Limit theorems for random walks on non-compact Grassmann manifolds with growing dimensions

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M.Sc.-Math. Merdan Artykov

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Limit theorems for random walks on non-compact Grassmann manifold with growing dimensions

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Erstgutachter: Prof. Dr. Michael Voit

Zweitgutachter: Prof. Dr. Ivan Veselic

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CHAPTER 1

Introduction

Invariance under a group action is a central theme in mathematics. It is can be observed that on a homogeneous space $M \simeq G/K$ for some locally compact group G and some closed subgroup K a Banach subalgebra of bounded measures in $\mathcal{M}_b(G/K)$ which are invariant under the group action of K can be identified with the Banach algebra of bounded measures $\mathcal{M}_b(G//K)$ on the double coset space with a convolution $*$ inherited from the measure algebra $\mathcal{M}_b(G)$. We call the pair $(G//K, *)$ a double coset hypergroup. The above identification has an important consequence for time-homogeneous Markov processes on G/K with invariant transition probabilities under actions of G or simply $G-$invariant Markov processes: They can be identified with time-homogeneous Markov processes on the hypergroup $(G//K, *)$ via the canonical map from G/K to $G//K$. If the double coset hypergroup $(G//K, *)$ is commutative, i.e., the convolution $*$ on $\mathcal{M}(G//K)$ is commutative, then important tools of Fourier analysis are available, which allow to analyse the distributions of time-homogeneous Markov processes on $(G//K, *)$, in particular to derive some limit theorems.

In fact, (commutative) hypergroups have been studied in more generality encompassing properties of the double coset setting above. They have been idependently introduced by Dunkl [**D**] in 1973, Spector [**Sp**] in 1975 and Jewett [**J**] in 1975. The study of limit theorems on some particlular hypergroups began even in 1960's with Haldane's [**H2**] and Kingman's [**Ki**], where they studied methods which allowed the investigation of rotation invariant vectors and generalized them into non-integer valued "dimensions". With this goal they have introduced Bessel-Kingman hypergroups on $[0, \infty)$, which are closely related to the product formula for Bessel functions. Zeuner [**Z1**], [**Z2**] studied random walks Sturm-Liouville hypergroups on $[0, \infty)$, which are closely related with invariant random walks on the hyperbolic spaces. Limit theorems on various hypergroups was also derived by Voit, see [**V1**]- [**V7**].

In this thesis we present several limit theorems for G-invariant random walks on the non-compact Grassmann manifolds $\mathcal{G}_{p,q}(\mathbb{F}) = G/K$ over the (skew-) fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or quaternions $\mathbb{H}$ with rank $q \geq 1$ and dimension $p > q$, where, depending on $\mathbb{F}$, the group G is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q,p)$, $SU(q,p)$ or $Sp(q,p)$ with $K = SO(q)\times SO(p)$, $S(U(q)\times U(p))$ or $Sp(q)\times Sp(p)$, as subgroups. The double coset space $G//K$ with convolution $*$ can be identified with some subset C_q^B of $\mathbb{R}^q$ called the Weyl chamber of type B carrying certain convolution $*_{p,q}$. For $p, q \in \mathbb{N}$ with $p \geq 2q$ these convolutions on $*_{p,q}$ on the space $\mathcal{M}^1(C_q^B)$ are associative, commutative, and probability-preserving, i.e., convolution of two probability measures is again a probability measure, and they generate commutative hypergroups $(C_q^B, *_{p,q})$ in the sense of Jewett, Spector and Dunkl with $0 \in C_q^B$ above. There are two important observations which have been made about hypergroups $(C_q^B, *_{p,q})$:

(1) The convolutions $*_{p,q}$ of measures can be extended to $p \in [2q-1, \infty)$ in a way that the hypergroup structure of $(C_q^B, *_{p,q})$ is preserved.
(2) As $p \to \infty$ the hypergroup $(C_q, *_{p,q})$ tends to the double coset hypergroups structures of $GL(q, \mathbb{F})//U(q, \mathbb{F})$ in some way.

For both of the above observations the main tools are spherical functions of the symmetric space G/K i.e., the nontrivial, K-biinvariant, multiplicative continuous functions on G. After some reparametrisation, these functions correspond to multiplicative functions of commutative hypergroups $(C_q, *_{p,q})$ which are which are precisely the functions φ_λ^p on C_q^B, with $\lambda \in \mathbb{C}^q$ defined in [**R2**] for which

$$\varphi_\lambda^p(x)\varphi_\lambda^p(y) = \int_{C_q^B} \varphi_\lambda^p(t)\, d(\delta_x *_{p,q} \delta_y)(t)$$

holds for all $x, y \in C_q^B$. The first observation is based on Heckman-Opdam theory of hypergeometric functions associated with root systems. It generalizes the theory of spherical functions on Riemannian symmetric spaces; see [**H2**], [**HS**] and [**O**] for the general theory and [**R2**], [**RKV**], [**RV1**], [**Sch**], [**NPP**] for some recent developments. In this context the functions φ_λ^p correspond to hypergeometric functions F_{BC} associated with root systems of type BC_q. Using the above identification it was proved by Rösler [**R2**] that the functions φ_λ^p can be extended to $p \in [2q-1, \infty)$ by analytic continuation, which leads to an extension of the $*_{p,q}$ to $p \in [2q-1, \infty)$. The second observation was made in [**RV1**], [**RKV**], where it was proved that the functions φ_λ^p tend to the spherical functions of the spaces $GL(q, \mathbb{F})/U(q, \mathbb{F})$, which also correspond to the hypergeometric functions associated with root systems of type A_{q-1}.

Now, fix q and $d := \dim\mathbb{F}_{\mathbb{R}} = 1, 2, 4$. For $p \in (2q-1, \infty)$ consider random walks hypergroup the $(C_q^B, *_{p,q})$ (as it is well-defined by observation 1) as follows: Fix a probability measure $\mu \in \mathcal{M}^1(C_q^B)$, and consider a time-homogeneous Markov process $(\tilde{S}_k^p)_{k\geq 0}$ on C_q^B with start at the hypergroup identity $0 \in C_q^B$ and with the transition probability

$$P(\tilde{S}_{k+1}^p \in A|\, \tilde{S}_k^p = x) = (\delta_x *_{p,q} \mu)(A) \qquad (x \in C_q^B,\ A \subset C_q^B \quad \text{a Borel set}).$$

Such Markov processes are called random walks on the hypergroup $(C_q^B, *_{q,p})$ associated with the measure μ.

Notice that we use p as a superscript here, as this p may be variable below. We study limit theorems for $(\tilde{S}_n^p)_{n\geq 0}$ under two types of normalization procedures. We obtain results fixed p as well as for the case when p tends to infinity in some coupled way with n.

We first consider "outer" normalization where we study the limiting distribution of random variables $\tilde{S}_n^p/n^\varepsilon, \varepsilon \geq 1/2$ as $n \to \infty$. In the case where p is fixed central limit theorem (CLT) and strong law of large numbers (LLN) results were obtained in [**V2**]. We focus here on the limit theorems for growing p. It turns out that under suitable moment conditions on μ and for any sequence $(p_n)_n \subset [2q, \infty)$ with $p_n \to \infty$, there are normalizing vectors $m(n) \in \mathbb{R}^q$ such that $(\tilde{S}_n^{p_n} - m(n))/\sqrt{n}$ tends in distribution to some classical q-dimensional normal distribution $N(0, \Sigma^2)$ and $(S_n^{p_n} - m(n))/n^\varepsilon$ for $\varepsilon > 1/2$ tend to 0 in probability, where the norming vectors $m(n)$ and the covariance matrix Σ^2 are explicitly known and depend on μ. For $q = 1$, CLTs of this kind were given in [**Gr1**] and [**V1**] by completely different methods. Both proofs for $q = 1$, however, are based on the fact that for $p \to \infty$ the hypergroup structures $(C_1^B = [0, \infty), *_p)$ converge to some commutative semigroup structure on $C_1^B = [0, \infty)$ which is isomorphic with the additive semigroup $([0, \infty), +)$. This observation finally shows that for large p, $(\tilde{S}_n^{p_n})_n$ behaves like a sum of i.i.d. random variables which then leads to the CLT. For $q \geq 2$, the situation is much more involved as here for $p \to \infty$ the hypergroup structures $(C_q^B, *_{p,q})$ converge to the double coset structures $G//K$ in the case A_{q-1}, as mentioned in the second observation above. As for $q \geq 2$, this limit structure is more complicated than for $q = 1$, the details of the CLT and will be more involved than in [**Gr1**] and [**V1**]. In fact, we will need stronger conditions either on the moments of μ or on the rate of convergence of $(p_n)_n$ to ∞ than in [**Gr1**]; see Theorems 6.4, 6.7 below. We remark that the CLTs in [**Gr1**], [**V1**], and here for the non-compact Grassmannians are related to other CLTs for radial random walks on Euclidean spaces of large dimensions in [**Gr2**] and references cited there. We also point out that our CLTs for $p \to \infty$ are closely related to a CLT in the case A_{q-1} proved in [**V2**] which depends heavily on the concept of moment functions on commutative hypergroups; see [**BH**] and [**Z1**] for the general background. In fact, we shall need these moment functions for the hypergroups $(C_q^B, *_{p,q})$ as well as for the limit cases associated with the case A_{q-1}. These moment function will be essential to describe the norming vectors $m(n)$ and the covariance matrix Σ^2 above. We point out that our CLTs for $p \to \infty$ are related to the research in [**B**] on the limit behaviour of Brownian motions on hyperbolic spaces and noncompact Grassmannians.

We next consider limit theorems with "inner" normalisation. We start with a probability measure $\mu \in \mathcal{M}^1(C_q^B)$ with second moments. For each constant $c \in [0, 1]$ we consider the compression mapping $D_c(x) := cx$ on C_q^B as well as the compressed probability measures $\mu_c := D_c(\mu) \in \mathcal{M}^1(C_q^B)$ and the associated random walks $(S_k^{(p,c)})_{k\geq 0}$. We shall prove that for fixed p, $S_n^{(p,n^{-1/2})}$ converges for $n\to\infty$ in distribution to some kind of "Gaussian" measure $\gamma_{t_0}\in\mathcal{M}^1(C_q^B)$ which depends on p, where the time parameter $t_0\geq 0$ can be computed via second moment of μ. Triangular CLTs of this type are well-known in probability theory on groups and hypergroups. We refer here in particular to [**BH**] and references there for several results in this direction for Sturm-Liouville hypergroups on $[0,\infty[$. Moreover, for integers $p\geq 2q$, our result is similar to a known CLT for biinvariant random walks on noncompact Grassmannians; see e.g. [**G1**], [**G2**], [**T1**], [**T2**], [**Ri**]. Finally, we shall prove that when $p_n\to\infty$ faster than n we obtain a weak LLN result for $S_n^{(p,n^{-1/2})}$ i.e., $S_n^{(p,n^{-1/2})}$ tends to some vector in $\mathbb{R}^q$ in probability.

This thesis is organised in the following way: In Chapter 2 we give a necessary background on hypergroups. In particular, we collect information on the class of double coset hypergroups. We then give an explicit description for the double coset hypergroups corresponding to A_{q-1} and BC root systems. In Chapter 3 we give a brief introduction to Markov processes on hypergroups and homogeneous spaces. We also introduce moments on hypergroups which are crucial in stating limit theorems on hypergroups. In Chapter 4 we briefly explain connection between Heckman-Opdam theory of hypergeometric functions and spherical functions associated with symmetric groups. Furthermore, we recapitulate the Harish-Chandra integral representation for spherical functions $(C_q^B,*_{p,q})$ which we use to prove limit theorems in the forthcoming chapters. Chapters 5 and 6 contain limit theorems for random walks on hypergroups $(C_q^B,*_{p,q})$ for fixed p and growing p, respectively.

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- M. Artykov, M. Voit, Some central limit theorems for random walks associated with hypergeometric functions of type BC. Preprint arXiv:1802.05147

CHAPTER 2

Hypergroups

We follow axiomatic work by Jewett where he introduced hypergroups as "convos". We will be mainly concerned with commutative hypergroups.
Commutatative hypergoups generalize the class of locally compact abelian groups; for extensive reference for harmonic analysis on locally compact abelian groups see [**Di**]. Roughly said, a commutative hypergroup is a locally compact Hausdorff space with commutative convolution structure $* : X \times X \to \mathcal{M}^1(X)$ and an involution $\sim : X \to X$. The convolution on a hypergroup generalizes the convolution on a group, and the involution in the group case is given by group inversion.

1. Preliminaries

In order to give the definition for hypergroups we need to lay down some simple notation and especially get to know the Michael topology on the set of compact subsets of X.

Let X be a locally compact Hausdorff space. Denote by $\mathfrak{B}(X)$ the space of Borel measurable functions on X and denote by $\mathfrak{B}_b(X)$ the space of bounded Borel functions. By $\mathcal{C}(X)$ we denote the subspace of $\mathcal{B}(X)$ consisting of continuous functions. We consider distinguished subsets of $\mathcal{C}(X)$ including $\mathcal{C}_b(X), \mathcal{C}_0(X)$ and $\mathcal{C}_c(X)$ consisting of continuous functions which are bounded, continuous functions vanishing at infinity, and continuous functions with compact support, respectively. The positive cones of above spaces are denoted by superscript $+$. $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ are topologized by uniform norm $\|\cdot\|_\infty$ whereas $\mathcal{C}_c(X)$ will be topologized as the inductive limit of the spaces $\mathcal{C}_K := \{f \in \mathcal{C}_c : supp(f) \subset K\}$, with $K \subset X$ compact, each of which carries uniform norm.
Denote the set of Borel measures on X by $\mathcal{M}(X)$. Moreover, denote by $\mathcal{M}_b(X)$, $\mathcal{M}_c(X)$, $\mathcal{M}^+(X)$ and $\mathcal{M}^1(X)$ spaces of bounded Borel measures, measures with compact support, positive measures, and probability measures respectively.

Definition 2.1. Let X be a locally compact Hausdorff space. Denote by $\mathscr{C}(X)$ the set of nonempty compact subsets of X. The Michael topology on $\mathscr{C}(X)$ is the topology generated by the subbasis $\{\mathscr{U}_{U,V} : U, V$ open subsets of $K\}$, where $\mathscr{U}_{U,V} := \{A \subset \mathscr{C}(X), A \cap U \neq \emptyset$ and $A \subset V\}$.

We note that if X is metrizable then the Michael topology is stronger than the Hausdorff topology on $\mathscr{C}(X)$ given by Hausdorff metric, see 1.1.1 in [**BH**].

Definition 2.2. Let X be a nonempty locally compact Hausdorff space. The pair $(X, *)$ with a bilinear associative operation $*$ on $\mathcal{M}_b(X)$ is called a *hypergroup* if the following conditions are satisfied:

(1) The map $(\mu, \nu) \mapsto \mu * \nu$ is weakly continuous.
(2) For all $x, y \in X$ the so called convolution $\delta_x * \delta_y$ of point measures is a compactly supported probability measure on X.
(3) The mapping $(x, y) \mapsto supp(\delta_x * \delta_y)$ from $X \times X$ into the space of compact subsets of X is continuous w.r.t Michael topology.
(4) There exists a unique element $e \in X$ satisfying $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$, $\forall x \in X$. This element is called the *identity element* of X.
(5) There is a continuous involutive homeomorphism $x \mapsto \bar{x}$ on X such that $\delta_{\bar{x}} * \delta_{\bar{y}} = (\delta_x * \delta_y)^-$ and $x = \bar{y} \iff e \in supp(\delta_x * \delta_y)$, where for $\mu \in \mathcal{M}_b(X)$, the measure μ^- is given by $\mu^-(A) = \mu(\bar{A})$ for all Borel sets $A \subset X$.

A hypergroup $(X, *)$ is called commutative if $*$ is commutative.

We collect some elementary observations about hypergroups $(X, *)$; for the proofs see [**J**].

(1) $(M_b(X), *, \|\cdot\|_{TV})$ is a Banach-*-algebra with the involution $\mu \mapsto \mu^*$ such that $\mu^*(A) := \overline{\mu(\bar{A})}$ and the identity δ_e. Here $\|\cdot\|_{TV}$ denotes the total variation norm.
(2) It is well known that the span of point measures is weakly dense in $\mathcal{M}_b(X)$. This, together with the bilinearity of the convolution implies that the definition of the convolutions $(\delta_x * \delta_y)$ $(x, y \in K)$ of arbitrary point measures defines the convolution $*$ on $\mathcal{M}_b(X)$ completely, and thus the hypergroup $(X, *)$.
(3) The weak continuity of the convolution $*$ on $\mathcal{M}_b(X)$ ensures that for all $\mu, \nu \in \mathcal{M}_b(X)$

$$\mu * \nu = \int_X \int_X \delta_x * \delta_y d\mu(x) d\nu(y).$$

This means that for all $f \in \mathcal{C}_b(X)$

$$\int_X f d(\mu * \nu) = \int_X \int_X \int_X f d(\delta_x * \delta_y) d\mu(x) d\nu(y).$$

Example 2.3. *Let $(G, \cdot)$ be a locally compact group with identity e. The usual convolution on $\mathcal{M}_b(X)$ is defined by*

$$\mu * \nu(A) := \int_X \int_X \mathbb{1}_A(x + y) d\mu(x) d\mu(y)$$

for $A \in \mathcal{B}(G), \mu, \nu \in \mathcal{M}_b(G)$. Then $(G,*)$ is a hypergroup with identity e. The involution is given by

$$\mu^-(A) = \mu(A^{-1}),$$

where $A^{-1} = \{g^{-1} : g \in A\} \subset G$ for all $A \in \mathcal{B}(G), \mu \in \mathcal{M}_b(G)$.
*It can be easily seen that the group $(G,\cdot)$ is abelian if and only if the hypergroup is commutative, see [**J**].*

We notice that an arbitrary hypergroup $(X,*)$ is not necessarily directly connected with an algebraic structure of X.
Nevertheless, many concepts from harmonic analysis on some locally compact groups can be transferred to hypergroups.
Let $(X,*)$ be a hypergroup. For a function $f \in \mathfrak{B}_b(X)$ and $x \in X$ we define

$$f(x*y) := \int_X f d(\delta_x * \delta_y)$$

if the integral exists.
Furthermore, let $T_x f(y) := f(x*y)$ and $T^x(y) := f(y*x)$ be the *right x-translate* of f at y and *left x-translate* of f at y, respectively.

The following basic facts can be found in [**J**]:
(1) For $f \in \mathfrak{B}_b(X), x \in X$, $T_x f, T^x f \in \mathfrak{B}_b(X)$.
(2) For $f \in \mathcal{C}_b(X), x \in X$, $T_x f, T^x f \in \mathcal{C}_b(X)$.
(3) For $f \in \mathcal{C}(X), x \in X$, $T_f \in \mathcal{C}(X)$ and the map $(x,y) \mapsto f(x*y)$ is continuous on $X \times X$.

Also the concept of a Haar measure can be transferred to hypergroups.

DEFINITION 2.4. Let $(X,*)$ be a hypergroup. A nonzero measure $\omega_X \in \mathcal{M}^+(X)$ is called a *left Haar measure* or *right Haar measure* if for all $x \in X$ and $f \in \mathcal{C}_c(X)$ it holds that

$$\int_X T_x f d\omega_X = \int_X f d\omega_X \text{ or } \int_X T^x f d\omega_X = \int_X f d\omega_X$$

respectively.
A left and right Haar measure is called a *Haar measure*.

It can be observed that for a commutative hypergroup a left Haar measure is also a right Haar measure and a right Haar measure is also a left Haar measure, see [**BH**].

THEOREM 2.5. *Let $(X,*)$ be a commutative hypergroup.Then there exists a unique Haar measure up to a multiplicative constant.*

PROOF. See Theorems 1.3.15, 1.3.22, 1.3.28 in [**BH**]. □

From now on let $(X, *)$ be a commutative hypergroup with a Haar measure ω_X. Then the convolution and the involution for measurable functions are given by

$$f * g(x) = \int_X f(y) g(x * \bar{y}) d\omega_X(y)$$

and $f^*(x) = \overline{f(\bar{x})}$, $(x \in X)$ respectively.
We list some basic properties of the convolution of functions from [**J**]:

(1) If $f, g \in \mathcal{C}_c(X)$, then $f * g, f^* \in \mathcal{C}_c(X)$ for all $x \in X$.
(2) If $f, g \in L^1(X) := L^1(X, \omega_X)$, then $f * g(x)$ exists for w_X-almost all $x \in X$, and $f^*, f * g \in L^1(X)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Moreover, $f^* \in L^1(X)$ with $\|f^*\|_1 = \|f\|_1$.
(3) $(L^1(X), *, ^*, \|\cdot\|_1)$ is a commutative Banach-*-algebra. This Banach-*-algebra can be identified with the commutative Banach-*-subalgebra of all absolutely continuous measures in $(\mathcal{M}_b(X), *)$ via $f \mapsto f \cdot \omega_X$.
(4) For $p \geq 1$ the translation above extends to $L^p(X) := L^p(X, \omega_X)$, and for $f \in L^p(X), g \in L^1(X)$ the convolutions obeys $\|f * g\|_p \leq \|f\|_p \|g\|_1$.

Definition 2.6. Let $(X, *)$ be a commutative hypergroup. We define the spaces

$$\chi(X) := \{\varphi \in \mathcal{C}(X) : \varphi \neq 0, \varphi(x * y) = \varphi(x)\varphi(y) \quad \forall x, y \in X\};$$

$$\chi_b(X) := \chi(X) \cap \mathcal{C}_b(X);$$

$$\hat{X} := \{\varphi \in \chi_b(X) : \varphi(\bar{x}) = \overline{\varphi(x)} \ \forall x \in X\}.$$

The functions in $\chi(X)$ are called *semicharacters.* The space $\hat{X}$ is called *dual* of X, and its elements are called *characters.*

The spaces $\chi(X), \chi_b(X)$ and $\hat{X}$ are endowed with the topolgy of uniform convergence on compact sets. $\hat{X}$ is homeomorphic to the symmetric spectrum of the Banach-*-algebra $\Delta^*(L^1(X))$ of $L^1(X)$ via

$$\hat{X} \ni \varphi \leftrightarrow L_\varphi \in \Delta^*(L^1(X)) \text{ with } L_\varphi(f) = \int_X f \cdot \bar{\varphi} dw_X,$$

see Theorem 2.2.2 in [**BH**]. This shows in particular that $\hat{X}$ is a locally compact Hausdorff space.
If $(X, *)$ is a locally compact group (in the sense of Example 2.3), then $\hat{X}$ carries again the group structure w.r.t pointwise multiplication of characters, see [**Di**]. It is well-known that such a dual algebraic structure is not available on $\hat{X}$ for arbitrary commutative hypergroups $(X, *)$, for examples see [**J**]. This is in particular the case for the examples considered below.
We next turn to the Fourier transform on hypergroups:

Definition 2.7. For $\mu \in \mathcal{M}_b(X)$ and $f \in L^1(X)$ the (hypergroup) Fourier-(Stieljes) transforms $\hat{\mu}$ and $\hat{f}$ are defined by

$$\widehat{f}(\varphi) := \int_X f(x)\overline{\varphi(x)}d\omega_X(x) \text{ and } \widehat{\mu}(\varphi) := \int_X \overline{\varphi(x)}d\mu(x) \quad (\varphi \in \hat{X}).$$

We list several well-known properties of these Fourier transforms:

Theorem 2.8. *Let X be a commutative hypergroup. Then the following statements are true.*

(i) *For $\mu, \nu \in \mathcal{M}_b(X)$ the Fourier transform satisfies $\widehat{(\mu * \nu)} = \hat{\mu}\hat{\nu}$, $\widehat{\mu^*} = \overline{\hat{\mu}}$. Moreover, $\hat{\mu} \in C_b(\hat{X})$ with $\|\hat{\mu}\|_\infty \leq \|\mu\|_{TV}$.*
(ii) *For $f, g \in L^1(X)$ the Fourier transform satisfies $\widehat{(f * g)} = \hat{f}\hat{g}$, $\widehat{f^*} = \overline{\hat{f}}$.*
(iii) *Riemann-Lebesgue lemma: for $f \in L^1(X)$, $\hat{f} \in \mathcal{C}_0(\hat{X})$ with $\|\hat{f}\|_\infty \leq \|f\|_1$.*
(iv) *The maps $f \mapsto \hat{f}$, $\mu \mapsto \hat{\mu}$ are injective.*
(v) *$\{\hat{f} : f \in \mathcal{C}_c(X)\}$ is dense in $\mathcal{C}_0(\hat{X})$.*

Proof. See Theorems 2.2.2 and 2.2.4 in [**BH**]. □

The Fourier transform leads to an L^2-isometry between $L^2(X) := L^2(X, w_X)$ and some L^2-space on $\hat{X}$ with respect to some Plancherel measure on $\hat{X}$. More precisely:

Theorem 2.9. ***(Levitan-Plancharel)*** *Let $(X, *)$ be a commutative hypergroup. Then, there exists a unique measure $\pi_X \in \mathcal{M}^+(\hat{X})$, which is called Plancherel measure such that for all $f \in L^2(X) \cap L^1(X)$ the identity*

$$\int_X |f|^2 d\omega_X = \int_{\hat{X}} |\hat{f}|^2 d\pi_X \tag{1}$$

is satisfied. The map $f \mapsto \hat{f}$ extends to an isometric isomorphism from $L^2(X, \omega)$ to $L^2(\hat{X}, \pi_X)$.

