}\sum_{\substack{f\in\mathbb{N}\\ \frac{a}{2}<f\le k}}\frac{1}{f^{2s}} &= \frac{\left(\frac{a}{2}\right)^{1-2s}-k^{1-2s}}{2s-1} - \varphi^{1,0}_{\frac{a}{2},k}(2s) + \frac{\frac{a}{2}-\lfloor\frac{a}{2}\rfloor}{\left(\frac{a}{2}\right)^{2s}} \\ &= \frac{2^{2s-1}}{(2s-1)a^{2s-1}} + (a \bmod 2)\cdot\frac{2^{2s-1}}{a^{2s}} - \varphi^{1,0}_{\frac{a}{2},k}(2s) + \frac{1}{(2s-1)k^{2s-1}}\ .\end{aligned}$$

For $\sigma > \frac{1}{2}$ and $k \to \infty$ the last expression of the last line vanishes and we end up with

$$F_{G_{24,p,2}}(s) = \frac{2^{2s-1}}{3^s}\left(\frac{1}{2s-1}\sum_{a=1}^{\infty}\frac{1}{a^{3s-1}} + \sum_{\substack{a=1\\ a\text{ odd}}}^{\infty}\frac{1}{a^{3s}}\right) - \underbrace{\frac{1}{3^s}\sum_{a=1}^{\infty}\frac{1}{a^s}\lim_{k\to\infty}\varphi^{1,0}_{\frac{a}{2},k}(2s)}_{R_{24,p,2}(s):=}\ .$$

We can express the first two series by the Riemann zeta function if we assume $\sigma > \frac{2}{3}$. Regarding the Dirichlet series $\zeta(s) = \sum_{a=1}^{\infty} \frac{1}{a^s}$ we know that the subseries over all even a and odd a have the values $\frac{1}{2^s}\zeta(s)$ and $(1 - \frac{1}{2^s})\zeta(s)$, respectively. Then

$$\begin{aligned} F_{G_{24,p,2}}(s) &= \frac{2^{2s-1}}{3^s}\left(\frac{1}{2s-1}\zeta(3s-1) + \left(1 - \frac{1}{2^{3s}}\right)\zeta(3s)\right) - R_{24,p,2}(s) \\ &= \frac{2^{2s-1}}{3^s(2s-1)}\zeta(3s-1) + \frac{8^s - 1}{2 \cdot 6^s}\zeta(3s) - R_{24,p,2}(s) \end{aligned}$$

with

$$\begin{aligned} |R_{24,p,2}(s)| &\le \frac{1}{3^\sigma}\sum_{a=1}^{\infty}\frac{1}{a^\sigma}\lim_{k\to\infty}\left|\varphi^{1,0}_{\frac{a}{2},k}(2s)\right| \le \frac{1}{3^\sigma}\sum_{a=1}^{\infty}\frac{1}{a^\sigma}\lim_{k\to\infty}\frac{|2s|}{2\sigma}\cdot\left|\frac{1}{\left(\frac{a}{2}\right)^{2\sigma}} - \frac{1}{k^{2\sigma}}\right| \\ &= \frac{1}{3^\sigma}\cdot\frac{|s|}{\sigma}\sum_{a=1}^{\infty}\frac{2^{2\sigma}}{a^\sigma a^{2\sigma}} = \frac{4^\sigma}{3^\sigma}\cdot\frac{|s|}{\sigma}\zeta(3\sigma)\ . \end{aligned}$$

$\boldsymbol{G_{16,p,1}}$ If a primitive tetragonal lattice has four shortest vectors, its Gram matrix has the form

$$B_{16,p,1} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

with $a \neq c$. Together with the reduction conditions we have $0 < a < c$ and determine the Dirichlet series as

$$F_{G_{16,p,1}}(s) = \sum_{\substack{a,c\in\mathbb{N} \\ a<c}} \frac{1}{(a^2c)^s} = \sum_{c=1}^{\infty}\frac{1}{c^s}\sum_{a=1}^{c-1}\frac{1}{a^{2s}} = \sum_{c=1}^{\infty}\frac{1}{c^s}\left(\sum_{a=1}^{\infty}\frac{1}{a^{2s}} - \sum_{a=c}^{\infty}\frac{1}{a^{2s}}\right).$$

We have to employ the last transformation because an application of Corollary 4.3 on $\sum_{a=1}^{c-1}\frac{1}{a^{2s}}$ would give an error term too big for the rest of the calculation. Next, we use Corollary 4.3 for $(x, y, \alpha, \beta) = (c, k, 1, 0)$ and assume $\sigma > 1$. This yields

$$\begin{aligned} F_{G_{16,p,1}}(s) &= \zeta(s)\zeta(2s) - \sum_{c=1}^{\infty}\frac{1}{c^s}\lim_{k\to\infty}\left(\frac{1}{c^{2s}} + \frac{c^{1-2s} - k^{1-2s}}{2s-1} - \varphi^{1,0}_{c,k}(2s)\right) \\ &= \zeta(s)\zeta(2s) - \zeta(3s) - \frac{\zeta(3s-1)}{2s-1} + \underbrace{\sum_{c=1}^{\infty}\frac{1}{c^s}\lim_{k\to\infty}\varphi^{1,0}_{c,k}(2s)}_{R_{16,p,1}(s):=} \end{aligned}$$

and

$$|R_{16,p,1}(s)| \le \sum_{c=1}^{\infty}\frac{1}{c^\sigma}\lim_{k\to\infty}\left|\varphi^{1,0}_{c,k}(2s)\right| \le \sum_{c=1}^{\infty}\frac{1}{c^\sigma}\lim_{k\to\infty}\frac{|2s|}{2\sigma}\cdot\left|\frac{1}{c^{2\sigma}} - \frac{1}{k^{2\sigma}}\right| = \frac{|s|}{\sigma}\zeta(3\sigma)$$

for the remainder term.

$\boldsymbol{G_{16,p,2}}$ A primitive tetragonal lattice with two shortest vectors is given by Gram matrices of the form

$$B_{16,p,2} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$$

with $a \neq b$. Thus, the independent parameters must fulfil $0 < a < b$. For $\sigma > \frac{1}{2}$ we get

$$\begin{aligned}
F_{G_{16,p,2}}(s) &= \sum_{\substack{a,b\in\mathbb{N}\\ a<b}} \frac{1}{(ab^2)^s} = \sum_{a=1}^{\infty} \frac{1}{a^s} \sum_{b=a+1}^{\infty} \frac{1}{b^{2s}} \\
&= \sum_{a=1}^{\infty} \frac{1}{a^s} \lim_{k\to\infty} \left(\frac{a^{1-2s} - k^{1-2s}}{2s-1} - \varphi_{a,k}^{1,0}(2s) \right) \\
&= \frac{\zeta(3s-1)}{2s-1} - \sum_{a=1}^{\infty} \frac{1}{a^s} \lim_{k\to\infty} \varphi_{a,k}^{1,0}(2s) = \frac{\zeta(3s-1)}{2s-1} - R_{16,p,1}(s) \ .
\end{aligned}$$

Since both primitive tetragonal cases involve the same remainder term with a different sign, we in particular have the exact result

$$F_{G_{16,p}}(s) = F_{G_{16,p,1}}(s) + F_{G_{16,p,2}}(s) = \zeta(s)\zeta(2s) - \zeta(3s)$$

for the Dirichlet series of all primitive tetragonal lattices.

$\boldsymbol{G_{16,i,1}}$ The first body-centered tetragonal case deals with Gram matrices of the form

$$B_{16,i,1} = \begin{pmatrix} 2e & 0 & e \\ 0 & 2e & e \\ e & e & c \end{pmatrix} .$$

Reducedness implies $2e \leq c$. For the special case $2e = c$ the boundary conditions would require $e = 0$, so this case is prohibited. We would have a face-centered cubic lattice (see the geometric derivation in Appendix A) which we do not want to count here anyway. We calculate $\det B_{16,i,1} = 4e^2(c-e)$. Then we have

$$\begin{aligned}
F_{G_{16,i,1}}(s) &= \sum_{\substack{c,e\in\mathbb{N}\\ 2e<c}} \frac{1}{(4e^2(c-e))^s} = \frac{1}{4^s} \sum_{e=1}^{\infty} \frac{1}{e^{2s}} \sum_{c=2e+1}^{\infty} \frac{1}{(c-e)^s} = \frac{1}{4^s} \sum_{e=1}^{\infty} \frac{1}{e^{2s}} \sum_{c=e+1}^{\infty} \frac{1}{c^s} \\
&= \frac{1}{4^s} \sum_{e=1}^{\infty} \frac{1}{e^{2s}} \lim_{k\to\infty} \sum_{c=e+1}^{k} \frac{1}{c^s} = \frac{1}{4^s} \sum_{e=1}^{\infty} \frac{1}{e^{2s}} \lim_{k\to\infty} \left(\frac{e^{1-s} - k^{1-s}}{s-1} - \varphi_{e,k}^{1,0}(s) \right) \\
&= \frac{\zeta(3s-1)}{4^s(s-1)} - \underbrace{\frac{1}{4^s} \sum_{e=1}^{\infty} \frac{1}{e^{2s}} \lim_{k\to\infty} \varphi_{e,k}^{1,0}(s)}_{R_{16,i,1}(s):=}
\end{aligned}$$

for $\sigma > 1$ and furthermore,

$$\begin{aligned}
|R_{16,i,1}(s)| &\leq \frac{1}{4^\sigma} \sum_{e=1}^{\infty} \frac{1}{e^{2\sigma}} \lim_{k\to\infty} \left|\varphi_{e,k}^{1,0}(s)\right| \leq \frac{1}{4^\sigma} \sum_{e=1}^{\infty} \frac{1}{e^{2\sigma}} \lim_{k\to\infty} \frac{|s|}{\sigma} \cdot \left| \frac{1}{e^\sigma} - \frac{1}{k^\sigma} \right| \\
&= \frac{1}{4^\sigma} \cdot \frac{|s|}{\sigma} \zeta(3\sigma) \ .
\end{aligned}$$

$\boldsymbol{G_{16,i,2}}$ The next Gram matrix

$$B_{16,i,2} = \begin{pmatrix} 2d-f & d & d \\ d & 2d-f & f \\ d & f & 2d-f \end{pmatrix}$$

has the determinant $\det B_{16,i,2} = 4d(d-f)^2$.

First, the essential reduction conditions imply $f \le 0 \le d$. Because of $a = b$, the boundary conditions necessitate $|f| \le d$ in addition. Thus,

$$d = 0 \;\Rightarrow\; f = 0.$$

But we also have

$$f = 0 \;\Rightarrow\; d = 0$$

since $f = 0$ would mean $a = 2d$, requiring $e \le f$ in turn. Therefore, we have to claim $-d \le f < 0 < d$. Finally, we want $-d \ne f$, in order to prevent from counting body-centered cubic lattices. For the Dirichlet series it follows that

$$\begin{aligned}
F_{G_{16,i,2}}(s) &= \sum_{\substack{d,f\in\mathbb{Z}\\ -d<f<0<d}} \frac{1}{(4d(d-f)^2)^s} = \frac{1}{4^s}\sum_{d=1}^{\infty}\frac{1}{d^s}\sum_{\substack{f\in\mathbb{Z}\\ -d<f<0}}\frac{1}{(d-f)^{2s}}\\
&= \frac{1}{4^s}\sum_{d=1}^{\infty}\frac{1}{d^s}\sum_{f=d+1}^{2d-1}\frac{1}{f^{2s}} = \frac{1}{4^s}\sum_{d=1}^{\infty}\frac{1}{d^s}\sum_{f=d+1}^{2d}\frac{1}{f^{2s}} - \frac{\zeta(3s)}{16^s}\\
&= \frac{1}{4^s}\sum_{d=1}^{\infty}\frac{1}{d^s}\left(\frac{d^{1-2s}-(2d)^{1-2s}}{2s-1} - \varphi^{1,0}_{d,2d}(2s)\right) - \frac{\zeta(3s)}{16^s}\\
&= \frac{1-2^{1-2s}}{4^s(2s-1)}\zeta(3s-1) - \frac{\zeta(3s)}{16^s} - \underbrace{\frac{1}{4^s}\sum_{d=1}^{\infty}\frac{1}{d^s}\varphi^{1,0}_{d,2d}(2s)}_{R_{16,i,2}(s):=}\\
&= \left(\frac{1}{4^s}-\frac{2}{16^s}\right)\frac{\zeta(3s-1)}{2s-1} - \frac{\zeta(3s)}{16^s} - R_{16,i,2}(s)
\end{aligned}$$

with the estimation

$$\begin{aligned}
|R_{16,i,2}(s)| &\le \frac{1}{4^\sigma}\sum_{d=1}^{\infty}\frac{1}{d^\sigma}\left|\varphi^{1,0}_{d,2d}(2s)\right| \le \frac{1}{4^\sigma}\sum_{d=1}^{\infty}\frac{1}{d^\sigma}\cdot\frac{|2s|}{2\sigma}\cdot\left|\frac{1}{d^{2\sigma}}-\frac{1}{(2d)^{2\sigma}}\right|\\
&= \left(\frac{1}{4^\sigma}-\frac{1}{16^\sigma}\right)\frac{|s|}{\sigma}\zeta(3\sigma)\;.
\end{aligned}$$

for the error term.

$\boldsymbol{G_{16,i,3}}$ For the Gram matrix

$$B_{16,i,3} = \begin{pmatrix} d+2e & d & e\\ d & d+2e & -e\\ e & -e & d+2e\end{pmatrix}$$

we have to consider the essential condition $0 \le d \le 2e$ and the boundary condition $e \le d$. So, again one parameter equals zero if and only if the other does, yielding the zero matrix in this case. To exclude body-centered cubic lattices, we must assume $d \ne e$. Altogether, we have $0 < e < d < 2e$. The determinant values $4e(d+e)^2$. We determine the Dirichlet series:

$$
\begin{aligned}
F_{G_{16,i,3}}(s) &= \sum_{\substack{d,e\in\mathbb{N}\\ e<d<2e}} \frac{1}{(4e(d+e)^2)^s} = \frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\sum_{d=e+1}^{2e-1}\frac{1}{(d+e)^{2s}} \\
&= \frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\left(\sum_{d=2e+1}^{3e}\frac{1}{d^{2s}} - \frac{1}{(3e)^{2s}}\right) \\
&= \frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\left(\frac{(2e)^{1-2s}-(3e)^{1-2s}}{2s-1} - \varphi^{1,0}_{2e,3e}(2s)\right) - \frac{\zeta(3s)}{36^s} \\
&= \left(\frac{2}{16^s}-\frac{3}{36^s}\right)\frac{\zeta(3s-1)}{2s-1} - \frac{\zeta(3s)}{36^s} - \underbrace{\frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\varphi^{1,0}_{2e,3e}(2s)}_{R_{16,i,3}(s):=} \ ,
\end{aligned}
$$

and give a bound for the remainder term:

$$
\begin{aligned}
|R_{16,i,3}(s)| &\le \frac{1}{4^\sigma}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\left|\varphi^{1,0}_{2e,3e}(2s)\right| \le \frac{1}{4^\sigma}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\cdot\frac{|2s|}{2\sigma}\cdot\left|\frac{1}{(2e)^{2\sigma}}-\frac{1}{(3e)^{2\sigma}}\right| \\
&= \left(\frac{1}{16^\sigma}-\frac{1}{36^\sigma}\right)\frac{|s|}{\sigma}\zeta(3\sigma) \ .
\end{aligned}
$$

$\boldsymbol{G_{16,i,4}}$ In the last body-centered tetragonal case we consider Gram matrices of the form

$$
B_{16,i,4} = \begin{pmatrix} 4f & 2f & 2f \\ 2f & b & f \\ 2f & f & b \end{pmatrix} .
$$

The essential reduction conditions yield $0 < 4f \le b$. All boundary conditions are already included in this. In the special case $b = 4f$ the lattice has no additional symmetries and is still body-centered tetragonal. Hence, $b = 4f$ is admissible here. The determinant is given by $\det B_{16,i,4} = 4f(b-f)^2$, leading to

$$
\begin{aligned}
F_{G_{16,i,4}}(s) &= \sum_{\substack{b,f\in\mathbb{N}\\ 4f<b}} \frac{1}{(4f(b-f)^2)^s} = \frac{1}{4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\sum_{b=4f}^{\infty}\frac{1}{(b-f)^{2s}} \\
&= \frac{1}{4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\left(\frac{1}{(3f)^{2s}} + \lim_{k\to\infty}\sum_{b=3f+1}^{k}\frac{1}{b^{2s}}\right) \\
&= \frac{\zeta(3s)}{36^s} + \frac{1}{4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\lim_{k\to\infty}\left(\frac{(3f)^{1-2s}-k^{1-2s}}{2s-1} - \varphi^{1,0}_{3f,k}(2s)\right) \\
&= \frac{\zeta(3s)}{36^s} + \frac{3}{36^s(2s-1)}\zeta(3s-1) - \underbrace{\frac{1}{4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\lim_{k\to\infty}\varphi^{1,0}_{3f,k}(2s)}_{R_{16,i,4}(s):=}
\end{aligned}
$$

with the remainder

$$
\begin{aligned}
|R_{16,i,4}(s)| &\le \frac{1}{4^\sigma}\sum_{f=1}^{\infty}\frac{1}{f^\sigma}\lim_{k\to\infty}\left|\varphi^{1,0}_{3f,k}(2s)\right| \le \frac{1}{4^\sigma}\sum_{f=1}^{\infty}\frac{1}{f^\sigma}\lim_{k\to\infty}\frac{|2s|}{2\sigma}\cdot\left|\frac{1}{(3f)^{2\sigma}}-\frac{1}{k^{2\sigma}}\right| \\
&= \frac{1}{36^\sigma}\cdot\frac{|s|}{\sigma}\zeta(3\sigma) \ .
\end{aligned}
$$

For the Dirichlet series of all body-centered tetragonal lattices we conclude

$$\begin{aligned} F_{G_{16,i}}(s) &= \sum_{j=1}^{4} F_{G_{16,i,j}}(s) \\ &= \left(\frac{1}{s-1} + \frac{1}{2s-1}\right)\frac{\zeta(3s-1)}{4^s} - \left(\frac{1}{16^s} + \frac{2}{36^s}\right)\zeta(3s) - \sum_{j=1}^{4} R_{16,i,j}(s) \ . \end{aligned}$$

$\boldsymbol{G_{12,r,1}}$ The first rhombohedral trigonal case has the Gram matrix

$$B_{12,r,1} = \begin{pmatrix} 2d & d & d \\ d & 2d & d \\ d & d & c \end{pmatrix} .$$

The reducedness of $B_{12,r,1}$ requires $0 < 2d \le c$. The additional condition $2d \neq c$ ensures that we do not count face-centered cubic lattices. With the determinant $\det B_{12,r,1} = d^2(3c-2d)$ and the assumption $\sigma > 1$ we get the Dirichlet series

$$\begin{aligned} F_{G_{12,r,1}}(s) &= \sum_{\substack{c,d\in\mathbb{N} \\ 2d<c}} \frac{1}{(d^2(3c-2d))^s} = \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \lim_{k\to\infty} \sum_{c=2d+1}^{k} \frac{1}{(3c-2d)^s} \\ &= \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \lim_{k\to\infty} \left(\frac{(4d)^{1-s} - (3k-2d)^{1-s}}{3(s-1)} - \varphi_{2d,k}^{3,-2d}(s) \right) \\ &= \frac{4}{3\cdot 4^s(s-1)}\zeta(3s-1) - \underbrace{\sum_{d=1}^{\infty} \frac{1}{d^{2s}} \lim_{k\to\infty} \varphi_{2d,k}^{3,-2d}(s)}_{R_{12,r,1}(s):=} \ . \end{aligned}$$

The remainder term can be bounded by

$$\begin{aligned} |R_{12,r,1}(s)| &\le \sum_{d=1}^{\infty} \frac{1}{d^{2\sigma}} \lim_{k\to\infty} \left|\varphi_{2d,k}^{3,-2d}(s)\right| \le \sum_{d=1}^{\infty} \frac{1}{d^{2\sigma}} \lim_{k\to\infty} \frac{|s|}{\sigma} \cdot \left| \frac{1}{(4d)^\sigma} - \frac{1}{(3k-2d)^\sigma} \right| \\ &= \frac{1}{4^\sigma} \cdot \frac{|s|}{\sigma}\zeta(3\sigma) \ . \end{aligned}$$

$\boldsymbol{G_{12,r,2}}$ Next, we deal with Gram matrices of the form

$$B_{12,r,2} = \begin{pmatrix} a & d & d \\ d & a & d \\ d & d & a \end{pmatrix} .$$

The reduction conditions amount to $0 < a$ and $0 \le 2d \le a$. Since $d = 0$ and $2d = a$ would give primitive and face-centered cubic lattices, respectively, we claim the strict inequalities $0 < 2d < a$. The determinant is given by $\det B_{12,r,2} = (a+2d)(a-d)^2$.

Then

$$
\begin{aligned}
F_{G_{12,r,2}}(s) &= \sum_{\substack{a,d\in\mathbb{N}\\2d<a}} \frac{1}{((a+2d)(a-d)^2)^s} = \sum_{\substack{a,d\in\mathbb{N}\\d<a}} \frac{1}{((a+3d)a^2)^s} \\
&= \sum_{a=1}^{\infty} \frac{1}{a^{2s}} \sum_{d=1}^{a-1} \frac{1}{(3d+a)^s} = \sum_{a=1}^{\infty} \frac{1}{a^{2s}} \left(\sum_{d=1}^{a} \frac{1}{(3d+a)^s} - \frac{1}{(4a)^s} \right) \\
&= \sum_{a=1}^{\infty} \frac{1}{a^{2s}} \left(\frac{a^{1-s}-(4a)^{1-s}}{3(s-1)} - \varphi_{0,a}^{3,a}(s) \right) - \frac{\zeta(3s)}{4^s} \\
&= \left(\frac{1}{3} - \frac{4}{3\cdot 4^s} \right) \frac{\zeta(3s-1)}{s-1} - \frac{\zeta(3s)}{4^s} - \underbrace{\sum_{a=1}^{\infty} \frac{1}{a^{2s}} \varphi_{0,a}^{3,a}(s)}_{R_{12,r,2}(s):=} \ .
\end{aligned}
$$

Here, we have used the Euler-Maclaurin formula from Lemma 3.9 in the version from Corollary 4.3 for sums starting at $x = 0$. This is correct even though in Lemma 3.9 we assume $x > 0$ and in Corollary 4.3 even $x \in \mathbb{N}$. We see this by looking at $x \to 0$ for the formula from 3.9. On the left-hand side $x < 1$ implies

$$
\sum_{\substack{n\in\mathbb{N}\\x<n\le y}} f(n) = \sum_{\substack{n\in\mathbb{N}\\0<n\le y}} f(n) \ .
$$

On the right-hand side all expression depend continuously on x for $x \to 0$, since for $f(n) = \frac{1}{(3n+a)^s}$ the terms $f(t), (t - \lfloor t \rfloor) f'(t)$ and $(t - \lfloor t \rfloor) f(t)$ possess no singularities at $t = 0$ and since the value of an integral depends continuously on the integration boundaries. Thus, this generalisation of the Euler-Maclaurin formula holds, as long as the function f is suitable. For Corollary 4.3 we need $\beta \neq 0$ in particular, since otherwise the estimate of the error term would involve dividing by zero.

For the error term we have

$$
\begin{aligned}
|R_{12,r,1}(s)| &\le \sum_{a=1}^{\infty} \frac{1}{a^{2\sigma}} \left| \varphi_{0,a}^{3,a}(s) \right| \le \sum_{a=1}^{\infty} \frac{1}{a^{2\sigma}} \cdot \frac{|s|}{\sigma} \cdot \left| \frac{1}{a^{\sigma}} - \frac{1}{(4a)^{\sigma}} \right| \\
&= \left(1 - \frac{1}{4^{\sigma}} \right) \frac{|s|}{\sigma} \zeta(3\sigma) \ .
\end{aligned}
$$

$\boldsymbol{G_{12,r,3}}$ Gram matrices of the form

$$
B_{12,r,3} = \begin{pmatrix} a & d & d \\ d & a & -d \\ d & -d & a \end{pmatrix}
$$

possess the determinant $(a - 2d)(a + d)^2$. The essential reduction conditions are $0 < a$ and $0 \le 3d \le a$. They are not sharpened by the boundary conditions. The special case $d = 0$ must be excluded as well as $a = 3d$, since this would lead to a body-centered cubic lattice.

Then

$$
\begin{aligned}
F_{G_{12,r,3}}(s) &= \sum_{\substack{a,d\in\mathbb{N}\\ 3d<a}} \frac{1}{\left((a-2d)(a+d)^2\right)^s} = \sum_{\substack{a,d\in\mathbb{N}\\ d<a}} \frac{1}{\left(a(a+3d)^2\right)^s} \\
&= \sum_{a=1}^{\infty} \frac{1}{a^s} \sum_{d=1}^{a-1} \frac{1}{(3d+a)^{2s}} = \sum_{a=1}^{\infty} \frac{1}{a^s} \left(\sum_{d=1}^{a} \frac{1}{(3d+a)^{2s}} - \frac{1}{(4a)^{2s}} \right) \\
&= \sum_{a=1}^{\infty} \frac{1}{a^s} \left(\frac{a^{1-2s}-(4a)^{1-2s}}{3(2s-1)} - \varphi_{0,a}^{3,a}(2s) \right) - \frac{\zeta(3s)}{16^s} \\
&= \left(\frac{1}{3} - \frac{4}{3\cdot 16^s} \right) \frac{\zeta(3s-1)}{2s-1} - \frac{\zeta(3s)}{16^s} - \underbrace{\sum_{a=1}^{\infty} \frac{1}{a^s} \varphi_{0,a}^{3,a}(2s)}_{R_{12,r,3}(s):=}
\end{aligned}
$$

with

$$
\begin{aligned}
|R_{12,r,3}(s)| &\le \sum_{a=1}^{\infty} \frac{1}{a^\sigma} \left|\varphi_{0,a}^{3,a}(2s)\right| \le \sum_{a=1}^{\infty} \frac{1}{a^\sigma} \cdot \frac{|2s|}{2\sigma} \cdot \left| \frac{1}{a^{2\sigma}} - \frac{1}{(4a)^{2\sigma}} \right| \\
&= \left(1 - \frac{1}{16^\sigma}\right) \frac{|s|}{\sigma} \zeta(3\sigma) \ .
\end{aligned}
$$

As in the last case we have applied the Euler-Maclaurin formula for $x = 0$ here.

$\boldsymbol{G_{12,r,4}}$ The last type of rhombohedral trigonal lattices is represented by Gram matrices

$$
B_{12,r,4} = \begin{pmatrix} 3d & d & d \\ d & d-2f & f \\ d & f & d-2f \end{pmatrix} .
$$

Reducedness requires $0 < d \le -f$. The second inequality has to be strict as well in order to prevent the lattice from being body-centered cubic. Having calculated $\det B_{12,r,4} = (d-3f)^2$ and assuming $\sigma > \frac{1}{2}$, we get

$$
\begin{aligned}
F_{G_{12,r,4}}(s) &= \sum_{\substack{d,f\in\mathbb{Z}\\ 0<d<-f}} \frac{1}{\left(d(d-3f)^2\right)^s} = \sum_{d=1}^{\infty} \frac{1}{d^s} \sum_{f=d+1}^{\infty} \frac{1}{(3f+d)^{2s}} \\
&= \sum_{d=1}^{\infty} \frac{1}{d^s} \lim_{k\to\infty} \left(\frac{(4d)^{1-2s}-(3k+d)^{1-2s}}{3(2s-1)} - \varphi_{d,k}^{3,d}(2s) \right) \\
&= \frac{4}{3\cdot 16^s(2s-1)} \zeta(3s-1) - \underbrace{\sum_{d=1}^{\infty} \frac{1}{d^s} \lim_{k\to\infty} \varphi_{d,k}^{3,d}(2s)}_{R_{12,r,4}(s):=}
\end{aligned}
$$

with

$$
\begin{aligned}
|R_{12,r,4}(s)| &\le \sum_{d=1}^{\infty} \frac{1}{d^\sigma} \lim_{k\to\infty} \left|\varphi_{d,k}^{3,d}(s)\right| \le \sum_{d=1}^{\infty} \frac{1}{d^\sigma} \lim_{k\to\infty} \frac{|2s|}{2\sigma} \cdot \left| \frac{1}{(4d)^{2\sigma}} - \frac{1}{(3k+d)^{2\sigma}} \right| \\
&= \frac{1}{16^\sigma} \cdot \frac{|s|}{\sigma} \zeta(3\sigma) \ .
\end{aligned}
$$

Adding up all trigonal lattices leads to

$$
\begin{aligned}
F_{G_{12,r}}(s) &= \sum_{j=1}^{4} F_{G_{12,r,j}}(s) \\
&= \left(\frac{1}{s-1} + \frac{1}{2s-1}\right) \frac{\zeta(3s-1)}{3} - \left(\frac{1}{4^s} + \frac{1}{16^s}\right) \zeta(3s) - \sum_{j=1}^{4} R_{12,r,j}(s) \ .
\end{aligned}
$$

***G*₈,ₚ** The Gram matrix of a primitive orthorhombic lattice is given by an (arbitrary) diagonal matrix. So, from now on we have to consider three parameters. A lattice with Gram matrix

$$
B_{8,p} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}
$$

could be tetragonal or even cubic unless we prescribe the parameters a, b, c to be pairwise different. Furthermore, reducedness implies $0 < a < b < c$.

For $\sigma > 1$ all of the following series converge and we have

$$
\sum_{a,b,c\in\mathbb{N}} \frac{1}{(abc)^s} = 6 \sum_{\substack{a,b,c\in\mathbb{N} \\ a<b<c}} \frac{1}{(abc)^s} + 3 \sum_{\substack{a,b,c\in\mathbb{N} \\ a=b<c}} \frac{1}{(abc)^s} + 3 \sum_{\substack{a,b,c\in\mathbb{N} \\ a<b=c}} \frac{1}{(abc)^s} + \sum_{\substack{a,b,c\in\mathbb{N} \\ a=b=c}} \frac{1}{(abc)^s} \ .
$$

We rewrite this as

$$
\zeta(s)^3 = 6F_{G_{8,p}}(s) + 3F_{G_{16,p,1}}(s) + 3F_{G_{16,p,2}}(s) + F_{G_{48,p}}(s) \ .
$$

Taking into account the results that we have already established for the primitive tetragonal and cubic cases, we conclude

$$
F_{G_{8,p}}(s) = \frac{\zeta(s)^3}{6} - \frac{\zeta(s)\zeta(2s) - \zeta(3s)}{2} + \frac{\zeta(3s)}{6} = \frac{\zeta(s)^3}{6} - \frac{\zeta(s)\zeta(2s)}{2} - \frac{\zeta(3s)}{3} \ .
$$

In particular, we have an exact formula here.

***G*₈,f,₁** The first face-centered orthorhombic case corresponds to the Gram matrix

$$
B_{8,f,1} = \begin{pmatrix} 4f & 2f & 2f \\ 2f & b & f \\ 2f & f & c \end{pmatrix}
$$

with $b \neq c$ and $\det B_{8,f,1} = 4f(b-f)(c-f)$. The essential reduction conditions are given by $0 < 4f \leq b \leq c$. Boundary conditions are already fulfilled. We assume $\mathrm{Re}(s) > 1$ and start with

$$
F_{G_{8,f,1}}(s) = \sum_{\substack{b,c,f\in\mathbb{N} \\ 4f\leq b<c}} \frac{1}{\left(4f(b-f)(c-f)\right)^s} = \frac{1}{4^s} \sum_{f=1}^{\infty} \frac{1}{f^s} \sum_{b=4f}^{\infty} \frac{1}{(b-f)^s} \sum_{c=b+1}^{\infty} \frac{1}{(c-f)^s} \ .
$$

We want to apply Corollary 4.3 with $(x, y, \alpha, \beta) = (b, k, 1, -f)$ to the innermost sum but have to use its second approximation here since the first one would include an error term that would turn out to be too big. We get

$$\begin{aligned} F_{G_{8,f,1}}(s) &= \frac{1}{4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\sum_{b=4f}^{\infty}\frac{1}{(b-f)^s}\lim_{k\to\infty}\Bigg(\frac{(b-f)^{1-s}-(k-f)^{1-s}}{s-1} \\ &\quad +\frac{1}{2(k-f)^s}-\frac{1}{2(b-f)^s}-\chi_{b,k}^{1,-f}(s)\Bigg) \\ &= \frac{1}{4^s(s-1)}\sum_{f=1}^{\infty}\frac{1}{f^s}\sum_{b=4f}^{\infty}\frac{1}{(b-f)^{2s-1}}-\frac{1}{2\cdot 4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\sum_{b=4f}^{\infty}\frac{1}{(b-f)^{2s}} \qquad (41) \\ &\quad \underbrace{-\frac{1}{4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\sum_{b=4f}^{\infty}\frac{1}{(b-f)^s}\lim_{k\to\infty}\chi_{b,k}^{1,-f}(s)}_{r_1(s):=}\,. \end{aligned}$$

Next, we approximate the sums over b:

$$\begin{aligned} \sum_{b=4f}^{\infty}\frac{1}{(b-f)^x} &= \lim_{k\to\infty}\sum_{b=3f}^{k}\frac{1}{b^x} = \frac{1}{(3f)^x}+\lim_{k\to\infty}\sum_{b=3f+1}^{k}\frac{1}{b^x} \\ &= \frac{1}{(3f)^x}+\lim_{k\to\infty}\left(\frac{(3f)^{1-x}-k^{1-x}}{x-1}+\frac{1}{2k^x}-\frac{1}{2(3f)^x}-\chi_{3f,k}^{1,0}(x)\right) \\ &= \frac{1}{2(3f)^x}+\frac{1}{(x-1)(3f)^{x-1}}-\lim_{k\to\infty}\chi_{3f,k}^{1,0}(x)\ . \end{aligned}$$

Including this twice in line (41) yields

$$\begin{aligned} F_{G_{8,f,1}}(s) &= \frac{1}{2\cdot 4^s\cdot 3^{2s-1}(s-1)}\sum_{f=1}^{\infty}\frac{1}{f^{3s-1}}+\frac{1}{4^s\cdot 3^{2s-2}(s-1)(2s-2)}\sum_{f=1}^{\infty}\frac{1}{f^{3s-2}} \\ &\quad -\frac{1}{4^s(s-1)}\underbrace{\sum_{f=1}^{\infty}\frac{1}{f^s}\lim_{k\to\infty}\chi_{3f,k}^{1,0}(2s-1)}_{r_2(s):=}-\frac{1}{4\cdot 4^s\cdot 3^{2s}}\sum_{f=1}^{\infty}\frac{1}{f^{3s}} \\ &\quad -\frac{1}{2\cdot 4^s\cdot 3^{2s-1}(2s-1)}\sum_{f=1}^{\infty}\frac{1}{f^{3s-1}}+\underbrace{\frac{1}{2\cdot 4^s}\sum_{f=1}^{\infty}\frac{1}{f^s}\lim_{k\to\infty}\chi_{3f,k}^{1,0}(2s)}_{r_3(s):=}-r_1(s)\ . \end{aligned}$$

We simplify and express the remaining series with the help of the Riemann zeta function:

$$\begin{aligned} F_{G_{8,f,1}}(s) &= \frac{9\cdot\zeta(3s-2)}{2\cdot 36^s(s-1)^2}+\frac{3s\cdot\zeta(3s-1)}{2\cdot 36^s(s-1)(2s-1)}-\frac{\zeta(3s)}{4\cdot 36^s} \\ &\quad -r_1(s)-\frac{1}{4^s(s-1)}r_2(s)+r_3(s)\ . \end{aligned}$$

The first two terms possess a pole at $\alpha = 1$ of order 3 and 1, respectively. There is no pole with a bigger real part in the explicit expressions. We estimate the three error terms. The first one can be bounded by

$$|r_1(s)| \le \frac{1}{4^\sigma} \sum_{f=1}^{\infty} \frac{1}{f^\sigma} \sum_{b=4f}^{\infty} \frac{1}{(b-f)^\sigma} \lim_{k\to\infty} \frac{|s|\cdot|s+1|}{8(\sigma+1)} \left(\frac{1}{(b-f)^{\sigma+1}} - \frac{1}{(k-f)^{\sigma+1}} \right)$$
$$= \frac{|s|\cdot|s+1|}{2\cdot 4^{\sigma+1}(\sigma+1)} \sum_{f=1}^{\infty} \frac{1}{f^\sigma} \sum_{b=3f}^{\infty} \frac{1}{b^{2\sigma+1}} \;.$$

The sum over b was already determined above. We get

$$|r_1(s)| \le \frac{|s|\cdot|s+1|}{2\cdot 4^{\sigma+1}(\sigma+1)} \sum_{f=1}^{\infty} \frac{1}{f^\sigma} \left(\frac{1}{2(3f)^{2\sigma+1}} + \frac{1}{2\sigma(3f)^{2\sigma}} - \lim_{k\to\infty} \chi^{1,0}_{3f,k}(2\sigma+1) \right)$$
$$\le \frac{|s|\cdot|s+1|}{16\cdot 36^\sigma(\sigma+1)} \left(\frac{\zeta(3\sigma+1)}{3} + \frac{\zeta(3\sigma)}{\sigma} \right)$$
$$+ \frac{|s|\cdot|s+1|}{2\cdot 4^{\sigma+1}(\sigma+1)} \sum_{f=1}^{\infty} \frac{1}{f^\sigma} \lim_{k\to\infty} \frac{|2\sigma+1|\cdot|2\sigma+2|}{8(2\sigma+2)} \left(\frac{1}{(3f)^{2\sigma+2}} - \frac{1}{k^{2\sigma+2}} \right)$$
$$= \frac{|s|\cdot|s+1|}{16\cdot 36^\sigma(\sigma+1)} \left(\frac{\zeta(3\sigma+1)}{3} + \frac{\zeta(3\sigma)}{\sigma} + \frac{|2\sigma+1|\cdot\zeta(3\sigma+2)}{36} \right),$$

from which we infer that $r_1(s)$ is holomorphic for $\sigma > \frac{1}{3}$. For the second error expression we have

$$|r_2(s)| \le \sum_{f=1}^{\infty} \frac{1}{f^\sigma} \lim_{k\to\infty} \frac{|2s-1|\cdot|2s|}{8\cdot 2\sigma} \left(\frac{1}{(3f)^{2\sigma}} - \frac{1}{k^{2\sigma}} \right) = \frac{|2s-1|\cdot|s|\cdot\zeta(3\sigma)}{8\sigma\cdot 9^\sigma} \;.$$

Hence, $r_2(s)$ is holomorphic for $\sigma > \frac{1}{3}$, too. Finally, the same holds for $r_3(s)$ since

$$|r_3(s)| \le \frac{1}{2\cdot 4^\sigma} \sum_{f=1}^{\infty} \frac{1}{f^\sigma} \lim_{k\to\infty} \frac{|2s|\cdot|2s+1|}{8(2\sigma+1)} \left(\frac{1}{(3f)^{2\sigma+1}} - \frac{1}{k^{2\sigma+1}} \right)$$
$$= \frac{|s|\cdot|2s+1|\cdot\zeta(3\sigma+1)}{24\cdot 36^\sigma(2\sigma+1)} \;.$$

We define

$$R_{8,f,1}(s) := \frac{-r_2(s)}{4^s} \quad \text{and} \quad S_{8,f,1}(s) := r_3(s) - r_1(s) - \frac{\zeta(3s)}{4\cdot 36^s} \;.$$

Then, $R_{8,f,1}$ and $S_{8,f,1}$ are holomorphic for $\sigma > \frac{1}{3}$ and fulfil

$$F_{G_{8,f,1}}(s) = \frac{9\cdot\zeta(3s-2)}{2\cdot 36^s(s-1)^2} + \frac{3s\cdot\zeta(3s-1)}{2\cdot 36^s(s-1)(2s-1)} + \frac{R_{8,f,1}(s)}{s-1} + S_{8,f,1}(s)$$

for $\sigma > 1$. We can see now that the rightmost pole of $F_{G_{8,f,1}}(s)$ is given by $\alpha = 1$ and has order 3. We will calculate $\lim_{\substack{s\to 1\\ \sigma>1}} (s-1)^3 F_{G_{8,f,1}}(s)$ in the proof of the following Proposition 4.5.

$\boldsymbol{G_{8,f,2}}$ In the second face-centered orthorhombic case we have Gram matrices of the form

$$B_{8,f,2} = \begin{pmatrix} d+2e & d & e \\ d & d+2e & -e \\ e & -e & c \end{pmatrix}$$

with essential conditions $d \le 2e < d+2e \le c$. Assume $d = 2e$. Then $2d = d+2e$ and the boundary conditions would require $e \le -e$. But $e > 0$ and thus, $d = 2e$ is not possible. To exclude lattices of type $16, i, 3$ we suppose $d + 2e \neq c$. Then, the other boundary conditions give no further restrictions. The determinant $\det B_{8,f,2} = 4e(c-e)(d+e)$ leads to the Dirichlet series

$$F_{G_{8,f,2}}(s) = \sum_{\substack{c,d,e\in\mathbb{N}\\ 2d<d+2e<c}} \frac{1}{\left(4e(c-e)(d+e)\right)^s} = \frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^s}\sum_{c=d+e+1}^{\infty}\frac{1}{c^s}\,.$$

We suppose $\mathrm{Re}(s) > 1$ and apply Corollary 4.3 with $(x, y, \alpha, \beta) = (d + e, k, 1, 0)$ to the sum over c which yields

$$\begin{aligned} F_{G_{8,f,2}}(s) &= \frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^s}\lim_{k\to\infty}\Bigg(\frac{(d+e)^{1-s}-k^{1-s}}{s-1} \\ &\quad + \frac{1}{2k^s} - \frac{1}{2(d+e)^s} - \chi^{1,0}_{d+e,k}(s)\Bigg) \\ &= \frac{1}{4^s(s-1)}\sum_{e=1}^{\infty}\frac{1}{e^s}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^{2s-1}} - \frac{1}{2\cdot 4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^{2s}} \\ &\quad \underbrace{- \frac{1}{4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^s}\lim_{k\to\infty}\chi^{1,0}_{d+e,k}(s)}_{r_1(s):=}\,. \end{aligned} \tag{42}$$

We approximate the explicit sums over d

$$\begin{aligned} \sum_{d=1}^{2e-1}\frac{1}{(d+e)^x} &= \sum_{d=e+1}^{3e}\frac{1}{d^x} - \frac{1}{(3e)^x} = \frac{e^{1-x}-(3e)^{1-x}}{x-1} - \frac{1}{2(3e)^x} - \frac{1}{2e^x} - \chi^{1,0}_{e,3e}(x) \\ &= \frac{e^{1-x}}{x-1}\left(1-3^{1-x}\right) - \frac{1}{2e^x}\left(1+3^{-x}\right) - \chi^{1,0}_{e,3e}(x) \end{aligned}$$

and include the result twice in line (42) to get

$$\begin{aligned} F_{G_{8,f,2}}(s) &= \frac{1-3^{2-2s}}{4^s(s-1)(2s-2)}\sum_{e=1}^{\infty}\frac{1}{e^{3s-2}} - \frac{1+3^{1-2s}}{2\cdot 4^s(s-1)}\sum_{e=1}^{\infty}\frac{1}{e^{3s-1}} \\ &\quad - \frac{1}{4^s(s-1)}\underbrace{\sum_{e=1}^{\infty}\frac{1}{e^s}\cdot\chi^{1,0}_{e,3e}(2s-1)}_{r_2(s):=} - \frac{1-3^{1-2s}}{2\cdot 4^s(2s-1)}\sum_{e=1}^{\infty}\frac{1}{e^{3s-1}} \\ &\quad + \frac{1+3^{-2s}}{4\cdot 4^s}\sum_{e=1}^{\infty}\frac{1}{e^{3s}} + \underbrace{\frac{1}{2\cdot 4^s}\sum_{e=1}^{\infty}\frac{1}{e^s}\cdot\chi^{1,0}_{e,3e}(2s)}_{r_3(s):=} - r_1(s)\,. \end{aligned}$$

We convert this to

$$\begin{aligned} F_{G_{8,f,2}}(s) &= \frac{9^s-9}{2\cdot 36^s(s-1)^2}\cdot\zeta(3s-2) - \frac{9^s(3s-2)+3s}{2\cdot 36^s(s-1)(2s-1)}\cdot\zeta(3s-1) \\ &\quad + \frac{9^s+1}{4\cdot 36^s}\cdot\zeta(3s) - r_1(s) - \frac{1}{4^s(s-1)}r_2(s) + r_3(s)\,. \end{aligned}$$

As in the first face-centered orthorhombic case, the first two expressions have a pole at $\alpha = 1$ of order 3 and 1, respectively, and no pole with a bigger real part occurs in the explicit terms. We will show, that the domain of analyticity of the three error terms is large enough for the application of Delange's Theorem. We start with

$$|r_1(s)| \le \frac{1}{4^\sigma}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^\sigma}\lim_{k\to\infty}\frac{|s|\cdot|s+1|}{8(\sigma+1)}\left(\frac{1}{(d+e)^{\sigma+1}}-\frac{1}{k^{\sigma+1}}\right)$$
$$= \frac{|s|\cdot|s+1|}{2\cdot 4^{\sigma+1}(\sigma+1)}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\sum_{d=1}^{2e-1}\frac{1}{(d+e)^{2\sigma+1}}\ .$$

The sum over d has been calculated above. We get

$$|r_1(s)| \le \frac{|s|\cdot|s+1|}{2\cdot 4^{\sigma+1}(\sigma+1)}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\left(\frac{1-3^{-2\sigma}}{2\sigma e^{2\sigma}}-\frac{1+3^{-2\sigma-1}}{2e^{2\sigma+1}}-\chi^{1,0}_{e,3e}(2\sigma+1)\right)$$
$$\le \frac{|s|\cdot|s+1|}{16\cdot 36^\sigma(\sigma+1)}\left(\frac{\zeta(3\sigma)}{\sigma}\left(3^{2\sigma}-1\right)-\zeta(3\sigma+1)\left(3^{2\sigma}+3^{-1}\right)\right)$$
$$+\frac{|s|\cdot|s+1|}{2\cdot 4^{\sigma+1}(\sigma+1)}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\cdot\frac{|2\sigma+1|\cdot|2\sigma+2|}{8(2\sigma+2)}\left(\frac{1}{e^{2\sigma+2}}-\frac{1}{(3e)^{2\sigma+2}}\right)$$
$$= \frac{|s|\cdot|s+1|}{16\cdot 36^\sigma(\sigma+1)}\left(\frac{\zeta(3\sigma)}{\sigma}\left(3^{2\sigma}-1\right)-\zeta(3\sigma+1)\left(3^{2\sigma}+3^{-1}\right)\right.$$
$$\left.+\frac{|2\sigma+1|\cdot\zeta(3\sigma+2)}{36}\left(3^{2\sigma}-1\right)\right).$$

We conclude that $r_1(s)$ is holomorphic for $\sigma > \frac{1}{3}$. Next, we show that the same is true for the second error term:

$$|r_2(s)| \le \sum_{e=1}^{\infty}\frac{1}{e^\sigma}\cdot\frac{|2s-1|\cdot|2s|}{8\cdot 2\sigma}\left(\frac{1}{e^{2\sigma}}-\frac{1}{(3e)^{2\sigma}}\right) = \frac{|2s-1|\cdot|s|\cdot\zeta(3\sigma)}{8\sigma\cdot 9^\sigma}\left(9^\sigma-1\right).$$

It remains to estimate the third error. We have

$$|r_3(s)| \le \frac{1}{2\cdot 4^\sigma}\sum_{e=1}^{\infty}\frac{1}{e^\sigma}\cdot\frac{|2s|\cdot|2s+1|}{8(2\sigma+1)}\left(\frac{1}{e^{2\sigma+1}}-\frac{1}{(3e)^{2\sigma+1}}\right)$$
$$= \frac{|s|\cdot|2s+1|\cdot\zeta(3\sigma+1)}{24\cdot 36^\sigma(2\sigma+1)}\left(3^{2\sigma+1}-1\right)$$

and see that $r_3(s)$ is holomorphic for $\sigma > \frac{1}{3}$ as well. We define new error terms

$$R_{8,f,2}(s) := \frac{-r_2(s)}{4^s} \quad\text{and}\quad S_{8,f,2}(s) := r_3(s) - r_1(s) + \frac{9^s+1}{4\cdot 36^s}\zeta(3s)$$

that are holomorphic for $\sigma > \frac{1}{3}$. With this definitions and $\sigma > 1$, we get

$$F_{G_{8,f,2}}(s) = \frac{(9^s-9)\,\zeta(3s-2)}{2\cdot 36^s(s-1)^2} - \frac{(9^s(3s-2)+3s)\,\zeta(3s-1)}{2\cdot 36^s(s-1)(2s-1)} + \frac{R_{8,f,2}(s)}{s-1} + S_{8,f,2}(s)\,.$$

As in the first face-centered orthorhombic case the rightmost pole is at $\alpha = 1$.