Proof. See Theorem 2.2.13 in [**BH**]. □

If $(G, \cdot)$ is a locally compact group, then the Plancherel measure π_G on $\hat{G}$ is just "the" Haar measure on the locally compact group $\hat{G}$. In particular, in this case, the support $\text{supp}(\pi_G)$ of π_G is equal to $\hat{G}$. For a commutative hypergroup however, the support of π_X may be a proper subset of $\hat{X}$. Examples of such hypergroups will be considered below.

Definition 2.10. (i) The inverse Fourier transform of $f \in L^1(\hat{X}) := L^1(\hat{X}, \pi_X)$ is defined as

$$\check{f}(x) = \int_{\hat{X}} f(\varphi)\varphi(x)d\pi(\varphi) \quad (x \in X).$$

(ii) The inverse Fourier transform of $\mu \in \mathcal{M}_b(\hat{X})$ is defined as

$$\check{\mu}(x) = \int_{\hat{X}} \mu(\varphi) d\mu(\varphi) \quad (x \in X).$$

We list some properties of these inverse Fourier transforms:

THEOREM 2.11. ***(Fourier inversion theorem)***
Let $(X, *)$ be a commutative hypergroup with dual $\hat{X}$. Then the following statements are true.

(i) *The Riemann-Lebesgue lemma is satisfied, i.e. for $f \in L^1(\hat{X})$, $\check{f} \in \mathcal{C}_0(X)$.*
(ii) *For $\mu \in \mathcal{M}(\hat{X})$, $\check{\mu} \in \mathcal{C}_b(X)$.*
(iii) *The maps $\mu \mapsto \check{\mu}, f \mapsto \check{f}$ are injective.*
(iv) *For all $\mu \in \mathcal{M}_b(\hat{X}), \nu \in \mathcal{M}_b(X)$, $\nu = \check{\mu}\omega_X$ if and only if $\mu = \hat{\nu}\pi$.*
(v) *For all $f \in \mathcal{C}(X) \cap L^1(X)$ with $\hat{f} \in L^1(\hat{X})$, $f = (\hat{f})^\vee$.*
(vi) *The set $\{\check{f} : f \in L^1(\hat{X}, \pi_X)\}$ is $\| \cdot \|_\infty$-dense in $\mathcal{C}_0(X)$.*

PROOF. See Theorems 2.2.35 and 2.2.36 in [**BH**]. □

We are now ready to state a "hypergroup" version of Lévy's continuity theorem which allows to recover some classical theorems in probability theory such as central limit theorems in the case of hypergroups.

THEOREM 2.12. ***(Lévy continuity theorem)***
Let $\mu \in \mathcal{M}^1(X)$ and let $(\mu_n)_{n\geq 1}$ be a sequence in $\mathcal{M}^1(X)$. Then the following statements are true:

(i) *If μ_n converges to μ weakly, then $\hat{\mu}_n \to \hat{\mu}$ locally uniformly in $\hat{X}$.*
(ii) *If $\hat{\mu}_n \to \hat{\mu}$ pointwise on $S \subset \hat{X}$ for $\mu \in \mathcal{M}^1(X)$, then μ_n converges to μ weakly.*
(iii) *If there exists $f \in \mathcal{C}(\hat{X})$ satisfying $\lim_{n\to\infty} \hat{\mu}_n = f$ pointwise, then there exists $\mu \in \mathcal{M}_b^+(X)$ such that $\hat{\mu} = f$ and $\mu_n \to \mu$ weakly.*
(iv) *If there exists $f \in \mathcal{C}(\hat{X})$ satisfying $\lim_{n\to\infty} \hat{\mu}_n = f$ pointwise on $\operatorname{supp}(\pi_X)$, then there exists a unique $\mu \in \mathcal{M}_b^+(X)$ such that $f = \hat{\mu}$ π_X-almost everywhere and $\mu_n \to \mu$ vaguely. Moreover, if in addition $1 \in \operatorname{supp}(\pi_X)$ and f is continuous at 1, then $\mu_n \to \mu$ weakly.*

PROOF. See Theorems 4.2.2 and 4.2.4(iv) and 4.2.11 in [**BH**]. □

2. Double coset hypergroups and Gelfand pairs

In this section we study an important class of hypergroups which are related to the group theory and which are commutative in particular cases. These examples are called double coset hypergroups. To introduce these examples let G be a locally compact group and let K be some compact subgroup of G. Moreover, let

$$\mathcal{M}_b(G|K) := \{\mu \in \mathcal{M}_b(G) : \mu * \delta_y = \mu \quad \forall y \in K\} \tag{2}$$

and

(3) $$\mathcal{M}_b(G\|K) := \{\mu \in \mathcal{M}_b(G) : \delta_x * \mu * \delta_y = \mu \quad \forall x, y \in K\}$$

be the spaces of K-invariant and K-biinvariant bounded measures on G, respectively. Then, clearly the space $\mathcal{M}_b(G|K)$ is Banach-subalgebra of $\mathcal{M}_b(G)$. Moreover, $\mathcal{M}_b(G\|K)$ is a Banach-*-subalgebra of $\mathcal{M}_b(G)$.
Let ω_G be some left Haar measure of G and ω_K be the normalized left Haar measure of K. The measure ω_K will be regarded also as probability measure on G.
Next, let $G/K = \{gK : g \in G\}$ and $G//K := K\backslash G/K = \{KgK : g \in G\}$ be the spaces of left and double cosets, respectively. Moreover, consider the canonical projections

$$\tilde{\pi} : G \to G/K, \quad g \mapsto gK;$$
$$\pi : G \to G//K, \quad g \mapsto KgK.$$

We now equip G/K and $G//K$ with the quotient topologies. It can be easily observed that G/K and $G//K$ are locally compact Hausdorff spaces, and that π and $\tilde{\pi}$ are continuous and open mappings, see Chapter 8 in [**J**].
The canonical projection $\tilde{\pi}$ induces a map $\tilde{\pi}^* : \mathcal{M}_b(G|K) \to \mathcal{M}_b(G/K)$ by taking images of measures w.r.t $\tilde{\pi}$. We next define the convolution for point measures on G/K by

$$\delta_{xK} *_{\tilde{\pi}} \delta_{yK} := \int_K \delta_{xkyK} d\omega_K(k) \quad (x, y \in G).$$

The general convolution $*_{\tilde{\pi}}$ on the Banach space $\mathcal{M}_b(G/K)$ is defined via unique bilinear, weakly continuous extension. When doing so, $(\mathcal{M}_b(G/K), *_{\tilde{\pi}}, \|\cdot\|_{TV})$ becomes a Banach-algebra, and $\tilde{\pi}^* : \mathcal{M}_b(G|K) \to \mathcal{M}_B(G/K)$ is an isometric isomorphism of Banach algebras, see e.g Chapter 8 in [**J**]. Furthermore, let $L^1(G) := L^1(G, \omega_G)$ and let

$$L^1(G|K) := \{f \in L^1(G) : f(xy) = f(x) \quad \forall x \in G \text{ and } y \in K\}$$

be the space of K-invariant integrable functions on G. It is well known that $\omega_{G/K} := \tilde{\pi}^*(\omega_G) \in \mathcal{M}^+(G/K)$ is a left Haar measure on G/K, see Proposition 8.1B in [**J**]. The map $\tilde{\pi}$ induces an isometric isomorphism $\tilde{\pi}_\# : L^1(G|K) \to L^1(G/K, \omega_{G/K})$, see e.g Chapter 8 of [**RS**].

Similarly, the canonical projection π induces a map $\pi^* : \mathcal{M}_b(G\|K) \to \mathcal{M}_b(G//K)$. We define the convolution of point measures by

$$\delta_{KxK} *_\pi \delta_{KyK} := \int_K \delta_{KxkyK} d\omega_K(k) \quad (x, y \in G).$$

The general convolution $*_\pi$ on the Banach space $\mathcal{M}_b(G//K)$ is then defined via unique bilinear, weakly continuous extension. Moreover, define an involution * : $\mathcal{M}_b(G//K) \to \mathcal{M}_b(G//K)$ such that $\mu^*(A) = \overline{\mu(A^{-1})}$ for all $\mu \in \mathcal{M}_b(G//K)$ and $A \in \mathcal{B}(G)$. Then $(\mathcal{M}_b(G//K), *_\pi, ^*, \|\cdot\|_{TV})$ becomes a Banach-*-algebra, and

$\pi^* : \mathcal{M}_b(G||K) \to \mathcal{M}_b(G//K)$ is an isometric isomorphism of Banach-*-algebras, see e.g Chapter 8 in [**RS**]. We can observe that $\omega_{G//K} := \pi^*(\omega_G)$ is a left Haar measure on $G//K$, see Proposition 8.2B in [**J**]. Let

$$L^1(G||K) := \{f \in L^1(G) : f(xyx) = f(x) \quad \forall x \in G \text{ and } y \in K\}$$

be the space of K-invariant integrable functions on G. Then π induces an isometric isomorphism $\pi_\# : L^1(G||K) \to L^1(G//K, \omega_{G//K})$.
We have the following well-known result (see [**J**]):

Theorem 2.13. *Let K be a compact subgroup of a locally compact group G. Then $(G//K, *_\pi)$ is a hypergroup with the identity $K = KeK$ and involution $KgK \mapsto Kg^{-1}K$ $(g \in G)$.*

We next study the case where $\mathcal{M}_b(G//K)$ is a commutative Banach algebra. For this we define:

Definition 2.14. Let G be a locally compact group and $K \subset G$ be a compact subgroup. Then the pair (G, K) is called *Gelfand pair* if $\mathcal{M}_b(G\|K)$ is commutative.

We note that if (G, K) is a Gelfand pair, then the group G is unimodular, i.e. ω_G is also a right Haar measure, see Proposition 6.1.2 in [**Di**]. Furthermore, in this case, the Banach-*-algebra $L^1(G||K)$ is also commutative.

For Gelfand pairs (G, K) we introduce spherical functions:

Definition 2.15. Let (G, K) be a Gelfand pair. Then $\varphi \in \mathcal{C}(G)$ is called a *spherical function* of (G, K) or *spherical function* of G/K if φ is K-biinvariant, $\varphi \not\equiv 0$ and if φ satisfies the product formula

$$\int_K \varphi(gkh)d\omega_K(k) = \varphi(g)\varphi(h) \text{ for all } g, h \in G. \tag{4}$$

If in addition φ is bounded and $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in G$, then φ is called a *spherical character*.

The spherical functions (or spherical characters) on G can be identified with multiplicative functions (or characters) on the commutative double coset hypergroup $(G//K, *_\pi)$ as follows:

Lemma 2.16. *Let (G, K) be a Gelfand pair. Then, for a K-biinvariant function $f \in \mathcal{C}(G)$ the following statements are equivalent:*

(i) *f is a spherical function;*
(ii) *f has the form $f = \varphi \circ \pi$ for some multiplicative function $\varphi \in \chi(G//K, *_\pi)$;*
(iii) *for all $\mu, \nu \in \mathcal{M}_b(G\|K)$ the multiplicativity property*

$$\int_G f(x)d(\mu * \nu)(x) = \int_G f(x)d\mu(x) \int_G f(y)d\nu(y)$$

is satisfied.

Proof. See Chapter 6.1 in [**Di**]. □

There are several equivalent descriptions of Gelfand pairs among which we quote criteria from [**GV**].

Theorem 2.17. *Let K be a compact subgroup of some locally compact group G. Then under each of the following conditions (G, K) is a Gelfand pair.*

(i) *There exists a continuous involutive automorphism θ on G satisfying $x^{-1} \in K\theta(x)K$ for all $x \in G$.*

(ii) *There exists a continuous involutive automorphism θ on G satisfying $\theta(k) = k$ for all $k \in K$ and $G = K \cdot P$ with $P := \{x \in G : \theta(x) = x^{-1}\}$.*

(iii) *There exists an involutive automorphism θ on G and an abelian subgroup $A \subset G$ such that $G = KAK$, i.e. every element $x \in G$ has a unique decomposition $= k_1ak_2$ with $k_1, k_2 \in K$ and $a \in A$, where $\theta(k) \in K$ for all $k \in K$ and $\theta(a) = a^{-1}$ for all $a \in A$.*

Proof. See Proposition 1.5.2, Corollary 1.5.3 and Corollary 1.5.4 in [**GV**] □

3. Two examples of Gelfand pairs

In this section we consider two classes of Gelfand pairs which are central to our work. Furthermore we give the explicit description for the product formula for spherical functions on these Gelfand pairs.

For the first example consider for $q \in \mathbb{N}$ the general linear group $G = GL(q, \mathbb{F})$ and a compact subgroup $K = U(q, \mathbb{F})$ of unitary matrices taken over one of the (skew-)fields $\mathbb{F} := \mathbb{R}, \mathbb{C}$ and the quaternions $\mathbb{H}$.

Let $M_q(\mathbb{F})$ denote the space of $q \times q$ matrices over the field $\mathbb{F}$ and let

$$\mathcal{P}_q(\mathbb{F}) := \{x \in M_q(\mathbb{F}) : x = x^*, x \text{ positive semi-definite}\}$$

denote the cone of positive semi-definite Hermitian matrices in $M_q(\mathbb{F})$.

It is well-known by the theory of symmetric spaces (see [**GV**], [**H1**]) that in these cases, (G, K) are Gelfand pairs. As this is central to our work, we recapitulate an elementary proof of this fact.

Proposition 2.18. *For $q \in \mathbb{N}$, the pair $(G, K) = (GL(q, \mathbb{F}), U(q, \mathbb{F}))$ over the (skew-)fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and quaternions $\mathbb{H}$ is a Gelfand pair.*

Proof. We prove the result for $\mathbb{F} = \mathbb{C}$. The result for $\mathbb{R}, \mathbb{H}$ follows similarly. Consider the automorphism $\theta(x) := (x^*)^{-1} := (\bar{x}^T)^{-1}$ on $GL(p, \mathbb{C})$, then $\theta(k) = k$ for all $k \in U(p, \mathbb{C})$. For $x = (x_1, ..., x_q)$ denote by $\underline{x}$ the diagonal matrix with entries $x_1, ..., x_q$ and set

$$A := \{\underline{x} = \operatorname{diag}(x_1, .., x_q) : x_1, ..., x_q \in (0, \infty)\}$$

which is an abelian subgroup of $GL(q, \mathbb{C})$. Every element in $g \in GL(q, \mathbb{C})$ has a polar decomposition $g = ry$ with some $r \in \mathcal{P}_q(\mathbb{C})$ and some $y \in U(p, \mathbb{C})$. Moreover, every $r \in \mathcal{P}_q(\mathbb{C})$ has the decomposition $r = zaz^{-1}$ for some $a \in A, z \in U(q,\mathbb{C})$. Thus every $g \in GL(q,\mathbb{C})$ has $G = KAK$ representation, i.e. $x = k_1 a k_2$ for some $k_1, k_2 \in U(p,\mathbb{F}), a \in A$. Now, the part (iii) of the Theorem 2.17 above yields the required assertion. □

For the above pair the left coset space G/K can be identified with the cone $P_q(\mathbb{F})$ via the map

$$gK \mapsto I(g) := gg^* \in P_q(\mathbb{F}), \quad (g \in G), \tag{5}$$

where G acts on $P_q(\mathbb{F})$ via $a \mapsto gag^*$. Let $\sigma_{sing}(g) \in C_q^B$ denote the singular spectrum of $g \in M_q(\mathbb{F})$, where the singular values of g, i.e. the square roots of eigenvalues of the positive definite matrix gg^*, are ordered by size. Then the map

$$KgK \mapsto \ln \sigma_{sing}(g) = \frac{1}{2} \ln \sigma(gg^*) \tag{6}$$

leads to identification of $G//K$ with

$$C_q^A := \{x = (x_1, ..., x_q) \in \mathbb{R}^q : x_1 \geq x_2 ... \geq x_q\}. \tag{7}$$

We shall now obtain the formula for the convolution on $G//K$, where we identify $G//K$ with C_q^A. As spherical functions are K-biinvariant functions we can regard spherical functions φ on G as functions on C_q^A. In this way a spherical function $\varphi \in \mathcal{C}(G)$ corresponds to some $\psi \in \mathcal{C}(C_q^A)$ via $\varphi(\underline{x}) = \psi(x)$ for all $x \in C_q^A$ in one-to-one way. Let $g \in G$ be arbitrary, then via the map (6) g has the form $g = ue^{\underline{x}}\tilde{u}$ for $x \in C_q^A$ and $u, \tilde{u} \in K$, where $e^{\underline{x}} := \operatorname{diag}(e^{x_1}, ..., e^{x_q})$. We thus obtain

$$x = \frac{1}{2} \ln \sigma_{sing}(g)$$

Thus, for the function $\psi \in \mathcal{C}(C_q^B)$ above, the product formula (4) writes

$$\psi(x)\psi(y) = \int_K \psi(\frac{1}{2}(\sigma_{sing}((e^{\underline{x}} k e^{\underline{y}}))))d\omega_K(k) \tag{8}$$

for all $x, y \in C_q^B$. With this product formula in mind, we can now define the convolution on C_q^B, which characterizes the convolution $*_\pi$ on $G//K$. Then, for $x, y \in C_q^B$ we define the convolution of Dirac measures δ_x, δ_y by

$$\delta_x *_q \delta_y(f) := \int_K f(\frac{1}{2}(\sigma_{sing}((e^{\underline{x}} k e^{\underline{y}}))d\omega_K(k). \tag{9}$$

We now present our second example: Fix $p, q \in \mathbb{N}$ with $p > q \geq 1$ and let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, as above. We recapitulate the indefinite orthogonal, unitary and symplectic groups with dimensions p, q. For this we define

$$I_{p,q} = \begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix}. \tag{10}$$

Then the groups

$$O(p,q) = \{A \in GL(p+q,\mathbb{R}) : A^* I_{p,q} A = I_{p,q}\};$$
$$U(p,q) = \{A \in GL(p+q,\mathbb{F}) : A^* I_{p,q} A = I_{p,q}\};$$
$$Sp(p,q) = \{A \in GL(p+q,\mathbb{H}) : A^* I_{p,q} A = I_{p,q}\}$$

are called indefinite orthogonal, indefinite unitary and indefinite symplectic groups, repectively. Furthermore, let $SO(p,q) := O(p,q) \cap SL(p+q,\mathbb{R})$, $SU(p,q) := U(p,q) \cap SL(p+q,\mathbb{C})$ and $Sp(p,q) := O(p,q) \cap SL(p+q,\mathbb{H})$, where $SL(p+q,\mathbb{F})$ denotes in all cases the $(p+q)\times(p+q)$ matrices with determinant 1. Moreover, let $SO_0(p,q) \subset SO(p,q)$ is the connected component in $SO(p,q)$ containing the identity. The groups $SU(p,q)$ and $Sp(p,q)$ are simply-connected, semisimple linear Lie groups. Now let G be one of the groups $SO(p,q), SU(p,q)$ or $Sp(p,q)$ for $p \geq q$. We choose the groups $K = SO(p)\times SO(q), S(U(p)\times U(q))$ or $Sp(p)\times Sp(q)$ as maximal subgroups of groups G, respectively for all of the above classes.

The homogeneous spaces $\mathcal{G}_{p,q} := G/K$ are called the *non-compact Grassmann manifolds* over the (skew-)fields $\mathbb{F} := \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

As in the preceding example, it is well known that the pair (G,K) is a Gelfand pair; see Theorem 8.6, Chapter VII in [**H1**]. As above we give an elementary proof. For this we use the diagonal matrix notations:

$\cosh \underline{x} := \mathrm{diag}(\cosh x_1, ..., \cosh x_q), \quad \sinh \underline{x} := \mathrm{diag}(\sinh x_1, ..., \sinh x_q)$ for $x \in \mathbb{R}^q$.

Proposition 2.19. *Let G and K be defined as above. Then (G,K) is a Gelfand pair. Moreover, the double coset space $G//K$ can be identified with the Weyl chamber*

$$C_q^B := \{x \in \mathbb{R}^q : x_1 \geq x_2 \geq ... \geq x_q \geq 0\}$$

as follows: C_q^B will be identified with the set

$$\left\{ a_x := \begin{pmatrix} \cosh \underline{x} & \sinh \underline{x} & 0 \\ \sinh \underline{x} & \cosh \underline{x} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} : x \in C_q^B \right\} \subset G, \tag{11}$$

via $C_q^B \ni x \leftrightarrow a_x$, where this set is a set of representatives of the K-double cosets in G.

Proof. We prove the result for $(G,K) = (SU(p,q), S(U(p)\times U(q)))$, the other two cases follow similarly. Analogously to the Example 2.18 above we set $\theta(x) := (x^*)^{-1}$. Clearly θ is an automorphism on G. Moreover, for all $k \in K_2$ we have $\theta(k) = k$. We now determine an abelian subgroup $A \subset G$ such that the representation $G = KAK$ is satisfied. Consider the $(p+q)\times(p+q)$ matrix

$$H_x = \begin{pmatrix} 0_{q\times q} & \underline{x} & 0_{q\times(p-q)} \\ \underline{x} & 0_{q\times q} & 0_{q\times p-q} \\ 0_{(p-q)\times q} & 0_{(p-q)\times q} & 0_{(p-q)\times(p-q)} \end{pmatrix}.$$

We define the exponential of a matrix X as $e^X := \sum_{k=1}^{\infty} \frac{X^k}{k!}$. Then it is easy to observe that

$$e^{H_x} = a_x = \begin{pmatrix} \cosh \underline{x} & \sinh \underline{x} & 0 \\ \sinh \underline{x} & \cosh \underline{x} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix},$$

see e.g. Lemma 15 in [**S**]. Since we have $H_x + H_y = H_{x+y}$ and $H_x H_y = H_y H_x$ it follows that

$$e^{H_x + H_y} = e^{H_x} e^{H_y} = e^{H_y} e^{H_x}.$$

Thus, $A := \{a_x : x \in \mathbb{R}^q\}$ is an abelian subgroup of G with $\theta(a_x) = a_x^{-1}$ for all $a_x \in A$.
We next prove :

① Every element $g \in G$ can be written as $g = k_1 a_x k_2$ for some $k_1, k_2 \in K$ and $x \in \mathbb{R}^q$.
② $K a_x K = K a_y K$ if and only if x can be derived from y by permutation of components and multiplication of each component by ± 1.

Then, ① and ② together with Theorem 2.17(iii) imply that (G, K) is a Gelfand pair and that

$$G//K \simeq C_q^B.$$

We begin with the proof of ①. The inclusion $\supseteq$ can easily be verified by showing $g^T I_{p,q} g = I_{p,q}$ for

$$g = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix} a_x \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}, \tag{12}$$

where $u_1, u_2 \in U(p), v_1, v_2 \in U(q)$ and $a_x \in A$ with

$$\det(u_1)\det(v_1) = 1 \text{ and } \det(u_2)\det(v_2) = 1.$$

To show the opposite direction choose any $g \in G \subset GL(p+q, \mathbb{C})$. Then by the same argument as above, there exist $k_1, k_2 \in U(p) \times U(q)$ such that

$$k_1 g k_2 = \tilde{g} = \begin{pmatrix} y_1 & a \\ b & y_2 \end{pmatrix}, \tag{13}$$

where $d_1 \in \mathbb{R}^{q\times q}, d_2 \in \mathbb{R}^{p \times p}$ are diagonal matrices and $a \in \mathbb{C}^{q\times p}, b \in \mathbb{C}^{p\times q}$. We have then

$$\begin{aligned} I_{p,q} &= \begin{pmatrix} d_1 & b^* \\ a^* & d_2 \end{pmatrix} I_{p,q} \begin{pmatrix} d_1 & a \\ b & y_2 \end{pmatrix} \\ &= \begin{pmatrix} d_1^2 - b^* b & d_1 a - b^* d_2 \\ a^* d_1 - d_2 b & a^* a - d_2^2 \end{pmatrix} \end{aligned}$$

which implies that $a^* a = d_2^2 - I_p$ is a diagonal matrix. On the other hand, the matrix $a^* a$ is positive semi-definite and has the rank $\leq q$. Thus, at least $p - q$

entries of a^*a are zero and all the entries of d_2 are greater or equal to 1. Without loss of generality we can assume that

$$d_2 = \begin{pmatrix} \cosh \underline{y} & 0 \\ 0 & I_{p-q} \end{pmatrix}$$

for some $y \in \mathbb{R}^q$. Moreover, a has the form $a = (Z \quad 0_{q\times p-q})$, where the columns of Z are orthogonal. Thus a can be written as $a = (u \sinh \underline{y} \quad 0_{q\times p-q})$ for some $u \in U(q)$. Similarly, one can show that there exists $x \in \mathbb{R}^q$ such that $d_1 = \cosh \underline{x}$ and $b = Y \cdot \sinh \underline{x}$, where Y has orthogonal columns. On the other hand, the equation $d_1 a - b^* d_2 = 0$ implies that $b = \binom{\tilde{u}}{0_{(p-q)\times q}} \sinh \underline{x}$ and

$$\cosh \underline{x} u \sinh \underline{y} = \sinh \underline{x} \tilde{u} \cosh \underline{y} \tag{14}$$

for $u, \tilde{u} \in U(q, \mathbb{C})$ as above.
Now, with elementary computation we can conclude that the vectors x, y are equal up to permutations and multiplication with ± 1. Without loss of generalization assume that $x = y$. This yields that $u = \tilde{u}$. Returning to the Eq. (13) we conclude that the matrix $\tilde{g}$ has without loss of generalization, the representation:

$$\tilde{g} = \begin{pmatrix} \cosh \underline{x} & u \sinh \underline{x} & 0 \\ \sinh \underline{x} u^* & \cosh \underline{x} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \tag{15}$$
$$= \begin{pmatrix} u & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \begin{pmatrix} \cosh \underline{x} & \sinh \underline{x} & 0 \\ \sinh \underline{x} & \cosh \underline{x} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \begin{pmatrix} u^* & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix}.$$

② follows similarly to ①. □

We shall now define the convolution on the double coset hypergroup $G//K$, where we identify $G//K$ with C_q^B as above. This convolution is equivalent to the product formula (4) for the associated spherical function. Here, we follow closely Section 2 in [**R2**].
As spherical functions are K-biinvariant functions on G, in view of Proposition 2.19 they can be regarded as functions on the set $\{a_x : x \in C_q^B\}$ of representatives of the double cosets and also as continuous functions on C_q^B. Thus, a spherical function $\varphi \in \mathcal{C}(G)$ corresponds to some $\psi \in \mathcal{C}(C_q^B)$ via $\varphi(a_x) = \psi(x)$ for all $x \in C_q^B$, in one-to-one way. Now let $g \in G$ be arbitrary, then g has the form

$$g = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} a_x \begin{pmatrix} \tilde{u} & 0 \\ 0 & \tilde{v} \end{pmatrix}, \tag{16}$$

with $u, \tilde{u} \in U(q), v, \tilde{v} \in U(p)$ and $x \in C_q^B$. The vector $x \in C_q^B$ here can be determined uniquely for $g \in G$ as follows: Denote the upper left $q \times q$ block of g by $A(g)$. Then, with a short calculation we obtain that

$$A(g) = u \cosh \underline{x} \tilde{u}, \text{ where } u, \tilde{u} \in U(q) \tag{17}$$

and thus $\sigma_{sing}(A(g)) = (\cosh x_1, ..., \cosh x_q) =: \cosh x$. Therefore,

$$x = \operatorname{arccosh}(\sigma_{sing}(A(g))) \text{ for } g \in K a_x K, x \in C_q^B, \tag{18}$$

where arccosh is taken componentwise.
As noticed above it suffices to calculate the product formula (4) for arguments $g = a_x, h = a_y$, where $x, y \in C_q^B$. Thus, for the function $\psi \in \mathcal{C}(C_q^B)$ above, the product formula (4) writes

$$\psi(x)\psi(y) = \int_K \psi(\operatorname{arccosh}(\sigma_{sing}(A(a_x k a_y))))d\omega_K(k) \tag{19}$$

for all $x, y \in C_q^B$.
We would like to achieve further simplification for the product formula. For this we introduce the following notation from [**R1**]:

Let $d = 1, 2, 4$ be the dimension of $\mathbb{F}$ over $\mathbb{R}$, and let

$$B_q := \{w \in M_q(\mathbb{F}) : ww^* < I\} \subset M_q(\mathbb{F})$$

denote the matrix unit ball, where the partial ordering $A < B$ means that $A - B$ is strictly positive definite.
Furthermore, define the following probability measure $d\mathbf{m}_p(w)$ on B_q s

$$d\mathbf{m}_p(w) := \frac{1}{\kappa_{pd/2}} \cdot \Delta(I - w^*w)^{pd/2-\gamma} dw, \tag{20}$$

where

$$\gamma := d(q - \frac{1}{2}) + 1,$$

Δ denotes the determinant on the cone $\mathcal{P}_q(\mathbb{F})$, dw is the Lebesgue measure on the ball B_q, and

$$\kappa_{pd/2} = \int_{B_q} \Delta(I - w^*w)^{pd/2-\gamma} \mathrm{d}w$$

is the normalization factor.

Theorem 2.20. *[**Proposition 2.2 in [R1]**]*
Let $p \geq 2q$ and γ be defined as above. If we regard a spherical function φ as a function ψ on C_q^B, then ψ satisfies the product formula

$$\psi(x)\psi(y) = \int_{U_q} \int_{B_q} \psi(d(\underline{x}, \underline{y}, u, w)) d\mathbf{m}_p(w) du, \tag{21}$$

where du is a Haar measure on U_q, $d\mathbf{m}_p(w)$ is defined as in (20),

$$d(\underline{x}, \underline{y}, u, w) = \operatorname{arccosh}(\sigma_{sing}(\cosh \underline{x} u \cosh \underline{y} + \sinh \underline{x} \eta^* v \eta \sinh \underline{x})) \tag{22}$$

and $\eta = \binom{I_q}{0_{(p-q)\times q}}$.

As one can see, the domain of integration is independent of $p \geq 2q$ in the product formula above. With this product formula in mind, we can now define the convolution on C_q^B, which characterizes the convolution $*_\pi$ on $G//K$.

DEFINITION 2.21. Let f be a continuous function on C_q^B. Then, for $x, y \in C_q^B$ define the convolution of Dirac measures δ_x, δ_y by

$$\delta_x *_{p,q} \delta_y(f) := \int_{U_q} \int_{B_q} f(d(\underline{x}, \underline{y}, u, w)) d\mathbf{m}_p(w) du, \tag{23}$$

where the probability measure $d\mathbf{m}_p(w)$ is defined as in (20) and $d(\underline{x}, \underline{y}, u, w)$ is defined as in (22).

We summarize the preceding results :

THEOREM 2.22. *Let $p, q \in \mathbb{N}$, $p \geq 2q$ and let $*_{p,q}$ be as in Definition 2.21. Then, $(C_q^B, *_{p,q})$ is a commutative hypergroup which can be identified with the double coset hypergroup $(G//K, *_\pi)$.*

REMARK 2.23. *The hypergroups $(C_q^B, *_{p,q})$ in the case $q = 1$ were extensively studied by Koornwinder in [**K**]. Here, the spherical functions φ parametrized by $\lambda \in \mathbb{C}$ are given by Jacobi functions*

$$\varphi_\lambda^{\alpha,\beta}(x) := {}_2F_1((\alpha+\beta+1-i\lambda)/2, (\alpha+\beta+1+i\lambda)/2; \alpha+1; -\sinh^2 x) \quad (\lambda \in \mathbb{C})$$

with

$$\alpha = pd/2 - 1, \quad \beta = d/2 - 1 \text{ where } d := dim_{\mathbb{R}}\mathbb{F} = 1, 2, 4.$$

*Moreover, double coset convolutions $*_{\alpha,\beta}$ on $[0, \infty)$ can be regraded as special case of Jacobi convolutions defined by Koornwinder in [**K**]. For $\alpha > \beta \geq -1/2$, these convolutions are given by*

$$\delta_x *_{\alpha,\beta} \delta_y(f) := \int_0^1 \int_0^\pi f(\operatorname{arccosh}|\cosh x \cosh y + re^{i\phi} \sinh x \sinh y|) d\mathbf{m}_{\alpha,\beta}(r, \phi)$$

where the probability measure $d\mathbf{m}_{\alpha,\beta}(r, \phi)$ is given by

$$d\mathbf{m}_{\alpha,\beta}(r, \phi) := \frac{2\Gamma(\alpha+1)(1-r^2)^{\alpha-\beta-1}(r \sin\phi)^{2\beta} \cdot r\mathrm{d}r\mathrm{d}\phi}{\Gamma(1/2)\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}. \tag{24}$$

*Furthermore, the following intergral representation for Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ was obtained in [**K**] :*

$$\varphi_\lambda^{(\alpha,\beta)}(x) = \int_0^1 \int_0^\pi |\cosh x + re^{i\phi} \sinh x|^{i\lambda - \rho} \mathrm{d}\mathbf{m}_{\alpha,\beta}(r, \varphi)$$

with $\mathrm{d}\mathbf{m}_{\alpha,\beta}(r, \varphi)$ introduced in (24) and

$$\rho := \alpha + \beta + 1.$$

It was shown by Rösler [**R2**] that the convolution $*_{p,q}$ on bounded measures on C_q^B can be extended by analytic continuation from $p \in \{p \in \mathbb{N} : p \geq 2q\}$ to all $p \in (2q-1, \infty)$, so that $(C_q^B, *_{p,q})$ in the extended case is still a commutative hypergroup. For non-integer p these hypergroups are no longer isomorphic to double coset hypergroups. As this result is central for Chapter 4, we will state it separately in Theorem 4.18.

CHAPTER 3

Markov processes on hypergroups and homogeneous spaces

In this chapter we first recapitulate the concept of some Markov processes on hypergroups which are motivated by the concept of Markov process on groups. We also introduce random walks on homogeneous spaces for some Gelfand pairs (G, K) and study their connection with Markov processes on double coset hypergroups $(G//K, *_\pi)$. We later recapitulate the concept of moments on hypergroups which among other things allows to state limit theorems and construct some martingales from Markov processes on hypergroups.

1. Random walks and Lévy processes on hypergroups and homogeneous spaces

1.1. Random walks and Lévy processes on hypergroups. Let $(G, \cdot)$ be a locally compact group. We first recapitulate the notion of random walks on groups. Let $(Y_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables. Then, consider the stochastic process $(S_n := Y_1 \cdots Y_n)_{n\geq 0}$ (with the convention $S_0 = e$) on G. It is well known that $(S_n)_{n\geq 1}$ is a time-homogeneous Markov process with independent, stationary increments. More precisely:

(i) For all $k \in \mathbb{N}$ and $t_1, ..., t_k \in \mathbb{N}$, with $0 = t_0 < t_1... < t_k$, the random variables $S_{t_0}, S_{t_0}^{-1}S_{t_1}, ..., S_{t_{k-1}}^{-1}S_{t_k}$ are independent.

(ii) For $k, n \in \mathbb{N}$ with $k \geq n$, $\mathbb{P}_{S_n^{-1}S_k}$ depends only on $k - n$.

(iii) The transition probability is given as follows:

$$P(S_{n+1} \in A | S_n = x) = (\delta_x * \mu)(A) \tag{25}$$

for all $n \in \mathbb{N}_0, x \in G, A \in \mathcal{B}(G)$. Here, $\mu \in \mathcal{M}^1(G)$ is the distribution of Y_n (independent of $n \in \mathbb{N}$). Clearly, the probability measure μ determines the distribution of $(S_n)_{n\geq 0}$ uniquely.

The Markov process $(S_n)_{n\geq 0}$ on G is called a (right) *random walk* on G associated with $\mu \in \mathcal{M}^1(G)$.

We now extend this notion from groups to commutative hypergroups $(X, *)$. As we do not have an algebraic operation on X, we cannot use the concept of products of i.i.d. random variables above. However, we can use Eq. (25) to define random walks on commutative hypergroups:

Definition 3.1. Let $(X, *)$ be a commutative hypergroup. Let $\mu \in \mathcal{M}^1(X)$ be an arbitrary probability measure. Then a time-homogeneous Markov process $(S_n)_{n\geq 0}$ with $S_0 = e$ is called a random walk on $(X, *)$ associated with $\mu \in \mathcal{M}^1(X)$ if

$$P(S_{n+1} \in A | S_n = x) = (\delta_x * \mu)(A) \tag{26}$$

for all $n \in \mathbb{N}, x \in X, A \in \mathcal{B}(X)$.

Now we introduce random walks in continuous time. We start with the group case: Let $(G, \cdot)$ be a locally compact group. A family of probability measures $(\mu_t)_{t\in[0,\infty)} \subset \mathcal{M}^1(G)$ is called a convolution semigroup on G, if the following conditions are satisfied:

(i) $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in [0, \infty)$;
(ii) the map $[0, \infty) \to \mathcal{M}^1(G)$, $t \mapsto \mu_t$ with $\mu_0 = \delta_e$ is weakly continuous.

Now let $(\mu_t)_{t\in[0,\infty)} \subset \mathcal{M}^1(G)$ be a convolution semigroup and let $(S_t)_{t\in[0,\infty)}$ be a (time-continuous) stochastic process with transition probabilities:

$$P(S_t \in A | S_s = x) = (\delta_x * \mu_{t-s})(A) \tag{27}$$

for all $s, t \in [0, \infty)$ with $s \leq t$, $x \in G$ and $A \in \mathcal{B}(G)$. Then it is easy to observe that $(S_t)_{t\in(0,\infty)}$ has independent and stationary increments, that is :

(i) For all $t_0, t_1, ..., t_k \in [0, \infty)$ with $0 = t_0 < t_1 ... < t_k$, the random variables $S_{t_0}, S_{t_0}^{-1} S_{t_1}, ..., S_{t_{k-1}}^{-1} S_{t_k}$ are independent.
(ii) For $s, t \geq 0$ with $t \geq s$, $\mathbb{P}_{S_s^{-1} S_t}$ depends only on $t - s$.

Conversely, if the G-valued process $(S_t)_{t\in(0,\infty)}$ has independent and stationary increments, then $(\mu_t := \mathbb{P}_{X_h^{-1} X_{t+h}})_{t\in[0,\infty)}$ forms a convolution semigroup on G.
The process $(S_t)_{t\in(0,\infty)}$ is called Lévy process associated with $(\mu_t)_{t\in[0,\infty)}$.

Motivated by this we now define Lévy processes on hypergroups:

Definition 3.2. Let $(X, *)$ be a commutative hypergroup. Then:

(i) A family of probability measures $(\mu_t)_{t\in[0,\infty)} \subset \mathcal{M}^1(X)$ is called a convolution semigroup on X, if $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in [0, \infty)$ and if the map $[0, \infty) \to \mathcal{M}^1(X)$, $t \mapsto \mu_t$ is weakly continuous.
(ii) Let $(\mu_t)_{t\in[0,\infty)} \subset \mathcal{M}^1(X)$ be a convolution semigroup. Then a (time-continuous) stochastic process $(S_t)_{t\in[0,\infty)}$ is called a X-valued Lévy process associated to $(\mu_t)_{t\in[0,\infty)}$ if

$$P(S_t \in A | S_s = x) = (\delta_x * \mu_{t-s})(A) \tag{28}$$

for all $s, t \in [0, \infty)$ with $s \leq t$, $x \in X$ and $A \in \mathcal{B}(X)$.

We now turn our attention to the double coset hypergroups $(G//K, *)$ in Chapter 2 and discuss examples of random walks and Lévy processes on $(G//K, *)$. There are two constructions for these processes. In both cases let $I = \mathbb{N}_0$ or $[0, \infty)$

be the parameter space.

In the first construction we start either with some K-biinvariant measure $\mu \in \mathcal{M}^1(G||K) \subset \mathcal{M}^1(G)$ (for $I = \mathbb{N}_0$) or with a K-biinvariant convolution semigroup $(\mu_t)_{t\in[o,\infty)} \subset \mathcal{M}^1(G||K) \subset \mathcal{M}^1(G)$.

Now, let $(S_t)_{t\in I}$ be an associated random walk or Lévy process on G.

THEOREM 3.3. *Let π be the canoncial projection from G to $G//K$. Then the process $(\pi(S_t))_{t\in I}$ is a Markov process on the hypergroup $(G//K, *_\pi)$. More precisely:*

(1) *for $I = \mathbb{N}_0$,*

$$\mathbb{P}(\pi(S_{n+1}) \in A|\pi(S_n) = z) = (\delta_z *_\pi \pi(\mu))(A)$$

for all $n \in \mathbb{N}_0, z \in G//K, A \in \mathcal{B}(G//K)$.

(2) *For $I = [0, \infty)$,*

$$\mathbb{P}(\pi(S_t) \in A|\pi(S_s) = z) = (\delta_z *_\pi \pi(\mu_{t-s}))(A)$$

for all $s, t \in [0, \infty)$ with $t \geq s$, $z \in G//K$ and $A \in \mathcal{B}(G//K)$.

PROOF. We prove the statement for the continuous-time case $I = [0, \infty)$. The result for the discrete time case $I = \mathbb{N}_0$ it follows similarly. Let $(\mathcal{F}_t)_{t\in I}$ be the canonical filtration of the process $((S_t))_{t\in I}$ and $(\hat{\mathcal{F}}_t)_{t\in I}$ be the canonical filtration of the process $(\pi(S_t))_{t\in I}$. We note that for all $s \leq t \in I$ and $A \in \mathcal{B}(G//K)$ that the function $x \mapsto (\delta_x * \mu_{t-s})(\pi^{-1}(A))$ is K-biinvariant. Since π^* is an isomorphism from $\mathcal{M}_b(G\|K)$ to $\mathcal{M}_b(G//K)$ as noticed in Chapter 2, we have that

$$(\delta_x * \mu_{t-s})(\pi^{-1}(A)) = (\delta_{\pi(x)} *_\pi \pi^*(\mu_{t-s}))(A)$$

for all $x \in G//K, A \in \mathcal{B}(G//K)$. Now, by Markov property of $(S_t)_{t\in I}$ we deduce that

$$\begin{aligned} P(\pi(S_t) \in A|\mathcal{F}_s) &= P(S_t \in \pi^{-1}(A)|S_s) \\ &= (\delta_{S_s} * \mu_{t-s})(\pi^{-1}(A)) \\ &= (\delta_{\pi(S_s)} *_\pi \pi^*(\mu_{t-s}))(A). \end{aligned}$$

Therefore, we have

$$P(\pi(S_t) \in A|\mathcal{F}_s) = (\delta_{\pi(S_s)} *_\pi \pi(\mu_{t-s}))(A) = P(\pi(S_t) \in A|\pi(S_s)) \quad \text{a.s.}$$

As $\sigma(\pi(S_s)) \subset \hat{\mathcal{F}}_s \subset \mathcal{F}_s$, we also have

$$P(\pi(S_t) \in A|\hat{\mathcal{F}}_s) = P(\pi(S_t) \in A|\pi(S_s)) \quad \text{a.s.}$$

This yields the claim. □

1.2. Random walks and Lévy processes on homogeneous spaces. We now turn to the Markov processes on homogeneous spaces for G. We closely follow [**L**].
We first recapitulate homogeneous spaces. A topological space M is called a *homogeneous space for group* G, if G acts transitively on M. Fix a point o in M. Then the *stabilizer*

$$K := \{g : g = go\}$$

of o in M is a closed subgroup of G. It is easy to see that M is homeomorphic to G/K under the map $go \mapsto gK$ and the G-action on M is just the natural action of G on G/K. Conversely, given a coset space G/K, it is a homogeneous space for G with a distinguished point, namely coset of the identity. Thus a homogeneous space can be thought of as a coset space without the choice of origin.
In this way, a Markov process on M, invariant under the transitive action of G may be regarded as a Markov process on the homogeneous space G/K invariant under the natural action. We will now give some basic properties of measures on G/K before discussing the G-invariant Markov processes in G/K.
Recall that a measure μ on G/K is called a K-invariant (invariant under action of K) measure, if $k(\mu) = \mu$ for all $k \in K$ where k acts on G/K as usual. It is clear that the Haar measure $\omega_{G/K}$ is K-invariant. Moreover, it can be observed that for all $\mu, \nu \in \mathcal{M}_b(G/K)$ and $k \in K$

$$k(\mu *_{\tilde{\pi}} \nu) = k(\mu) *_{\tilde{\pi}} k(\nu),$$

see c.f. Chapter 1 in [**L**]. This means that if $\mu, \nu \in \mathcal{M}_b(G/K)$ are K-invariant, then $\mu * \nu$ is K-invariant as well. Therefore the set of K-invariant measures in $\mathcal{M}_b(G/K)$ is a Banach subalgebra of $\mathcal{M}_b(G/K)$. Denote this space by $\mathcal{M}_{b,K}(M)$. Furthermore, recall the canonical projection $\tilde{\pi} : G \to G/K$ and let $\tilde{\pi}^* : \mathcal{M}_b(G|K) \to \mathcal{M}_b(G/K)$ be the map induced by taking images of measures w.r.t $\tilde{\pi}$. he following result provides the relation between K-(bi)invariant measures on G and on $M = G/K$.