The proof of Proposition 4.4 is finished. □

Proposition 4.5. *For every index i Table 3 gives a function f_i with $f_i = f_{G_i}$, that is $f_i(x) \sim H_{G_i}(x)$ for $x \to \infty$.*

Proof. For every index i the Theorem of Delange (Proposition 4.2) will be applied to $F_i(s)$.

$\boldsymbol{G_{48,p}}$ The Dirichlet series $F_{G_{48,p}}(s) = \zeta(3s)$ has a pole of order one at $\alpha = \frac{1}{3}$. We want to use Proposition 4.2 for $n = 0$. If s tends to 1 from the right-hand side, that is, with $\sigma > 1$, we have

$$1 = \lim_{\substack{s\to 1\\ \sigma>1}} (s-1)\zeta(s) = \lim_{\substack{s\to \frac{1}{3}\\ \sigma>\frac{1}{3}}} (3s-1)\zeta(3s).$$

We deduce

$$\lim_{\substack{s\to \frac{1}{3}\\ \sigma>\frac{1}{3}}} \left(s - \frac{1}{3}\right) F_{G_{48,p}}(s) = \frac{1}{3}\ .$$

Thus, there is a function $g(s)$ with $g(\alpha) = \frac{1}{3}$, which is holomorphic (in this case in the complete complex plane $\mathbb{C}$) such that $F_{48,p}(s)$ has a singularity of the form

$$F_{48,p}(s) = \frac{g(s)}{\left(s - \frac{1}{3}\right)}$$

at $\alpha = \frac{1}{3}$. Applying Delange's Theorem yields

$$f_{G_{48,p}}(x) = x^{\frac{1}{3}}.$$

$\boldsymbol{G_{48,i}}$ The rightmost singularity of $F_{G_{48,i}}(s) = \frac{1}{16^s}\zeta(3s)$ is a pole of order one at $\alpha = \frac{1}{3}$. We have

$$\lim_{\substack{s\to \frac{1}{3}\\ \sigma>\frac{1}{3}}} \left(s - \frac{1}{3}\right) F_{G_{48,i}}(s) = \frac{1}{16^{\frac{1}{3}}} \cdot \frac{1}{3} = 2^{-\frac{4}{3}}3^{-1}.$$

As for the primitive case, there is a holomorphic function $g(s)$ with $g(\alpha) = 2^{-\frac{4}{3}}3^{-1}$ and

$$F_{48,i}(s) = \frac{g(s)}{\left(s - \frac{1}{3}\right)}$$

at the pole α. Delange's Theorem for $n = 0$ and $\alpha = \frac{1}{3}$ gives us

$$f_{G_{48,i}}(x) = 2^{-\frac{4}{3}}x^{\frac{1}{3}}.$$

$\boldsymbol{G_{48,f}}$ An analogous calculation for $F_{G_{48,f}}(s) = \frac{1}{4^s}\zeta(3s)$ shows that

$$f_{G_{48,i}}(x) = 2^{-\frac{2}{3}}x^{\frac{1}{3}}.$$

$\boldsymbol{G_{24,p,1}}$ The Dirichlet series for the first hexagonal case is given by

$$F_{24,p,1}(s) = \frac{2}{6^s(s-1)}\zeta(3s-1) + \frac{1}{6^s}\zeta(3s) - R_{24,p,1}(s).$$

The first expression on the right-hand side has a pole at $\alpha = 1$ whereas the second and the third expression converge for $\sigma > \frac{1}{3}$. We have

$$\lim_{\substack{s\to 1\\ \sigma>1}} (s-1)\, F_{G_{24,p,1}}(s) = \frac{1}{3}\zeta(2)\,.$$

Thus, there is a function $g(s)$ with $g(1) = \frac{1}{3}\zeta(2)$ that is holomorphic in the halfplane to the right of $s = \frac{2}{3}$ (the second rightmost singularity of $F_{24,p,1}(s)$). We can apply Proposition 4.2 with $(n, \alpha, g(\alpha)) = \left(0, 1, \frac{1}{3}\zeta(2)\right)$ to obtain

$$f_{G_{24,p,1}}(x) = \frac{1}{3}\zeta(2)\, x\,.$$

***G*₂₄,ₚ,₂** In the second hexagonal case we have

$$F_{24,p,2}(s) = \frac{2^{2s-1}}{3^s(2s-1)}\zeta(3s-1) + \frac{8^s-1}{2\cdot 6^s}\zeta(3s) - R_{24,p,2}(s)\,,$$

and once again the second and the third expression converge for $\sigma > \frac{1}{3}$. Regarding the first term, the pole of the zeta function at $\alpha = \frac{2}{3}$ has a bigger real part than the zero of the denominator. From

$$1 = \lim_{\substack{s\to 1\\ \sigma>1}}(s-1)\zeta(s) = \lim_{\substack{s\to 2\\ \sigma>2}}(s-2)\zeta(s-1) = \lim_{\substack{s\to \frac{2}{3}\\ \sigma>\frac{2}{3}}}(3s-2)\zeta(3s-1) \tag{43}$$

we infer

$$\lim_{\substack{s\to \frac{2}{3}\\ \sigma>\frac{2}{3}}} \left(s-\frac{2}{3}\right) F_{G_{24,p,2}}(s) = \frac{1}{3}\cdot\frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\cdot\frac{1}{3}} = 2^{\frac{1}{3}}3^{-\frac{2}{3}}\,.$$

Applying Delange's Theorem to the case $(n, \alpha, g(\alpha)) = \left(0, \frac{2}{3}, 2^{\frac{1}{3}}3^{-\frac{2}{3}}\right)$ yields

$$f_{G_{24,p,2}}(x) = \sqrt[3]{\frac{3}{4}}\cdot x^{\frac{2}{3}}\,.$$

$G_{16,p,1}$ For the first primitive tetragonal case we have determined the Dirichlet series

$$F_{16,p,1}(s) = \zeta(s)\zeta(2s) - \frac{1}{2s-1}\zeta(3s-1) - \zeta(3s) + R_{16,p}(s)\,.$$

The crucial expression on the right-hand side is the first one, having a simple pole at $\alpha = 1$, since all the other terms converge for $\sigma > \frac{2}{3}$. As before, we can deduce the existence of a holomorphic function g with

$$g(1) = \lim_{\substack{s\to 1\\ \sigma>1}}(s-1)F_{16,p,1}(s) = \zeta(2)\,.$$

With the Theorem of Delange we see that

$$f_{G_{16,p,1}}(x) = \zeta(2)\, x\,.$$

$\boldsymbol{G_{16,p,2}}$ We start with

$$F_{16,p,2}(s) = \frac{1}{2s-1}\zeta(3s-1) - R_{16,p}(s)\,.$$

The rightmost singularity is the pole of $\zeta(3s-1)$ at $\alpha = \frac{2}{3}$. Using the result from line (43), we get

$$\lim_{\substack{s\to\frac{2}{3}\\ \sigma>\frac{2}{3}}} \left(s-\frac{2}{3}\right) F_{G_{16,p,2}}(s) = \frac{1}{3}\cdot\frac{1}{\frac{1}{3}} = 1\,,$$

and, applying Proposition 4.2 to $(n,\alpha,g(\alpha)) = \left(0,\frac{2}{3},1\right)$,

$$f_{G_{16,p,2}}(x) = \frac{3}{2}\,x^{\frac{2}{3}}$$

for the asymptotic frequency of primitive tetragonal lattices of the second type.

$\boldsymbol{G_{16,i,1}}$ From Proposition 4.4 we have

$$F_{16,i,1}(s) = \frac{1}{4^s(s-1)}\zeta(3s-1) - R_{16,i,1}(s)\,.$$

For $(n,\alpha,g(\alpha)) = \left(0,1,\frac{1}{4}\zeta(2)\right)$ Delange's Theorem gives us

$$f_{G_{16,i,1}}(x) = \frac{1}{4}\zeta(2)\,x\,.$$

$\boldsymbol{G_{16,i,2}}$ The rightmost singularity of the Dirichlet series

$$F_{16,i,2}(s) = \left(\frac{1}{4^s}-\frac{2}{16^s}\right)\frac{1}{2s-1}\zeta(3s-1) - \frac{1}{16^s}\zeta(3s) - R_{16,i,2}(s)$$

is the simple pole of $\zeta(3s-1)$ at $\alpha = \frac{2}{3}$. Considering (43) again, we have

$$\lim_{\substack{s\to\frac{2}{3}\\ \sigma>\frac{2}{3}}} \left(s-\frac{2}{3}\right) F_{G_{16,i,2}}(s) = \frac{1}{3}\left(\frac{1}{2^{\frac{4}{3}}}-\frac{2}{2^{\frac{8}{3}}}\right)\frac{1}{\frac{1}{3}} = \frac{1}{\sqrt[3]{16}} - \frac{1}{\sqrt[3]{32}}$$

which leads to

$$f_{G_{16,i,2}}(x) = \frac{3}{2}\left(\frac{1}{\sqrt[3]{16}}-\frac{1}{\sqrt[3]{32}}\right)x^{\frac{2}{3}} = \frac{3}{8}\left(\sqrt[3]{4}-\sqrt[3]{2}\right)x^{\frac{2}{3}}\,.$$

$\boldsymbol{G_{16,i,3}}$ The Dirichlet series

$$F_{16,i,3}(s) = \left(\frac{2}{16^s}-\frac{3}{36^s}\right)\frac{1}{2s-1}\zeta(3s-1) - \frac{1}{36^s}\zeta(3s) - R_{16,i,3}(s)$$

is very similar to $F_{16,i,2}(s)$ from the last case. We end up with

$$\lim_{\substack{s\to\frac{2}{3}\\ \sigma>\frac{2}{3}}} \left(s-\frac{2}{3}\right) F_{G_{16,i,3}}(s) = \frac{2}{2^{\frac{8}{3}}} - \frac{3}{2^{\frac{4}{3}}\cdot 3^{\frac{4}{3}}} = \frac{1}{\sqrt[3]{32}} - \frac{1}{\sqrt[3]{48}}$$

and

$$f_{G_{16,i,3}}(x) = \frac{3}{2}\left(\frac{1}{\sqrt[3]{32}}-\frac{1}{\sqrt[3]{48}}\right)x^{\frac{2}{3}} = \frac{3}{8}\left(\sqrt[3]{2}-\sqrt[3]{\frac{4}{3}}\right)x^{\frac{2}{3}}\,.$$

$\boldsymbol{G_{16,i,4}}$ The next Dirichlet series also resembles very much the ones of the last two cases:

$$F_{16,i,4}(s) = \frac{3}{36^s(2s-1)}\zeta(3s-1) + \frac{1}{36^s}\zeta(3s) - R_{16,i,4}(s)\,.$$

Again, the relevant singularity is the simple pole of $\zeta(3s-1)$ at $\alpha = \frac{2}{3}$. From

$$\lim_{\substack{s\to\frac{2}{3}\\ \sigma>\frac{2}{3}}} \left(s - \frac{2}{3}\right) F_{G_{16,i,4}}(s) = \frac{3}{2^{\frac{4}{3}}\cdot 3^{\frac{4}{3}}} = \frac{1}{\sqrt[3]{48}}$$

we get to

$$f_{G_{16,i,4}}(x) = \frac{3}{8}\sqrt[3]{\frac{4}{3}}\; x^{\frac{2}{3}}$$

with the help of Delange's Theorem.

$\boldsymbol{G_{12,r,1}}$ For the first rhombohedral trigonal case we have calculated

$$F_{12,r,1}(s) = \frac{4}{3\cdot 4^s(s-1)}\zeta(3s-1) - R_{12,r,1}(s)\,.$$

With $(n,\alpha,g(\alpha)) = \big(0,1,\frac{1}{3}\zeta(2)\big)$ we get

$$f_{G_{12,r,1}}(x) = \frac{1}{3}\zeta(2)\,x\,.$$

$\boldsymbol{G_{12,r,2}}$ Next, we have to deal with

$$F_{12,r,2}(s) = \left(\frac{1}{3} - \frac{4}{3\cdot 4^s}\right)\frac{1}{s-1}\zeta(3s-1) - \frac{1}{4^s}\zeta(3s) - R_{12,r,2}(s)\,.$$

The rightmost singularity lies at $s=1$. Because of

$$\begin{aligned}\lim_{s\to 1}\left(\frac{1}{3} - \frac{4}{3\cdot 4^s}\right)\frac{\zeta(3s-1)}{s-1} &= \zeta(2)\lim_{s\to 1}\frac{\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{3} - \frac{4}{3\cdot 4^s}\right)}{\frac{\mathrm{d}}{\mathrm{d}s}(s-1)}\\ &= \zeta(2)\lim_{s\to 1}\left(-\frac{4}{3\cdot 4^s}\right)\log(4)(-1) = \frac{2\log 2}{3}\zeta(2)\,,\end{aligned}$$

this singularity, however, is removable. Hence, we have to look at $\alpha = \frac{2}{3}$, the simple pole of $\zeta(3s-1)$. With (43) we conclude

$$\lim_{\substack{s\to\frac{2}{3}\\ \sigma>\frac{2}{3}}} \left(s - \frac{2}{3}\right) F_{G_{12,r,2}}(s) = \frac{1}{3}\left(\frac{1}{3} - \frac{4^{\frac{1}{3}}}{3}\right)\frac{1}{-\frac{1}{3}} = \frac{1}{3}\left(\sqrt[3]{4}-1\right).$$

Applying Proposition 4.2 yields

$$f_{G_{12,r,2}}(x) = \frac{1}{2}\left(\sqrt[3]{4}-1\right)x^{\frac{2}{3}}\,.$$

$\boldsymbol{G_{12,r,3}}$ The Dirichlet series for the third rhombohedral trigonal case

$$F_{12,r,3}(s) = \left(\frac{1}{3} - \frac{4}{3 \cdot 16^s}\right) \frac{1}{2s-1}\zeta(3s-1) - \frac{1}{16^s}\zeta(3s) - R_{12,r,3}(s)\,.$$

also leads to $\alpha = \frac{2}{3}$ and $n = 0$. We have seen several times already that

$$\lim_{\substack{s \to \frac{2}{3} \\ \sigma > \frac{2}{3}}} \left(s - \frac{2}{3}\right) \frac{1}{2s-1}\zeta(3s-1) = 1\,,$$

thus,

$$\lim_{\substack{s \to \frac{2}{3} \\ \sigma > \frac{2}{3}}} \left(s - \frac{2}{3}\right) F_{G_{12,r,3}}(s) = \frac{1}{3} - \frac{1}{3 \cdot 2^{\frac{2}{3}}} = \frac{1}{3}\left(1 - \frac{1}{\sqrt[3]{4}}\right).$$

Delange's Theorem gives us

$$f_{G_{12,r,3}}(x) = \frac{1}{2}\left(1 - \frac{1}{\sqrt[3]{4}}\right) x^{\frac{2}{3}}\,.$$

$\boldsymbol{G_{12,r,4}}$ The last trigonal case is similar to the third one. We have

$$F_{12,r,4}(s) = \frac{4}{3 \cdot 16^s(2s-1)}\zeta(3s-1) - R_{12,r,4}(s)$$

with a simple pole at $\alpha = \frac{2}{3}$ and

$$\lim_{\substack{s \to \frac{2}{3} \\ \sigma > \frac{2}{3}}} \left(s - \frac{2}{3}\right) F_{G_{12,r,4}}(s) = \frac{1}{3 \cdot 2^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{4}}\,.$$

Hence, the asymptotic frequency is given by

$$f_{G_{12,r,4}}(x) = \frac{1}{2\sqrt[3]{4}}\, x^{\frac{2}{3}} = \frac{1}{\sqrt[3]{32}}\, x^{\frac{2}{3}}\,.$$

$\boldsymbol{G_{8,p}}$ The Dirichlet series for the primitive orthorhombic case is

$$F_{8,p}(s) = \frac{1}{6}\zeta(s)^3 - \frac{1}{2}\zeta(s)\zeta(2s) - \frac{1}{3}\zeta(3s)\,.$$

Its rightmost singularity is a pole of order 3 at $\alpha = 1$. We have

$$\lim_{\substack{s \to 1 \\ \sigma > 1}} (s-1)^3\, F_{G_{8,p}}(s) = \frac{1}{6}\,.$$

We apply Proposition 4.2 with $(n, \alpha, g(\alpha)) = \left(2, 1, \frac{1}{6}\right)$ and end up with

$$f_{G_{8,p}}(x) = \frac{1}{12}\, x\, (\log x)^2\,.$$

$\boldsymbol{G_{8,f,1}}$ For $\sigma > 1$ we have

$$F_{G_{8,f,1}}(s) = \frac{9 \cdot \zeta(3s-2)}{2 \cdot 36^s (s-1)^2} + \frac{3s \cdot \zeta(3s-1)}{2 \cdot 36^s (s-1)(2s-1)} + \frac{R_{8,f,1}(s)}{s-1} + S_{8,f,1}(s)$$

with $R_{8,f,1}(s)$ and $S_{8,f,1}(s)$ being analytic around $\alpha = 1$. Furthermore,

$$\begin{aligned}\lim_{\substack{s\to 1\\ \sigma>1}} (s-1)\zeta(3s-2) &= \lim_{\substack{s\to 3\\ \sigma>3}} \left(\frac{s}{3} - 1\right) \zeta(s-2) = \lim_{\substack{s\to 1\\ \sigma>1}} \left(\frac{s+2}{3} - 1\right) \zeta(s) \\ &= \frac{1}{3} \lim_{\substack{s\to 1\\ \sigma>1}} (s-1)\zeta(s) = \frac{1}{3}\,,\end{aligned}$$

and thus

$$\lim_{\substack{s\to 1\\ \sigma>1}} (s-1)^3 F_{G_{8,f,1}}(s) = \lim_{\substack{s\to 1\\ \sigma>1}} \frac{3}{2 \cdot 36^s} = \frac{1}{24}\,.$$

Delange's Theorem for $(n, \alpha, g(\alpha)) = \left(2, 1, \frac{1}{24}\right)$ gives us

$$f_{G_{8,f,1}}(x) = \frac{1}{48}\, x \left(\log x\right)^2 .$$

$\boldsymbol{G_{8,f,2}}$ As in the last case we use

$$\lim_{\substack{s\to 1\\ \sigma>1}} (s-1)\zeta(3s-2) = \frac{1}{3}$$

to infer

$$\lim_{\substack{s\to 1\\ \sigma>1}} (s-1)^2 F_{G_{8,f,2}}(s) = \lim_{\substack{s\to 1\\ \sigma>1}} \frac{9^s - 9}{6 \cdot 36^s (s-1)} = \lim_{\substack{s\to 1\\ \sigma>1}} \frac{9^s \log 9}{6 \cdot 36^s} = \frac{\log 3}{12}\,.$$

With $(n, \alpha, g(\alpha)) = \left(1, 1, \frac{\log 3}{12}\right)$ we conclude

$$f_{G_{8,f,2}}(x) = \frac{\log 3}{12}\, x \log x\,.$$

□

In the rest of the chapter we determine the class numbers of type $H_G^L(D)$ of complete Bravais classes. We will not use the finer subdivision into isometry classes of Schiemann reduced lattices any longer since their class numbers seem to be harder to calculate.

Proposition 4.6. *The aggregated class numbers of the body-centered orthorhombic and base-centered orthorhombic lattices fulfil the following asymptotic equivalences:*

$$H_{8,i}(D) \sim \frac{D}{48} \left(\log D\right)^2 \text{ and } H_{8,b}(D) \sim \frac{D}{8} \left(\log D\right)^2 .$$

Proof. We begin with the body-centered case. We consider lattices (L, b) with b the usual dot product and

$$L := \mathbb{Z} \begin{pmatrix} \sqrt{a} \\ 0 \\ 0 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ \sqrt{b} \\ 0 \end{pmatrix} + \mathbb{Z}\, \frac{1}{2} \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \\ \sqrt{c} \end{pmatrix}, \ a, b, c \in \mathbb{N}\,.$$

Then, the given vectors form a basis for L and

$$\det L = \left| \begin{pmatrix} a & 0 & \frac{a}{2} \\ 0 & b & \frac{b}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{a+b+c}{4} \end{pmatrix} \right| = \frac{1}{4} \left| \begin{pmatrix} a & 0 & a \\ 0 & b & b \\ a & b & a+b+c \end{pmatrix} \right| = \frac{1}{4} abc \ .$$

The Gram matrix shows that for the lattice to be integral we must have $a, b \in 2\mathbb{Z}$ and $a+b+c \in 4\mathbb{Z}$. In order to exclude the cases of body-centered tetragonal and body-centered cubic lattices we assume a, b and c to be pairwise different. Interchanging the values of a, b and c yields isometric copies of the same lattice. Except for this ambiguity there are no further isometries between lattices of this type: Let

$$M := \mathbb{Z} \begin{pmatrix} \sqrt{x} \\ 0 \\ 0 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ \sqrt{y} \\ 0 \end{pmatrix} + \mathbb{Z} \frac{1}{2} \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \\ \sqrt{z} \end{pmatrix}, \ x, y, z \in 2\mathbb{N}, \ x+y+z \in 4\mathbb{N}$$

be another lattice of this kind. Without loss of generality, we can assume $a < b < c$ and $x < y < z$. Suppose that there is an isometry $F : L \to M$. We will prove that $L = M$ by showing that $(a, b, c) = (x, y, z)$. The square length of an arbitrary lattice vector

$$u = i \begin{pmatrix} \sqrt{a} \\ 0 \\ 0 \end{pmatrix} + j \begin{pmatrix} 0 \\ \sqrt{b} \\ 0 \end{pmatrix} + \frac{k}{2} \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \\ \sqrt{c} \end{pmatrix}, \ i, j, k \in \mathbb{Z}$$

is given by

$$|u|^2 = \left(i + \frac{k}{2}\right)^2 a + \left(j + \frac{k}{2}\right)^2 b + \frac{k^2}{4} \cdot c \ . \tag{44}$$

Let $u \neq 0$.
Suppose $k = 0$, then $|u|^2$ is minimal for $i = \pm 1$ and $j = 0$ resulting in $|u|^2 = a$. The corresponding vectors are the first basis vector and its negative. The shortest vectors that are not spanned by these arise for $i = 0$ and $j = \pm 1$ (second basis vector and its negative). For $k \neq 0$, the square length is bounded from below by $\frac{a+b+c}{4}$. The bound is (only) realised for $k = 1, \ i, j \in \{0, -1\}$ and $k = -1, \ i, j \in \{0, 1\}$. The vectors that correspond to these eight possibilities span the whole lattice.

We determine the successive minima of the lattice and count the vectors that realise these minima. Depending on the value of $\frac{a+b+c}{4}$ we distinguish the following cases (a visualisation can be found in the appendix on page 89 and onwards):

Case	**Condition**	**Successive Minima**	**# Minimal Vectors**
1	$\frac{a+b+c}{4} < a < b$	$(\frac{a+b+c}{4}, \frac{a+b+c}{4}, \frac{a+b+c}{4})$	$(8, 8, 8)$
2	$\frac{a+b+c}{4} = a < b$	(a, a, a)	$(10, 10, 10)$
3	$a < \frac{a+b+c}{4} < b$	$(a, \frac{a+b+c}{4}, \frac{a+b+c}{4})$	$(2, 8, 8)$
4	$a < \frac{a+b+c}{4} = b$	(a, b, b)	$(2, 10, 10)$
5	$a < b < \frac{a+b+c}{4}$	$(a, b, \frac{a+b+c}{4})$	$(2, 2, 8)$

Since isometries preserve square lengths and linear independence of vectors (therefore, the successive minima as well), L and M must belong to the same case (x, y, z replacing a, b, c in the table).

- In the last case the equality of the successive minima $(a, b, \frac{a+b+c}{4}) = (x, y, \frac{x+y+z}{4})$ yields $(a, b, c) = (x, y, z)$ at once.

- In Case 4 we start with $(a, b) = (x, y)$. The assumptions of the case being $b = \frac{a+b+c}{4}$ and $y = \frac{x+y+z}{4}$, we have $(a, b, \frac{a+b+c}{4}) = (x, y, \frac{x+y+z}{4})$ again.