Proposition 3.4. (1) *The map*

$$\mu \mapsto \nu = \tilde{\pi}^*(\mu)$$

is an isomorphism from $\mathcal{M}_b(G||K)$ to $\mathcal{M}_{b,K}(G/K)$.
(2) *The map π^* preserves the convolution in the sense that for any measures $\mathcal{M}_b(G|K)$*

$$\tilde{\pi}^*(\mu * \nu) = \tilde{\pi}^*(\mu) *_{\tilde{\pi}} \tilde{\pi}^*(\nu).$$

A convolution semigroup on G/K is defined in the same way as on G. A family of probability measures $(\mu_t)_{t\in[0,\infty)} \subset \mathcal{M}^1(G/K)$ is called a convolution semigroup on G/K, if the following conditions are satisfied:
(i) $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in [0, \infty)$;
(ii) the map $[0, \infty) \to \mathcal{M}^1(G/K)$, $t \mapsto \mu_t$ with $\mu_0 = \delta_{eK}$ is weakly continuous.

Proposition 3.5. *The map*

$$(\mu_t)_{t\in[0,\infty)} \mapsto (\nu_t)_{t\in[0,\infty)} = (\pi^*(\mu_t))_{t\in[0,\infty)}$$

is a bijection from the set of K-biinvariant convolution semigroups $(\mu_t)_{t\in[0,\infty)}$ on G onto the set of convolution semigroups $\nu_t)_{t\in[0,\infty)}$ on $M = G/K$.

Proof. See Proposition 1.9 in [**L**]. □

We can now define the G-invariant random walk and Lévy processes on $M = G/K$:

Definition 3.6. (1) A random walk $(Z_n)_{n\in\mathbb{N}}$ on $M = G/K$ with start in $o = eK$ is called G-invariant random walk if its transition kernel $\mathcal{K}$, i.e.

$$\mathcal{K}(x, A) := \mathbb{P}(Z_{n+1} \in A | Z_n = x)$$

$x \in G/K, A \in \mathcal{B}(G/K), n \in \mathbb{N}$, satisfies

$$\mathcal{K}(x, A) = \mathcal{K}(g(x), g(A))$$

for all $x \in G/K, A \in \mathcal{B}(G/K)$ and $g \in G$, where G acts on G/K in a canonical way.

(2) A Lévy process $(Z_t)_{t\in[0,\infty)}$ on $M = G/K$ is G-invariant if the transition semigroup $(P_t)_{t\in[0,\infty)}$, i.e.,

$$P_t(x, A) := \mathbb{P}(Z_{t+s} \in A | Z_s = x)$$

for $x \in M, A \in \mathcal{B}(M), s, t \in [0, \infty)$, satisfies

$$P_t(x, A) = P_t(g(x), g(A))$$

for $g \in G, x \in M, A \in \mathcal{B}(M), s, t \in [0, \infty)$.

It can be easily shown that for a G-invariant random walk $\tilde{\mu} := \mathcal{K}(eK, \cdot)$ is a K-invariant measure on G/K. Indeed, for all $k \in K$ and $A \in \mathcal{B}(G/K)$ we have

$$\begin{aligned} k(\tilde{\mu})(A) =& \tilde{\mu}(k^{-1}(A)) = \mathcal{K}(eK, k^{-1}(A)) \\ =& \mathcal{K}(k \cdot eK, A) = \mathcal{K}(eK, A) \\ =& \tilde{\mu}(A). \end{aligned}$$

By Proposition 3.4 there exists a unique K-binvariant measure μ such that $\tilde{\pi}^*(\mu) = \tilde{\mu}$. This means that for any random walk $(S_n)_{n\geq 0}$ associated with K-biinvariant probability measure, the process $(\pi(S_n))_{n\geq 0}$ is a $G-$invariant random walk on G/K. Conversely, any G-invariant random walk on $M = G/K$ can be obtained from a random walk with associated K-biinvariant measure above.
Similarly, a G-invariant Lévy process $(Z_t)_{t\in[0,\infty)}$ on $M = G/K$ is associated with a K-invariant convolution semigroup $(\tilde{\mu}_t)_{t\geq 0} := (P_t(eK, \cdot))_{t\geq 0}$. By Proposition 3.5 there exista a unique K-biinvariant convolution semigroup $(\mu_t)_{t\geq 0}$ on G such that $\pi^*(\tilde{\mu}_t))_{t\geq 0} = (\tilde{\mu}_t)_{t\geq 0}$. It can be also shown that for Lévy process $(S_t)_{t\in[0,\infty)}$ on G associated with K- biinvariant convolution semigroup, the process $(\tilde{\pi}(S_t))_{t\in[0,\infty)}$

is a $G-$invariant Lévy process on M, see Theorem 1.17 in [**L**]. Conversely, any G-invariant Lévy process on $M = G/K$ can be obtained from a Lévy process on G with an associated K-biinvariant convolution semigroup as above, see Theorem 3.10 in [**L**].

Now, consider the canonical projection

$$\tilde{\tilde{\pi}} : G/K \to G//K, \quad gK \mapsto KgK,$$

which is continuous and open. Let the map $\tilde{\tilde{\pi}}^* : \mathcal{M}_b(G/K) \to \mathcal{M}_b(G/K)$ be induced from $\tilde{\tilde{\pi}}$ by taking images of measures w.r.t $\tilde{\tilde{\pi}}$.
Then, since $\tilde{\pi}^* : \mathcal{M}_b(G||K) \to \mathcal{M}_{g,K}(G/K)$ and $\pi^* : \mathcal{M}_b(G||K) \to \mathcal{M}_b(G//K)$ are both bijections, it follows that the map $\mu \mapsto \tilde{\tilde{\pi}}(\mu)$ is also a bijection from $\mathcal{M}_{b,K}(G/K)$ onto $\mathcal{M}_b(G//K)$, with

$$\tilde{\tilde{\pi}}^*(\mu *_{\tilde{\pi}} \nu) = \tilde{\tilde{\pi}}^*(\mu) *_{\pi} \tilde{\tilde{\pi}}^*(\nu)$$

for all $\mu, \nu \in \mathcal{M}_{b,K}(G/K)$. Similarly, $\tilde{\tilde{\pi}}$ maps convolutions semigroups in $\mathcal{M}_{b,K}(G/K)$ onto convolutions semigroups in $\mathcal{M}_b(G//K)$ bijectively. In summary we obtain the following result:

PROPOSITION 3.7. (1) *Let $(Z_n)_{n\geq 0}$ be a G-invariant random walk on $M = G/K$ with $\mu = \mathcal{K}(eK, \cdot)$. Then $(\tilde{\tilde{\pi}}(Z_n))_{n\geq 0}$ is a random walk on the hypergroup $(G//K, *_\pi)$ with*

$$\mathbb{P}(\tilde{\tilde{\pi}}(Z_{n+1}) \in A | \tilde{\tilde{\pi}}(Z_n) = x) = (\delta_x *_\pi \tilde{\tilde{\pi}}(\mu))(A)$$

for all $n \in \mathbb{N}_0, x \in G//K, A \in G//K$.

(2) *Let $(Z_t)_{t\in[0,\infty)}$ be a G-invariant Lévy process on $M = G/K$ with $(\tilde{\mu}_t)_{t\geq 0} = (P_t(eK, \cdot))_{t\geq 0}$. Then $(\tilde{\tilde{\pi}}(Z_t))_{t\in[0,\infty)}$ is a Lévy process the hypergroup $(G//K, *_\pi)$ with*

$$\mathbb{P}(\tilde{\tilde{\pi}}(Z_t)) \in A | \tilde{\tilde{\pi}}(Z_s) = x) = (\delta_x *_\pi \tilde{\tilde{\pi}}(\mu_{t-s}))(A)$$

for all $t \in [0, \infty), x \in G//K, A \in G//K$.

PROOF. We prove the statement for the continuous-time case $I = [0, \infty)$. The result for the discrete time case $I = \mathbb{N}_0$ it follows similarly. Let $(\mathcal{F}_t)_{t\in I}$ be canonical filtration of the process $(Z_t)_{t\in I}$ and $(\tilde{\mathcal{F}}_t)_{t\in I}$ be the canonical filtration of the process $(\tilde{\tilde{\pi}}(Z_t))_{t\in I}$. We note that for all $s \leq t \in I$ and $A \in \mathcal{B}(G//K)$ that the function $x \mapsto (\delta_x *_{\tilde{\pi}} \mu_{t-s})(\tilde{\tilde{\pi}}^{-1}(A))$ is K-invariant (invariant under natural action of K). Since $\tilde{\tilde{\pi}}^*$ is an isomorphism from $\mathcal{M}_{b,K}(G/K)$ to $\mathcal{M}_b(G//K)$ by Proposition 3.4 we have that

$$((\delta_x *_{\tilde{\pi}} \mu_{t-s})(\tilde{\tilde{\pi}}^{-1}(A)) = (\delta_{\pi(x)} *_\pi \pi^*(\mu_{t-s}))(A)$$

for all $x \in G//K, A \in \mathcal{B}(G//K)$. Now, by Markov property of $(Z_t)_{t\in I}$ we deduce that

$$P(\tilde{\tilde{\pi}}(Z_t) \in A|\mathcal{F}_s) = P(Z_t \in \tilde{\tilde{\pi}}^{-1}(A)|Z_s)$$
$$= (\delta_{Z_s} *_{\tilde{\tilde{\pi}}} \mu_{t-s})(\tilde{\tilde{\pi}}^{-1}(A))$$
$$= (\delta_{\tilde{\tilde{\pi}}(Z_s)} *_{\tilde{\pi}} \tilde{\tilde{\pi}}^*(\mu_{t-s}))(A).$$

Therefore, we have

$$P(\pi(Z_t) \in A|\mathcal{F}_s) = (\delta_{\pi(Z_s)} *_\pi \pi(\mu_{t-s}))(A) = P(\pi(Z_t) \in A|\pi(Z_s)) \quad \text{a.s.}$$

As $\sigma(\pi(Z_s)) \subset \hat{\mathcal{F}}_s \subset \mathcal{F}_s$, we also have

$$P(\pi(Z_t) \in A|\tilde{\mathcal{F}}_s) = P(\pi(Z_t) \in A|\pi(Z_s)) \quad \text{a.s.}$$

This yields the claim. □

2. Moment functions on hypergroups

In this section we introduce the concept of moments on commutative hypergroups.These moment functions can be seen as analogues of multidimensional monomials on $\mathbb{R}^q, q \in \mathbb{N}$. Throughout this section we mainly follow [**RV2**].
To get the feeling of how the moment functions are defined, consider the Euclidean space $\mathbb{R}^q$. We regard $X = \mathbb{R}^q$ as the group $(\mathbb{R}^q, \cdot)$. Then $\hat{X} = \mathbb{R}^q$ and its characters are given by exponential functions $\varphi_\lambda(x) = e^{i\langle x,\lambda\rangle}$. The monomials $x^\kappa =: x_1^{\kappa_1}...x_q^{\kappa_q}$ for $x \in \mathbb{R}^q, \kappa \in \mathbb{Z}_+^q$ satisfy $x^\kappa = i^{|\kappa|}\partial_\lambda^\kappa \varphi_\lambda(x)|_{\lambda=0}$, where by

$$\partial_\lambda^\kappa = \partial_{\xi_1}^{\kappa_1}...\partial_{\xi_q}^{\kappa_q} \tag{29}$$

we denote partial derivatives. In particular, they satisfy the Leibniz rule

$$(x+y)^\kappa = \sum_{\eta\le\kappa} \binom{\kappa}{\eta} x^\eta x^{\kappa-\eta},$$

where, by $\eta \le \kappa$ we mean the partial ordering $\eta_i \le \kappa_i$ for all $i = 1, ..., q$, $\kappa - \eta =: (\kappa_1 - \eta_1, ..., \kappa_q - \eta_q)$ and $\binom{\kappa}{\eta} := \prod_{i=1}^q \binom{\kappa_i}{\eta_i}$.
The monomials play an important role in deriving limit theorems for Markov processes in $\mathbb{R}^q$. To demonstrate this, as a toy example consider the following setting: Let $S_n := \sum_{k=1}^n X_k$ be a $\mathbb{R}^q$-valued random walk, where $(X_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d variables with distribution $\nu \in \mathcal{M}^1(\mathbb{R}^q)$. Then, if the second moments $\int_{\mathbb{R}^n} x_i^2 d\nu$, $i = 1, ..., N$ exist, the distribution of $\frac{S_n}{\sqrt{n}} - \sqrt{n}\cdot m$ tends to the normal distribution $N(0,\Sigma)$, where the mean m and covariance matrix Σ are given by integrals related to monomials with $|\kappa| = 1, 2$:

$$m = \left(\int_{\mathbb{R}^q} x_i d\nu\right)_{1\le i\le q} \text{ and } \Sigma = \left(\int_{\mathbb{R}^q} x_i x_j d\nu - \int_{\mathbb{R}^q} x_i d\nu \int_{\mathbb{R}^q} x_j d\nu\right)_{1\le i,j\le q}.$$

In the following let $(X, *)$ be a commutative, second countable hypergroup. We now imitate the ideas above and introduce moment functions for hypergroups which were first introduced)in a slightly different form) by Zeuner in [**Z1**]. However, unlike the Euclidean case we need to make additional assumptions on $(X, *)$, namely:

(A1) The dual $\hat{X}$ can be identified with a closed subset of $\mathbb{R}^q$ for some $q \in \mathbb{N}$, where we assume that the identity character $\mathbb{1}$ corresponds to $0 \in \mathbb{R}^q$, i.e. we assume $\hat{e} = 0 \in \hat{X}$.

(A2) There exist linearly independent vectors $\xi_1, ..., \xi_n \in \mathbb{R}^q$ and $\varepsilon > 0$ with $t_1\xi_1 + ... + t_n\xi_n \in \hat{X}$ for $t_1, .., t_n \in [0, \varepsilon]$.

(A3) We now fix the vectors $\xi_1, ..., \xi_q \in \mathbb{R}^q$ as in (A2). Then each $\lambda \in \hat{X}$ can be written as $\lambda = t_1\xi_1 + ... + t_q\xi_q \in \hat{X}$. We use the same notation as in (29) for partial derivatives. Then, assume that for all $x \in X$ and $\kappa \in \mathbb{Z}_+^q$ the functions
$$\partial_\lambda^\kappa \varphi_\lambda(x)|_{\lambda=0}$$
exist, and the map $x \mapsto \partial_\lambda^\kappa \varphi_\lambda(x)|_{\lambda=0}$ is continuous on X.

Definition 3.8. Let $(X, *)$ be a second countable hypergroup. Suppose that the assumptions (A1), (A2) and (A3) are satisfied for X. Let $\mu \in \mathcal{M}^1(X)$ be a probability measure on X.

(i) For $\kappa \in \mathbb{N}_0^q$ the functions
$$m_\kappa(x) := i^{|\kappa|} \partial_\lambda^\kappa \varphi_\lambda(x)|_{\lambda=0} \quad (x \in X)$$
with convention $m_{(0,...,0)} \equiv 1$ are called *moment functions of order κ.*

(ii) If for $\kappa \in \mathbb{N}_0^q$, the integral $\int_X m_\kappa(x) d\mu(x) =: m_\kappa(\mu)$ exists, then it is called κ-th *moment of μ.*

(iii) We say that μ admits moments up to order k $(k \in \mathbb{N})$ if $m_\kappa(\mu) < \infty$ for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| \leq k$.

(iv) Denote the space of measures with moments up to order k by:
$$\mathfrak{M}_k^1(X) := \{\mu \in \mathcal{M}^1(X) : m_\kappa \in L^1(X, \mu) \text{ for all } \kappa \text{ with } |\kappa| \leq k|\}$$

Example 3.9. *The moment functions for the Jacobi hypergroups $(X, *_{\alpha,\beta})$ the moment functions for $k \in \mathbb{N}$ are given by*
$$m_k(x) = \int_0^1 \int_0^\pi \left(\ln|\cosh\phi + r \cdot e^{i\phi}\sinh(\phi)|\right)^k \mathrm{d}\mathbf{m}_{\alpha,\beta}(r, \phi).$$
where the measure $\mathrm{d}\mathbf{m}_{\alpha,\beta}(r, \phi)$ is as in (24).

We now return to the general theory where $(X, *)$ is a commutative hypergroup which satisfies the assumptions (A1), (A2) and (A3).

Lemma 3.10. *For $x, y \in X$ the moment functions satisfy the Leibniz rule*
$$\int_X m_\kappa(z) d(\delta_x * \delta_y)(z) = \sum_{\eta \leq \kappa} \binom{\kappa}{\eta} m_\eta(x) m_{\kappa-\eta}(y) \tag{30}$$
for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| \geq 1$.

Proof. Applying the product rule for the partial derivatives defined above and using the fact that $\varphi_0(x) = 1$, we obtain for $x, y \in X$ and $\kappa \in N_0^q$ with $|\kappa| > 0$:

$$\partial_\lambda^\kappa(\varphi_\lambda(x)\varphi_\lambda(x))|_{\lambda=0} = \sum_{\eta\leq\kappa}\binom{\kappa}{\eta}\partial_\lambda^\eta\varphi_\lambda(x)|_{\lambda=0}\cdot\partial_\lambda^{\kappa-\eta}\varphi_\lambda(x)|_{\lambda=0}$$
$$= \sum_{\eta\leq\kappa}\binom{\kappa}{\eta}m_\eta(x)m_{\kappa-\eta}(y).$$

Now, the result follows from above by definition of hypergroup characters and by exchanging order of integration and differentiation:

$$\int_X m_\kappa(z)d(\delta_x*\delta_y)(z) = \partial_\lambda^\kappa\Big(\int_X \varphi_\lambda(z)d(\delta_x*\delta_y)(z)\Big)\Big|_{\lambda=0}$$
$$= \partial_\lambda^\kappa(\varphi_\lambda(x)\varphi_\lambda(x))|_{\lambda=0}$$
$$= \sum_{\eta\leq\kappa}\binom{\kappa}{\eta}m_\eta(x)m_{\kappa-\eta}(y).$$

□

Proposition 3.11. *Let $(m_\kappa)_{\kappa\in\mathbb{N}_0^q}$ be the moment functions defined as in Definition 3.8. Then*

(a) *$m_\kappa(e) = 0$ for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| \geq 1$.*
(b) *Let $n \in \mathbb{N}$. Then $\mu * \nu \in \mathfrak{M}_n^1(X)$ if and only if $\mu, \nu \in \mathfrak{M}_n^1(X)$.*

Proof. (a) We prove this by induction on $n = |\kappa|$. For all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| = 1$ it follows that $m_\kappa(e) = 1$ by substituting $x = y = e$ in Eq. (30). We now assume that assertion is true for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| \leq n$. Denote j-th unit vector by e_j. Then, by substituting $x = y = e$ and $\kappa + e_j$ instead of κ in the Eq. (30), one can easily see that $m_{\kappa+e_j}(e) = 0$ for $j = 1, ..., q$, as claimed.
(b) We again prove this by induction on $n = |\kappa|$. We start with the case $n = 1$. Let $\kappa = e_j$ for $j = 1, ..., q$. Indeed, $\int_X |m_{e_j}(z)|d(\mu*\nu)(z) < \infty$ implies that

$$\int_X\int_X \big|m_{e_j}(x) + m_{e_j}(y)\big|\,d\mu(x)d\nu(y) = \int_X\int_X\left|\int_X m_{e_j}(z)d(\delta_x*\delta_y)(z)\right|d\mu(x)d\nu(y)$$
$$\leq \int_X\int_X\left(\int_X |m_{e_j}(z)|d(\delta_x*\delta_y)(z)\right)d\mu(x)d\nu(y)$$
$$= \int_X |m_{e_j}(z)|d(\mu*\nu)(z) < \infty.$$

Therefore, by Fubini's theorem there exists $y_0 \in X$ such that

$$\int_X \big|m_{e_j}(x) + m_{e_j}(y_0)\big|\,d\mu(x) < \infty.$$

This implies

$$\int_X \left|m_{e_j}(x)\right| d\mu(x) \leq \int_X \left(\left|m_{e_j}(x) + m_{e_j}(y_0)\right| + |m_{e_j}(y_0)|\right) d\mu(x) < \infty,$$

and by symmetry we get $\int_X |m_{e_j}(z)| d\nu(z) < \infty$. The reverse implication follows from

$$\begin{aligned}\int_X |m_{e_j}(z)| d(\mu * \nu)(z) &= \int_X \int_X \left|m_{e_j}(x) + m_{e_j}(y)\right| d\mu(x) d\nu(y) \\ &\leq \int_X |m_{e_j}(x)| d\mu(x) + \int_X |m_{e_j}(y)| d\nu(y) < \infty.\end{aligned}$$

Thus the case $n = 1$ is complete.

Now, assume that the assertion holds for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| < n$. The proof of the assertion for $|\kappa| = n$ is similar to the initial step. Assume that the integral $\int_X m_\kappa d(\mu * \nu)$ exists for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| \leq n$. Then we obtain

$$\begin{aligned}&\int_X \int_X \left|\sum_{\eta \leq \kappa} \binom{\kappa}{\eta} m_\eta(x) m_{\kappa-\eta}(y)\right| d\mu(x) d\nu(y) \\ &\qquad = \int_X \int_X \left|\int_X m_\kappa(z) d(\delta_x * \delta_y)(z)\right| d\mu(x) d\nu(y) \\ &\qquad \leq \int_X \int_X \left(\int_X |m_\kappa(z)| d(\delta_x * \delta_y)(z)\right) d\mu(x) d\nu(y) \\ &\qquad = \int_X |m_\kappa(z)| d(\mu * \nu)(z) < \infty.\end{aligned}$$

Thus, by Fubini's theorem there exists $y_0 \in X$ such that for all $\kappa \in \mathbb{N}_0^q$ with $|\kappa| \leq n$,

$$\int_X \left|\sum_{\eta \leq \kappa} \binom{\kappa}{\eta} m_\eta(x) m_{\kappa-\eta}(y_0)\right| d\mu(x) < \infty.$$

This implies that

$$\begin{aligned}\int_X |m_\kappa(x)|d\mu(x) \le \int_X \left|\sum_{\eta\le\kappa}\binom{\kappa}{\eta} m_\eta(x)m_{\kappa-\eta}(y_0)\right| d\mu(x)\\ + \int_X \left|\sum_{\eta<\kappa}\binom{\kappa}{\eta} m_\eta(x)m_{\kappa-\eta}(y_0)\right| d\mu(x)\\ \le \left|\sum_{\eta\le\kappa}\binom{\kappa}{\eta} m_{\kappa-\eta}(y_0)\right| \int_X |m_\eta(x)|d\mu(x)\\ + \sum_{\eta\le\kappa}\binom{\kappa}{\eta} m_{\kappa-\eta}(y_0)\int_X |m_\eta(x)|d\mu(x) < \infty,\end{aligned}$$

and by symmetry we have $\int_X |m_\kappa| d\nu(x) < \infty$. The reverse implication follows from:

$$\begin{aligned}\int_X |m_\kappa(z)|d(\mu*\nu)(z) &= \int_X\int_X \left|\int_X m_\kappa(z)d(\delta_x*\delta_y)(z)\right| d\mu(x)d\nu(y)\\ &\le \int_X\int_X \sum_{\eta\le\kappa}\binom{\kappa}{\eta}|m_\eta(x)||m_{\kappa-\eta}(y)|d\mu(x)d\nu(y)\\ &= \sum_{\eta\le\kappa}\binom{\kappa}{\eta}\int_X |m_\eta(x)|d\mu(x)\int_X |m_{\kappa-\eta}(y)|d\nu(y).\end{aligned}$$

□

We now construct a martingale from the Markov process $(S_t)_{t\in I}$ using the moment functions:

PROPOSITION 3.12. *Let $(X;*)$ be second countable hypergroup and let $I = \mathbb{N}$ or $[0,\infty)$. Moreover, let $(S_t)_{t\in I}$ a Lévy process or random walk on $(X,*)$ defined as in Lemma 3.3.*
(a) If $(S_t)_{t\ge 0}$ admits first moments i.e $E(m_{e_j}(S_t)) < \infty$ for all $t \ge 0$ and $j = 1,...,N$ then $(m_{e_j}(S_t) - E(m_{e_j}(S_t)))_{t\ge0}$ is a martingale .
(b) If the $(S_t)_{t\ge0}$ admits second moments for $j,k = 1,...,N$ then

$$\begin{aligned}(m_{e_j+e_k}(S_t) - m_{e_j}(S_t)E(m_{e_k}(S_t)) - m_{e_k}(S_t)E(m_{e_j}(S_t))\\ + E(m_{e_j}(S_t))E(m_{e_k}(S_t)) - E(m_{e_j+e_k}(S_t)))_{t\ge0}\end{aligned}$$

is a martingale.

PROOF. See Theorem 4.37 in [**RV2**]. □

CHAPTER 4

Spherical functions on noncompact Grassmann manifolds and hypergreometric functions

In this chapter we collect some properties of spherical function on symmetric spaces. We will look at identifications of spherical functions with hypergeometric functions associated with root systems, which were studied by Heckman and Opdam (see [**HS**]). We also recapitulate Harish-Chandra integral representation for spherical functions. In particular, we focus on the above properties of spherical functions on Grassmann manifolds $\mathcal{G}_{p,q}(\mathbb{F})$. We also study spherical functions on $GL(q,\mathbb{F})/U(q,\mathbb{F})$ which appear the as the limit of spherical functions on Grassmann manifolds above.
For convenience of the reader we give a short survey on this subject based on [**R2**]. We shall see from the results of [**R1**] that these functions lead to a larger classes of commutative hypergroups than just the double coset hypergroups $(G//K,*)$ associated with non-compact symmetric space G/K. .

1. Root systems, Cherednik operators and hypergeometric functions

The basic ingredients in the theory of hypergeometric functions are root systems and finite reflection groups acting on some Euclidean space. Let $\mathfrak{a}$ be Euclidean space with inner product $\langle \cdot, \cdot \rangle$. We extend this inner product to a complex bilinear form on the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$. For $\alpha \in \mathfrak{a} \setminus \{0\}$ we denote by r_α the orthogonal reflection on the hyperplane

$$H_\alpha = \{x \in \mathfrak{a} : \langle x, \alpha \rangle = 0\}$$

perpendicular to α, i.e r_α is given by

$$r_\alpha(x) := x - \frac{2\langle x, \alpha \rangle}{|\alpha|^2}\alpha.$$

Definition 4.1. A finite subset $R \subset \mathfrak{a}$ is called an abstract root system, if $\mathfrak{a}$ is spanned by R and $r_\alpha(R) = R$ for all $\alpha \in R$. Moreover,

- R is called reduced if for all $\alpha, \beta \in R$

$$\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}.$$

- R is called crystallographic if $\alpha \in R$ implies $2\alpha \notin R$.

The group W generated by reflections $\{r_\alpha, \alpha \in R\}$ is called *Weyl group* associated with R. The dimension of $span_{\mathbb{R}} R$ is called rank of R. If R is crystallographic, then $span_{\mathbb{Z}} R$ forms a *root lattice* $Q = R.\mathbb{Z}$ which is stabilized by the action of associated Weyl group. The root systems occurring in Lie theory and in a geometric context associated with Riemannian symmetric spaces are always crystallographic, see [**HS**].

Next, we lay out some well known facts about root systems.

LEMMA 4.2. (i) *If* $\alpha \in R$ *then also* $\alpha \in -R$.
(ii) *For any root system* R *in* $\mathfrak{a}$ *the Weyl group* W *is finite.*
(iii) *The set of reflections contained in* W *is exactly* $\{r_\alpha : \alpha \in R\}$.
(iv) $\omega r_\alpha \omega = r_{\omega\alpha}$ *for all* $\omega \in W$ *and* $\alpha \in R$.

PROOF. See Lemma 2.2 in [**RV2**]. □

As one can see from (i) above, one can write R as a disjoint union $R = R^+ \cup R^-$, where R^+ and R^- are separated by hyperplane H_α. We call R^+ a positive subsystem. Furthermore, we call a root *simple*, if it cannot be written as a sum of two positive roots. There are exactly $q = \dim \mathfrak{a}$ simple roots and they are linearly independent. Let $\{\alpha_1, ..., \alpha_q\}$ be the basis generated by the simple roots. Then, every root $\beta \in R$ can be written as linear combination $\beta = x_1\alpha_1 + ... + x_q\alpha_q$ of $\alpha_1, ..., \alpha_q$, where all of x_i's have the same sign. For details and proofs we refer to [**Hu**].

We call $\lambda \in \mathfrak{a}$ *dominant* if $\langle \alpha_i, \lambda \rangle \geq 0$ for $i = 1, ..., q$, and strictly dominant if the inequality is strict. The set of all strictly dominant vectors generates a *Weyl chamber*

$$C := \{\lambda \in \mathfrak{a} : \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in R\}.$$

It can be shown that the topological closure $\bar{C}$ of the Weyl chamber C is a fundamental domain, i.e. $\bar{C}$ is naturally homeomorphic to the space $(\mathfrak{a})^W$ of all W orbits of in $\mathfrak{a}$, endowed with quotient topology.
We now give a list of important examples of root systems (see [**RV2**]):

EXAMPLE 4.3. • *The root system* $\mathbf{A_{q-1}}$. *Let* S_q *denote the symmetric group of q elements. It acts faithfully on* $\mathfrak{a}$ *by permuting the standard basis vectors* $e_1, ..., e_q$. *Each transposition* (ij) *acts as a reflection* r_{ij} *sending* $e_i - e_j$ *to its negative. It is a finite reflection group, since* S_q *is generated by transpositions. The root system of* S_q *is called* A_{q-1} *and is given by*

$$A_{q-1} = \{\pm(e_i - e_j), 1 \leq i < j \leq q\}.$$

This root system is crystallographic. Its span is the orthogonal complement of the vector $e_1 + ... + e_q$, *and thus the rank is* $q - 1$.
A positive subsystem is given by

$$R_+^A = \{(e_j - e_i), 1 \leq i < j \leq q\}$$

and the associated Weyl chamber is

$$C_q^A := \{x \in \mathbb{R}^q : x_1 \geq x_2 ... \geq x_q\}.$$

- ***The root system*** $\mathbf{B_q}$. *Here W is the reflection group in $\mathfrak{a}$ generated by the transpositions (ij) as above, as well as by the sign changes $r_i : e_i \mapsto -e_i$ in all coordinates $i = 1, ..., q$. The corresponding root system is called B_q; it is given by*

  $$B_q = \{\pm e_i, 1 \leq i \leq q\} \cap \{\pm(e_i \pm e_j), 1 \leq i < j \leq q\}.$$

  B_q is crystallographic and has rank q.
  Here a positive subsystem is given by

  $$R_+^B = \{e_i, 1 \leq i \leq q\} \cap \{(e_i \pm e_j), 1 \leq i < j \leq q\}$$

  and the associated Weyl chamber is

  $$C_q^B = \{t \in \mathbb{R}^q : x_1 \geq x_2 ... > x_q \geq 0\}$$

- ***The root system*** $\mathbf{BC_q}$ *is given by*

  $$BC_q := \{\pm e_i, \pm(2e_i)\} \cup \{\pm(e_i \pm e_j), 1 \leq i < j \leq q\}.$$

  Here a positive subsystem is given by

  $$R_+^{BC} := \{e_i, 2e_i\} \cup \{e_i \pm e_j, 1 \leq i < j \leq q\}$$

  and its associated Weyl chamber is the same as the Weyl chamber associated with the root system B_q.

Definition 4.4. Let R be a root system and W be its Weyl group. A W-invariant map $m : R \to \mathbb{C}, \alpha \mapsto m_\alpha$ is called a *multiplicity function.* Denote the set of multiplicity functions by $\mathcal{M}$ and define the *half sum of roots* by

$$\rho = \rho(m) := \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha \alpha. \tag{31}$$

The set of multiplicity functions forms a $\mathbb{C}$-vector space whose dimension is equal to the number of W-orbits in R. We are now ready to introduce the main object in the theory of this section, namely Cherednik operators. For extended information on Cherednik operators see [**HS**].

Definition 4.5. Let $\xi \in \mathfrak{a}_\mathbb{C}$ and m be a multiplicity function. The *Cherednik operator* is given by

$$T_\xi = T(\xi, m) := \partial_\xi + \sum_{\alpha \in R^+} m_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-2\alpha}} (1 - r_\alpha) - \langle \rho, \xi \rangle, \tag{32}$$

where ∂_ξ denotes a directional derivative corresponding to ξ and e^λ is the exponential function $e^\lambda(\xi) := e^{\langle \alpha, \xi \rangle}$ for $\lambda, \xi \in \mathfrak{a}_\mathbb{C}$.

For $m = 0$ reflection terms vanish and the Cherdnik operator becomes simply the directional derivative ∂_ξ in direction ξ.

REMARK 4.6. *We notice that in the works of Heckman and Opdam (cf. [**HS**]) parameters appear in slightly different normalization, namely root system R there, corresponds to $2R$ in our notation and multiplicities $k_{2\alpha}$ there correspond to $\frac{1}{2}m_\alpha$ in our notation. However, definition of ρ does not change since*

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+} m_\alpha \alpha = \frac{1}{2}\sum_{2\alpha\in 2R^+} k_{2\alpha}2\alpha.$$

DEFINITION 4.7. Denote the space of finite linear combinations of functions $e^\lambda, \lambda \in \Lambda$ by

$$\mathbb{C}[e^P] =: \{\sum a_\lambda e^\lambda : \lambda \in \Lambda, a_\lambda \in \mathbb{C}\}.$$

We call $\mathbb{C}[e^P]$ the space of *trigonometric polynomials.*

Basic algebraic computations show that $T_\xi, \xi \in \mathfrak{a}_\mathbb{C}$, maps $\mathbb{C}[e^P]$ into itself, for the proof see Proposition 2.10 in [**O**]. This property extends to the algebra of complex polynomials $\mathcal{P}(\mathfrak{a}_\mathbb{C}), \mathcal{C}^\infty(\mathfrak{a}_\mathbb{C})$ on $\mathfrak{a}_\mathbb{C}$.

One can also consider an analogue of Laplace operator in the Cherednik setting:

DEFINITION 4.8. The *Heckman-Opdam Laplacian* is given by

$$\Delta_m f(x) := \sum_{i=1}^{n} T_{\xi_i}^2 f(x) - \langle\rho,\rho\rangle, \tag{33}$$

where $\{\xi_1, ..., \xi_q\}$ is an arbitrary orthogonal basis of $\mathfrak{a}$.

The Heckman-Opdam Laplacian is independent from the choice of the basis, see Proposition 1.2.3 in [**HS**]. Explicitly, Δ_m is given as follows:

$$\Delta_m f(x) = \Delta f(x) + \sum_{\alpha\in R^+} m_\alpha \coth\langle\alpha, x\rangle\partial_\alpha f(x) - \sum_{\alpha\in R^+} m_\alpha \frac{|\alpha|^2}{2\sinh\langle\alpha, x\rangle}(f(x) - f(r_\alpha x)), \tag{34}$$

where Δ denotes the euclidean Laplace operator on $\mathfrak{a}$. The action of W on functions $f : \mathfrak{a} \to \mathbb{C}$ is given by

$$w \cdot f(x) := f(w^{-1}x) \text{ for all } x \in \mathfrak{a}.$$

The Cherednik operators T_ξ do not commute under the actions of $w \in W$ in general. However, one has has the following weak W-equivariance property:

PROPOSITION 4.9. *Let T_ξ be the Cherednik operator associated with the root system R and the Weyl group W. Then T_ξ is weakly W-equivariant. This means that for all $\xi \in \mathfrak{a}_\mathbb{C}$ and $w \in W$,*

$$(w \circ T_\xi \circ w^{-1})f(x) = T_{w\xi} f(x) + \sum_{\alpha\in R^+\cap wR^-} m_\alpha\langle\alpha, w\xi\rangle f(r_\alpha x). \tag{35}$$

Proof. See Proposition 1.1 in [**O**]. □

The second fundamental property of the Cherednik operator is commutativity.

Theorem 4.10. *Fix a multiplicity m. Then*

$$T_\eta \circ T_\xi = T_\xi \circ T_\eta \text{ for all } \eta, \xi \in \mathfrak{a}_{\mathbb{C}}.$$

The proof was given by Heckman where he used simultaneous diagonalization methods for some polynomials constructed using trigonometric polynomials, see Corollary 2.7. [**O**].
The commutativity property for $T_\xi, \xi \in \mathfrak{a}_C$ implies that $\xi \mapsto T_\xi$ can be extended to a homomorphism from the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ to the commutative algebra of differential-reflection operators which is generated by T_ξ. Since the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ can be identified with the space of polynomials $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})$ over $\mathfrak{a}_{\mathbb{C}}$, this leads to the notion of Cherednik operators T_p for every $p \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})$. Let us denote by $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$ the subalgebra of $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})$ consisting of the polynomials which are W-invariant. We obtain from Proposition 4.9 that for any W-invariant polynomial $p \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$ and each $f \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$, $T_p f \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$ holds. As $T_p \circ T_q = T_{pq}$ for all $p, q \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})$, in particular we have that

$$\tilde{T}_p \circ \tilde{T}_q = \tilde{T}_{pq}$$

for all $p, q \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$. It has been shown in [**HS**] that for $p \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$ the operators $\tilde{T}_p$ are differential operators on $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$, where the degree of $\tilde{T}_p$ is equal to the degree of polynomial p with coefficients from $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$. In particular, the Heckman-Opdam Laplacian can be regarded as Cherednik operator

$$\Delta_m = T_p, \text{ with } p(x) = |x|^2.$$

As $p \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$, it follows from (35) that the restriction of Δ_m to $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$ is given by

$$\tilde{\Delta}_m = \Delta + \sum_{\alpha \in R^+} \coth\langle \alpha, \cdot \rangle \partial_\alpha. \tag{36}$$

Notice that the operator $\tilde{\Delta}_m$ is singular on reflection hyperplanes $H_\alpha = \{x \in \mathfrak{a} : \langle x, \alpha \rangle = 0\}$.

The next theorem is the basis for the main result for this section. It was proved by Heckman and Opdam in a series of papers, cf. [**HS**] or Theorem 3.5 in [**O**].

Theorem 4.11. *There exists a set $\mathcal{M}^{reg} \subseteq \mathcal{M}$ with*

$$\{m \in \mathcal{M} : \Re(m) \geq 0\} \subseteq \mathcal{M}^{reg}$$

such that for every $m \in \mathcal{M}^{reg}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}$, the system

$$\begin{aligned} T_\xi f &= \langle \lambda, \xi \rangle f, \quad \xi \in \mathfrak{a} \\ f(0) &= 1 \end{aligned} \tag{37}$$

has a unique solution $f(x) = G(\lambda, m; x)$ on $\mathfrak{a}$, which is called ***Opdam hypergeometric function****. Furthermore, there exists a W-invariant tubular neighbourhood U of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbb{C}}$, such that the solution*

$$(\lambda, x) \mapsto G(\lambda, m; x)$$

is a holomorphic function $G(\lambda, m; x)$ of $\lambda \in \mathfrak{a}_{\mathbb{C}}, x \in U$ and $m \in \mathcal{M}^{reg}$.

We now define a mean of the Opdam hypergeometric function with respect to Weyl group W:

DEFINITION 4.12. The Heckman-Opdam hypergeometric function is the average

$$F(\lambda, m; x) := \frac{1}{|W|} \sum_{w \in W} G(\lambda, m; wx). \tag{38}$$

In light of the extension of $\xi \mapsto T_\xi, \xi \in \mathfrak{a}_{\mathbb{C}}$, to $p \mapsto T_p, p \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}})$, $F(\lambda, m; x)$ can be characterized by the following system of differential-reflection equations.

COROLLARY 4.13. *The Heckmann-Opdam hypergeometric function $F(\lambda, m; \cdot)$ is the unique solution of differential equations*

$$\begin{aligned} &\tilde{T}_p f = p(\lambda) f \text{ for all } p \in \mathcal{P}(\mathfrak{a})^W \\ &f_\lambda(0) = 1. \end{aligned} \tag{39}$$

In view of (39) one can also consider the differential equation for Heckman-Opdam Laplacian. In this spirit the hypergeometric function F_λ is the unique solution of system of differential equations

$$\Delta_m f = (|\lambda|^2 - |\rho|^2) f. \tag{40}$$

where $|\lambda|^2 = \sum \lambda_i^2$ is the Euclidean norm.

2. Spherical functions on symmetric spaces

In this section we give a necessary background on the theory of symmetric spaces. We will focus on different characterizations of spherical functions in the sense of the Definition 2.15. The most important property of spherical functions that they can be identified by hypergeometric functions. We shall also give a famous result by Harish-Chandra which states that the spherical functions admit an integral representation.

2.1. Root system identification for symmetric spaces. Consider a semisimple connected Lie group G with some maximal compact subgroup K. We first describe the root system associated with G/K. We follow here Chapter 2 in [**HS**], for more background see [**H1**].
Let $\mathfrak{g}$ be the Lie algebra of Lie group G. Since G is semisimple the Killing form B

on $\mathfrak{g}$ is non-degenerate. As in chapter 2, let θ be an involutive automorphism on G, such that

$$K = \{g \in G : \theta(g) = g\}.$$

As an automorphism of G, θ fixes the identity element, and hence, by differentiating at the identity it induces an automorphism of the Lie algebra $\mathfrak{g}$ of G, which we will also denote by θ, whose square is identity. It follows that the eigenvalues of θ are 1,-1. Let $\mathfrak{k}$ be the Lie algebra of K. Then, $\mathfrak{k} \subset \mathfrak{g}$ is defined as the eigenspace of 1, i.e.

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}.$$

Denote by $\mathfrak{q}$ the eigenspace of -1, i.e.

$$\mathfrak{q} = \{X \in \mathfrak{g} : \theta(X) = -X\}.$$

Since θ is an automorphism on $\mathfrak{g}$ with $\theta^2 = id$, this gives the direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$$

which is also called the *Cartan decomposition.* Here, the Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{q}$. In particular, $(\mathfrak{q}, B)$ is isomorphic to a Euclidean space with finite dimensions, where B is Killing form on $\mathfrak{g}$.
Choose now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{q}$. Then choose $q \in \mathbb{N}$ such that $(\mathfrak{a}, B) \simeq (\mathbb{R}^q, \langle\cdot,\cdot\rangle)$ and let $\mathfrak{a}_{\mathbb{C}}$. Let $\mathfrak{a}^*$ be the dual vector space to $\mathfrak{a}$. For each $\alpha \not\equiv 0$ in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Those $\alpha \neq 0$ with $\mathfrak{g}_\alpha \neq \{0\}$ are called the *restricted roots* of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ or the roots of $(\mathfrak{g}, \mathfrak{a})$. The *geometric multiplicity* m_α is defined as the dimension of $\mathfrak{a}$. Given the root system $R(\mathfrak{g}, \mathfrak{a})$ we can define a positive subsystem $R^+ := R^+(\mathfrak{g}, \mathfrak{a})$ in the same as in the case of abstract root decomposition. Similarly, we define the *half sum of restricted roots* by

$$\rho = \rho(m) := \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha \alpha.$$

Now, let $\mathfrak{g}_0$ be the centralizer of $\mathfrak{a}$. Then, the simultaneous diagonalization of the commuting operators $adH, H \in \mathfrak{a}$ leads to the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in R} \mathfrak{g}_\alpha, \tag{41}$$

In summary, for all pairs (G, K) as above we can find can find associated triple $(\mathfrak{a}, R, m)$. If we identify $\mathfrak{a}^*$ with $\mathfrak{a}$, then the geometric root system $R(\mathfrak{g}, \mathfrak{a})$ can be identified with some abstract root system from Definition 4.1, for details cf Theorem 2.6 in [**HS**]. We denote the set of restricted roots by $R(\mathfrak{g}, \mathfrak{a})$. These roots correspond to reflections which r_α which generates Weyl-group W.

2.2. Spherical functions and hypergeometric functions. We shall now give two different properties for spherical functions of non-compact symmetric spaces G/K, where G is a connected non-compact Lie group. The first property involves an integral representation which traces back to Harish-Chandra. The second property involves the connection between spherical functions and hypergeometric functions.

In order to give the integral formula for spherical function we recapitulate the Iwasawa decomposition of Lie groups. Let G be a semisimple connected non-compact Lie group G. Then G admits Iwasawa decomposition $G = KAN$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where K, A and N are compact, abelian and nilpotent subgroups of G, and $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ are their respective Lie algebras, c.f. [**GV**]. In the Propositions 2.18 and 2.19 we have considered this decomposition the cases $G = GL(q, \mathbb{F})$ and $SU(p, q, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ and $p, q \in \mathbb{N}$ with $p \geq q$. This decomposition $G = KAN$ is called the *Iwasawa decompostion.* It induces a diffeomorphism

$$\begin{aligned} K \times A \times N &\to G, \\ (k, a, n) &\mapsto kan. \end{aligned}$$

Now, let $\exp : \mathfrak{a} \to A$ be an exponential map. This map is an isomorphism with inverse $\log : A \mapsto \mathfrak{a}$. We are now ready to present the integral representation for spherical functions for (G, K):

THEOREM 4.14. *Let G be defined above. For $g \in G$, let $H(g) \in \mathfrak{a}, k(g) \in K$ be the unique elements such that $g \in N \exp H(g) k(g)$. Then, as λ runs through $\mathfrak{a}^*_{\mathbb{C}}$ the functions*

$$\varphi_\lambda(g) := \int_K e^{(i\lambda - \rho)(H(gk)))} d\omega_K(k) \tag{42}$$

exhaust the class of spherical functions on (G, K), where ρ denotes the half sum of the roots. Moreover $\varphi_\lambda = \varphi_{\lambda'}$ if and only $\lambda = w\lambda'$ for some $w \in W$.

PROOF. See Theorem 4.3 in [**Hel2**]. □

Theorem 4.14 leads to a parametrization of the space of spherical functions of (G, K). From now on, when we write φ_λ, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is a spherical function, we mean the function φ_λ indexed as in Eq. (42) above. Since the spherical functions are K-biinvariant φ_λ is uniquely determined by the values $\varphi_\lambda(a), a \in A$.