- If the lattices L and M belong to Case 3, the assumptions are $a = x$ and $\frac{a+b+c}{4} = \frac{x+y+z}{4}$. Therefore, we have $b + c = y + z$, and we are done if we can show that $b = y$. We look at the second basis vector of L and its image under F:

$$b = \left| \begin{pmatrix} 0 \\ \sqrt{b} \\ 0 \end{pmatrix} \right|^2 = \left| F(\begin{pmatrix} 0 \\ \sqrt{b} \\ 0 \end{pmatrix}) \right|^2 = \left| F(\underbrace{\frac{1}{2}\begin{pmatrix} \sqrt{a} \\ \sqrt{b} \\ \sqrt{c} \end{pmatrix}}_{v:=}) + F(\underbrace{\frac{1}{2}\begin{pmatrix} -\sqrt{a} \\ \sqrt{b} \\ -\sqrt{c} \end{pmatrix}}_{w:=}) \right|^2.$$

The vectors v and w of the right-hand side are two of the eight ones that realise the second successive minimum of L. The isometry F maps them to corresponding vectors of M, but without further knowledge of F, we do not know the signs of the vector entries. So, we have

$$b = \left| \frac{1}{2}\begin{pmatrix} e_1\sqrt{x} \\ e_2\sqrt{y} \\ e_3\sqrt{z} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} f_1\sqrt{x} \\ f_2\sqrt{y} \\ f_3\sqrt{z} \end{pmatrix} \right|^2 = \frac{(e_1+f_1)^2 x + (e_2+f_2)^2 y + (e_3+f_3)^2 z}{4}$$

with $e_i, f_i \in \{1, -1\}$ for $i = 1, 2, 3$. Hence, $(e_i + e_j)^2 \in \{0, 4\}$ and

$$b \in \Big\{0, x, y, z, x+y, x+z, y+z, x+y+z\Big\}.$$

The assumptions $0 < a < b$ and $a = x$ imply $b \neq 0$ and $b \neq x$. Now suppose, $b \geq z$. Having $b < c$ and $y < z$, we can conclude

$$b + c > 2b \geq 2z > y + z,$$

in contradiction to $b + c = y + z$. Therefore, $b < z$, and only the possibilities $b = y$ and $b = x + y$ remain. Analogous considerations for F^{-1} yield $y \in \{b, a+b\}$. But for $b \neq y$ we would have $b = x + y = x + a + b$, which is impossible.

- In the second case, the beginning of our considerations is exactly the same. The first difference occurs when we look at $F(v)$ and $F(w)$. Here, because of $a = \frac{a+b+c}{4}$, it is also possible that these image vectors equal the first basis vector of M or its negative (that is, $e_1 = \pm 2,\ e_2 = e_3 = 0$, for example).

  If none of them does, we have the above possibilities for the value of b.

  If one does and the other not, we get $e_1 + f_1 \in \{-3, -1, 1, 3\}$ and hence $b = \frac{x+y+z}{4}$ or $b = \frac{9x+y+z}{4}$. Because of

$$b > a = \frac{a+b+c}{4} = \frac{x+y+z}{4},$$

  the first case is impossible. The second case would imply

$$b = 2x + \frac{x+y+z}{4} = 3a = b + c,$$

which is a contradiction as well .

Since v and w are linearly independent, they cannot both be mapped on multiples of the first basis vector of M. This leaves us is in the situation of Case 3 and finishes the argument for Case 2.

- In Case 1 we only have $a+b+c=x+y+z$ to begin with, corresponding to the eight shortest vectors of L and M. But, looking at the square length of an arbitrary lattice vector of L in line (44), we see that the coefficients for a, b and c are either $\frac{1}{4}$ or at least 1, in which case we would have $|u|^2 \geq a$. Therefore, the second smallest length can only occur for $k=0$ and is given by a, only realised by the first basis vector and its negative. So, these vectors must be mapped to the corresponding ones of M, which gives us $a=x$ in this case as well. Now the argument of Case 3 can be applied.

We have shown $L=M$ for all five cases, and thus we can count as follows:

$$h_{8,i}(d) = \frac{1}{6} \sum_{\substack{a,b,c \in 2\mathbb{N} \\ a+b+c \in 4\mathbb{N} \\ abc=4d \\ a \neq b \neq c \neq a}} 1 \; .$$

For appropriate $s \in \mathbb{C}$ the corresponding Dirichlet series $F_{G_{8,i}}(s)$ converges, in which case we can change the order of summation:

$$F_{G_{8,i}}(s) = \sum_{d=1}^{\infty} \frac{h_{G_{8,i}}(d)}{d^s} = \sum_{d=1}^{\infty} \frac{1}{d^s} \left(\frac{1}{6} \sum_{\substack{a,b,c \in 2\mathbb{N} \\ a+b+c \in 4\mathbb{N} \\ abc=4d \\ a \neq b \neq c \neq a}} 1 \right) = \frac{1}{6} \sum_{\substack{a,b,c \in 2\mathbb{N} \\ a+b+c \in 4\mathbb{N} \\ a \neq b \neq c \neq a}} \frac{1}{\left(\frac{1}{4}abc\right)^s} \; .$$

It is easier to determine the sum if we drop the inequality condition on a, b and c, thus we define

$$\widetilde{F}_{G_{8,i}}(s) := \frac{1}{6} \sum_{\substack{a,b,c \in 2\mathbb{N} \\ a+b+c \in 4\mathbb{N}}} \frac{1}{\left(\frac{1}{4}abc\right)^s} = \frac{1}{6} \sum_{a,b \in \mathbb{N}} \sum_{\substack{c \in \mathbb{N} \\ c \equiv_2 a+b}} \frac{1}{(2abc)^s} \; .$$

The new series $\widetilde{F}_{G_{8,i}}$ comprises the body-centered tetragonal and cubic lattices in addition:

$$\widetilde{F}_{G_{8,i}} = F_{G_{8,i}} + 3F_{G_{16,i}} + F_{G_{48,i}}.$$

We already know the asymptotic behaviour of the tetragonal and cubic lattices. Their number grows linearly with and like the cube root of the determinant, respectively. Hence, the error that occurs if we allow a, b and c to partly coincide is small enough not to affect the result we want to show. To deal with the parity condition on c we have to look at the parities of a and b which yields four different cases:

$$\widetilde{F}_{G_{8,i}}(s) = \frac{1}{6 \cdot 2^s} \left(\sum_{\substack{a \in \mathbb{N} \\ a \equiv_2 0}} \sum_{\substack{b \in \mathbb{N} \\ b \equiv_2 0}} \sum_{\substack{c \in \mathbb{N} \\ c \equiv_2 0}} \frac{1}{(abc)^s} + \sum_{\substack{a \in \mathbb{N} \\ a \equiv_2 0}} \sum_{\substack{b \in \mathbb{N} \\ b \equiv_2 1}} \sum_{\substack{c \in \mathbb{N} \\ c \equiv_2 1}} \frac{1}{(abc)^s} + \sum_{\substack{a \in \mathbb{N} \\ a \equiv_2 1}} \sum_{\substack{b \in \mathbb{N} \\ b \equiv_2 0}} \sum_{\substack{c \in \mathbb{N} \\ c \equiv_2 1}} \frac{1}{(abc)^s} \right.$$

$$\left. + \sum_{\substack{a \in \mathbb{N} \\ a \equiv_2 1}} \sum_{\substack{b \in \mathbb{N} \\ b \equiv_2 1}} \sum_{\substack{c \in \mathbb{N} \\ c \equiv_2 0}} \frac{1}{(abc)^s} \right) .$$

The last three sums are equal because their summands are symmetric in a, b and c. We substitute even numbers by their halves and odd numbers by the smallest integer greater than their half:

$$\begin{aligned}\widetilde{F}_{G_{8,i}}(s) &= \frac{1}{6\cdot 2^s}\left(\frac{1}{8^s}\sum_{a,b,c\in\mathbb{N}}\frac{1}{(abc)^s}+\frac{3}{2^s}\sum_{a,b,c\in\mathbb{N}}\frac{1}{(a(2b-1)(2c-1))^s}\right)\\ &= \frac{1}{6\cdot 16^s}\left(\sum_{a=1}^{\infty}\frac{1}{a^s}\right)^3+\frac{1}{2\cdot 4^s}\sum_{a=1}^{\infty}\frac{1}{a^s}\left(\sum_{b=1}^{\infty}\frac{1}{(2b-1)^s}\right)^2 .\end{aligned}$$

The remaining series can be expressed with the help of the Riemann zeta function which leaves us with

$$\widetilde{F}_{G_{8,i}}(s) = \frac{\zeta(s)^3}{6\cdot 16^s}+\frac{\zeta(s)}{2\cdot 4^s}\left(\left(1-\frac{1}{2^s}\right)\zeta(s)\right)^2 = \frac{\zeta(s)^3}{6\cdot 16^s}\left(1+3\left(2^s-1\right)^2\right).$$

Now, we apply Proposition 4.2. The rightmost pole of the last expression is at $\alpha = 1$ and has order 3, hence $n = 2$. We can choose

$$g(s) := (s-1)^3\frac{\zeta(s)^3}{6\cdot 16^s}\left(1+3\left(2^s-1\right)^2\right)$$

to meet the requirements of Proposition 4.2. We observe

$$g(1) = \frac{1}{6\cdot 16^s}\left(1+3\left(2^s-1\right)^2\right) = \frac{1}{24}\ .$$

Then, we get

$$\sum_{d\leq D}\Big(h_{8,i}(d)+3h_{16,i}(d)+h_{48,i}(d)\Big) \sim \frac{g(1)}{2!}D\left(\log D\right)^2 = \frac{1}{48}D(\log D)^2$$

since $\widetilde{F}_{G_{8,i}}$ corresponds to the sum on the left-hand side. Finally, we also have

$$H_{8,i}(D) = \sum_{d\leq D}h_{8,i}(d) \sim \frac{1}{48}D(\log D)^2$$

because of the smaller growth of the number of body-centered tetragonal and cubic lattices. This proves the first formula of the Proposition.

The result for the base-centered orthorhombic lattices can be shown quite similarly. The lattices have the form

$$L := \mathbb{Z}\begin{pmatrix}\sqrt{a}\\0\\0\end{pmatrix}+\mathbb{Z}\begin{pmatrix}0\\\sqrt{b}\\0\end{pmatrix}+\mathbb{Z}\,\frac{1}{2}\begin{pmatrix}0\\\sqrt{b}\\\sqrt{c}\end{pmatrix},\ a,b,c\in\mathbb{N}\ ,$$

which implies

$$\det L = \left|\begin{pmatrix}a&0&0\\0&b&\frac{b}{2}\\0&\frac{b}{2}&\frac{b+c}{4}\end{pmatrix}\right| = \frac{1}{4}\left|\begin{pmatrix}a&0&0\\0&b&b\\0&b&b+c\end{pmatrix}\right| = \frac{1}{4}abc\ .$$

Integrality of the lattice requires $b \in 2\mathbb{Z}$ and $b+c \in 4\mathbb{Z}$. Again, we start with assuming pairwise different parameters a, b and c to exclude lattices of higher symmetry but can forget this restriction as soon as it poses a problem. This time, we can only swap the values of b and c if we want to remain in the same isometry class of lattices. Hence, we must divide by 2 instead of 6 and get

$$h_{8,b}(d) = \frac{1}{2} \sum_{\substack{a,c\in\mathbb{N}, b\in 2\mathbb{N} \\ b+c\in 4\mathbb{N} \\ abc=4d \\ a\neq b\neq c\neq a}} 1 \text{ and } F_{G_{8,b}}(s) = \frac{1}{2} \sum_{\substack{a,c\in\mathbb{N}, b\in 2\mathbb{N} \\ b+c\in 4\mathbb{N} \\ a\neq b\neq c\neq a}} \frac{1}{\left(\frac{1}{4}abc\right)^s} = \frac{1}{2} \sum_{\substack{a\in\mathbb{N}, b,c\in 2\mathbb{N} \\ \frac{b}{2}\equiv_2 \frac{c}{2} \\ a\neq b\neq c\neq a}} \frac{1}{\left(\frac{1}{4}abc\right)^s} .$$

We substitute b and c by their halves and omit the inequality condition:

$$\widetilde{F}_{G_{8,b}}(s) := \frac{1}{2} \sum_{\substack{a,b,c\in\mathbb{N} \\ b\equiv_2 c}} \frac{1}{(abc)^s} = \frac{1}{2} \sum_{a=1}^{\infty} \frac{1}{a^s} \left(\sum_{b,c\in 2\mathbb{N}} \frac{1}{(bc)^s} + \sum_{b,c\in 2\mathbb{N}+1} \frac{1}{(bc)^s} \right).$$

The first sum equals $\zeta(s)$. We compute the other sum with the help of the same substitutions as in the previous case:

$$\begin{aligned}
\widetilde{F}_{G_{8,b}}(s) &= \frac{\zeta(s)}{2} \left(\frac{1}{4^s} \sum_{b,c\in\mathbb{N}} \frac{1}{(bc)^s} + \sum_{b,c\in\mathbb{N}} \frac{1}{(2b-1)^s(2c-1)^s} \right) \\
&= \frac{\zeta(s)}{2} \left(\frac{\zeta(s)^2}{4^s} + \zeta(s)^2 \left(1 - \frac{1}{2^s}\right)^2 \right) = \frac{\zeta(s)^3}{2\cdot 4^s} \left(1 + (2^s-1)^2\right).
\end{aligned}$$

With regard to Proposition 4.2 we have $\alpha = 1$ and $n = 2$ as before. We set

$$g(s) := (s-1)^3 \frac{\zeta(s)^3}{2\cdot 4^s} \left(1 + (2^s-1)^2\right).$$

Then $g(1) = \frac{1}{4}$ and

$$H_{8,b}(D) \sim \frac{1}{8} D (\log D)^2.$$

Here, the difference between $\widetilde{F}_{G_{8,b}}$ and $F_{G_{8,b}}$ is irrelevant for our computation because of the same reasons as before. □

For the remaining Bravais types (groups of order 4) the method of calculating Dirichlet series appeared to be less fruitful. Rather, we use a direct approach for the aggregated class number $H(D)$.

Proposition 4.7. *For the aggregated class number of three-dimensional lattices of primitive monoclinic type we have the equation*

$$H_{4,p}(D) = \frac{\pi\zeta\left(\frac{3}{2}\right)}{9} D^{\frac{3}{2}} - \frac{3}{16} D \left(\log D\right)^2 + \mathcal{O}\left(D \log D\right).$$

Proof. The calculation of the class number of primitive monoclinic lattices uses the result for two-dimensional lattices with trivial automorphism groups. With the help of Proposition 3.6, part (b) we get

$$
\begin{aligned}
H_{4,p}(D) &= \sum_{a\le D} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le \frac{D}{a}}} 1 = \sum_{a\le D} H_0\left(\frac{D}{a}\right) = \sum_{a\le D}\left(\frac{\pi}{9}\left(\frac{D}{a}\right)^{\frac{3}{2}} - \frac{3}{8}\cdot\frac{D}{a}\log\left(\frac{D}{a}\right) + \mathcal{O}\left(\frac{D}{a}\right)\right)\\
&= \frac{\pi D^{\frac{3}{2}}}{9}\sum_{a\le D}\frac{1}{a^{\frac{3}{2}}} - \frac{3D\log D}{8}\sum_{a\le D}\frac{1}{a} + \frac{3D}{8}\sum_{a\le D}\frac{\log a}{a} + \mathcal{O}\left(D\sum_{a\le D}\frac{1}{a}\right)\\
&= \frac{\pi D^{\frac{3}{2}}}{9}\left(\frac{-2}{\sqrt{D}} + \zeta\left(\frac{3}{2}\right) + \mathcal{O}\left(D^{-\frac{3}{2}}\right)\right) - \frac{3D\log D}{8}\left(\log D + C + \mathcal{O}\left(D^{-1}\right)\right)\\
&\quad + \frac{3D}{8}\left(\frac{1}{2}\left(\log D\right)^2 + \mathcal{O}(1)\right) + \mathcal{O}\left(D\log D\right)\\
&= \frac{\pi\zeta\left(\frac{3}{2}\right)}{9}D^{\frac{3}{2}} - \frac{3}{16}D\left(\log D\right)^2 + \mathcal{O}(D\log D)\,.
\end{aligned}
$$

□

Proposition 4.8. *In the base-centered monoclinic case the following formula holds:*

$$
H_{4,b}(D) = \frac{\pi\zeta\left(\frac{3}{2}\right)}{6\sqrt{2}}D^{\frac{3}{2}} - \frac{7}{32}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right).
$$

To prepare the proof we start as in Proposition 4.6. We assume that the bilinear form b is given by the usual scalar product. Let $u, v, w \in \mathbb{Q}^3$ such that (v, w) is a reduced basis of the two-dimensional lattice $\langle v, w\rangle_{\mathbb{Z}}$ in the subspace $\langle v, w\rangle_{\mathbb{Q}}$ and $u \perp \langle v, w\rangle_{\mathbb{Q}}$. We suppose that $\mathcal{O}(\langle v, w\rangle_{\mathbb{Z}}) = \{\pm\,\mathrm{id}\}$. Let a, b, c be the square lengths of u, v, w, respectively, and let $f := \langle v, w\rangle$. We assume that $a, b, c, f \in \mathbb{Z}$. Then every base-centered monoclinic lattice is isometric to exactly one lattice of the ones with the following bases:

$$
\begin{aligned}
\mathcal{B}_1 &= \left(\frac{1}{2}(u+v), v, w\right),\\
\mathcal{B}_2 &= \left(\frac{1}{2}(u+w), v, w\right),\\
\mathcal{B}_3 &= \left(\frac{1}{2}(u+v-w), v, w\right).
\end{aligned}
$$

We want to count all the possible bases of this kind (or more precisely their Gram matrices) in order to determine the class number for base centered monoclinic lattices. However, if $3a = 3f = b$ or $3a = 3f = c$, then the corresponding lattice is trigonal and not monoclinic. Thus, we will count too many lattices. We already know that the asymptotic frequency of the trigonal lattices is given by

$$
\frac{1}{3}\zeta(2)D + \mathcal{O}(D^{\frac{2}{3}})\,.
$$

For the class number in the base-centered monoclinic case we only claim accuracy up to multiples of $D\log D$, hence the error of counting additional lattices is neglectable.

The corresponding Gram matrices for the three bases are

$$B_1 = \begin{pmatrix} \frac{a+b}{4} & \frac{b}{2} & \frac{f}{2} \\ \frac{b}{2} & b & f \\ \frac{f}{2} & f & c \end{pmatrix}, B_2 = \begin{pmatrix} \frac{a+c}{4} & \frac{f}{2} & \frac{c}{2} \\ \frac{f}{2} & b & f \\ \frac{c}{2} & f & c \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} \frac{a+b+c}{4} - \frac{f}{2} & \frac{b-f}{2} & \frac{f-c}{2} \\ \frac{b-f}{2} & b & f \\ \frac{f-c}{2} & f & c \end{pmatrix}.$$

The determinant of all three matrices is given by $\frac{1}{4}a\left(bc - f^2\right)$. Since $bc - f^2$ is the determinant of a two-dimensional sublattice, this expression is at least one. Therefore, $a \leq 4D$. Moreover, we have $0 < 2f < b < c$ because of the reducedness of (v, w). We want to find an upper bound on c as well. We begin with

$$\frac{1}{4}a\left(bc - f^2\right) \leq D \;\Leftrightarrow\; bc \leq \frac{4D}{a} + f^2 \;\Leftrightarrow\; c \leq \frac{4D}{ab} + \frac{f^2}{b}\,.$$

Using $2f < b$ and $b < c$ gives us

$$c < \frac{4D}{ab} + \frac{b}{4} < \frac{4D}{ab} + \frac{c}{4} \;\Rightarrow\; \frac{3c}{4} < \frac{4D}{ab} \;\Leftrightarrow\; c < \frac{16D}{3ab}\,.$$

We can weaken this inequality to

$$c < \frac{16D}{3a}\,.$$

In each of the three cases some additional conditions must be fulfilled to ensure the integrity of the lattice. In the first case we need $b, f \in 2\mathbb{N}$ and $a + b \in 4\mathbb{N}$, for example. We sum over the possible values for a, b, c, f and calculate the three sums in the following lemma.

Lemma 4.9.

(a)
$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b,f\in 2\mathbb{N}\\ a+b\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}}D^{\frac{3}{2}} - \frac{3}{32}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right),$$

(b)
$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ c,f\in 2\mathbb{N}\\ a+c\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}}D^{\frac{3}{2}} - \frac{5}{64}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right),$$

(c)
$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}}D^{\frac{3}{2}} - \frac{3}{64}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right).$$

Before we prove this lemma, we state two auxiliary results from analytic number theory and the calculation of a certain sum that arises in all three cases.

Lemma 4.10. *Let x and s be real numbers with $x \geq 2$ and $s > 0, s \neq 1$. Then*

(a)
$$\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + \mathcal{O}\left(x^{-s}\right),$$

(b)
$$\sum_{n\leq x} \frac{\log n}{n} = \frac{1}{2}\left(\log x\right)^2 + \mathcal{O}(1).$$

Proof. For the first formula see [Apo76], Theorem 3.2.

Part (b) follows from Euler's summation formula (see Lemma 3.9) and is formulated as an exercise in a sharper version in [Apo76](Exercise 1(a) on page 70). □

Lemma 4.11.

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}}} 1 = \frac{2\sqrt{2}}{9}\pi\zeta\left(\frac{3}{2}\right) D^{\frac{3}{2}} - \frac{3}{8}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right).$$

Proof. We begin with substituting a by $\frac{a}{2}$ and observe

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}}} 1 = \sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{2D}{a}}} 1 = \sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} H_0\left(\frac{2D}{a}\right),$$

that is, the inner sum over b, c and f can be interpreted as a class number of two-dimensional lattices with trivial automorphism group. For comparison, see line (20) in the proof of Proposition 3.6 . We apply part (b) of this proposition and get

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}}} 1 = \sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \left(\frac{\pi}{9}\left(\frac{2D}{a}\right)^{\frac{3}{2}} - \frac{3}{8}\left(\frac{2D}{a}\right)\log\left(\frac{2D}{a}\right) + \mathcal{O}\left(\frac{D}{a}\right)\right)$$

$$= \frac{2\sqrt{2}}{9}\pi D^{\frac{3}{2}} \sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} a^{-\frac{3}{2}} - \frac{3D}{4}\log\left(2D\right)\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \frac{1}{a} + \frac{3D}{4}\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \frac{\log a}{a} + \mathcal{O}\Bigg(\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \frac{D}{a}\Bigg). \tag{45}$$

Next, we approximate the four remaining sums. We obtain

$$\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} a^{-\frac{3}{2}} = \frac{(2D)^{1-\frac{3}{2}}}{1-\frac{3}{2}} + \zeta\left(\frac{3}{2}\right) + \mathcal{O}\left((2D)^{-\frac{3}{2}}\right) = \zeta\left(\frac{3}{2}\right) + \mathcal{O}\left(D^{-\frac{1}{2}}\right), \quad \text{(cf. 4.10, (a))}$$

$$\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \frac{1}{a} = \log(2D) + C + \mathcal{O}\left(\frac{1}{2D}\right) = \log D + \mathcal{O}(1), \quad \text{(cf. 3.4, (a))}$$

$$\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \frac{\log a}{a} = \frac{1}{2}\left(\log\left(2D\right)\right)^2 + \mathcal{O}(1) = \frac{1}{2}\left(\log D\right)^2 + \mathcal{O}\left(\log D\right), \quad \text{(cf. 4.10, (b))}$$

and for the error term

$$\mathcal{O}\Bigg(\sum_{\substack{a\in \mathbb{N}\\ a\leq 2D}} \frac{D}{a}\Bigg) = \mathcal{O}\Big(D\big(\log D + \mathcal{O}(1)\big)\Big) = \mathcal{O}\left(D\log D\right).$$

We include these results in line (45) and get

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}}} 1 = \frac{2\sqrt{2}}{9}\pi\zeta\left(\frac{3}{2}\right) D^{\frac{3}{2}} - \frac{3D}{4}\log D\log(2D) + \frac{3D}{8}\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right)$$

$$= \frac{2\sqrt{2}}{9}\pi\zeta\left(\frac{3}{2}\right) D^{\frac{3}{2}} - \frac{3}{8}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right).$$

□

Proof of Lemma 4.9. We note that all three sums are finite since $2f < b < c < \frac{16D}{9}$ and $a \leq 4D$. Therefore, we can always change the order of summation.