THEOREM 4.15. *Let G be simply connected, semisimple non-compact Lie group with maximal compact subgroup K. Moreover let $(\mathfrak{a}, R, m)$ be the corresponding triple to (G, K) as above. Then for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $x \in \mathfrak{a}$*

$$\varphi_\lambda(\exp x) = F(i\lambda, m; x) \tag{43}$$

PROOF. See Theorem 5.2.2 in [**HS**]. □

3. Spherical functions of non-compact Grassmann manifold

In view of Heckman-Opdam theory the (restricted) root system decomposition of the noncompact Grassmann manifold $\mathcal{G}_{p,q}$ corresponds to the abstract root system BC_q. Moreover, the the corresponding double coset hypergroup $(C_q^B, *_{p,q})$ can be extended to all $p \in (2q-1, \infty)$ using the identification (43) of spherical functions φ^p with hypergeometric functions.
The symmetric space $\mathcal{G}_q \simeq \mathcal{P}_q(\mathbb{F})$ is closely related with the abstract root system A_{q-1} introduced in Definition 4.3. For both symmetric spaces we give explicit integral representations for the spherical functions in the sense of Theorem 4.14. The symmetric spaces $\mathcal{G}_{p,q}$ and $\mathcal{G}_q$ have a close relationship: As $p \to \infty$ spherical functions on $GL(q, \mathbb{F})/U(q, \mathbb{F})$ converge to spherical function on $\mathcal{G}_q$. Throughout this section we follow [**RV1**], [**R2**], closely.

3.1. Spherical functions of noncompact Grassmannian. Let $\mathcal{G}_{p,q} = G/K$ be Grassmann manifold where G is one of the groups $SO_0(p,q)$, $SU(p,q)$ and $Sp(p,q)$, and subgroup K is one of the groups $SO(p) \times SO(q)$, $S(U(p) \times U(q))$ and $Sp(p) \times Sp(q)$, respectively.
The Lie algebra $\mathfrak{g}$ of G is given by the matrices $X \in M_{p+q}$ of the block form

$$X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

where $A \in M_q(\mathbb{F})$ and $D \in M_p(\mathbb{F})$ are skew-Hermitian matrices with the property $trA + trD = 0$, and $B \in M_{q,p}(\mathbb{F})$. Let $\mathfrak{k}$ be the Lie algebra of K and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the associated Cartan decomposition of $\mathfrak{g}$. Then the $\mathfrak{q}$ consists of block matrices $X \in M_{p+q}$

$$X = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$$

where $C \in M_{q,p}$.
In accordance with [**S**] (see also Proposition 2.19) we can identify a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{q}$ with $\mathbb{R}^q$ by the matrices

$$H_x = \begin{pmatrix} 0_{q\times q} & \underline{x} & 0_{q\times(p-q)} \\ \underline{x} & 0_{q\times q} & 0_{q\times p-q} \\ 0_{(p-q)\times q} & 0_{(p-q)\times q} & 0_{(p-q)\times(p-q)} \end{pmatrix}$$

where $\underline{x} = \mathrm{diag}(x_1, ..., x_q)$ is the diagonal matrix corresponding to $x = (x_1, ..., x_q) \in \mathbb{R}^q$. The abelian group A in the Iwasawa decompostion $G = KAN$ consists of elements

$$a_x = \exp(H_x) = \begin{pmatrix} \cosh \underline{x} & \sinh \underline{x} & 0 \\ \sinh \underline{x} & \cosh \underline{x} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix}$$

as in Proposition 2.19.
The corresponding restricted root system $R(\mathfrak{g}, \mathfrak{a})$ is of type B_q if $\mathbb{F} = \mathbb{R}$ and of

type BC_q if $\mathbb{F} = \mathbb{C}, \mathbb{F}$ and the real rank of this symmetric space is q. The restricted roots $\alpha \in \mathfrak{a}^*$ are given by

$$\{\pm e_i, \pm 2e_i, \pm e_i \pm e_j : 1 \leq i, j \leq q\}.$$

In this case we have 3 classes of roots (c.f. Example 4.3(3)) which correspond to 3 different root spaces in the root space decomposition (41). These roots with corresponding geometric multiplicities $m_\alpha = m_\alpha(p, q, d)$ are give in the table below.

Root α	Multipilcity m_α
$\alpha(H_x) = \pm x_i = \pm e_i \cdot x,\ 1 \leq i \leq q$	$d(p-q)$
$\alpha(H_x) = \pm 2x_i = \pm 2e_i \cdot x,\ 1 \leq i \leq q$	$d-1$
$\alpha(H_x) = \pm t_i \pm x_j = \pm(e_i \pm e_j) \cdot x,\ 1 \leq i \leq q$	d

Here d denotes the real dimension of the underlying field $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. For the explicit description of root space decomposition see [**RV1**], [**S**].
Here a positive subsystem can be identified by

$$R^+ = \{e_i, 2e_i : 1 \leq i \leq q\} \cup \{e_i \pm e_j : 1 \leq i < j \leq q\}.$$

The sum of positive half roots is

$$\rho^{BC} = \rho^{BC}(p) = \frac{1}{2} \sum_{\alpha \in R^+}^{q} m_\alpha \alpha = \sum_{i=1}^{q} \left((\frac{d}{2}(p+q+2-2i) \right) e_i. \tag{44}$$

Denote the triplet of multiplicities by

$$m_p = (d(p-q), (d-1)/2, d/2).$$

In view of Theorem 2.16 for integers $p \geq 2q$ the spherical functions associated with $\mathcal{G}_{p,q}(\mathbb{F})$ are given by

$$\varphi_\lambda^p(a_x) = F_{BC}(i\lambda, m_p, x),$$

where F_{BC} is the hypergeometric function of type BC_q.

We now give explicit description of Harisch Chandra integral formula (42 for the spherical functions φ_λ^p. For this we need to introduce some notation. For a square matrix $A = (a_{i,j})_{1 \leq i \leq j \leq q}$ over $\mathbb{F}$ we denote by $\Delta_r(A) = \det((a_{i,j})_{1 \leq i \leq j \leq r})$ the r-th principal minor of A. Here, for $\mathbb{F} = \mathbb{H}$ the determinant is understood in the sense of Dieudonné, i.e. $\det(A) = (\det_{\mathbb{C}}(A))^{1/2}$ when A is considered as a complex matrix. See [**A**] for more information about Dieudonné determinant. Moreover, for $\lambda \in \mathbb{C}^q \simeq \mathfrak{a}_{\mathbb{C}}^*$ and $x \in \mathcal{P}_q(\mathbb{F})$, we define

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1 - \lambda_2} \Delta_2(x)^{\lambda_2 - \lambda_2} ... \Delta_q(x)^{\lambda_q}. \tag{45}$$

Theorem 4.16 (Corollary 2.2, [**RV1**]). *Let $\mathcal{G}_{p,q}(\mathbb{F})$ be a non-compact Grassmannian. Assume that $p \geq 2q$ is an integer. Then the spherical functions are* *given by*

$$\varphi_\lambda^p(a_x) = \int_{U_0(q,\mathbb{F})} \int_{B_q} \Delta_{(i\rho_{BC}-\lambda)}(g_x(u,w))\mathrm{d}\mathbf{m}_p(w)\mathrm{d}u, \tag{46}$$

where $\mathrm{d}u$ *is the normalized Haar measure on* $U_0(q,\mathbb{F})$, $\mathrm{d}\mathbf{m}_p(w)$ *is the probability measure defined in (20) and*

$$g_x(u,w) = u^{-1}(\cosh \underline{x} + \sinh \underline{x}w)^*(\cosh \underline{x} + \sinh \underline{x}w)u.$$

We now identify $x \in C_q^B$ with matrices $a_x \in A$ above and regard φ_λ^p as a function of $x \in C_q^B$. We note that the integral formula (46) can be extended to $p \in]2q-1,\infty[$. Notice that the domain of integration in the integral formula (46) above is independent of p. This means that right hand side of the Eq. (46) remains well defined for all $p > 2q-1$. On the other hand, from Theorem 4.11 we know that hypergeometric functions F_{BC} are well defined for all multiplicities m with $\Re(m) \geq 0$. In our case this means that F_{BC_q} is well defined for all multiplicities

$$m_p = (d(p-q)/2, (d-1)/2, d/2) \tag{47}$$

correponding to the roots $\pm e_i, \pm 2e_i$ and $(\pm e_i \pm e_j)$ for $p \in \mathbb{C}^q$ with $\Re p \geq q$. Now, for $p \in (2q-1,\infty)$ define the functions

$$\varphi_\lambda^p(x) := F_{BC_q}(i\lambda, m_p, x).$$

Then, for integers p the the functions φ_λ^p are precisely the spherical functions in (46). The extension ofthe integral formula (46) to $p \in (2q-1,\infty)$ can be obtained by analytic continuation using Carlson's theorem below.

Theorem 4.17. *(Carlson) Let f be a function which is a holomorphic in a neighborhood of $\{z \in \mathbb{C} : \Re z \geq 0\}$ satisfying $f(z) = O(e^{c|z|})$ for some constant $c < \pi$. Suppose that $f(n) = 0$ for all $n \in \mathbb{N}_0$. Then f is identically zero.*

For the detailed proof extension for integral formula we refer to Theorem 2.4 in [**RV1**].
We now return to the hypergroup $(C_q^B, *_{p,q})$ given in Definition 2.21, where $p, q \in \mathbb{N}$ with $p \geq 2q-1$. As we pointed out in Chapter 2 we can can extend the convolution $*_{p,q}$ to all $p \in (2q-1,\infty)$, where the hypergroup structure remains preserved. This extension is made with similar techniques by analytic continuation using Carlson's theorem above. More precisely we have:

Theorem 4.18. *Let $q \in \mathbb{N}$ and $p \in (2q-1,\infty)$.*

(1) *The point measures δ_x, δ_y for $x, y \in C_q^B$ with convolution*

$$\delta_x *_{p,q} \delta_y(f) := \frac{1}{\kappa_{pd/2}} \int_{U_q} \int_{B_q} f(d(\underline{x},\underline{y},u,w))\Delta(I-w^*w)^{pd/2-\gamma} dudw, \tag{48}$$

for all $f \in \mathcal{C}^\infty(C_q^B)$ define a commutative hypergroup $(C_q^B, *_{p,q})$. Here, the neutral element is 0 and involution is the identity mapping.

(2) *For all* $x, y \in C_q^B$

$$\mathrm{supp}(\delta_x *_{p,q} \delta_y) \subset \{z \in C_q^B : \|z\|_{\max} \leq \|x\|_{\max} + \|y\|_{\max}\}$$

is satisfied, where $\|\cdot\|_{\max}$ *denotes the maximum norm in* $\mathbb{R}^q$.

PROOF. See Theorem 4.1 in [**R1**]. □

We note that the functions $\varphi_\lambda^p, \lambda \in \mathbb{C}^q$ exhaust all multiplicative functions for $(C_q^B, *_{p,q})$ i.e., if $\phi(x)\phi(y) = \phi(x *_{p,q} y)$ for all $x, y \in C_q^B$, then there exists $\lambda \in \mathbb{C}$ such that $\varphi_\lambda^p = \phi$, see Lemma 5.3 in [**R2**]. In fact, the set of multiplicative characters and the dual space for C_q^B can be explicitly determined:

THEOREM 4.19. *[5.4 in [**R1**]] Let* $p \in (2q-1, \infty)$. *The set of multiplicative functions and the dual of the hypergroup* $(C_q^B, *_{p,q})$ *are given by*

$$\chi((C_q^B, *_{p,q}) = \{\varphi_\lambda^p : \lambda \in C_q^B + iC_q^B\},$$

$$(\widehat{C_q^B, *_{p,q}}) = \{\varphi_\lambda^p \in \chi((C_q^B, *_{p,q}) : \bar{\lambda} \in W^B.\lambda \text{ and } \Im\lambda \in \mathrm{co}(W^B.\rho_{BC})\}$$

respectively, where ρ_{BC} *is the half sum of multiplicities and* $\mathrm{co}(W^B.\rho_{BC})$ *denotes the convex hull of Weyl orbit* $W^B.\rho_{BC}$.

3.2. Spherical functions of $GL(q,\mathbb{F})/U(q,\mathbb{F}))$ **and limit transition.** Let $\mathcal{G}_q = G/K$ be symmetric space with $(G, K) = (GL(q,\mathbb{F}), U(q,\mathbb{F}))$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. It is well known that G has Iwasawa decomposition $G = KAN$ where the abelian group is given by $A = exp(\mathfrak{a})$ with

$$\mathfrak{a} = \{H_x = \underline{x} : x = (x_1, ..., x_q) \in \mathbb{R}^q\}$$

and the unique nilpotent group N consists of upper diagonal matrices with entries 1 in the diagonal. We can identify $\mathfrak{a}$ through the map $x \mapsto H_x$ with $\mathbb{R}^q$. The restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type A_{q-1}. The restricted roots $\alpha \in \mathfrak{a}^*$ are given by

$$\alpha(H_x) = \pm(x_i - x_j) = \pm(e_i - e_j) \cdot x \text{ for } i, j \in \{1, ..., q.\}$$

Then, an abstract positive root subsystem is given by

$$R_+ = \{e_i - e_j : 1 \leq i < j \leq q\}.$$

The sum of positive half roots is

$$\rho^A = \frac{1}{2} \sum_{\alpha \in R^+}^{q} m_\alpha \alpha = \sum_{i=1}^{q} \left(\frac{d}{2}(q + 1 - 2i)\right) e_i$$

The following explicit integral representation for spherical functions of type A_{q-1} was obtained in [**RV1**] using the Harish-Chandra representation (42).

THEOREM 4.20. *The spherical function* φ_λ^A *of* $(G,K) = (GL(q,\mathbb{F}), U(q,\mathbb{F}))$ *admits an integral representation*

$$\varphi_\lambda^A(e^{\underline{x}}) = \int_{U(q,\mathbb{F})} \Delta_{(\lambda - i\rho_A)/2}(ue^{\underline{2x}}u^{-1})\mathrm{d}u \tag{49}$$

for all $\lambda \in \mathbb{C}^q$ *and* $x \in \mathbb{R}^q$.

PROOF. See Section 3 in [**RV1**]. □

The spherical function φ_λ^p converges to φ_λ^A as $p \to \infty$. For convenience we define

$$\psi_\lambda(x) := \varphi_\lambda^A(\cosh \underline{x}) = \int_{U(q,\mathbb{F})} \Delta_{(\lambda - i\rho_A)/2}(u \cosh^2 \underline{x} u^{-1})\mathrm{d}u$$

for $x \in \mathbb{R}^q$. The following convergence result was obtained in [**RV2**]:

THEOREM 4.21. *Let* $p > 2q-1, x \in C_q^B$ *and* $\lambda \in \mathbb{C}^q$ *such that* $\Im\lambda - \rho_{BC}$ *is contained in* $\mathrm{co}(W.\rho_A)$, *i.e.* $\varphi_{\lambda - i\rho_{BC}}$ *is bounded in* C_q^B. *Then, there exists a universal constant* $C = C(\mathbb{F}, q)$ *such that*

$$|\varphi_{\lambda - i\rho_{BC}}^p(x) - \psi_{\lambda - i\rho_A}(x)| \leq C \cdot \frac{\|\lambda\|_1 \cdot \tilde{x}}{p^{1/2}} \tag{50}$$

where $\tilde{x} = \min(x_1, 1)$. *In particular, for these spectral parameters* λ *the order of convergence is uniform of order* $p^{-1/2}$ *in* $x \in C_q^B$.

PROOF. See Theorem 4.2 in [**RV2**]. □

CHAPTER 5

Limit theorems with fixed dimensions

In this chapter we study several limit theorems for time-homogeneous random walks on hypergroups $(C_q^A, *_q)$ and $(C_q^B, *_{p,q})$ for fixed $p, q \in \mathbb{N}$ with $p > q$, which can be identified with double coset hypergroups $(G//K, *_\pi)$ for symmetric spaces $G/K = GL(q, \mathbb{F})/U(q, \mathbb{F})$ corresponding to root system of type A_{q-1} and $G/K := \mathcal{G}_{p,q}(\mathbb{F})$ corresponding to root system of type BC, respectively. By the extension in Theorem 4.18 results are valid for random walks on hypergroups $(C_q^B, *_{p,q})$ for all $p \in [2q-1, \infty)$. We consider here two kinds limit theorems under two different normalization procedures: inner and outer normalizations. These normalizations yield different limiting distributions. We note that the results with outer normalizations were derived in [**R2**], we state these results without proofs to have a full picture and to compare with the results for growing dimensions p in Chapter 6. We start with limit theorems on $(C_q^A, *_q)$ as A-case can be in the Heckman-Opdam theory appears a limit of the BC-case.

1. Limit theorems on the hypergroup $(C_q^A, *_q)$

Let $(\tilde{S}_n)_{n\geq 0}$ be a random walk on the hypergroup $(C_q^A, *)$ associated with some probability measure μ i.e. $(\tilde{S}_n)_{n\geq 0}$ a time-homogeneous Markov process with start in $0 \in C_q^A$ and transition probabilities

$$(51)\quad \mathbb{P}(\tilde{S}_{n+1} \in A|\,(\tilde{S}_n = x) = (\delta_x *_q \mu)(A) \qquad (x \in C_q^A,\ A \subset C_q^A \quad \text{a Borel set}).$$

Remark 5.1. *In view of Theorem 3.3 the random walk $(\tilde{S}_n)_{n\geq 0}$ above, can be identified with random walk on $G := GL(q, \mathbb{F})$ in the following way: Let $(S_n)_{n\geq 0}$ on G with a K-biinvariant associated probability measure μ_G, then the process*

$$(\tilde{S}_n)_{n\geq 0} := (\ln \sigma_{sing}(S_n))_{n\geq 0}$$

is a random walk on with associated $(C_q^A, *_q)$ associated with the probability measure $\tilde{\mu}$, which is the image of μ_G under the map $\ln \sigma_{sing} : G \to C_q^A$.

We present strong LLN and CLT with outer normalization for the random walk $(\tilde{S}_n)_{n\geq 0}$ under some moment conditions for the associated measure μ, that is

$$\frac{\tilde{S}_n}{n} \to m_{\mathbf{1}}^A(\mu) \text{ almost surely,}$$

for some vector $m_{\mathbf{1}}^A(\mu)$, and the distributions of

$$\frac{1}{\sqrt{n}}(\tilde{S}_n - n \cdot m_{\mathbf{1}}^A(\mu))$$

converge to some normal distribution $N(0, \Sigma^A(\nu))$.

We now give the precise formulas for the vector $m_{\mathbf{1}}^A(\mu)$ and covariance matrix $\Sigma^A(\mu)$ via moment functions of the hypergroup $(C_q^A, *_q)$. Let m_l^A, $l \in \mathbb{N}_0^q$ be the the moment functions on $(C_q^A, *_q)$ as in Definition 3.8. By definition these moment functions are given by partial derivatives the spherical function φ_λ^A. More precisely, for multiindices $l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q$ the moment functions m_l^A are given by

$$\begin{aligned} m_l^A(x) := & \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{-i\rho-i\lambda}^A(x)\Big|_{\lambda=0} = \frac{\partial^{|l|}}{(\partial\lambda_1)^{l_1} \cdots (\partial\lambda_n)^{l_q}} \varphi_{-i\rho-i\lambda}^A(x)\Big|_{\lambda=0} \\ = & \frac{1}{2^{|l|}} \int_K (\ln \Delta_1(u^{-1}e^{2\underline{x}}\,u))^{l_1} \cdot \left(\ln\left(\frac{\Delta_2(u^{-1}e^{2\underline{x}}\,u)}{\Delta_1(u^{-1}e^{2\underline{t}}\,u)}\right)\right)^{l_2} \cdot \\ & \cdots \left(\ln\left(\frac{\Delta_q(u^{-1}e^{2\underline{x}}\,u)}{\Delta_{q-1}(u^{-1}e^{2\underline{x}}\,u)}\right)\right)^{l_q} du \end{aligned} \tag{52}$$

of order $|l| := l_1 + \cdots + l_q$ for $x \in C_q^A$. The last equality in (52) follows from (49) by interchanging integration and derivatives. We denote the j-th unit vector by $e_j \in \mathbb{N}^q$ and the moment functions of order 1 and 2 by $m_{e_j}^A$ and $m_{e_j+e_k}^A$ $(j, k = 1, .., q)$. The q moment functions of first order lead to the vector-valued moment function

$$m_{\mathbf{1}}^A(x) := (m_{e_1}^A(x), ..., m_{e_q}^A(x)) \tag{53}$$

of first order. Moreover, the moment functions of second order can be grouped by

$$m_{\mathbf{2}}^A(x) := \begin{pmatrix} m_{2e_1}^A(x) & \cdots & m_{e_1+e_q}^A(x) \\ \vdots & & \vdots \\ m_{e_q+e_1}^A(x) & \cdots & m_{2e_q}^A(x) \end{pmatrix} \qquad \text{for} \quad x \in C_q^A.$$

We now form the $q \times q$-matrices $\Sigma^A(x) := m_{\mathbf{2}}^A(x) - m_{\mathbf{1}}^A(x)^t \cdot m_{\mathbf{1}}^A(x)$. These moment functions have the following basic properties; see Section 2 of [**V2**]:

Lemma 5.2. (1) *There is a constant $C = C(q)$ such that for all $x \in C_q^A$, $\|m_{\mathbf{1}}^A(x) - x\| \leq C$.*
(2) *For each $x \in C_q^A$, $\Sigma^A(x)$ is positive semidefinite.*
(3) *For $x = c \cdot (1, \ldots, 1) \in C_q^A$ with $c \in \mathbb{R}$, $\Sigma^A(t) = 0$. For all other $x \in C_q^A$, $\Sigma^A(x)$ has rank $q - 1$.*
(4) *All second moment functions $m_{e_i+e_j}^A(x)$ are growing at most quadratically, and $m_{2e_1}^A(x)$ and $m_{2e_q}^A(x)$ are in fact growing quadratically.*

(5) *There exists a constant $C = C(p)$ such that for all $x \in C_q^A$ and $\lambda \in \mathbb{R}^q$,*

$$|\varphi^A_{-i\rho^A-\lambda}(x) - e^{i\langle\lambda, m_{\mathbf{1}}^A(x)\rangle}| \leq C||\lambda||^2.$$

Let $\mu \in \mathcal{M}^1(C_q^A)$. In accordance with Definition 3.8, for $k \in \mathbb{N}$ we say that μ admits all moments of type A up to order k if for all $l \in \mathbb{N}_0^q$ with $|l| \leq k$ the moment condition $m_l^A \in L^1(C_q^A, \mu)$ holds. Now, by Lemma 5.2(1) it follows that μ admits all moments of type A up to order 1 if the usual moments $\int_{C_q^A} x_i d\mu(x)$ $(i = 1, ..., q)$ of order 1 exist. Similarly, Lemma 5.2(4) implies that μ admits all moments of type A up to order 2 if the usual second order moments $\int_{C_q^A} x_i^2 d\mu(x)$ $(i = 1, ..., q)$ exist. This means that if μ admits second moments, then the second order moment matrix $m_{\mathbf{2}}^A(\mu)$ and the covariance matrix $\Sigma^A(\mu)$ exist.
We are now ready to present the strong of law large numbers and central limit theorem for the random walk $(\tilde{S}_n)_{n\geq 1}$ on $(C_q^A, *_q)$ with associated measure μ which was obtained in [**V2**].

THEOREM 5.3. *(Theorem 2.4 in [**V2**])*

(1) *If μ admits first moments, then for $n \to \infty$,*

$$\frac{\tilde{S}_n}{n} \to m_{\mathbf{1}}^A(\mu) \text{ almost surely.}$$

(2) *If μ admits second moments, then for all $\varepsilon > 1/2$ and $n \to \infty$,*

$$\frac{1}{n^\varepsilon}(\tilde{S}_n - n \cdot m_{\mathbf{1}}^A(\mu)) \to 0 \text{ almost surely.}$$

THEOREM 5.4. *(Theorem 2.5 in [**V2**]) If $\mu \in \mathcal{M}^1(C_q^A)$ admits finite second moments, then for $n \to \infty$*

$$\frac{1}{\sqrt{n}}(\tilde{S}_n - n \cdot m_{\mathbf{1}}^A(\mu)) \to N(0, \Sigma^A(\mu)) \text{ in distribution.}$$

2. Limit theorems for random walks on $(C_q^B, *_{p,q})$

Let $p \in (2q-1, \infty)$ and consider a random walk $(\tilde{S}_n^p)_{n\geq 0}$ on the hypergroup $(C_q^B, *)$ associated with some probability measure μ i.e. $(\tilde{S}_n^p)_{n\geq 0}$ a time-homogeneous Markov process with start in $0 \in C_q^B$ and transition probabilities

$$\mathbb{P}(\tilde{S}_{n+1}^p \in A|\, (\tilde{S}_n^p = x) = (\delta_x *_{p,q} \mu)(A) \qquad (x \in C_q^B,\ A \subset C_q^B \quad \text{a Borel set}). \tag{54}$$

REMARK 5.5. *In view of Theorem 3.3, for integers $p \geq 2q$, the random walk $(\tilde{S}_n)_{n\geq 0}$ above, can be identified with a random walk on $G := SU(p, q, \mathbb{F})$ in the following way: Let $(S_n^p)_{n\geq 0}$ be a random walk on G associated with K-biinvariant probability measure μ_G, then the process*

$$(\tilde{S}_n^p)_{n\geq 0} := ((\text{arccosh}(A(S_n^p))_{n\geq 0}$$

is a random walk on $(C_q^B, *_{p,q})$ *with associated with the probability measure* $\tilde{\mu}$*, which is the image of* μ_G *under the map* $\operatorname{arccosh}(\sigma_{sing}(A(\cdot)) : G \to C_q^B$*, where* $A(\cdot)$ *is given as in Eq. (17).*

We consider limit theorems for $(\tilde{S}_n^p)_{n\geq 0}$ under two different normalization procedures:
The first case with outer normalization: We present a CLT and strong LLN results for the random variable $\tilde{S}_n^p$ under some moment conditions for the associated measure μ, that is

$$\frac{\tilde{S}_n^p}{n} \to m_{\mathbf{1}}^p(\mu) \text{ almost surely,}$$

for some vector $m_{\mathbf{1}}^p(\mu)$, and the distributions of the random variable

$$\frac{1}{\sqrt{n}}(\tilde{S}_n^p - n \cdot m_{\mathbf{1}}^p(\mu))$$

converge to some normal distribution $\mathcal{N}(0, \Sigma^p(\mu))$.
The second case is the inner normalization, consider the following setting: Fix some nontrivial probability measure $\mu \in \mathcal{M}^1(C_q^B)$ with some moment condition and for $d \in (0,1)$ consider the component-wise compression map $D_c : x \mapsto c \cdot x$ on C_q^B as well as compressed measure $\mu_c := D_c(\mu) \in \mathcal{M}^1(C_q^B)$. For given μ and c we consider the random walk $(S_n^{(p,c)})_{n\geq 0}$ associated with μ_c. We investigate the limiting behavior of $(S_n^{(p,n^{-1/2})})_{n\geq 1}$. The limit theorem for $(S_n^{(p,n^{-1/2})})_{n\geqslant 1}$ in the rank 1 case was studied by Zeuner [**Z1**]. In the group cases, this CLT is related with the CLTs in [**G1**], [**G2**], [**T1**], [**T2**], [**Ri**].

We start with the first case and give the precise formulas for the vector $m_{\mathbf{1}}^p(\mu)$ and the covariance matrix $\Sigma^p(\mu)$ via moment functions of the hypergorup $(C_q^B, *_{p,q})$. Let $m_l^p, l \in \mathbb{N}^q$ be the moment functions of the hypergroup $(C_q^B, *_{p,q})$ as in Definition 3.8. These moments are given by spherical functions φ_λ^p. Thus, using integral representation (46) for φ_λ^p the moment functions m_l^p for $l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q$ are given by :

$$m_l^p(x) := \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{-i\rho^{BC}-i\lambda}^p(x)\Big|_{\lambda=0} := \frac{\partial^{|l|}}{(\partial \lambda_1)^{l_1} \cdots (\partial \lambda_q)^{l_q}} \varphi_{-i\rho^{BC}-i\lambda}^p(x)\Big|_{\lambda=0}$$

$$= \frac{1}{2^{|l|}} \int_{B_q} \int_{U(q,\mathbb{F})} (\ln \Delta_1(g(x,u,w)))^{l_1} \cdot \left(\ln \frac{\Delta_2(g(x,u,w))}{\Delta_1(g(x,u,w))} \right)^{l_2} \cdot \tag{55}$$

$$\cdots \left(\ln \frac{\Delta_q(g(x,u,w))}{\Delta_{q-1}(g(x,u,w))} \right)^{l_q} \, du \, dm_p(w) \tag{56}$$

for $x \in C_q^B$. We also form the vector-valued first moment function $m_{\mathbf{1}}^p$, the matrix-valued second moment function $m_{\mathbf{2}}^p$, as well as $\Sigma^p(x) := m_{\mathbf{2}}^p(t) - (m_{\mathbf{1}}(x)^p)^t \cdot m_{\mathbf{1}}^p(x)$ as above.

We have the following basic properties; see Section 3 of [**V2**]:

LEMMA 5.6. (1) *There is a constant* $C = C(p,q)$ *such that for all* $x \in C_q^B$,

$$\|m_{\mathbf{1}}^p(x) - x\| \leq C.$$

(2) *For each* $x \in C_q^B$, $\Sigma^p(x)$ *is positive semidefinite.*
(3) $\Sigma^p(0) = 0$, *and for* $t \in C_q^B \setminus \{0\}$, $\Sigma^p(x)$ *has full rank* q.
(4) *All second moment functions* $m_{e_j+e_l}^p(x)$ *are growing at most quadratically, and* $m_{2e_1}^p$ *is growing quadratically.*
(5) *There exists a constant* $C = C(p,q)$ *such that for all* $x \in C_q^B$ *and* $\lambda \in \mathbb{R}^q$,

$$|\varphi_{-i\rho-\lambda}^p(t) - e^{i\langle\lambda, m_{\mathbf{1}}^p(t)\rangle}| \leq C||\lambda||_2^2.$$

Similarly to the A-case, for $\nu \in \mathcal{M}^1(C_q^B)$ we define l-th BC(p) multivariate moments of $\nu \in \mathcal{M}^1(C_q^B)$ for $l \in \mathbb{N}_0^q$ as $m_l^p(\nu) := \int_{C_q^B} m_l^p(x) d\nu(t)$.

THEOREM 5.7. *(Theorem 3.5 in [**R2**])*
(1) *If* μ *admits first moments, then for* $n \to \infty$,

$$\frac{\tilde{S}_n^p}{n} \longrightarrow m_{\mathbf{1}}^p(\mu) \textit{ almost surely.}$$

(2) *If* μ *admits second moments, then for all* $\varepsilon > 1/2$ *and* $n \to \infty$

$$\frac{1}{n^\varepsilon}(\tilde{S}_n^p - n \cdot m_{\mathbf{1}}^p(\mu)) \longrightarrow 0 \textit{ almost surely.}$$

THEOREM 5.8. *(Theorem 3.6 in [**R2**])*
If μ *admits finite second moments, then for* $n \to \infty$

$$\frac{1}{\sqrt{n}}(\tilde{S}_n^p - n \cdot m_{\mathbf{1}}^p(\mu)) \longrightarrow \mathcal{N}(0, \Sigma^p(\nu)) \textit{ in distribution.}$$

We now turn to the second case. In order to state the limit theorem in the we need to introduce some notation. We first define the hypergroup Fourier transform on the hypergroup in accordance with Definition 2.7.

DEFINITION 5.9. Let $\mu \in \mathcal{M}^1(C_q^B)$. Define the BC-type spherical (or hypergroup) Fourier transform in the sense of Definition 2.7 by

$$\mathcal{F}_{BC}^p(\mu)(\lambda) := \int_{C_q^B} \varphi_\lambda^p(x) d\mu(x)$$

for $\lambda \in \{\lambda \in \mathbb{C}^q : \Im\lambda \in co(W_q^B \cdot \rho)\}$.

We note that the above hypergroup spherical transform is well defined by Theorem 2.7 since φ_λ^p is bounded for all $\lambda \in \{\lambda \in \mathbb{C}^q : \Im\lambda \in co(W_q^B \cdot \rho_{BC})\}$. The dual space $(\widehat{C_q^B, *_{p,q}})$ can be parametrized by the set $\{\varphi_\lambda^p : \lambda \in C_q^B \textit{ or } \lambda \in i \cdot co(W_q^B \cdot \rho_{BC})\}$. The support of Plancherel measure is parametrized by the set

$\{\varphi_\lambda^p : \lambda \in C_q^B\}$.

DEFINITION 5.10. Let $\mu \in \mathcal{M}^1(C_q^B)$. The BC-type spherical (or hypergroup) Fourier transform is given by

$$\mathcal{F}_{BC}^p(\mu)(\lambda) := \int_{C_q^B} \varphi_\lambda^p(x) d\mu(x)$$

for $\lambda \in \{\lambda \in \mathbb{C}^q : \Im\lambda \in co(W_q^B \cdot \rho_{BC})\}$.

We now give some estimates on spherical functions and Fourier transforms from [**V2**].

LEMMA 5.11. *For all $x \in C_q^B$, $\lambda \in \mathbb{R}^q$, and $l \in \mathbb{N}_0^q$,*

$$\left| \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{\lambda - i\rho}^p(x) \right| \leqslant m_l^p(x)$$

LEMMA 5.12. *Let $k \in \mathbb{N}_0$ and assume that $\mu \in \mathcal{M}^1(C_q^B)$ admits finite k-th modified moments. Then, for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda \in co(W_q^B \cdot \rho_{BC})$, $\mathcal{F}_{BC}^p(\mu)(\cdot)$ is k-times continuously differentiable, and for all $l \in \mathbb{N}_0^n$ with $|l| \leqslant k$,*

$$\frac{\partial^{|l|}}{\partial \lambda^l} \mathcal{F}_{BC}^p(\mu)(\lambda) = \int_{C_q^B} \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_\lambda^p(x) d\mu(x). \tag{57}$$

In particular,

$$\frac{\partial^{|l|}}{\partial \lambda^l} \mathcal{F}_{BC}(\mu)(-i\rho) = \int_{C_q^B} m_l^p(x) d\mu(x). \tag{58}$$

REMARK 5.13. *There are corresponding results to the Lemmas 5.11 and 5.12 for the A-case with the corresponding moment functions m_l^A for $l \in \mathbb{N}_0^q$ and the Fourier transform $\mathcal{F}_A$ and $\mu \in \mathcal{M}^1(C_q^A)$; see Lemmas 6.1, 6.2 in [**V2**].*

We now define a version of Gaussian measure in connection with the above hypergroup Fourier transform.

DEFINITION 5.14. Let $p \geq 2q-1$ and $t \geq 0$. A probability measure $\gamma_t = \gamma_t(p) \in \mathcal{M}^1(C_q^B)$ (if it exists) is called *BC(p)-Gaussian* with time parameter t and shape parameter p if

$$\mathcal{F}_{BC}^p(\gamma_t)(\lambda) = \exp\left(\frac{-t(\lambda_1^2 + \ldots + \lambda_q^2 + \|\rho\|_2^2)}{2}\right)$$

for all $\lambda \in C_q^B \cup i \cdot co(W_q^B \cdot \rho) \subset \mathbb{C}^q$.

The existence of the measures γ_t for $t > 0$ is not quite obvious at the beginning, but we shall see from the proof of the following CLT that γ_t indeed exists. We notice that by injectivity of the hypergroup Fourier transform, if the measures γ_t

exist, then they are determined uniquely. Using properties of hypergroup Fourier transform one can easily show that $(\gamma_t)_{t\geq 0}$ form a convolution semigroup , i.e. for all $s,t\geq 0$ we have $\gamma_s *_{p,q}\gamma_t=\gamma_{s+t}$ and $\gamma_0=\delta_0$. Moreover, the map $t\to\gamma_t$ is weakly continuous. Indeed, for a sequence $(t_n)_{n\geq 1}\subset\mathbb{R}_+$ with $\lim_{n\to} t_n=0$ we have $\lim_{n\to\infty}\mathcal{F}^p_{BC}(\gamma_{t_n})(\lambda)=1$ for all $\lambda\in C_q^B\cup i\cdot co(W_q^B\cdot\rho)$. Thus, by Lévy's continuity theorem it follows that $\lim_{t\downarrow 0}\gamma_t=\delta_0$. We denote the associated Lévy processes on the hypergroup $(C_q^B,*_{p,q})$ in the sense of Definition 3.2 by $(X_t^p)_{t\geq 0}$.

THEOREM 5.15. *Let $\mu\in\mathcal{M}^1(C_q^B)$ with $\nu\neq\delta_0$ and with finite second moments. Let*

$$t_0:=\frac{2}{qd}\int_{C_q^B}\|x\|_2^2 d\mu(x).$$

Then,

$$S_n^{(p,n^{-1/2})}\to X^p_{\frac{t_0}{(p+1)}}\ \textit{in distrubtion.}$$

For the proof we need some information on φ_λ^p:

LEMMA 5.16. *Let $p\in[2q-1,\infty[$ be fixed. Then:*

(1) *For all $i,j=1,2,...,q$ with $i\neq j$ and all $\lambda\in\mathbb{C}^q$,*

$$\frac{\partial}{\partial x_i}\varphi_\lambda^p(0)=0\ \textit{and}\ \frac{\partial^2}{\partial x_i\partial x_j}\varphi_\lambda^p(0)=0 \tag{59}$$

(2) *For all $i=1,2,...,q,$ and $\lambda\in C_q^B\cup i\cdot co(W_q\cdot\rho)$,*

$$\frac{\partial^2}{\partial x_i^2}\varphi_\lambda^p(0)=-\frac{(\lambda_1^2+...+\lambda_q^2+\|\rho\|_2^2)}{(p+1)qd}<0.$$

PROOF. The spherical functions $\varphi_\lambda^p(x)$ are invariant under the action of the Weyl group of of type BC w.r.t. x. Therefore, $\varphi_\lambda^p(x_1,..,x_q)$ is even in each x_i, which leads to (1). Moreover, as $\varphi_\lambda^p(x_1,....,x_q)$ is invariant under the permutations of x_i, $\frac{\partial^2}{\partial x_i^2}\varphi_\lambda^p(0)$ is independent of i. To complete the proof of (2), we recall from Corollary 4.13 and Eq. 40 that for all $\lambda\in\mathbb{C}^q$ the hypergeometric function $F_{BC}(\lambda,k_p,\cdot)$ is the unique solution to the eigenvalue problem

$$Lf=-(\lambda_1^2+...+\lambda_q^2+\|\rho\|_2^2)f \tag{60}$$

for $x \in int(C_q^B) = \{x \in C_q^B : x_1 > x_2 > ... > x_q > 0\}$ with $f(0) = 1$ where the differential operator L is defined as

$$(61)\quad L = \Delta f(x) + \sum_{\alpha \in R^+} m_\alpha \coth\langle \alpha, x\rangle \partial_\alpha f(x)$$
$$= \sum_{1 \le i \le q} \left[\frac{\partial_i^2}{\partial x_i^2} + (m_1 \coth(x_i) + 2m_2 \coth(2x_i)) \frac{\partial_i}{\partial x_i} \right]$$
$$+ m_3 \sum_{1 \le i < j \le q} \left[\coth(x_i + x_j) \left(\frac{\partial_i}{\partial x_i} + \frac{\partial_j}{\partial x_j} \right) + \coth(x_i - x_j) \left(\frac{\partial_i}{\partial x_i} - \frac{\partial_j}{\partial x_j} \right) \right]$$

where $(m_1, m_2, m_3) = (d(p-q)/2, (d-1)/2, d/2)$ as in (47). Notice here that the factor 2 of the multiplicity m_2 originates from the directional derivatives w.r.t the roots in Eq. (34).
Now, using part (1), $\varphi_\lambda^p(x) = F_{BC}(i\lambda, m_p, x)$, and the Taylor expansion of coth around 0, we have

$$\begin{aligned} -(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)\varphi_\lambda^p(0) &= \lim_{\|x\| \to 0} L\varphi_\lambda^p(x) \\ &= (q + qm_1 + 2qm_2 + q(q-1)m_3) \left. \frac{\partial_1^2}{\partial x_1^2} \varphi_\lambda^p(x) \right|_{x=0} \\ &= \frac{(p+1)qd}{2} \cdot \left. \frac{\partial_1^2}{\partial x_1^2} \varphi_\lambda^p(x) \right|_{x=0} \end{aligned}$$

for all $\lambda \in \mathbb{C}^q$. Finally, as $co(W_q^B \cdot \rho)$ is contained in $\{x \in \mathbb{R}^q : \|x\|_2 \le \|\rho\|_2\}$, the final statement of (2) is also clear. □

Proof of Theorem 5.15. Lemma 5.16 and $\varphi_\lambda^p(x) \le 1$ for $x \in C_q^B$ ensure that there exists $c > 0$ such that

$$1 - c(x_1^2 + x_2^2 + ... + x_q^2) \leqslant \varphi_\lambda^p(x) \text{ for all } x \in C_q^B.$$

Consequently by Taylor expansion,

$$n \left| \varphi_\lambda^p(\frac{x}{\sqrt{n}}) - 1 + \frac{\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2}{(p+1)qd} \cdot \frac{\|x\|_2^2}{n} \right| \le C\|x\|_2^2$$

for some constant $C > 0$ where $\|x\|_2^2$ is integrable w.r.t μ by our assumption. Thus, dominated convergence theorems yields that

$$\lim_{n\to\infty} n \int_{C_q^B} \left(\varphi_\lambda^p(\frac{x}{\sqrt{n}}) - 1 + \frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{(p+1)qd} \cdot \frac{\|x\|_2^2}{n} \right) d\mu(x) = 0.$$

Rewriting this relation as

$$\int_{C_q^B} \varphi_\lambda^p(\frac{x}{\sqrt{n}}) d\mu(x) = 1 - \frac{1}{n} \frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{(p+1)qd} \cdot \int_{C_q^B} \|x\|_2^2 d\mu(x) + o(\frac{1}{n})$$

we obtain

$$\mathcal{F}_{BC}^{p}(\mathbb{P}_{S_n^{(p,n^{-1/2})}})(\lambda) = \int_{C_q^B} \varphi_\lambda^p(\frac{x}{\sqrt{n}})d\mu^{(n)}(x) = \left[\int_{C_q^B} \varphi_\lambda^p(\frac{x}{\sqrt{n}})d\mu(x)\right]^n$$

$$= \left(1 - \frac{1}{n} \cdot \frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{(p+1)qd} \int_{C_q^B} \|x\|_2^2 d\mu(x) + o(\frac{1}{n})\right)^n$$

which implies

$$\lim_{n\to\infty} \mathcal{F}_{BC}^{p}(\mathbb{P}_{S_n^{(p,n^{-1/2})}})(\lambda) = \exp\left(-\frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{(p+1)qd} \cdot \int_{C_q^B} \|x\|_2^2 d\mu(x)\right)$$

$$= \exp\left(-\frac{t_0(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2(p+1)}\right)$$

for all $\lambda \in \mathbb{R}^q \cup i \cdot co(W_q^B \cdot \rho)$. Hence, by Theorem 2.12(iii) there exists a bounded positive measure in $\mu \in \mathcal{M}_b^+(C_q^B)$ with

$$\mathcal{F}_{BC}^{p}(\mu)(\lambda) = \exp\left(-\frac{t_0(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2(p+1)}\right) \tag{62}$$

for all $\lambda \in \mathbb{R}^q$, and $(\mathbb{P}_{S_n^{n^{-1/2}}})_{n\geq 1}$ converges to μ weakly.
Moreover, since we have $\mathcal{F}_{BC}^{p}(\mu)(-i\rho) = 1$, the limiting positive measure μ is indeed a probability measure. This implies that $(\mathbb{P}_{S_n^{(p,n^{-1/2})}})_{n\geq 1}$ converges weakly to $\mu = \gamma_{\frac{t}{(p+1)}}$ as desired. □

REMARK 5.17. *The considerations in the above proof imply that the probability measures* γ_t *in Definition* 5.14 *above indeed exist.*

CHAPTER 6

Central limit theorems for growing parameters

1. Limit thereoms with for growing parameters with outer normalization

In this section we derive two CLTs for random walks when the time and the dimension parameter p tend to infinity. Unlike the case of fixed parameters p for growing parameters it is not possible to obtain limit theorems without having restriction either on the moment conditions or on the growth rate of p. In this section we present limit theorem with varying moment conditions and growth rate condition for parameter p.
In the first case we show a CLT and a weak LLN results with second moment conditions for the associated measure μ, as in Chapter 5, but with restriction on the growth rate for p_n coupled with n i.e., we show that as $p_n, n \to \infty$ coupled, the sequence of random variables

$$\frac{1}{\sqrt{n}}(\tilde{S}_n^{p_n} - n \cdot m_{\mathbf{1}}^{p_n}(\mu))$$

converge to some normal distribution $N(0, \tilde{\Sigma}(\mu))$, and

$$\frac{\tilde{S}_n^{p_n} - n \cdot m_{\mathbf{1}}^{p_n}(\mu)}{n} \longrightarrow 0 \text{ in probability,}$$

for the drift vector $m_{\mathbf{1}}^{p}(\mu)$ as in Chapter 5 depending on p and with the some covariance matrix $\tilde{\Sigma}(\mu)$.
In the second case we show a CLT and a weak LLN results without restriction on the growth rate for p with but higher (fourth) moment conditions for the associated measure μ, i.e we show that as $p_n, n \to \infty$ the distributions of the random variable

$$\frac{1}{\sqrt{n}}(\tilde{S}_n^{p_n} - n \cdot \tilde{m}_{\mathbf{1}}(\mu))$$

converge to some normal distribution $\mathcal{N}(0, \tilde{\Sigma}(\mu))$, and also

$$\frac{\tilde{S}_n^{p_n}}{n} \longrightarrow \tilde{m}_{\mathbf{1}}(\mu) \text{ in probability,}$$

for some drift vector $\tilde{m}_{\mathbf{1}}(\mu)$ and with the same covariance matrix $\tilde{\Sigma}(\mu)$ as above.