Part (a)
We want to apply Lemma 4.11. Therefore, we have to get rid of all equivalence conditions on the summation indices except for $a \in 2\mathbb{N}$. We begin with choosing an order of summation for the first sum:

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b,f\in 2\mathbb{N}\\ a+b\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a}} \sum_{\substack{f\in 2\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}}} 1\,. \tag{46}$$

To calculate the innermost sum from the right-hand side we want to get rid of the parity condition on f. We multiply by $\frac{1}{2}$ instead and determine the arising error:

$$\sum_{\substack{f\in 2\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}}} 1 = \frac{1}{2}\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}}} 1 + \frac{1}{2}\Bigg(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 0}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 1}} 1\Bigg).$$

Now, we change the expression within the brackets by leaving out the condition on the determinant D and subtract corresponding sums with the reverse inequality:

$$\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 0}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 1}} 1 = \Bigg(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f\equiv_2 0}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f\equiv_2 1}} 1\Bigg) - \Bigg(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f^2<bc-\frac{4D}{a}\\ f\equiv_2 0}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f^2<bc-\frac{4D}{4}\\ f\equiv_2 1}} 1\Bigg).$$

The value of each sum over odd f is equal to or greater by one than the corresponding sum over even f, because the upper bounds are the same and the smallest natural number is odd. Hence, the bracketed differences equal 0 or -1.

For $bc - \frac{4D}{a} \leq 1$ in particular, the second difference is 0 since both sums are empty in this case.

The first difference depends on the remainder of b modulo 4. To be more specific, we have to check if the greatest integer $f_{\max}$ smaller than the upper bound $\frac{b}{2}$ is even or odd:

$$\begin{aligned}
b \equiv_4 0 &\;\Rightarrow\; \tfrac{b}{2} \in 2\mathbb{Z} &&\Rightarrow\; f_{\max} \text{ is odd,}\\
b \equiv_4 1 &\;\Rightarrow\; \tfrac{b}{2} \in 2\mathbb{Z} + \tfrac{1}{2} &&\Rightarrow\; f_{\max} \text{ is even,}\\
b \equiv_4 2 &\;\Rightarrow\; \tfrac{b}{2} \in \mathbb{Z}\setminus 2\mathbb{Z} &&\Rightarrow\; f_{\max} \text{ is even,}\\
b \equiv_4 3 &\;\Rightarrow\; \tfrac{b}{2} \in (\mathbb{Z}\setminus 2\mathbb{Z}) + \tfrac{1}{2} &&\Rightarrow\; f_{\max} \text{ is odd.}
\end{aligned}$$

The cases $b \equiv_4 1$ and $b \equiv_4 3$ cannot occur here because of $b \equiv_4 a \in 2\mathbb{N}$. Nevertheless, we consider them here since they will be used later on in part (b) of this lemma. We have

$$\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 0}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 1}} 1 = \left\{\begin{array}{rl} -1, & b\equiv_4 0,3\\ 0, & b\equiv_4 1,2\end{array}\right\} - \left\{\begin{array}{rl} \mathcal{O}(1), & bc-\frac{4D}{a}>1\\ 0, & bc-\frac{4D}{a}\leq 1\end{array}\right\},$$

and

$$\sum_{\substack{f\in 2\mathbb{N}\\ 2f<b\\ bc-f^2\le\frac{4D}{a}}} 1 = \frac{1}{2}\left(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\le\frac{4D}{a}}} 1 + \left\{\begin{array}{rl} -1, & b\equiv_4 0,3\\ 0, & b\equiv_4 1,2\end{array}\right\} - \left\{\begin{array}{rl} \mathcal{O}(1), & bc-\frac{4D}{a}>1\\ 0, & bc-\frac{4D}{a}\le 1\end{array}\right\}\right). \tag{47}$$

We insert this result into line (46):

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b,f\in 2\mathbb{N}\\ a+b\in 4\mathbb{N}\\ a(bc-f^2)\le 4D}} 1 = \frac{1}{2}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 a}} 1 - \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ b\equiv_4 0,3}} 1 + \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ bc-\frac{4D}{a}>1}} \mathcal{O}(1)\right). \tag{48}$$

The three double sums on the right-hand side will be dealt with separately. First, we will show that the order of magnitude of the last one does not exceed $D\log D$. Since the summands are non-negative, and we want to find an upper bound, we can ignore the congruence conditions on a and b:

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ bc-\frac{4D}{a}>1}} \mathcal{O}(1) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c\\ \frac{4D}{ab}+\frac{1}{b}<c<\frac{4D}{ab}+\frac{b}{4}}} 1\right) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}}\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}}\sum_{\substack{c\in\mathbb{N}\\ \frac{1}{b}<c<\frac{b}{4}}} 1\right).$$

For the last transformation we have used that $b<c$ and $c<\frac{4D}{ab}+\frac{b}{4}$ imply $b<4\sqrt{\frac{D}{3a}}$. Because of

$$\sum_{\substack{c\in\mathbb{N}\\ \frac{1}{b}<c<\frac{b}{4}}} 1 = \mathcal{O}(b),\quad \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} b = \mathcal{O}\left(\frac{D}{a}\right) \text{ and } \sum_{\substack{a\in\mathbb{N}\\ a\le 4D}} \frac{D}{a} = \mathcal{O}\left(D\log D\right)$$

we have

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ bc-\frac{4D}{a}>1}} \mathcal{O}(1) = \mathcal{O}\left(D\log D\right). \tag{49}$$

Now, we deal with the second double sum from line (48). As above, we first sum over c and deduce $b<4\sqrt{\frac{D}{3a}}$ as a weakened upper bound on b that is independent of c. This yields

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ b\equiv_4 0,3}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a\\ b\equiv_4 0,3}}\sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a\\ b\equiv_4 0,3}} \left(\frac{4D}{ab}+\mathcal{O}(b)\right).$$

We consider the congruence conditions. Since $b\equiv_4 a\equiv_2 0$, the case $b\equiv_4 3$ cannot occur.

Therefore, $a \equiv_4 b \equiv_4 0$ and

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ b\equiv_4 0,3}} 1 = \sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b\in 4\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} \left(\frac{4D}{ab} + \mathcal{O}(b)\right) = \sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<\sqrt{\frac{D}{3a}}}} \left(\frac{D}{ab} + \mathcal{O}(b)\right)$$

$$= \sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \sum_{\substack{b\in\mathbb{N}\\ b<\sqrt{\frac{D}{12a}}}} \left(\frac{D}{4ab} + \mathcal{O}(b)\right) = \frac{D}{4}\sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \left(\frac{1}{a} \sum_{\substack{b\in\mathbb{N}\\ b<\sqrt{\frac{D}{12a}}}} \frac{1}{b}\right) + \sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \mathcal{O}\left(\frac{D}{a}\right).$$

We apply the asymptotic formula for the sum over the reciprocals of the natural numbers up to a given bound:

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ b\equiv_4 0,3}} 1 = \frac{D}{4}\sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \frac{1}{a}\left(\log\left(\sqrt{\frac{D}{12a}}\right) + \mathcal{O}(1)\right) + \mathcal{O}\left(D\log D\right)$$

$$= \frac{D}{8}\sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \frac{\log D - \log 12 - \log a}{a} + \mathcal{O}\left(D\log D\right)$$

$$= \frac{D\log D}{8}\sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \frac{1}{a} - \frac{D}{8}\sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \frac{\log a}{a} + \mathcal{O}\left(D\log D\right).$$

For the second sum in the last line we need the second formula of Lemma 4.10, whose appliance results in

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a\\ b\equiv_4 0,3}} 1 = \left(\frac{1}{8} - \frac{1}{16}\right) D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right) = \frac{D\left(\log D\right)^2}{16} + \mathcal{O}\left(D\log D\right). \tag{50}$$

Finally, we look at the first double sum from line (48). We want to avoid the congruence condition $b \equiv_4 a$, so we take the following approach:

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 a}} 1 = \frac{1}{4}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \left(4\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 a}} 1 - \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1\right)\right),$$

and subdivide the last sum in four parts according to the remainder of $b-a$ modulo 4. Note that the difference in the interior brackets vanishes for $x=0$, so only three cases remain. We get

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 a}} 1 = \frac{1}{4}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{x\in\{\pm1,2\}} \left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 a}} 1 - \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 a+x}} 1\right)\right).$$

The first double sum is known thanks to Lemma 4.11. Therefore, we have

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{9\sqrt{2}} D^{\frac{3}{2}} - \frac{3}{32} D\left(\log D\right)^2 + \frac{1}{4}\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{x\in\{\pm1,2\}} \left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a}} 1 - \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a+x}} 1\right)$$
$$+\mathcal{O}\left(D\log D\right).$$

We include this preliminary result as well as those from lines (49) and (50) in line (48) which gives us

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b,f\in 2\mathbb{N}\\ a+b\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}} D^{\frac{3}{2}} - \frac{3}{64} D\left(\log D\right)^2 + \frac{1}{8}\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{x\in\{\pm1,2\}} \left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a}} 1 - \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a+x}} 1\right)$$
$$-\frac{D(\log D)^2}{32} + \mathcal{O}\left(D\log D\right).$$

We simplify to

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b,f\in 2\mathbb{N}\\ a+b\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}} D^{\frac{3}{2}} - \frac{5}{64} D\left(\log D\right)^2 + \frac{1}{8}\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{x\in\{\pm1,2\}} \left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a}} 1 - \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a+x}} 1\right)$$
$$+\mathcal{O}\left(D\log D\right),$$

and see that it remains to show that

(51) $$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{x\in\{\pm1,2\}} \left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a+x}} 1 - \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a}} 1\right) = \frac{D\left(\log D\right)^2}{8} + \mathcal{O}\left(D\log D\right).$$

We begin with determining the sums of the inner difference:

$$\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq \frac{4D}{a}\\ b\equiv_4 a+x}} 1 = \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a+x}} \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq \frac{4D}{a}}} 1 = \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a+x}} \left(\left\lfloor \frac{b-1}{2}\right\rfloor - \begin{cases} \left\lceil \sqrt{bc-\frac{4D}{a}}\,\right\rceil - 1, & bc-\frac{4D}{a} > 1\\ 0, & bc-\frac{4D}{a}\leq 1\end{cases}\right)$$

(52) $$= \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a+x}} \left\lfloor \frac{b-1}{2}\right\rfloor - \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 a+x}} \left(\left\lceil \sqrt{bc-\frac{4D}{a}}\,\right\rceil - 1\right).$$

These two sums will be calculated separately. We start with the first one:

$$\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a+x}} \left\lfloor \frac{b-1}{2}\right\rfloor = \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} \left\lfloor \frac{b-1}{2}\right\rfloor = \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \left\lfloor \frac{b-1}{2}\right\rfloor \left(\frac{4D}{ab} - \frac{3b}{4} + \mathcal{O}(1)\right).$$

If we just approximated the value of the floor function by its argument, the error would be too big to obtain line (51). Therefore, we have to do a case-by-case analysis with regard to x. We can assume $a \equiv_2 0$ since we want to insert the result in line (51). That yields

$$\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a+x}} \left\lfloor \frac{b-1}{2} \right\rfloor = \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \left\{ \begin{array}{ll} \frac{b}{2}-1, & x=0,2 \\ \frac{b-1}{2}, & x=\pm 1 \end{array} \right\} \left(\frac{4D}{ab} - \frac{3b}{4} + \mathcal{O}(1) \right)$$

$$= \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \left(\frac{2D}{a} - \frac{3b^2}{8} + \mathcal{O}(b) - \left\{ \begin{array}{ll} 4, & x=0,2 \\ 2, & x=\pm 1 \end{array} \right\} \frac{D}{ab} \right)$$

(53) $$= \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \left(\frac{2D}{a} - \frac{3b^2}{8} - \left\{ \begin{array}{ll} 4, & x=0,2 \\ 2, & x=\pm 1 \end{array} \right\} \frac{D}{ab} \right) + \mathcal{O}\left(\frac{D}{a}\right).$$

Since we have to subtract two sums in line (51) that only differ with regard to x, all expressions that are independent of x will be irrelevant later on. We split up line (53) in three sums and begin with

$$\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \frac{2D}{a} = \frac{2D}{a}\left(\frac{1}{4} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} 1 + \mathcal{O}(1) \right) = \frac{2D}{a}\left(\sqrt{\frac{D}{3a}} + \mathcal{O}(1) \right)$$

(54) $$= \frac{2}{\sqrt{3}} \left(\frac{D}{a}\right)^{\frac{3}{2}} + \mathcal{O}\left(\frac{D}{a}\right).$$

Next, we let $y \in \{1,2,3,4\}$ with $y \equiv_4 a+x$. Then, by substituting $b = 4k+y$ we have

$$\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \frac{3b^2}{8} = \frac{3}{8} \sum_{\substack{k\in\mathbb{N}_0\\ 4k+y<4\sqrt{\frac{D}{3a}}}} (4k+y)^2 = \frac{3}{8} \sum_{\substack{k\in\mathbb{N}_0\\ k<\sqrt{\frac{D}{3a}}-\frac{y}{4}}} \left(16k^2+8ky+y^2\right)$$

(55) $$= 6 \sum_{\substack{k\in\mathbb{N}_0\\ k<\sqrt{\frac{D}{3a}}-\frac{y}{4}}} k^2 + \mathcal{O}\left(\frac{D}{a}\right) = 6 \sum_{\substack{k\in\mathbb{N}_0\\ k<\sqrt{\frac{D}{3a}}}} k^2 + \mathcal{O}\left(\frac{D}{a}\right).$$

We apply the same substitution to the following sum:

$$\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \frac{1}{b} = \sum_{\substack{k\in\mathbb{N}_0\\ 4k+y<4\sqrt{\frac{D}{3a}}}} \frac{1}{4k+y} = \sum_{\substack{k\in\mathbb{N}\\ 1<k\leq\sqrt{\frac{D}{3a}}-\frac{y}{4}}} \frac{1}{4k+y} + \mathcal{O}(1)\,.$$

The Euler-Maclaurin formula (Lemma 3.9) yields

$$\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \frac{1}{b} = \int_1^{\sqrt{\frac{D}{3a}}-\frac{y}{4}} \frac{1}{4t+y}\mathrm{d}t + \int_1^{\sqrt{\frac{D}{3a}}-\frac{y}{4}} \frac{-4\left(t-\lfloor t\rfloor\right)}{(4t+y)^2}\mathrm{d}t$$

$$+ \frac{1}{4\sqrt{\frac{D}{3a}}} \left(\left\lfloor \sqrt{\frac{D}{3a}} - \frac{y}{4} \right\rfloor - \left(\sqrt{\frac{D}{3a}} - \frac{y}{4} \right) \right) - \frac{1}{4+y}\left(\lfloor 1\rfloor - 1\right) + \mathcal{O}(1)\,.$$

Because of $a \leq 4D$ we have $\mathcal{O}\left(\sqrt{\frac{a}{D}}\right) = \mathcal{O}(1)$. Thus, the terms of the last line can be neglected, and we have

$$\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \frac{1}{b} = \left[\frac{1}{4}\log\left(4t+y\right)\right]_1^{\sqrt{\frac{D}{3a}}-\frac{y}{4}} + \mathcal{O}\left(\int_1^{\sqrt{\frac{D}{3a}}-\frac{y}{4}} \frac{-4}{\left(4t+y\right)^2}\mathrm{d}t\right) + \mathcal{O}(1)$$

$$= \frac{1}{4}\log\left(4\sqrt{\frac{D}{3a}}\right) - \frac{1}{4}\log(4+y) + \mathcal{O}\left(\left[\frac{1}{4t+y}\right]_1^{\sqrt{\frac{D}{3a}}-\frac{y}{4}}\right) + \mathcal{O}(1)$$

$$= \frac{1}{8}\log\left(\frac{D}{3a}\right) + \mathcal{O}(1) = \frac{1}{8}\log\left(\frac{D}{a}\right) + \mathcal{O}(1)\,. \tag{56}$$

Inserting the last results (54),(55) and (56) in line (53) gives us

$$\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 a+x}} \left\lfloor\frac{b-1}{2}\right\rfloor = \frac{2}{\sqrt{3}}\left(\frac{D}{a}\right)^{\frac{3}{2}} - 6\sum_{\substack{k\in\mathbb{N}_0\\ k<\sqrt{\frac{D}{3a}}}} k^2 - \left\{\begin{array}{ll}\frac{1}{2}, & x=0,2\\ \frac{1}{4}, & x=\pm 1\end{array}\right\}\frac{D}{a}\log\left(\frac{D}{a}\right) + \mathcal{O}\left(\frac{D}{a}\right) \tag{57}$$

for the first sum of line (52). Recall, that only the third summand will be relevant since the others do not depend on x.

Now, we want to determine the second sum of line (52). We want to get rid of the ceiling function and choose to sum over c first:

$$\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 a+x}} \left(\left\lceil\sqrt{bc-\frac{4D}{a}}\right\rceil - 1\right) = \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1}} \left(\sqrt{bc-\frac{4D}{a}} + \mathcal{O}(1)\right)$$

$$= \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \left(\sum_{\substack{c\in\mathbb{N}\\ b<c\leq\frac{4D}{ab}+\frac{b}{4}\\ \frac{4D}{ab}+\frac{1}{b}<c}} \sqrt{bc-\frac{4D}{a}} + \mathcal{O}\left(b\right)\right)$$

$$= \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \sum_{\substack{c\in\mathbb{N}\\ b<c\leq\frac{4D}{ab}+\frac{b}{4}\\ \frac{4D}{ab}+\frac{1}{b}<c}} \sqrt{bc-\frac{4D}{a}} + \mathcal{O}\left(\frac{D}{a}\right).$$

To calculate the interior sum over c with the help of the Euler-Maclaurin formula (Lemma 3.9) we need to know which of the lower bounds on c is the bigger one. That clearly depends on b. We have $b \leq \frac{4D}{ab} + \frac{1}{b} \Leftrightarrow b \leq \sqrt{\frac{4D}{a}+1}$ and therefore

$$\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 a+x}} \left(\left\lceil\sqrt{bc-\frac{4D}{a}}\right\rceil - 1\right) = \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\leq\sqrt{\frac{4D}{a}+1}\\ b\equiv_4 a+x}} \sum_{\substack{c\in\mathbb{N}\\ \frac{4D}{ab}+\frac{1}{b}<c\leq\frac{4D}{ab}+\frac{b}{4}}} \sqrt{bc-\frac{4D}{a}}$$

$$+ \sum_{\substack{b\in\mathbb{N}\\ \sqrt{\frac{4D}{a}+1}<b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \sum_{\substack{c\in\mathbb{N}\\ b<c\leq\frac{4D}{ab}+\frac{b}{4}}} \sqrt{bc-\frac{4D}{a}} + \mathcal{O}\left(\frac{D}{a}\right).$$

(Note that the second sum over b is empty if $a > \frac{4}{3}D$.) The interior sums only differ in their lower bounds on the summation index c. To calculate them efficiently let A denote a general lower bound on c and define $f(t) := \sqrt{bt - \frac{4D}{a}}$. The Euler-Maclaurin formula yields

$$\sum_{\substack{c\in\mathbb{N}\\ A<c\leq \frac{4D}{ab}+\frac{b}{4}}} f(c) = \int_A^{\frac{4D}{ab}+\frac{b}{4}} f(t)\mathrm{d}t + \int_A^{\frac{4D}{ab}+\frac{b}{4}} \underbrace{(t-\lfloor t\rfloor)}_{=\mathcal{O}(1)} f'(t)\mathrm{d}t$$
$$+ f\left(\frac{4D}{ab}+\frac{b}{4}\right)\underbrace{\left(\left\lfloor \frac{4D}{ab}+\frac{b}{4}\right\rfloor - \left(\frac{4D}{ab}+\frac{b}{4}\right)\right)}_{=\mathcal{O}(1)} - f(A)\underbrace{(\lfloor A\rfloor - A)}_{=\mathcal{O}(1)}.$$

We find a primitive function for f to calculate the first integral and roughly estimate the rest by the value of f at the summation bounds:

$$\begin{aligned}\sum_{\substack{c\in\mathbb{N}\\ A<c\leq \frac{4D}{ab}+\frac{b}{4}}} f(c) &= \left[\frac{2}{3b}\left(bt-\frac{4D}{a}\right)^{\frac{2}{3}}\right]_A^{\frac{4D}{ab}+\frac{b}{4}} + \mathcal{O}\left(f\left(A\right)\right) + \mathcal{O}\left(f\left(\frac{4D}{ab}+\frac{b}{4}\right)\right)\\ &= \frac{2}{3b}\left(\frac{b^2}{4}\right)^{\frac{2}{3}} - \frac{2}{3b}\left(bA-\frac{4D}{a}\right)^{\frac{2}{3}} + \mathcal{O}\left(f\left(A\right)\right) + \mathcal{O}\left(\frac{b}{2}\right)\\ &= -\frac{2}{3b}\left(bA-\frac{4D}{a}\right)^{\frac{2}{3}} + \mathcal{O}\left(f\left(A\right)\right) + \mathcal{O}\left(b\right).\end{aligned}$$

We have $f\left(\frac{4D}{ab}+\frac{1}{b}\right) = 1$ and $f(b) = \sqrt{b^2 - \frac{4D}{a}} = \mathcal{O}(b)$. Thus,

$$\begin{aligned}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 a+x}} \left(\left\lceil\sqrt{bc-\frac{4D}{a}}\right\rceil - 1\right) &= \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\leq\sqrt{\frac{4D}{a}+1}\\ b\equiv_4 a+x}} \left(-\frac{2}{3b} + \mathcal{O}\left(b\right)\right)\\ &+ \sum_{\substack{b\in\mathbb{N}\\ \sqrt{\frac{4D}{a}+1}<b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \left(-\frac{2}{3b}\left(b^2-\frac{4D}{a}\right)^{\frac{2}{3}} + \mathcal{O}\left(b\right)\right) + \mathcal{O}\left(\frac{D}{a}\right)\\ &= \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 a+x}} \mathcal{O}\left(b\right) + \mathcal{O}\left(\frac{D}{a}\right) = \mathcal{O}\left(\frac{D}{a}\right).\end{aligned} \tag{58}$$

We look back on line (52) and plug in the findings from (57) and (58) for the two sums:

$$\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 a+x}} 1 = \frac{2}{\sqrt{3}}\left(\frac{D}{a}\right)^{\frac{3}{2}} - 6\sum_{\substack{k\in\mathbb{N}_0\\ k<\sqrt{\frac{D}{3a}}}} k^2 - \left\{\begin{array}{ll}\frac{1}{2}, & x=0,2\\ \frac{1}{4}, & x=\pm1\end{array}\right\}\frac{D}{a}\log\left(\frac{D}{a}\right) + \mathcal{O}\left(\frac{D}{a}\right).$$

We use this to start verifying line (51):

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{x\in\{\pm1,2\}}\left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 a+x}}1-\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 a}}1\right)=\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{x=\pm1}\left(\left(-\frac{1}{4}+\frac{1}{2}\right)\frac{D}{a}\log\left(\frac{D}{a}\right)+\mathcal{O}\left(\frac{D}{a}\right)\right)$$

$$=\frac{D}{2}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\frac{\log D-\log a}{a}+\mathcal{O}\left(D\log D\right).$$

We substitute a by $\frac{a}{2}$ and apply the usual asymptotic formulae to finally get

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{x\in\{\pm1,2\}}\left(\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 a+x}}1-\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 a}}1\right)=\frac{D}{2}\sum_{\substack{a\in\mathbb{N}\\ a\le 2D}}\frac{\log D-\log a-\log 2}{2a}+\mathcal{O}\left(D\log D\right)$$

$$=\frac{D\log D}{4}\Big(\log\left(2D\right)+\mathcal{O}(1)\Big)$$
$$-\frac{D}{4}\left(\frac{1}{2}\log\left(2D\right)^2+\mathcal{O}(1)\right)+\mathcal{O}\left(D\log D\right)$$
$$=\frac{D\left(\log D\right)^2}{4}-\frac{D\left(\log D\right)^2}{8}+\mathcal{O}\left(D\log D\right)$$
$$=\frac{D\left(\log D\right)^2}{8}+\mathcal{O}\left(D\log D\right).$$

This finishes the proof of part (a) of Lemma 4.9.

Part (b)
Our approach to part (b) of Lemma 4.9 is quite similar to part (a). We choose an order in which to carry out the summation and apply the result from line (47) to the sum over f:

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ c,f\in 2\mathbb{N}\\ a+c\in 4\mathbb{N}\\ a(bc-f^2)\le 4D}}1=\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a}}\sum_{\substack{f\in 2\mathbb{N}\\ 2f<b\\ bc-f^2\le\frac{4D}{a}}}1$$

$$=\frac{1}{2}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a}}\left(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\le\frac{4D}{a}}}1+\begin{cases}-1, & b\equiv_4 0,3\\ 0, & b\equiv_4 1,2\end{cases}-\begin{cases}\mathcal{O}(1), & bc-\frac{4D}{a}>1\\ 0, & bc-\frac{4D}{a}\le 1\end{cases}\right)$$

$$=\frac{1}{2}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ c\equiv_4 a}}1-\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}}1+\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ bc-\frac{4D}{a}>1}}\mathcal{O}(1)\right)\tag{59}$$

Next, we separately calculate the three double sums. Apart from the congruence condition $c\equiv_4 a$ of the inner sum, the last double sums in lines (48) and (59) are identical. Above, we ignored all congruence conditions since they were irrelevant for finding an upper bound on the expression. Thus, in doing the same here, we end up with the same result:

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ bc-\frac{4D}{a}>1}} \mathcal{O}(1) = \mathcal{O}\left(D\log D\right). \tag{60}$$

The second double sum from line (59) can be determined as follows:

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 0,3}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 0,3}} \left(\frac{1}{4}\sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1+\mathcal{O}(1)\right).$$

We will see that the error term of this estimate is small enough for our purpose. We start with

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 0,3}} \mathcal{O}(1) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} 1\right) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}} \sqrt{\frac{D}{a}}\right) = \mathcal{O}\left(\sqrt{D}\sum_{\substack{a\in\mathbb{N}\\ a\le D}} \frac{1}{a^{\frac{1}{2}}}\right),$$

and apply Lemma 4.10, part (a) which leads to

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 0,3}} \mathcal{O}(1) = \mathcal{O}\left(\sqrt{D}\left(2\sqrt{D}+\zeta\left(\frac{1}{2}\right)+\mathcal{O}\left(D^{-\frac{1}{2}}\right)\right)\right) = \mathcal{O}(D)\ .$$

Hence, we have

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 = \frac{1}{4}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{y\in\{0,3\}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1+\mathcal{O}(D)\ .$$

We want to get rid of the congruence condition $b\equiv_4 y$. In order to achieve this we switch the summations over b and c using $\frac{16D}{3a}$ as an upper bound on c (see Remarks after Proposition 4.8) Then, the summands of the sum over b are all equal to one, and we can replace the congruence condition by a multiplication by $\frac{1}{4}$ as done above for the sum over c. We get

$$\begin{aligned}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 &= \frac{1}{4}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{y\in\{0,3\}} \sum_{\substack{c\in\mathbb{N}\\ c<\frac{16D}{3a}}} \sum_{\substack{b\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 y}} 1+\mathcal{O}(D)\\ &= \frac{1}{4}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}} \sum_{y\in\{0,3\}} \sum_{\substack{c\in\mathbb{N}\\ c<\frac{16D}{3a}}} \left(\frac{1}{4}\sum_{\substack{b\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1+\mathcal{O}(1)\right)+\mathcal{O}(D)\ .\end{aligned}$$

For the upper bound on c see the remarks before the statement of this lemma. The summation index b of the innermost sum is constrained by $b<c$ and

$$c<\frac{4D}{ab}+\frac{b}{4} \ \Leftrightarrow\ \frac{b^2}{4}-cb+\frac{4D}{a}>0 \ \Leftrightarrow\ b^2-4bc+\frac{16D}{a}>0\ .$$

We interpret the left-hand side of the last inequality as a parabola function with variable b and determine its zeros:

$$b_{1/2} = 2c \pm \sqrt{4c^2 - \frac{16D}{a}} = 2\left(c \pm \sqrt{c^2 - \frac{D}{a}}\right).$$

For the inequality to be fulfilled we must have $b < b_1$ or $b > b_2$. The second case is impossible since $b < 2c < b_2$. Thus,

$$1 \leq b \leq \min\{c, b_1\}.$$

Hence, we sum over a discrete interval, and adding one at every fourth integer is essentially the same as counting all integers and dividing by four. This verifies the last approximation.

The remaining expression does not depend on y any longer. So the corresponding sum over y translates into a multiplication by 2. We switch back the order of summation and continue with

$$\begin{aligned}
\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 &= \frac{1}{8} \sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{c\in\mathbb{N}\\ c<\frac{16D}{3a}}} 1 \right) + \mathcal{O}(D) \\
&= \frac{1}{8} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \sum_{\substack{b\in\mathbb{N}\\ b<2\sqrt{\frac{2D}{3a}}}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{2D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\leq D}} \frac{D}{a} \right) + \mathcal{O}(D) \\
&= \frac{1}{8} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \sum_{\substack{b\in\mathbb{N}\\ b<2\sqrt{\frac{2D}{3a}}}} \left(\frac{2D}{ab} + \mathcal{O}(b) \right) + \mathcal{O}\left(D\log D\right).
\end{aligned}$$

Because of

$$\sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \sum_{\substack{b\in\mathbb{N}\\ b<2\sqrt{\frac{2D}{3a}}}} \mathcal{O}(b) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \sum_{\substack{b\in\mathbb{N}\\ b<2\sqrt{\frac{2D}{3a}}}} b \right) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \frac{D}{a} \right) = \mathcal{O}\left(D\log D\right),$$

the last estimate is good enough for us to have

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 = \frac{D}{4} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \left(\frac{1}{a} \sum_{\substack{b\in\mathbb{N}\\ b<2\sqrt{\frac{2D}{3a}}}} \frac{1}{b} \right) + \mathcal{O}\left(D\log D\right).$$

We apply Lemma 3.4, part (a):

$$\begin{aligned}
\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 &= \frac{D}{4} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \left(\frac{1}{a} \left(\log\left(2\sqrt{\frac{2D}{3a}} \right) + C + \mathcal{O}\left(\sqrt{\frac{a}{D}} \right) \right) \right) + \mathcal{O}\left(D\log D\right) \\
&= \frac{D}{4} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \left(\frac{1}{a} \left(\frac{\log D - \log a}{2} + \mathcal{O}(1) \right) \right) + \mathcal{O}\left(D\log D\right) \\
&= \frac{D\log D}{8} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \frac{1}{a} - \frac{D}{8} \sum_{\substack{a\in\mathbb{N}\\ a\leq 2D}} \frac{\log a}{a} + \mathcal{O}\left(D\log D\right).
\end{aligned}$$

We once again use the same formula for the first of the remaining sums and Lemma 4.10, part (b) for the second one:

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 a\\ b\equiv_4 0,3}} 1 = \frac{D\log D}{8}\Big(\log(2D)+\mathcal{O}(1)\Big) - \frac{D}{8}\left(\frac{1}{2}\log(2D)^2+\mathcal{O}(1)\right) + \mathcal{O}\left(D\log D\right)$$

$$(61)\qquad = \frac{D\left(\log D\right)^2}{8} - \frac{D\left(\log D\right)^2}{16} + \mathcal{O}\left(D\log D\right) = \frac{D\left(\log D\right)^2}{16} + \mathcal{O}\left(D\log D\right).$$

This concludes the computation of the second double sum from line (59).

The yet to be calculated first double sum only differs in the additional congruence condition $c\equiv_4 a$ from the sum in Lemma 4.11. If we avoid this condition in the usual way, that is, by multiplying by $\frac{1}{4}$ instead, the occurring error is sufficiently small. We begin with

$$\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ c\equiv_4 a}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,f\in\mathbb{N}\\ 2f<b\\ b^2-f^2\le\frac{4D}{a}}}\ \sum_{\substack{c\in\mathbb{N}\\ b<c\\ bc-f^2\le\frac{4D}{a}\\ c\equiv_4 a}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,f\in\mathbb{N}\\ 2f<b\\ b^2-f^2\le\frac{4D}{a}}}\left(\frac{1}{4}\sum_{\substack{c\in\mathbb{N}\\ b<c\le\frac{4D}{ab}+\frac{f^2}{b}}} 1 + \mathcal{O}(1)\right)$$

$$(62)\qquad = \frac{1}{4}\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}}} 1 + \mathcal{O}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,f\in\mathbb{N}\\ 2f<b\\ b^2-f^2\le\frac{4D}{a}}} 1\right).$$

Again, the inequalities $2f<b$ and $b^2-f^2\le\frac{4D}{a}$ imply $b^2<\frac{16D}{3a}$. Thus, the error can be bounded as follows:

$$\mathcal{O}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,f\in\mathbb{N}\\ 2f<b\\ b^2-f^2\le\frac{4D}{a}}} 1\right) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}}\ \sum_{\substack{f\in\mathbb{N}\\ 2f<b}} 1\right) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} b\right) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le D}}\frac{D}{a}\right)$$

$$= \mathcal{O}\left(D\log D\right).$$

Together with the application of Lemma 4.11 to the first summand of line (62) this yields

$$(63)\qquad \sum_{\substack{a\in 2\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ c\equiv_4 a}} 1 = \frac{\sqrt{2}}{18}\pi\zeta\left(\frac{3}{2}\right)D^{\frac{3}{2}} - \frac{3}{32}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right).$$

Finally, we insert the findings from (60),(61) and (63) in line (59) and end up with

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ c,f\in 2\mathbb{N}\\ a+c\in 4\mathbb{N}\\ a(bc-f^2)\le 4D}} 1 = \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}}D^{\frac{3}{2}} - \frac{5}{64}D\left(\log D\right)^2 + \mathcal{O}\left(D\log D\right),$$

which finishes the proof of part (b).

Part (c)

One condition on the sum in part (c) is the parity of $b-f$ and $f-c$. Therefore, we can split the sum according to the two different cases that b, c and f are either all even or all odd, respectively. Note, that $a+b+c-2f \in 4\mathbb{N}$ requires a to be even in both cases. We have

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in 2\mathbb{N}\\ 2f<b<c\\ a+b+c\equiv_4 0\\ bc-f^2\leq \frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in \mathbb{N}\setminus 2\mathbb{N}\\ 2f<b<c\\ a+b+c\equiv_4 2\\ bc-f^2\leq \frac{4D}{a}}} 1.$$

To simplify the congruence conditions we further subdivide the sums with regard to the remainder of a after division by 4. For the first sum, b and c are even. If a is divisible by 4, then

$$a+b+c\equiv_4 0 \Rightarrow b+c\equiv_4 0 \Rightarrow b\equiv_4 c.$$

If a is not divisible by 4 (but even), then

$$a+b+c\equiv_4 0 \Rightarrow b+c\equiv_4 2 \Rightarrow b\equiv_4 c+2.$$

For the second sum, b and c are odd. If a is divisible by 4, then

$$a+b+c\equiv_4 2 \Rightarrow b+c\equiv_4 2 \Rightarrow b\equiv_4 c.$$

If a is not divisible by 4 (but even), then

$$a+b+c\equiv_4 2 \Rightarrow b+c\equiv_4 0 \Rightarrow b\equiv_4 c+2.$$

Hence, we get

$$\begin{aligned}\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 &= \sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in 2\mathbb{N}\\ 2f<b<c\\ b\equiv_4 c\\ bc-f^2\leq \frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\setminus 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in 2\mathbb{N}\\ 2f<b<c\\ b\equiv_4 c+2\\ bc-f^2\leq \frac{4D}{a}}} 1 + \sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in \mathbb{N}\setminus 2\mathbb{N}\\ 2f<b<c\\ b\equiv_4 c\\ bc-f^2\leq \frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\setminus 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in \mathbb{N}\setminus 2\mathbb{N}\\ 2f<b<c\\ b\equiv_4 c+2\\ bc-f^2\leq \frac{4D}{a}}} 1\\ &= \sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in \mathbb{N}\\ 2f<b<c\\ b\equiv_2 c\equiv_2 f\\ b\equiv_4 c\\ bc-f^2\leq \frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\setminus 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c,f\in \mathbb{N}\\ 2f<b<c\\ b\equiv_2 c\equiv_2 f\\ b\equiv_4 c+2\\ bc-f^2\leq \frac{4D}{a}}} 1.\end{aligned}$$

The sums over f will be carried out first, since they coincide for both expressions:

(64)
$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 c}} \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 b}} 1 + \sum_{\substack{a\in 2\mathbb{N}\setminus 4\mathbb{N}\\ a\leq 4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 c+2}} \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 b}} 1.$$

For part (a) of this lemma we already calculated a sum which only differs from the present sum over f in the congruence condition. Then we had $f\equiv_2 0$ and now we have $f\equiv_2 b$. Therefore, the next steps are quite similar to those following line (46):

$$\begin{aligned}\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 b}} 1 &= \frac{1}{2}\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}}} 1 + \frac{1}{2}\Bigg(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 b}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}\\ f\equiv_2 b+1}} 1\Bigg)\\ &= \frac{1}{2}\Bigg(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\leq\frac{4D}{a}}} 1 + \Big(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f\equiv_2 b}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f\equiv_2 b+1}} 1\Big) - \Big(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f^2<bc-\frac{4D}{a}\\ f\equiv_2 b}} 1 - \sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ f^2<bc-\frac{4D}{a}\\ f\equiv_2 b+1}} 1\Big)\Bigg).\end{aligned}$$

To determine the first difference we have to take into account the remainder of b modulo 4 again:

$$\begin{aligned}
b \equiv_4 0 \quad &\Rightarrow \quad b \equiv_2 0, \tfrac{b}{2} \in 2\mathbb{Z} &&\Rightarrow \quad f_{\max} \text{ is odd,}\\
b \equiv_4 1 \quad &\Rightarrow \quad b \equiv_2 1, \tfrac{b}{2} \in 2\mathbb{Z} + \tfrac{1}{2} &&\Rightarrow \quad f_{\max} \text{ is even,}\\
b \equiv_4 2 \quad &\Rightarrow \quad b \equiv_2 0, \tfrac{b}{2} \in \mathbb{Z} \setminus 2\mathbb{Z} &&\Rightarrow \quad f_{\max} \text{ is even,}\\
b \equiv_4 3 \quad &\Rightarrow \quad b \equiv_2 1, \tfrac{b}{2} \in (\mathbb{Z} \setminus 2\mathbb{Z}) + \tfrac{1}{2} &&\Rightarrow \quad f_{\max} \text{ is odd.}
\end{aligned}$$

The difference to the calculation in part (a) is given by the commutation of minuend and subtrahend for $b \equiv_4 1$ and $b \equiv_4 3$.
The second difference is considered as an error term. Altogether, we have

$$\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\le\frac{4D}{a}\\ f\equiv_2 b}} 1 = \frac{1}{2}\left(\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ bc-f^2\le\frac{4D}{a}}} 1 + \left\{ \begin{array}{rl} -1, & b\equiv_4 0\\ 0, & b\equiv_4 1,2\\ 1, & b\equiv_4 3 \end{array}\right\} - \left\{ \begin{array}{rl} \mathcal{O}(1), & bc-\frac{4D}{a}>1\\ 0, & bc-\frac{4D}{a}\le 1 \end{array}\right\}\right).$$

Inserting this twice in line (64) yields

$$\begin{aligned}
\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in2\mathbb{Z}\\ a+b+c-2f\in4\mathbb{N}\\ a(bc-f^2)\le4D}} 1 = \frac{1}{2}\Bigg(&\sum_{\substack{a\in4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 c}} 1 + \sum_{\substack{a\in2\mathbb{N}\setminus4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\le\frac{4D}{a}\\ b\equiv_4 c+2}} 1\\
&- \sum_{\substack{a\in4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 c\equiv_4 0}} 1 + \sum_{\substack{a\in4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 c\equiv_4 3}} 1 - \sum_{\substack{a\in2\mathbb{N}\setminus4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ 0\equiv_4 b\equiv_4 c+2}} 1 + \sum_{\substack{a\in2\mathbb{N}\setminus4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ 3\equiv_4 b\equiv_4 c+2}} 1 \qquad (65)\\
&+ \sum_{\substack{a\in4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 c}} \mathcal{O}(1) + \sum_{\substack{a\in2\mathbb{N}\setminus4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 c+2}} \mathcal{O}(1)\Bigg). \qquad (66)
\end{aligned}$$

We claim that the error terms in line (66) do not grow faster than $D\log D$. We have

$$\sum_{\substack{a\in4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 c}} \mathcal{O}(1)\ ,\ \sum_{\substack{a\in2\mathbb{N}\setminus4\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ bc-\frac{4D}{a}>1\\ b\equiv_4 c+2}} \mathcal{O}(1) = \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le4D}} \sum_{\substack{b,c\in\mathbb{N}\\ b<c\\ \frac{4D}{ab}+\frac{1}{b}<c<\frac{4D}{ab}+\frac{b}{4}}} 1 \right) = \mathcal{O}\left(D\log D\right).$$

The last equality has been explained in detail in part (a). Now, we want to show that the four sums of line (65) are all equal up to this error term, and this line adds up to $\mathcal{O}\left(D\log D\right)$, thus. We note that all four sums share the following structure:

$$\sum_{\substack{a\in\mathbb{N}\\ a\le4D\\ a\equiv_4 x}} \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}} \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 z}} 1, \quad x,y,z\in\{0,1,2,3\}.$$

This triple sum can be determined sufficiently exactly and independently of x, y and z. The calculation is similar to that of the second double sum from line (59). That reflects the fact, that the expression from part (a) can be seen as a sum of four sums of the general type (all combinations of $x = 0, 2$ with $y = 0, 3$). We start with the elimination of z by

$$\begin{aligned}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}}\ \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 z}} 1 &= \sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}} \left(\frac{1}{4}\sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}(1)\right)\\ &= \frac{1}{4}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{c\in\mathbb{N}\\ c<\frac{16D}{3a}}}\ \sum_{\substack{b\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ b\equiv_4 y}} 1 + \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} 1\right)\end{aligned}$$

and continue with applying the same method to y:

$$\begin{aligned}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}}\ \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 z}} 1 &= \frac{1}{4}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{c\in\mathbb{N}\\ c<\frac{16D}{3a}}} \left(\frac{1}{4}\sum_{\substack{b\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}(1)\right) + \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}} \sqrt{\frac{D}{a}}\right)\\ &= \frac{1}{16}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}}\ \sum_{\substack{c\in\mathbb{N}\\ c<\frac{16D}{3a}}} 1\right) + \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}} \sqrt{\frac{D}{a}}\right).\end{aligned}$$

We once again change the order of summation to get rid of x and note that the first error term dominates the second one:

$$\begin{aligned}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}}\ \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 z}} 1 &= \frac{1}{16}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{b}+\frac{b}{4}}}\ \sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ c<\frac{4D}{ab}+\frac{b}{4}\\ a\equiv_4 x}} 1 + \mathcal{O}\left(\sum_{\substack{a\in\mathbb{N}\\ a\le 4D}} \frac{D}{a}\right)\\ &= \frac{1}{16}\sum_{\substack{b,c\in\mathbb{N}\\ b<c<\frac{4D}{b}+\frac{b}{4}}} \left(\frac{1}{4}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}(1)\right) + \mathcal{O}\left(D\log D\right).\end{aligned}$$

It remains to show that the new error term is small enough:

$$\begin{aligned}\sum_{\substack{a\in\mathbb{N}\\ a\le 4D\\ a\equiv_4 x}}\ \sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}\\ b\equiv_4 y}}\ \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{ab}+\frac{b}{4}\\ c\equiv_4 z}} 1 &= \frac{1}{64}\sum_{\substack{a,b,c\in\mathbb{N}\\ a\le 4D\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}\left(\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3}}}}\ \sum_{\substack{c\in\mathbb{N}\\ b<c<\frac{4D}{b}+\frac{b}{4}}} 1\right) + \mathcal{O}\left(D\log D\right)\\ &= \frac{1}{64}\sum_{\substack{a,b,c\in\mathbb{N}\\ a\le 4D\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}\left(\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3}}}} \frac{D}{b}\right) + \mathcal{O}\left(D\log D\right)\\ &= \frac{1}{64}\sum_{\substack{a,b,c\in\mathbb{N}\\ a\le 4D\\ b<c<\frac{4D}{ab}+\frac{b}{4}}} 1 + \mathcal{O}\left(D\log D\right).\end{aligned}$$

Having determined lines (65) and (66), we arrive at

$$\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 = \frac{1}{2}\left(\sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 c}} 1 + \sum_{\substack{a\in 2\mathbb{N}\setminus 4\mathbb{N}\\ a\leq 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 c+2}} 1\right) + \mathcal{O}\left(D\log D\right). \tag{67}$$

For the calculation of the interior triple sums let $y\in\{0,2\}$. We have

$$\begin{aligned}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}\\ b\equiv_4 c+y}} 1 &= \sum_{\substack{b,f\in\mathbb{N}\\ 2f<b\\ b^2-f^2<\frac{4D}{a}}}\sum_{\substack{c\in\mathbb{N}\\ b<c\\ bc-f^2\leq\frac{4D}{a}\\ c\equiv_4 b+y}} 1 = \sum_{\substack{b,f\in\mathbb{N}\\ 2f<b\\ b^2-f^2<\frac{4D}{a}}}\left(\frac{1}{4}\sum_{\substack{c\in\mathbb{N}\\ b<c\leq\frac{4D}{ab}+\frac{f^2}{b}}} 1+\mathcal{O}(1)\right)\\ &= \frac{1}{4}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1+\mathcal{O}\left(\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}}\sum_{\substack{f\in\mathbb{N}\\ 2f<b\\ b^2-f^2\leq\frac{4D}{a}}} 1\right) = \frac{1}{4}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1+\mathcal{O}\left(\sum_{\substack{b\in\mathbb{N}\\ b<4\sqrt{\frac{D}{3a}}}} b\right)\\ &= \frac{1}{4}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1+\mathcal{O}\left(\frac{D}{a}\right).\end{aligned}$$

We insert this in line (67):

$$\begin{aligned}\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 &= \frac{1}{8}\left(\sum_{\substack{a\in 4\mathbb{N}\\ a\leq 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1 + \sum_{\substack{a\in 2\mathbb{N}\setminus 4\mathbb{N}\\ a\leq 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1\right) + \mathcal{O}\left(\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}}\frac{D}{a}\right) + \mathcal{O}\left(D\log D\right)\\ &= \frac{1}{8}\sum_{\substack{a\in 2\mathbb{N}\\ a\leq 4D}}\sum_{\substack{b,c,f\in\mathbb{N}\\ 2f<b<c\\ bc-f^2\leq\frac{4D}{a}}} 1+\mathcal{O}\left(D\log D\right).\end{aligned}$$

We complete the proof of part (c) by applying Lemma 4.11 once more:

$$\begin{aligned}\sum_{\substack{a,b,c,f\in\mathbb{N}\\ 2f<b<c\\ b-f,f-c\in 2\mathbb{Z}\\ a+b+c-2f\in 4\mathbb{N}\\ a(bc-f^2)\leq 4D}} 1 &= \frac{1}{8}\left(\frac{2\sqrt{2}}{9}\pi\zeta\left(\frac{3}{2}\right)D^{\frac{3}{2}}-\frac{3}{8}D\left(\log D\right)^2\right)+\mathcal{O}\left(D\log D\right)\\ &= \frac{\pi\zeta\left(\frac{3}{2}\right)}{18\sqrt{2}}D^{\frac{3}{2}}-\frac{3}{64}D\left(\log D\right)^2+\mathcal{O}\left(D\log D\right).\end{aligned}$$

□

Altogether, adding up the three formulas of Lemma 4.9 and taking into account the preceding remarks finishes the proof of Proposition 4.8.

To finish this chapter, we give a review of the above results that were obtained by the different methods. We coarsen the findings by concentrating on complete Bravais classes. Moreover, we only specify the terms of highest order of magnitude.

Theorem 4.12. *For every index i, the function f_i of Table 4 fulfils the asymptotic equality $f_i(x) \sim H_i(x)$ for $x \to \infty$.*

Proof. The results for the cubic Bravais classes carry over from Table 3 unchanged. The same holds for the primitive orthorhombic case.

The formulas for the hexagonal, tetragonal and trigonal Bravais classes are obtained by summing up several results for the corresponding Schiemann classes from Table 3. The same holds for the face-centered orthorhombic case.

The body-centered and base-centered orthorhombic results are given in Proposition 4.6.

The monoclinic formulas are established (more precisely) in Propositions 4.7 and 4.8.

For the triclinic case, that is, for the lattices with trivial automorphism group, we use the Theorem of Minkowski (Theorem 3.5). For the three-dimensional case Minkowski has shown

$$H(D) = v_3 D^2 + \mathcal{O}\left(D^{\frac{5}{3}}\right),$$

where $H(D)$ is the class number of all three-dimensional lattices and v_3 is a certain constant. Since all the other Bravais classes grow with D at most as fast as some multiple of $D^{\frac{3}{2}}$, we can deduce that $v_3 D^2$ is the order of magnitude of the triclinic lattices. It remains to calculate the constant. We use $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ and $\zeta\left(\frac{3}{2}\right) = \frac{\pi^2}{6}$ to get

$$v_3 = \frac{2}{4\Gamma\left(\frac{1}{2}\right)^5} \prod_{k=2}^{3} \Gamma\left(\frac{k}{2}\right) \zeta(k) = \frac{1}{2\pi^{\frac{5}{2}}} \cdot \frac{\pi^2}{6} \cdot \frac{1}{2}\sqrt{\pi} \cdot \zeta(3) = \frac{1}{24}\zeta(3) .$$

This proves the last line of Table 4. □

Table 4: Asymptotic Class Numbers of Bravais Types

Crystal System	**Bravais Type**	**Index** i	$f_i(x)$
cubic	primitive	$48, p$	$x^{\frac{1}{3}}$
cubic	body-centered	$48, i$	$2^{-\frac{4}{3}}x^{\frac{1}{3}}$
cubic	face-centered	$48, f$	$2^{-\frac{2}{3}}x^{\frac{1}{3}}$
hexagonal	primitive	$24, p$	$\frac{1}{3}\zeta(2)x$
tetragonal	primitive	$16, p$	$\zeta(2)x$
tetragonal	body-centered	$16, i$	$\frac{1}{4}\zeta(2)x$
trigonal	rhombohedral	$12, r$	$\frac{1}{3}\zeta(2)x$
orthorhombic	primitive	$8, p$	$\frac{1}{12}x(\log x)^2$
orthorhombic	body-centered	$8, i$	$\frac{1}{48}x(\log x)^2$
orthorhombic	face-centered	$8, f$	$\frac{1}{48}x(\log x)^2$
orthorhombic	base-centered	$8, b$	$\frac{1}{8}x(\log x)^2$
monoclinic	primitive	$4, p$	$\frac{\pi}{9}\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}$
monoclinic	base-centered	$4, b$	$\frac{\pi}{6\sqrt{2}}\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}$
triclinic	primitive	$2, p$	$\frac{1}{24}\zeta(3)x^2$

A Bravais Classes of Lattices

We collect some properties of three-dimensional lattices. Each lattice has a certain Bravais type. There are 14 Bravais types, which belong to the following seven crystal systems: cubic, hexagonal, tetragonal, trigonal, orthorhombic, monoclinic, triclinic. Subsets of Euclidean vector spaces are called Bravais equivalent if their symmetry groups (distance preserving affine transformations of the surrounding vector space that leave the set invariant) are conjugated in the general affine group of the space. We will use another definition (specialised for lattices) that involves the orthogonal group (see Definition 1.6) of the lattice.

Definition A.1. Two lattices (L_1, b_2) and (L_2, b_2) are **Bravais equivalent** if there is a linear transformation $F \in \mathrm{GL}(V)$ with $F(L_1) = L_2$ and $F \circ \mathcal{O}(L_1, b_1) \circ F^{-1} = \mathcal{O}(L_2, b_2)$.

For the equivalence of the two definitions see [Sch12, Lemma 2.6.12].

Definition A.2. Two lattices (L_1, b_2) and (L_2, b_2) belong to the same crystal system if there is a linear transformation $F \in \mathrm{GL}(V)$ with $F \circ \mathcal{O}(L_1, b_1) \circ F^{-1} = \mathcal{O}(L_2, b_2)$, that is, if their orthogonal groups are conjugated in the linear group of the vector space.

For the fact that every three-dimensional lattice belongs to exactly one of the following crystal systems and Bravais types see [Kle82, Satz 12.9]. There, the hexagonal and trigonal crystal systems are merged.

Provided the isometry group of the lattice has order at least eight, we sketch an excerpt of the lattice containing a fundamental domain of the group's action on the lattice. We add the shortest vectors, a reduced basis with corresponding Gram matrix, and the isometry group of the lattice.

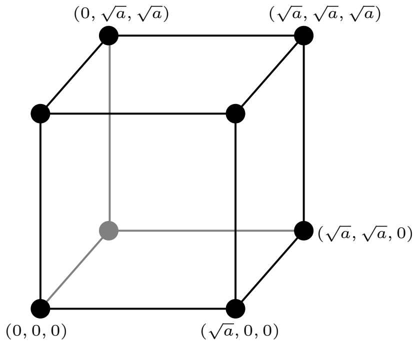
We use Schiemann's notion of a reduced basis here, so that *reduced* will be short for *Schiemann-reduced*.

Cubic Crystal System

The cubic crystal system (more precisely, the set of all isometry classes of cubic lattices) is subdivided in three Bravais classes: primitive, body-centered, face-centered.

Let $a > 0$ be the square length of a shortest lattice vector in the primitive case.

primitive



6 shortest vectors:

$$(\pm\sqrt{a}, 0, 0),$$
$$(0, \pm\sqrt{a}, 0),$$
$$(0, 0, \pm\sqrt{a}).$$

Reduced basis:

$$(\sqrt{a}, 0, 0),$$
$$(0, \sqrt{a}, 0),$$
$$(0, 0, \sqrt{a}).$$

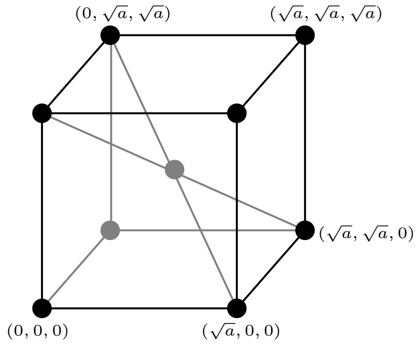
Corresponding Gram matrix:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Isometry group of order 48:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle =: G_{48,p} \ .$$

body-centered



8 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a}, \pm\sqrt{a}, \pm\sqrt{a}\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a}, \sqrt{a}, \sqrt{a}\right),$$
$$\frac{1}{2}\left(\sqrt{a}, -\sqrt{a}, \sqrt{a}\right),$$
$$\frac{1}{2}\left(\sqrt{a}, \sqrt{a}, -\sqrt{a}\right).$$

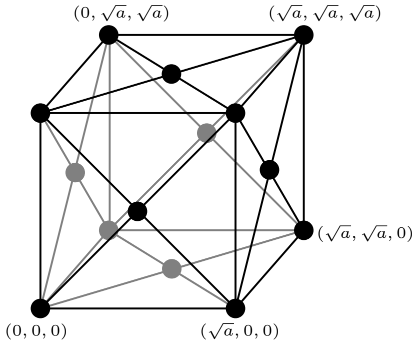
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 3a & a & a \\ a & 3a & -a \\ a & -a & 3a \end{pmatrix}.$$

Isometry group of order 48:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rangle =: G_{48,i} \ .$$

face-centered



12 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a},\pm\sqrt{a},0\right),$$
$$\frac{1}{2}\left(\pm\sqrt{a},0,\pm\sqrt{a}\right),$$
$$\frac{1}{2}\left(0,\pm\sqrt{a},\pm\sqrt{a}\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a},\sqrt{a},0\right),$$
$$\frac{1}{2}\left(\sqrt{a},0,\sqrt{a}\right),$$
$$\frac{1}{2}\left(0,\sqrt{a},\sqrt{a}\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix}2a & a & a\\ a & 2a & a\\ a & a & 2a\end{pmatrix}.$$

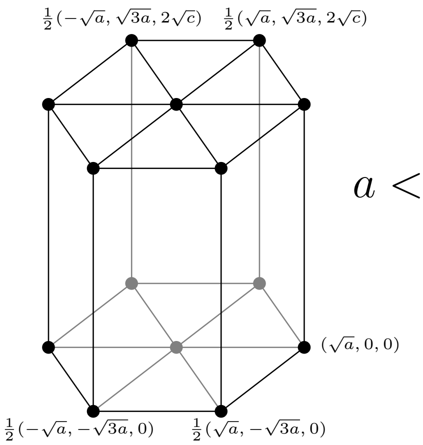
Isometry group of order 48:

$$\langle \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 1 & 1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{pmatrix}\rangle =: G_{48,f}\ .$$

Hexagonal Crystal System

This crystal system consists of only one Bravais class.

Again, let a be the square length of a shortest lattice vector. In the generic hexagonal case a reduced basis contains at least one vector of greater length. According to the common entries of the corresponding Gram matrix, we denote this second length by b, if there is only one vector of length a, and by c, otherwise.



$a < c$ *

6 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right),$$
$$\frac{1}{2}\left(\pm\sqrt{a}, \pm\sqrt{3a}, 0\right).$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right),$$
$$\frac{1}{2}\left(\sqrt{a}, \sqrt{3a}, 0\right),$$
$$\left(0, 0, \sqrt{c}\right).$$

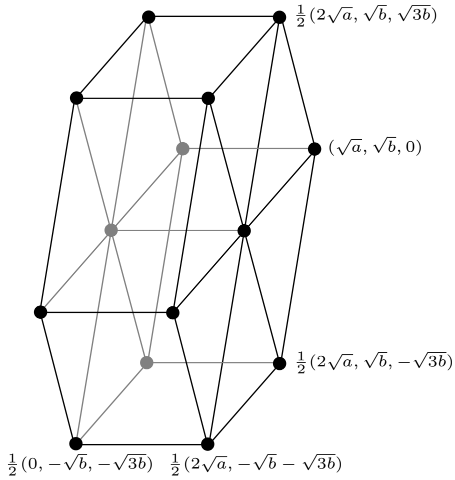
Corresponding Gram matrix:

$$\frac{1}{2}\begin{pmatrix} 2a & a & 0 \\ a & 2a & 0 \\ 0 & 0 & 2c \end{pmatrix}.$$

Isometry group of order 24:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{24,p,1}.$$

* The equality $a = c$ is admissible here as well. Then, however, there are eight shortest lattice vectors. In the second hexagonal case we need the strict inequality $a < b$ for the given vectors to form a reduced basis.



$a < b$

2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right).$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right),$$
$$\left(0, \sqrt{b}, 0\right),$$
$$\frac{1}{2}\left(0, \sqrt{b}, \sqrt{3b}\right).$$

Corresponding Gram matrix:

$$\frac{1}{2}\begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & b \\ 0 & b & 2b \end{pmatrix}.$$

Isometry group of order 24:

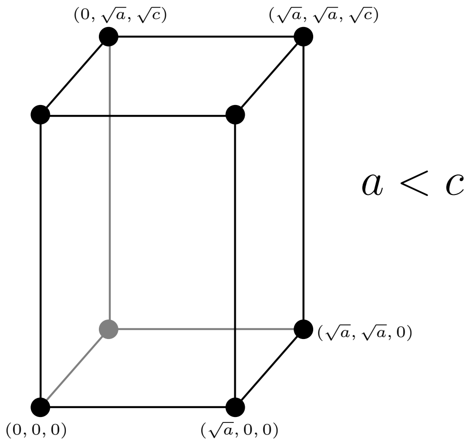
$$\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{24,p,2}.$$

Tetragonal Crystal System

This crystal system includes the primitive tetragonal and the body-centered tetragonal Bravais class.

There are two free parameters again. We denote them with a and b or c, following the same principle as in the hexagonal case above.

primitive



4 shortest vectors:

$$\left(\pm\sqrt{a},0,0\right),$$
$$\left(0,\pm\sqrt{a},0\right).$$

Reduced basis:

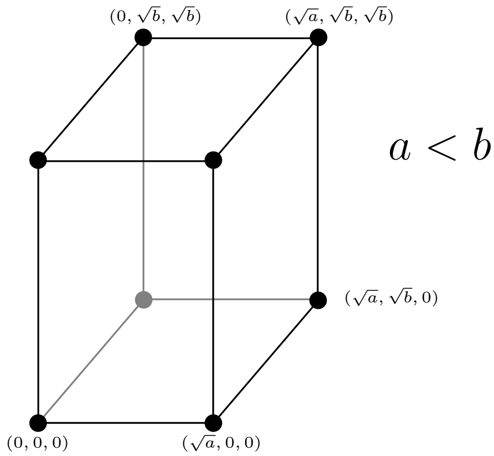
$$\left(\sqrt{a},0,0\right),$$
$$\left(0,\sqrt{a},0\right),$$
$$\left(0,0,\sqrt{c}\right).$$

Corresponding Gram matrix:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Isometry group of order 16:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{16,p,1} .$$



2 shortest vectors:

$$\left(\pm\sqrt{a},0,0\right).$$

Reduced basis:

$$\left(\sqrt{a},0,0\right),$$
$$\left(0,\sqrt{b},0\right),$$
$$\left(0,0,\sqrt{b}\right).$$

Corresponding Gram matrix:

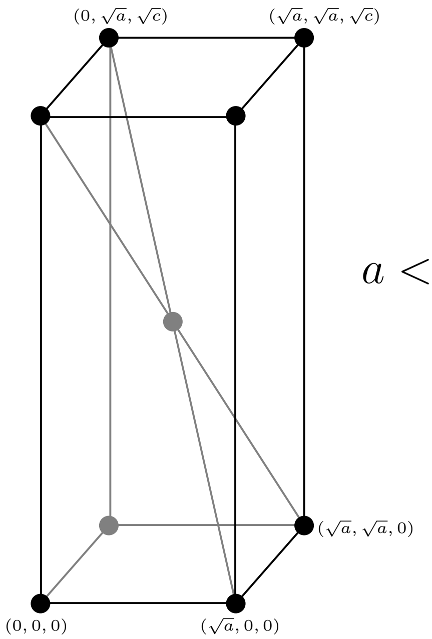
$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

Isometry group of order 16:

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle =: G_{16,p,2} .$$

For $a = c$ and $a = b$ the lattice would belong to the primitive cubic type.

body-centered



$$a < \tfrac{1}{2}\, c$$

4 shortest vectors:

$$\left(\pm\sqrt{a},0,0\right),$$
$$\left(0,\pm\sqrt{a},0\right).$$

Reduced basis:

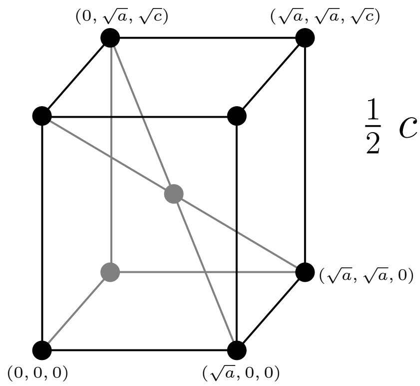
$$\left(\sqrt{a},0,0\right),$$
$$\left(0,\sqrt{a},0\right),$$
$$\frac{1}{2}\left(\sqrt{a},\sqrt{a},\sqrt{c}\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 0 & 2a \\ 0 & 4a & 2a \\ 2a & 2a & 2a+c \end{pmatrix}.$$

Isometry group of order 16:

$$\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{16,i,1}\ .$$



$$\tfrac{1}{2}\, c < a < c \quad *$$

8 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a},\pm\sqrt{a},\pm\sqrt{c}\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a},\sqrt{a},\sqrt{c}\right),$$
$$\frac{1}{2}\left(-\sqrt{a},\sqrt{a},\sqrt{c}\right),$$
$$\frac{1}{2}\left(\sqrt{a},-\sqrt{a},\sqrt{c}\right).$$

Corresponding Gram matrix:

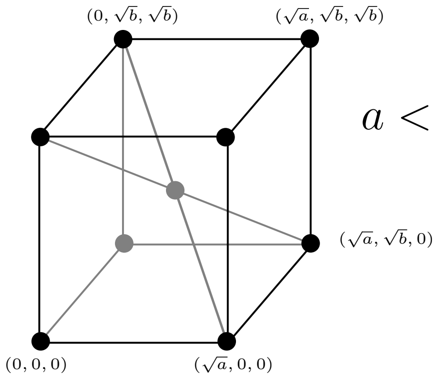
$$\frac{1}{4}\begin{pmatrix} 2a+c & c & c \\ c & 2a+c & -2a+c \\ c & -2a+c & 2a+c \end{pmatrix}.$$

Isometry group of order 16:

$$\langle \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{16,i,2}\ .$$

* For $a = c$ we get a body-centered cubic lattice.

In case of $2a = c$ there are twelve shortest vectors, the spanned lattice is face-centered cubic. It can be obtained from the above given representative by a rotation of $\frac{\pi}{4}$ around one of the space diagonals followed by a dilation with factor $\sqrt{2}$. None of the given bases remains reduced for $2a = c$.



$$a < b < \tfrac{3}{2}\, a$$

8 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a}, \pm\sqrt{b}, \pm\sqrt{b}\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, \sqrt{b}\right),$$
$$\frac{1}{2}\left(-\sqrt{a}, \sqrt{b}, \sqrt{b}\right),$$
$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, -\sqrt{b}\right).$$

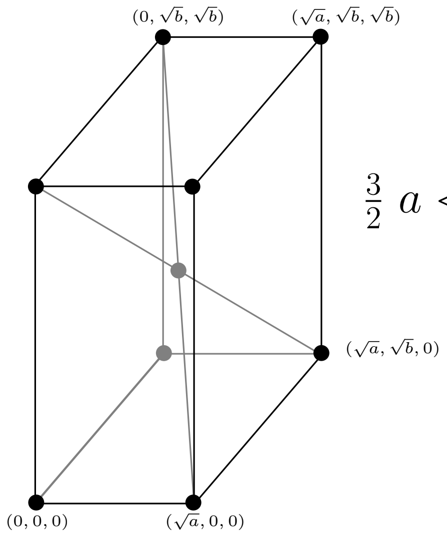
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} a+2b & -a+2b & a \\ -a+2b & a+2b & -a \\ a & -a & a+2b \end{pmatrix}.$$

Isometry group of order 16:

$$\langle \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{16,i,3}\ .$$

For $a = b$ the lattice is body-centered cubic.



$$\tfrac{3}{2}\, a < b \quad *$$

2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right).$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right),$$
$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, \sqrt{b}\right),$$
$$\frac{1}{2}\left(\sqrt{a}, -\sqrt{b}, \sqrt{b}\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 2a & 2a \\ 2a & a+2b & a \\ 2a & a & a+2b \end{pmatrix}.$$

Isometry group of order 16:

$$\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{16,i,4}\ .$$

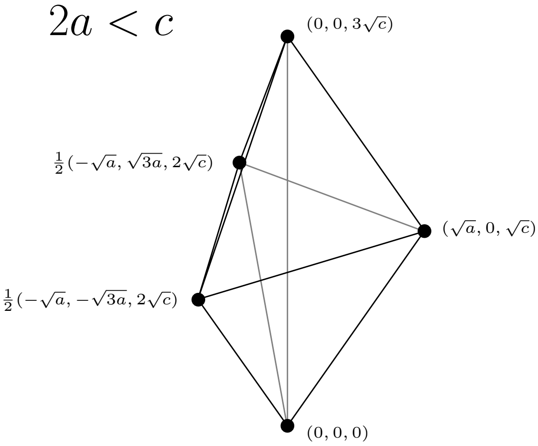
* The equality $3a = 2b$ is possible here, resulting in ten shortest vectors. In the preceding case however, we need the strict inequality for the given vectors to form a reduced basis.

Trigonal Crystal System

The trigonal crystal system consists of only one Bravais class, which is called rhombohedral.

rhombohedral

$2a < c$



6 shortest vectors:

$$\frac{1}{2}\left(\pm 3\sqrt{a}, \pm\sqrt{3a}, 0\right),$$
$$\left(0, \pm\sqrt{3a}, 0\right).$$

Reduced basis:

$$\frac{1}{2}\left(3\sqrt{a}, -\sqrt{3a}, 0\right),$$
$$\frac{1}{2}\left(3\sqrt{a}, \sqrt{3a}, 0\right),$$
$$\left(\sqrt{a}, 0, \sqrt{c}\right).$$

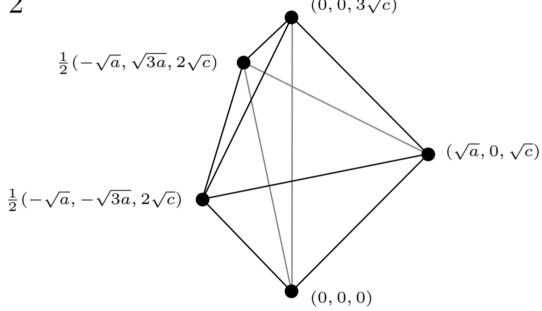
Corresponding Gram matrix:

$$\frac{1}{2}\begin{pmatrix} 6a & 3a & 3a \\ 3a & 6a & 3a \\ 3a & 3a & 2a+2c \end{pmatrix}.$$

Isometry group of order 12:

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle =: G_{12,r,1}\ .$$

$\frac{1}{2}\ a < c < 2a$



6 shortest vectors:

$$\pm\left(\sqrt{a}, 0, \sqrt{c}\right),$$
$$\pm\frac{1}{2}\left(-\sqrt{a}, \sqrt{3a}, 2\sqrt{c}\right),$$
$$\pm\frac{1}{2}\left(-\sqrt{a}, -\sqrt{3a}, 2\sqrt{c}\right).$$

Reduced basis:

$$\left(\sqrt{a}, 0, \sqrt{c}\right),$$
$$\frac{1}{2}\left(-\sqrt{a}, \sqrt{3a}, 2\sqrt{c}\right),$$
$$\frac{1}{2}\left(-\sqrt{a}, -\sqrt{3a}, 2\sqrt{c}\right).$$

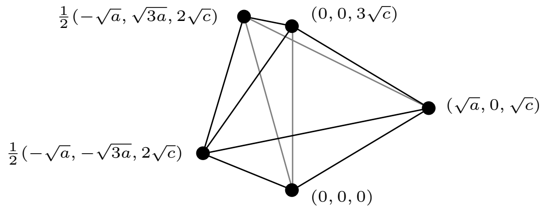
Corresponding Gram matrix:

$$\frac{1}{2}\begin{pmatrix} 2a+2c & -a+2c & -a+2c \\ -a+2c & 2a+2c & -a+2c \\ -a+2c & -a+2c & 2a+2c \end{pmatrix}.$$

Isometry group of order 12:

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle =: G_{12,r,2}\ .$$

$\frac{1}{8}\,a < c < \frac{1}{2}\,a$



6 shortest vectors:

$$\pm\left(\sqrt{a}, 0, \sqrt{c}\right), \quad \pm\frac{1}{2}\left(-\sqrt{a}, \sqrt{3a}, 2\sqrt{c}\right), \quad \pm\frac{1}{2}\left(-\sqrt{a}, -\sqrt{3a}, 2\sqrt{c}\right).$$

Reduced basis:

$$\left(-\sqrt{a}, 0, -\sqrt{c}\right), \quad \frac{1}{2}\left(-\sqrt{a}, \sqrt{3a}, 2\sqrt{c}\right), \quad \frac{1}{2}\left(-\sqrt{a}, -\sqrt{3a}, 2\sqrt{c}\right).$$

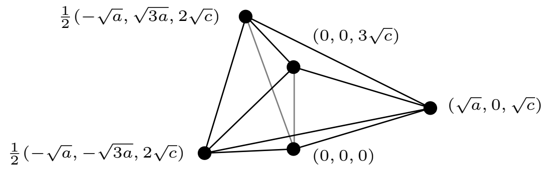
Corresponding Gram matrix:

$$\frac{1}{2}\begin{pmatrix} 2a+2c & a-2c & a-2c \\ a-2c & 2a+2c & -a+2c \\ a-2c & -a+2c & 2a+2c \end{pmatrix}.$$

Isometry group of order 12:

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle =: G_{12,r,3}.$$

$c < \frac{1}{8}\,a$



2 shortest vectors:

$$\left(0, 0, \pm 3\sqrt{c}\right).$$

Reduced basis:

$$\left(0, 0, 3\sqrt{c}\right), \quad \left(\sqrt{a}, 0, \sqrt{c}\right), \quad \frac{1}{2}\left(-\sqrt{a}, \sqrt{3a}, 2\sqrt{c}\right).$$

Corresponding Gram matrix:

$$\frac{1}{2}\begin{pmatrix} 18c & 6c & 6c \\ 6c & 2a+2c & -a+2c \\ 6c & -a+2c & 2a+2c \end{pmatrix}.$$

Isometry group of order 12:

$$\left\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle =: G_{12,r,4}.$$

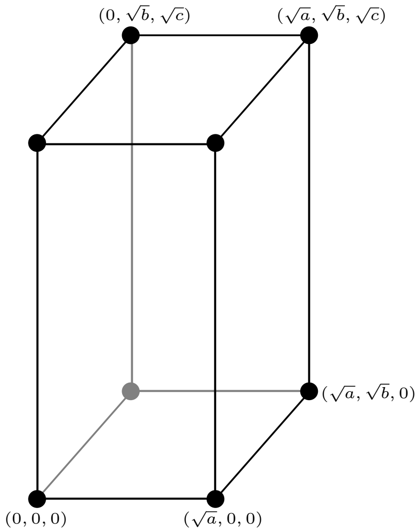
If we replace any of the classifying inequalities of the different rhombohedral trigonal cases by the corresponding equality, the given basis always remains reduced. The lattice becomes cubic. For $c = 2a$, $2c = a$ and $8c = a$ it is face-centered, primitive and body-centered, respectively.

Orthorhombic Crystal System

The orthorhombic crystal system's Bravais classes are the following four: primitive, body-centered, face-centered, base-centered.

Let a, b, c be the square lengths of the basis vectors in the primitive case. We assume $a < b < c$.

primitive



2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right) .$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right) ,$$
$$\left(0, \sqrt{b}, 0\right) ,$$
$$\left(0, 0, \sqrt{c}\right) .$$

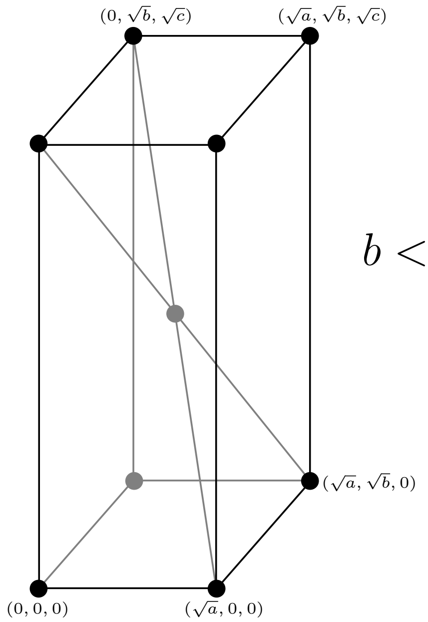
Corresponding Gram matrix:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} .$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,p} .$$

body-centered



$$b < \frac{a+b+c}{4}$$

2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right) .$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right) ,$$
$$\left(0, \sqrt{b}, 0\right) ,$$
$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right) .$$

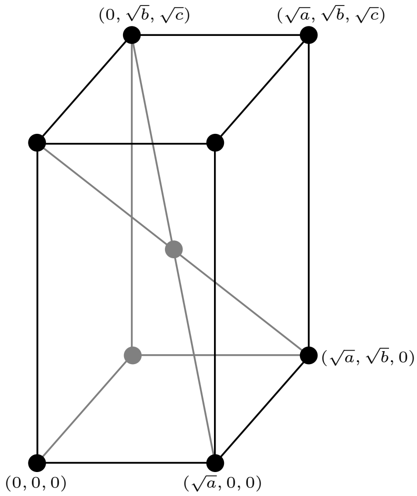
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 0 & 2a \\ 0 & 4b & 2b \\ 2a & 2b & a+b+c \end{pmatrix} .$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} , \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} ,$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,i,1} .$$

$$a < \frac{a+b+c}{4} \leq b \quad *$$



2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right).$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right), \quad \frac{1}{2}\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right), \quad \frac{1}{2}\left(\sqrt{a}, -\sqrt{b}, \sqrt{c}\right).$$

Corresponding Gram matrix:

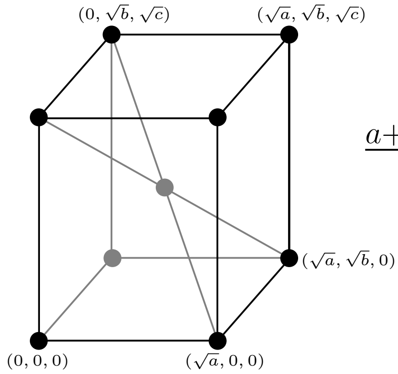
$$\frac{1}{4}\begin{pmatrix} 4a & 2a & 2a \\ 2a & a+b+c & a-b+c \\ 2a & a-b+c & a+b+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,i,2}.$$

* For $3a = b + c$ the basis remains reduced and we have ten shortest vectors. The basis of the following case would no longer be reduced in case of this equality.

$$\frac{a+b+c}{4} < a$$



8 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a}, \pm\sqrt{b}, \pm\sqrt{c}\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right), \quad \frac{1}{2}\left(-\sqrt{a}, \sqrt{b}, \sqrt{c}\right), \quad \frac{1}{2}\left(\sqrt{a}, -\sqrt{b}, \sqrt{c}\right).$$

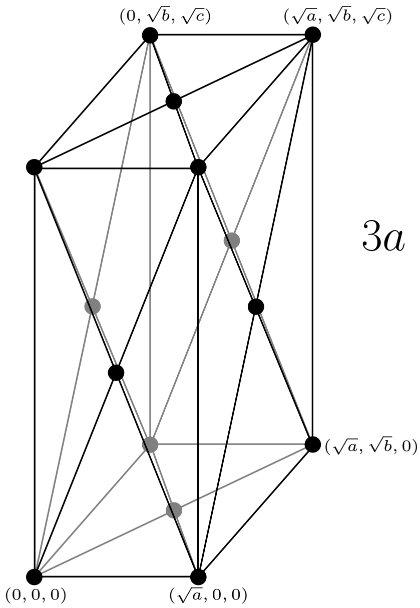
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} a+b+c & -a+b+c & a-b+c \\ -a+b+c & a+b+c & -a-b+c \\ a-b+c & -a-b+c & a+b+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,i,3}.$$

face-centered



$3a < b$ *

2 shortest vectors:

$$\left(\pm\sqrt{a},0,0\right).$$

Reduced basis:

$$\left(\sqrt{a},0,0\right), \quad \frac{1}{2}\left(\sqrt{a},\sqrt{b},0\right), \quad \frac{1}{2}\left(\sqrt{a},0,\sqrt{c}\right).$$

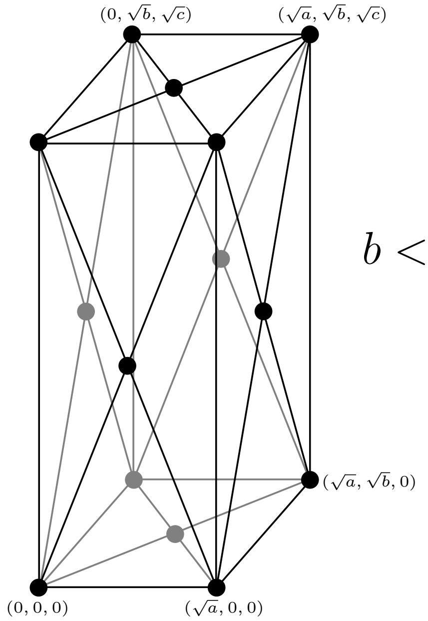
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 2a & 2a \\ 2a & a+b & a \\ 2a & a & a+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,f,1}.$$

* Here, as opposed to the next case, $3a = b$ is possible, resulting in six shortest vectors.



$b < 3a$

4 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a},\pm\sqrt{b},0\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a},\sqrt{b},0\right), \quad \frac{1}{2}\left(-\sqrt{a},\sqrt{b},0\right), \quad \frac{1}{2}\left(\sqrt{a},0,\sqrt{c}\right).$$

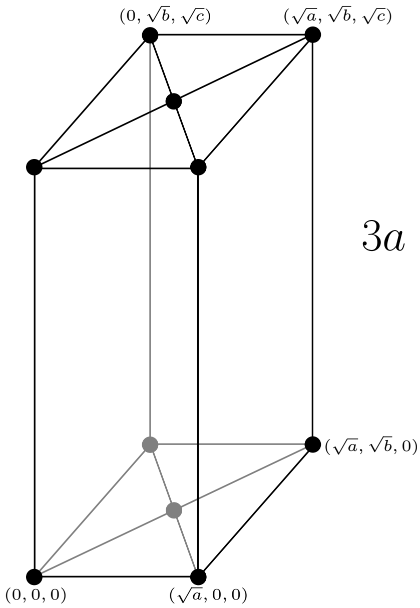
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} a+b & -a+b & a \\ -a+b & a+b & -a \\ a & -a & a+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,f,2}.$$

base-centered



$3a < b$

2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right).$$

Reduced basis:

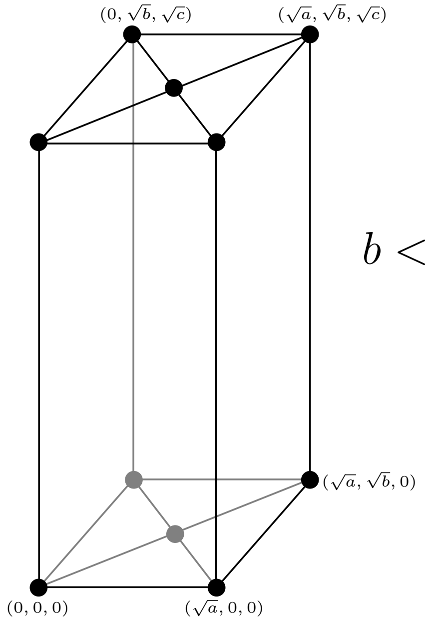
$$\left(\sqrt{a}, 0, 0\right),$$
$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, 0\right),$$
$$\left(0, 0, \sqrt{c}\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 2a & 0 \\ 2a & a+b & 0 \\ 0 & 0 & 4c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,b,1}.$$



$b < 3a$

4 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a}, \pm\sqrt{b}, 0\right).$$

Reduced basis:

$$\frac{1}{2}\left(\sqrt{a}, \sqrt{b}, 0\right),$$
$$\frac{1}{2}\left(-\sqrt{a}, \sqrt{b}, 0\right),$$
$$\left(0, 0, \sqrt{c}\right).$$

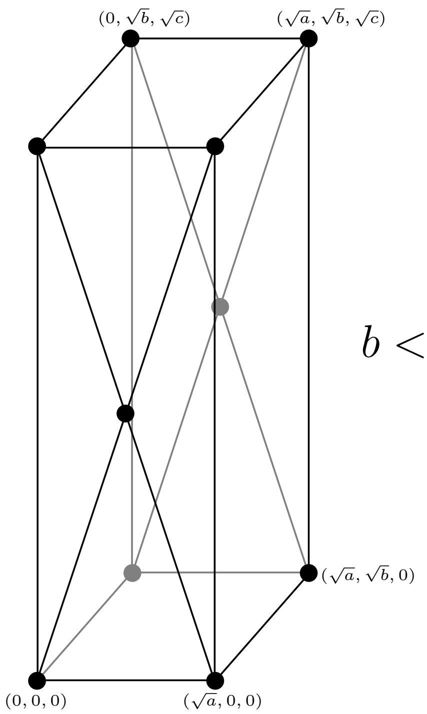
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} a+b & -a+b & 0 \\ -a+b & a+b & 0 \\ 0 & 0 & 4c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,b,2}.$$

In the limit case $3a = b$ the lattice is hexagonal and has six shortest vectors.



$$b < \tfrac{a+c}{4}$$

2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right).$$

Reduced basis:

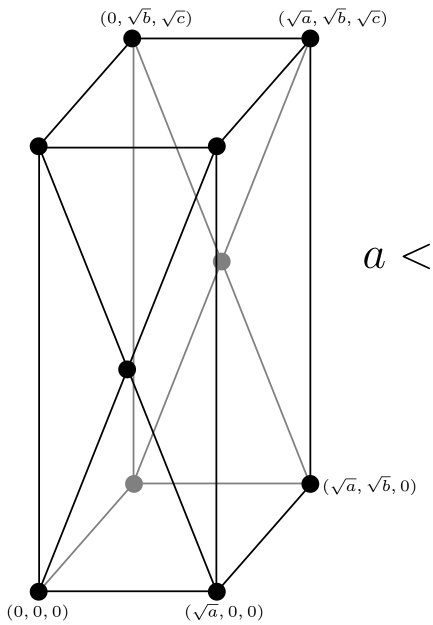
$$\left(\sqrt{a}, 0, 0\right),$$
$$\left(0, \sqrt{b}, 0\right),$$
$$\frac{1}{2}\left(\sqrt{a}, 0, \sqrt{c}\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 0 & 2a \\ 0 & 4b & 0 \\ 2a & 0 & a+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,b,3}\ .$$



$$a < \tfrac{a+c}{4} \leq b$$

2 shortest vectors:

$$\left(\pm\sqrt{a}, 0, 0\right).$$

Reduced basis:

$$\left(\sqrt{a}, 0, 0\right),$$
$$\frac{1}{2}\left(\sqrt{a}, 0, \sqrt{c}\right),$$
$$\left(0, \sqrt{b}, 0\right).$$

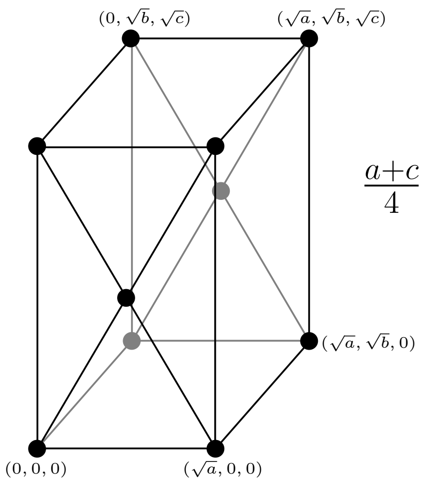
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 2a & 0 \\ 2a & a+c & 0 \\ 0 & 0 & 4b \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle = G_{8,b,1}\ .$$

For $3a = c$ the lattice is hexagonal again and has six shortest vectors. The basis of the following case remains reduced as well.



$\frac{a+c}{4} < a$

4 shortest vectors:

$$\frac{1}{2}\left(\pm\sqrt{a},0,\pm\sqrt{c}\right).$$

Reduced basis:

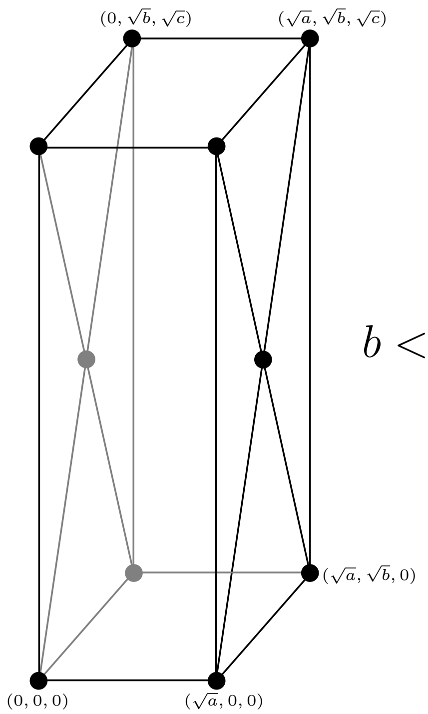
$$\frac{1}{2}\left(\sqrt{a},0,\sqrt{c}\right),$$
$$\frac{1}{2}\left(-\sqrt{a},0,\sqrt{c}\right),$$
$$\left(0,\sqrt{b},0\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} a+c & -a+c & 0 \\ -a+c & a+c & 0 \\ 0 & 0 & 4b \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle = G_{8,b,2}.$$



$b < \frac{b+c}{4}$

2 shortest vectors:

$$\left(\pm\sqrt{a},0,0\right).$$

Reduced basis:

$$\left(\sqrt{a},0,0\right),$$
$$\left(0,\sqrt{b},0\right),$$
$$\frac{1}{2}\left(0,\sqrt{b},\sqrt{c}\right).$$

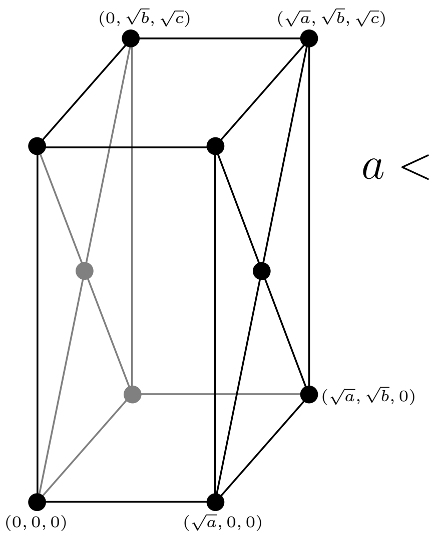
Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 0 & 0 \\ 0 & 4b & 2b \\ 0 & 2b & b+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,b,4}.$$

The equality $3b = c$ generates a hexagonal lattice with two shortest vectors. The same applies to the following case.



$$a < \frac{b+c}{4} < b$$

2 shortest vectors:

$$\left(\pm\sqrt{a},0,0\right).$$

Reduced basis:

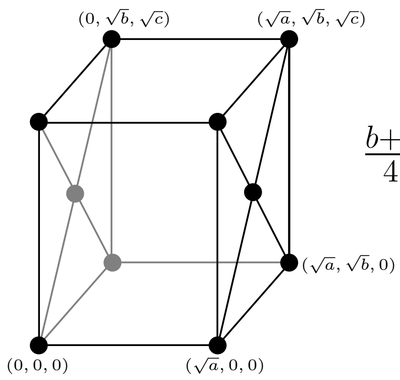
$$\left(\sqrt{a},0,0\right), \quad \frac{1}{2}\left(0,\sqrt{b},\sqrt{c}\right), \quad \frac{1}{2}\left(0,-\sqrt{b},\sqrt{c}\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} 4a & 0 & 0 \\ 0 & b+c & -b+c \\ 0 & -b+c & b+c \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: G_{8,b,5}.$$



$$\frac{b+c}{4} < a \quad *$$

4 shortest vectors:

$$\frac{1}{2}\left(0,\pm\sqrt{b},\pm\sqrt{c}\right).$$

Reduced basis:

$$\frac{1}{2}\left(0,\sqrt{b},\sqrt{c}\right), \quad \frac{1}{2}\left(0,-\sqrt{b},\sqrt{c}\right), \quad \left(\sqrt{a},0,0\right).$$

Corresponding Gram matrix:

$$\frac{1}{4}\begin{pmatrix} b+c & -b+c & 0 \\ -b+c & b+c & 0 \\ 0 & 0 & 4a \end{pmatrix}.$$

Isometry group of order 8:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle = G_{8,b,2}.$$

* The equality $4a = b + c$ is admissible here. We have six shortest vectors in this case.

In the following table we specify for each of the occurring groups G, which (additional) conditions on the entries of a reduced Gram matrix

$$B = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \in \mathbb{Z}^{3\times 3}$$

must be fulfilled in order to have $G =_B \mathrm{GL}_3(\mathbb{Z})$. Generally, all equalities and inequalities from Lemma 2.8 apply, since we assume the matrix to be reduced. Hence, if the table gives $a = c$ for example, this implies $a = b = c$. Because of the additional conditions, not all parameters are needed. We choose the independent parameters in such a way that no fractions arise in the Gram matrix.

Table 5: Gram Matrices of Three-Dimensional Lattices

Index	**Conditions**	**Gram Matrix**	**Determinant**
$48, p$	$a = c$, $d = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$	a^3
$48, i$	$a = c = 3d$, $d = -f$	$\begin{pmatrix} 3d & d & d \\ d & 3d & -d \\ d & -d & 3d \end{pmatrix}$	$16d^3$
$48, f$	$a = c = 2d$, $d = f$	$\begin{pmatrix} 2d & d & d \\ d & 2d & d \\ d & d & 2d \end{pmatrix}$	$4d^3$
$24, p, 1$	$a = b = 2d$, $e = 0$	$\begin{pmatrix} 2d & d & 0 \\ d & 2d & 0 \\ 0 & 0 & c \end{pmatrix}$	$3cd^2$
$24, p, 2$	$b = c = 2f$, $d = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & 2f & f \\ 0 & f & 2f \end{pmatrix}$	$3af^2$
$16, p, 1$	$a = b \neq c$, $d = e = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$	a^2c
$16, p, 2$	$a \neq b = c$, $d = f = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$	ab^2
$16, i, 1$	$b = 2e = 2f$, $d = 0$	$\begin{pmatrix} 2e & 0 & e \\ 0 & 2e & e \\ e & e & c \end{pmatrix}$	$4e^2(c-e)$
$16, i, 2$	$a = c = 2e - f$, $e \neq -f$	$\begin{pmatrix} 2d-f & d & d \\ d & 2d-f & f \\ d & f & 2d-f \end{pmatrix}$	$4d(d-f)^2$
$16, i, 3$	$a = c = d - 2f$, $d \neq -f$	$\begin{pmatrix} d+2e & d & e \\ d & d+2e & -e \\ e & -e & d+2e \end{pmatrix}$	$4e(d+e)^2$
$16, i, 4$	$b = c$, $a = 2e = 4f$	$\begin{pmatrix} 4f & 2f & 2f \\ 2f & b & f \\ 2f & f & b \end{pmatrix}$	$4f(b-f)^2$

Index	Conditions	Gram Matrix	Determinant
$12, r, 1$	$a = b = 2d \neq c$, $d = f$	$\begin{pmatrix} 2d & d & d \\ d & 2d & d \\ d & d & c \end{pmatrix}$	$d^2(3c - 2d)$
$12, r, 2$	$a = c \neq 2d$, $d = f \neq 0$	$\begin{pmatrix} a & d & d \\ d & a & d \\ d & d & a \end{pmatrix}$	$(a + 2d)(a - d)^2$
$12, r, 3$	$a = c \neq 3d$, $d = -f \neq 0$	$\begin{pmatrix} a & d & d \\ d & a & -d \\ d & -d & a \end{pmatrix}$	$(a - 2d)(a + d)^2$
$12, r, 4$	$a = 3d = 3e \neq b$, $b = c = d - 2f$	$\begin{pmatrix} 3d & d & d \\ d & d-2f & f \\ d & f & d-2f \end{pmatrix}$	$d(d - 3f)^2$
$8, p$	$a \neq b \neq c$, $d = e = f = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$	abc
$8, i, 1$	$a = 2e \neq b = 2f$, $d = 0$	$\begin{pmatrix} 2e & 0 & e \\ 0 & 2f & f \\ e & f & c \end{pmatrix}$	$2ef(2c - e - f)$
$8, i, 2$	$b = c$, $a = 2e \neq 4f$, $a \neq b \vee e \neq f$	$\begin{pmatrix} 2d & d & d \\ d & b & f \\ d & f & b \end{pmatrix}$	$2d(b - f)(b + f - d)$
$8, i, 3$	$a = c = d + e - f$, $d \neq e$, $e \neq -f$	$\begin{pmatrix} d+e-f & d & e \\ d & d+e-f & f \\ e & f & d+e-f \end{pmatrix}$	$2(d + e)(d-f)(e-f)$
$8, f, 1$	$a = 2d = 2e = 4f$, $b \neq c$	$\begin{pmatrix} 4f & 2f & 2f \\ 2f & b & f \\ 2f & f & c \end{pmatrix}$	$4f(b - f)(c - f)$
$8, f, 2$	$a = b = 2e+d \neq c$, $e = -f$	$\begin{pmatrix} d+2e & d & e \\ d & d+2e & -e \\ e & -e & c \end{pmatrix}$	$4e(c - e)(d + e)$
$8, b, 1$	$a = 2d \neq b$, $f = 0$	$\begin{pmatrix} 2d & d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{pmatrix}$	$cd(2b - d)$
$8, b, 2$	$a = b \neq 2d \neq 0$, $e = 0$	$\begin{pmatrix} a & d & 0 \\ d & a & 0 \\ 0 & 0 & c \end{pmatrix}$	$c(a + d)(a - d)$
$8, b, 3$	$a = 2e$, $f = 0$	$\begin{pmatrix} 2e & 0 & e \\ 0 & b & 0 \\ e & 0 & c \end{pmatrix}$	$be(2c - e)$
$8, b, 4$	$b = 2f \neq c$, $e = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & 2f & f \\ 0 & f & c \end{pmatrix}$	$af(2c - f)$
$8, b, 5$	$b = c \neq 2f \neq 0$, $d = 0$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & f \\ 0 & f & b \end{pmatrix}$	$a(b + f)(b - f)$

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