We now give the precise formula for the drift vector $\tilde{m}_{\mathbf{1}}(\mu)$ and covariance matrix $\tilde{\Sigma}(\mu)$. For this we consider the transformation

(63)
$$T : C_q^B \to C_q^B \subset C_q^A, \quad x = (x_1, ..., x_q) \mapsto \ln\cosh x := (\ln\cosh x_1, ..., \ln\cosh x_q).$$

We then define the modified moment functions $\tilde{m}_l(x) := m_l^A(T(x))$ which admit modified integral representations similar to (52). Moreover, for $\mu \in \mathcal{M}^1(C_q^B)$ we consider the image measure $T(\mu) \in \mathcal{M}^1(C_q^B) \subset \mathcal{M}^1(C_q^A)$. As $|x - \ln\cosh x| \leq \ln 2$ for all $x \in [0, \infty[$ by an elementary calculation, we see that for all multiindices l, the l-th moment of type A of μ exists if and only if the l-th modified of $T(\mu)$ exists. We put $\tilde{m}_l(\mu) := m_l^A(T(\mu))$ and $\tilde{\Sigma}(\mu) := \Sigma^A(T(\mu))$.

We now show that for a given $\mu \in \mathcal{M}^1(C_q^q)$ the existence of moments of some maximal order is independent from taking classical moments,modified moments, or moments of type BC. For our purpose it will be sufficient to restrict to the case when $|l|$ is even.
Let $k \in \mathbb{N}_0$ and $\mu \in \mathcal{M}^1(C_q^B)$. It is easy to see that μ admits finite modified moments of order at most $2k$ if

$$\tilde{m}_{2k\cdot e_1}, ..., \tilde{m}_{2k\cdot e_q} \in L^1(C_q^B, \mu).$$

Indeed, it follows immediately from the definition of moment functions in (52) and Hölder's inequality, that in this case all moments of order at most $2k$ are μ-integrable. Similarly, if

$$m^p_{2k\cdot e_1}, ..., m^p_{2k\cdot e_q} \in L^1(C_q^B, \mu)$$

then μ admits finite BC(p)-type moments of order at most $2k$.

1.1. Rate of convergence for the moment functions m^p for $p \to \infty$. We next derive the estimates for $|\tilde{m}_l(\mu) - m_l^p(\mu)|$ for $l \in \mathbb{N}_0^q$ and large p under the assumption that these moments exist.

PROPOSITION 6.1. *For $k \in \mathbb{N}$ and $\mu \in \mathcal{M}^1(C_q^B)$ the following statements are equivalent:*

(1) *μ admits all classical moments of order at most $2k$, i.e. $\int_{C_q^B} x_1^{l_1} \cdots x_q^{l_q} d\mu(x) < \infty$ for all $l = (l_1, ..., l_q) \in \mathbb{N}_0^q$ with $|l| \leq 2k$.*
(2) *μ admits all moments of type A of order at most $2k$.*
(3) *$T(\mu)$ admits all moments of type A of order at most $2k$.*
(4) *For each $p \geq 2q-1$, μ admits all moments of type BC(p) of order at most $2k$.*

We first recapitulate the following facts; see Lemmas 4.10 and 4.8 of [**RV1**]:

Lemma 6.2. (1) *Let* $d\mathbf{m}_p(w)$ *be the probability measures defined in (20). Then for each* $n \in \mathbb{N}$ *there exists a constant* $C := C(q, n, \mathbb{F})$ *such that all* $p \geqslant 2q$,

$$\int_{B_q} \frac{\sigma_1(w)^{2n}}{\Delta(I - w^*w)^{2n}} d\mathbf{m}_p(w) \leq \frac{C}{p^n}. \tag{64}$$

(2) *Let* $x \in C_q^B, w \in B_q, u \in U(q, \mathbb{F})$ *and* $r = 1, ..., q$. *Then*

$$\frac{\Delta_r(g(x,u,w))}{\Delta_r(g(x,u,0))} \in [(1 - \tilde{x}\sigma_1(w))^{2r}, (1 + \tilde{x}\sigma_1(w))^{2r}] \quad \textit{with} \quad \tilde{x} := \min(x_1, 1).$$

Proof Proposition 6.1. To show (1)⇒(2) it is sufficient to prove that $m^A_{2k\cdot e_1}, ..., m^A_{2k\cdot e_q} \in L^1(C_q^B, \mu)$. From (52) we have

$$m^A_{2k\cdot e_j}(\mu) = \frac{1}{2^{2k}} \int_{C_q^B} \int_{U(q,\mathbb{F})} \left(\ln \Delta_{j+1}(u^* e^{2\underline{x}} u) - \ln \Delta_j(u^* e^{2\underline{x}} u)\right)^{2k} du\, d\mu(x).$$

We now recall from Lemma 4.2 [**V2**] that $jx_q \leq \ln \Delta_j(u^* e^{2\underline{x}} u) \leq jx_1$ for $u \in U(q, \mathbb{F}), x \in C_q^B$, and $j = 1, ..., q$. Therefore, from elementary inequalities we obtain that

$$m^A_{2k\cdot e_j}(\mu) \leq \frac{1}{2^{2k}} \int_{C_q^B} |(j(x_1 - x_q) + x_q|^{2k} \mathrm{d}\mu(x) < \infty. \tag{65}$$

To prove (2)⇒(1) it is sufficient to show that $\int_{C_q^B} x_1^{2k}\, d\mu(x) < \infty$. It can be easily seen that for every $u \in U(q, \mathbb{F})$ there exist coefficients $c_i(u) \geq 0$ for $i = 1, ...q$ with $\sum_{i=1}^q c_i(u) = 1$ such that

$$\Delta_1(u^* e^{2\underline{x}} u) = \sum_{i=1}^q c_i(u) e^{2x_i} \geq c_1(u) e^{2x_1}.$$

Thus, using the elementary inequality $2^{2k}(a^{2k} + b^{2k}) \geq (a+b)^{2k}$ for $a = \ln(c_1(u)e^{2x_1})$ and $b = -\ln c_1(u)$ we have

$$\int_{U(q,\mathbb{F})} \int_{C_q^B} (\ln \Delta_1(u^* e^{2\underline{x}} u))^{2k}\, du\, d\mu(x) \geq \int_{U(q,\mathbb{F})} \int_{C_q^B} (\ln(c_1(u) e^{2x_1}))^{2k}\, du\, d\mu(x)$$
$$\geq - \int_{U(q,\mathbb{F})} (|\ln c_1(u)|)^{2k}\, du + \int_{C_q^B} x_1^{2k}\, d\mu(x).$$

Now, Lemma 5.1 and Proposition 4.9 of [**V2**] ensure that $\int_{U(q,\mathbb{F})} (|\ln c_1(u)|)^{2k}\, du$ is finite. Hence we have $\int_{C_q^B} x_1^{2k}\, d\mu(x) < \infty$ as desired.

The equivalence of (2) and (3) follows from

$$\frac{1}{4} u^* e^{2\underline{x}} u \leq u^* (\cosh \underline{x})^2 u \leq \frac{1}{2} u^* e^{2\underline{x}} u$$

which implies that

$$|\ln\Delta_j(u^*(\cosh\underline{x})^2u)-\ln\Delta_j(u^*e^{2\underline{x}}u)|\leq\ln 4.$$

To prove (3)⇒ (4) we recall from Lemma 6.4 in [**V2**] that

$$\begin{aligned}|\ln\Delta_j g(x,u,w)-\ln\Delta_j(u^*(\cosh\underline{x})u)| &\leq 2j\cdot\max(|\ln(1-\sigma_1(w))|,\ln(\sigma_1(w)+1))\\ &:= H_j(w). \qquad (66)\end{aligned}$$

It can be easily seen that $\int_{B_q}\ln(1+\sigma_1(w))^{2k}\mathrm{d}m_p(w)$ is finite.

Moreover, as $1\geq\sigma_1(w)\geq ....\geq\sigma_q(w)\geq 0$ for $w\in B_q$ we have

$$\frac{1}{1-\sigma_1(w)}\leq\frac{2}{1-\sigma_1(w)^2}\leq 2\prod_{r=1}^{q}\frac{1}{1-\sigma_r(w)^2}\leq\frac{2}{\Delta(I-w^*w)}. \qquad (67)$$

Now, from Lemma 6.2 and (67) together with the elementary inequality

$$|\ln(1+z)|\leqslant\frac{|z|}{1-|z|}\text{ for }|z|<1 \qquad (68)$$

we obtain that

$$\int_{B_q}|\ln(1-\sigma_1(w))|^{2k}\mathrm{d}\mathbf{m}_p(w)\leq 2^{2k}\int_{B_q}\sigma_1(w)^{2k}\cdot\Delta(I-w^*w)^{-2k}\mathrm{d}\mathbf{m}_p(w)<\infty. \qquad (69)$$

Hence, $\int_{B_q}|H_j(q)|^{2k}d\mathbf{m}_p(w)<\infty$ for $j=1,..,q$. Therefore, using the elementary inequality

$$3^{2k}(a^{2k}+b^{2k}+c^{2k})\geq(a+b+c)^{2k}$$

we have

(70)

$$\begin{aligned}m^p_{2k\cdot e_j}(\mu)\leq\left(\frac{3}{2}\right)^{2k}\int_{B_q\times U(q,\mathbb{F})\times C^B_q}&\Big(|\ln\Delta_{j+1}g(x,u,w)-\ln\Delta_{j+1}(u^*(\cosh\underline{x})u)|^{2k}+\\ &+|\ln\Delta_{j+1}(u^*(\cosh\underline{x})u)-\ln\Delta_j(u^*(\cosh\underline{x})u)|^{2k}+\\ &+|\ln\Delta_j g(x,u,w)-\ln\Delta_j(u^*(\cosh\underline{x})u)|^{2k}\Big)d\mathbf{m}_p(w)\,\mathrm{d}u\,\mathrm{d}\mu(x).\end{aligned}$$

We see that the right hand side of (70) is finite, from (66), (69) and the assumption that $m^A_{2k\cdot e_j}(\mu)$ is finite.

Finally, the converse statement (4)⇒(3) follows analogously from

(71)

$$\begin{aligned}m^A_{2k\cdot e_j}(\mu)\leq\left(\frac{3}{2}\right)^{2k}\int_{B_q\times U(q,\mathbb{F})\times C^B_q}&[|\ln\Delta_{j+1}(u^*(\cosh\underline{x})u)-\ln\Delta_{j+1}g(x,u,w)|^{2k}\\ &+|\ln\Delta_{j+1}g(x,u,w)-\ln\Delta_j g(x,u,w)|^{2k}\\ &+|\ln\Delta_j(u^*(\cosh\underline{x})u)-\ln\Delta_j g(x,u,w)|^{2k}]d\mathbf{m}_p(w)\mathrm{d}u\mathrm{d}\mu(x).\end{aligned}$$

$\square$

PROPOSITION 6.3. *Let $l=(l_1,....,l_q)\in\mathbb{N}_0^q$ with $|l|\geq 3$ and $\mu\in\mathcal{M}(C_q^B)$. Assume that μ admits finite moments of order $4(|l|-2)$. Then, there exists a constant $C:=C(|l|,q,\mu)$ such that*

$$|\tilde{m}_l(\mu)-m_l^p(\mu)|\leqslant\frac{C}{\sqrt{p}}. \tag{72}$$

PROOF. We consider the $|l|$ factors of the integrand in the integral representations (55) of the moment functions m_l^p and the modified version of (52) for $\tilde{m}_l$. For $i=1,2,...,|l|$ these factors have the form:

$$\begin{aligned}f_i(x,u,w)&:=\ln\Delta_r(g(x,u,w))-\ln\Delta_{r-1}(g(x,u,w)),\\ \tilde{f}_i(x,u,w)&:=\ln\Delta_r(g(x,u,0))-\ln\Delta_{r-1}(g(x,u,0))\end{aligned}$$

with the convention $\Delta_0\equiv 1$ where $r\in\{1,...,q\}$ is the smallest integer with $i\leq l_1+...+l_r$.
Then, from Lemma 6.2(2) and (68) for all $i=1,...,|l|, x\in C_q^B, u\in U(q,\mathbb{F}), w\in B_q$ we obtain that

$$\begin{aligned}|f_i(x,u,w)-\tilde{f}_i(x,u,w)|&\leq 2\max_{r=1,\dots,q}|\ln\Delta_r(g(x,u,w))-\ln\Delta_r(g(x,u,0))|\\ &\leqslant 4q\cdot\frac{\tilde{x}\sigma_1(w)}{1-\tilde{x}\sigma_1(w)}\leqslant 4q\tilde{x}\frac{\sigma_1(w)}{1-\sigma_1(w)}\end{aligned}$$

where $\tilde{x}=\min\{1,x\}$. Thus, by (67) we have

$$|f_i(x,u,w)-\tilde{f}_i(x,u,w)|\leq 8q\tilde{x}\frac{\sigma_1(w)}{\Delta(I-w^*w)}.$$

Now, notice that

$$\begin{aligned}&|\tilde{m}_l(\mu)-m_l^p(\mu)| \\ &\quad=\left|\frac{1}{2^{|l|}}\int_{B_q\times U(q,\mathbb{F})\times C_q^B}\left(\prod_{i=1}^{|l|}f_i(x,u,w)-\prod_{i=1}^{|l|}\tilde{f}_i(x,u,w)\right)dud\mathbf{m}_p(w)d\mu(x)\right|\end{aligned} \tag{73}$$

Therefore, by a telescopic sum,

$$
\begin{aligned}
|\tilde{m}_l(\mu) - m_l^p(\mu)| &= \\
&= \Big| \frac{1}{2^{|l|}} \sum_{i=1}^{|l|} \int_{B_q \times U(q,\mathbb{F}) \times C_q^B} \Big((f_i(x,u,w) - \tilde{f}_i(x,u,w)) \times \\
&\qquad \prod_{j=i+1}^{|l|} f_j(x,u,w) \prod_{k=1}^{i} \tilde{f}_k(x,u,w) \Big) du d\mathbf{m}_p(w) d\mu(x) \Big| \\
&\le \frac{1}{2^{|l|}} \sum_{i=1}^{|l|} \int_{B_q \times U(q,\mathbb{F}) \times C_q^B} \Big| (f_i(x,u,w) - \tilde{f}_i(x,u,w)) \times \\
&\qquad \prod_{j=i+1}^{|l|} f_j(x,u,w) \prod_{k=1}^{i} \tilde{f}_k(x,u,w) \Big| du d\mathbf{m}_p(w) d\mu(x)
\end{aligned} \tag{74}
$$

We estimate the summands of the expression of the last formula of (74) in two ways:
Summands for $i = 1$ and $|l|$:
From Cauchy-Schwarz inequality, (74) and Lemma 6.2 we obtain that

$$
\begin{aligned}
&\int_{B_q \times U(q,\mathbb{F}) \times C_q^B} \left| (f_1(x,u,w) - \tilde{f}_1(x,u,w)) \prod_{j=2}^{|l|} f_j(x,u,w) \right| du d\mathbf{m}_p(w) d\mu(x) \\
&\le \left(\int_{B_q \times U(q,\mathbb{F}) \times C_q^B} |f_i(x,u,w) - \tilde{f}_i(x,u,w)|^2 du d\mathbf{m}_p(w) d\mu(x) \right)^{1/2} \times \\
&\quad \times \left(\int_{B_q \times U_0(q,\mathbb{F}) \times C_q^B} \prod_{j=2}^{|l|} f_j(x,u,w)^2 du d\mathbf{m}_p(w) d\mu(x) \right)^{1/2} \\
&\le M_1 \cdot 8q \left(\int_{B_q} \frac{\sigma_1(w)^2}{\Delta(I - w^* w)^2} d\mathbf{m}_p(w) \right)^{1/2} \\
&\le M_1 \cdot \frac{C}{\sqrt{p}}
\end{aligned} \tag{75}
$$

where

$$M_1 := M_1(\mu, |l|, q) = 8q \cdot \max_{r \in \mathbb{N}_0^q, |r| \le 2(|l|-1)} \max\{\tilde{m}_r(\mu), m_r^p(\mu)\}$$

which is finite by initial assumption and Proposition 6.1. Similarly, we obtain same upper bound for the $|l|$th summand in (74).
Now, let $i = 2, ..., q-1$. Here, we apply Hölder's inequality twice and obtain with

the same arguments as above that

$$
\begin{aligned}
(76) \quad & \left| \int_{B_q \times U_0(q,\mathbb{F}) \times C_q^B} \Big((f_i(x,u,w) - \tilde{f}_i(x,u,w)) \right. \\
& \qquad \left. \times \prod_{j=i+1}^{|l|} f_j(x,u,w) \prod_{k=1}^{i-1} \tilde{f}_k(x,u,w) \Big) \, du d\mathbf{m}_p(w) d\mu(x) \right| \\
& \leq \left(\int_{B_q \times U_0(q,\mathbb{F}) \times C_q^B} |(f_i(x,u,w) - \tilde{f}_i(x,u,w)|^2 du d\mathbf{m}_p(w) d\mu(x) \right)^{1/2} \\
& \qquad \times \left(\int_{B_q \times U_0(q,\mathbb{F}) \times C_q^B} \prod_{j=i+1}^{|l|} |f_j(x,u,w)|^4 du d\mathbf{m}_p(w) d\mu(x) \right)^{1/4} \\
& \qquad \times \left(\int_{B_q \times U_0(q,\mathbb{F})} \prod_{k=1}^{i-1} |\tilde{f}_k(x,u,w)|^4 du d\mathbf{m}_p(w) d\mu(x) \right)^{1/4} \\
& \leq M_2 \cdot \frac{C}{\sqrt{p}}
\end{aligned}
$$

where

$$M_2 := M_2(\mu, |l|, q) = 8q \cdot \max_{r \in \mathbb{N}_0^q, |r| \leq 4(|l|-2)} \max\{\tilde{m}_r(\mu), m_r^p(\mu)\}$$

which is again finite by our assumption and Proposition 6.1. Thus, the estimates (75) and (76) give the desired assertion. □

We are now ready to present the limit theorems. :

1.2. Limit theorems for $p \to \infty$.

THEOREM 6.4. *Let $(p_n)_{n \geq 1} \subset (2q-1, \infty)$ be an increasing sequence with $\lim_{n\to\infty} n/p_n = 0$. Let $\mu \in \mathcal{M}^1(C_q^B)$ be with $\mu \neq \delta_0$ and second moments. Consider the associated random walks $(\tilde{S}_n^p)_{n \geqslant 0}$ on C_q^B for $p > 2q-1$. Then*

$$\frac{\tilde{S}_n^{p_n} - n \cdot \tilde{m}_{\mathbf{1}}(\mu)}{\sqrt{n}}$$

converges in distribution to $\mathcal{N}(0, \tilde{\Sigma}(\mu))$.

PROOF OF THEOREM 6.4. We know from Theorem 4.21 that there exists a constant $C > 0$ such that for all $p > 2q-1, x \in C_q^B, \lambda \in \mathbb{R}^q$,

$$|\varphi_{\lambda - i\rho}^p(x) - \varphi_{\lambda - i\rho^A}^A(\ln \cosh x)| \leqslant C \cdot \frac{\|\lambda\|_1 \cdot \tilde{x}}{p^{1/2}}$$

where $\|\lambda\|_1 := |\lambda_1| + \ldots\ |\lambda_q|$ and $\tilde{x} := min(x_1, 1) \geqslant 0$. Hence, denoting the half sums of positive roots of type BC associated with p_n by $\rho(n) := \rho^{BC}(p_n)$, for all $\nu \in \mathcal{M}^1(C_q^B)$, we get

$$\left| \int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)}(x) d\mu(x) - \int_{C_q^B} \varphi^A_{\lambda - i\rho^A}(\ln \cosh x) d\mu(x) \right| \leqslant C \cdot \frac{\|\lambda\|_1}{\sqrt{p_n}}. \tag{77}$$

Let $\mu^{(n,p)} \in \mathcal{M}^1(C_q^B)$ be the law of $\tilde{S}_n^p$. Then, $T(\tilde{S}_n^{p_n})$ has the distribution $T(\mu^{(n,p_n)})$ whose A-type spherical Fourier transform satisfies

$$\mathcal{F}_A(T(\mu^{(n,p_n)}))(\lambda - i\rho^A) = \int_{C_q^A} \varphi^A_{\lambda - i\rho^A}(x) dT(\mu^{(n,p_n)})(x) \tag{78}$$

$$= \int_{C_q^B} \varphi^A_{\lambda - i\rho^A}(\ln \cosh x) d\mu^{(n,p_n)}(x) \tag{79}$$

for $\lambda \in \mathbb{R}^q$. Therefore, by plugging $\mu^{(n,p_n)}$ into (77) we get

$$\begin{aligned}
\mathcal{F}_A(T(\mu^{(n,p_n)}))(\lambda - i\rho^A) &= \int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)} d\mu^{(n,p_n)}(x) + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \\
&= \mathcal{F}^{p_n}_{BC}(\mu^{(n,p_n)})(\lambda - \rho(n)) + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \\
&= (\mathcal{F}^{p_n}_{BC}(\mu)(\lambda - \rho(n)))^n + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \\
&= \left(\int_{C_q^B} \varphi^A_{\lambda - i\rho^A}(\ln \cosh x) d\mu(x) \right)^n + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \\
&= \left(\mathcal{F}_A(T(\mu))(\lambda - i\rho^A) + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \right)^n + O(\frac{\|\lambda\|_1}{p_n^{1/2}}).
\end{aligned} \tag{80}$$

Using the the initial moment assumption and Lemma 6.1 we see that the first and second modified moments $\tilde{m}_{\mathbf{1}}$ and $\tilde{m}_{\mathbf{2}}$ exist. Moreover, all entries of the modified covariance matrix

$$\tilde{\Sigma}(\mu) = \tilde{m}_{\mathbf{2}}(\mu) - \tilde{m}_{\mathbf{1}}(\mu)^t \cdot \tilde{m}_{\mathbf{1}}(\mu)$$

are finite.
By Lemma 5.12, the Taylor expansion of $\mathcal{F}_A(T(\mu))(\lambda - i\rho^A)$ for $|\lambda| \to 0$ is given by

$$\mathcal{F}_A(T(\mu))(\lambda - i\rho^A) = 1 - i\langle \lambda, \tilde{m}_{\mathbf{1}}(\mu) \rangle - \lambda \tilde{m}_{\mathbf{2}}(\mu) \lambda^t + o(|\lambda|^2). \tag{81}$$

Using the initial assumption that $O(1/\sqrt{np_n}) = o(1/n)$ we obtain

$$\begin{aligned}
&E(\varphi^A_{\frac{\lambda}{\sqrt{n}}-i\rho^A}(T(\tilde{S}_n^{p_n}))e^{i\langle\lambda,\sqrt{n}\tilde{m}_{\mathbf{1}}(\mu)\rangle} \\
&\qquad = \mathcal{F}_A(T(\mu^{(n,p_n)}))(\lambda/\sqrt{n} - i\rho^A)\cdot e^{i\langle\lambda,\sqrt{n}\tilde{m}_{\mathbf{1}}(\nu)\rangle} \\
&\qquad = \left[\left(\mathcal{F}_A(T(\mu))(\frac{\lambda}{\sqrt{n}} - i\rho^A) + O(\frac{\|\lambda\|_1}{\sqrt{np_n}})\right)^n + O(\frac{\|\lambda\|_1}{\sqrt{np_n}})\right]\cdot e^{i\langle\lambda,\frac{\tilde{m}_{\mathbf{1}}(\mu)}{\sqrt{n}}\rangle n} \\
&\qquad = \left[\left(1 - \frac{i\langle\lambda,\tilde{m}_{\mathbf{1}}(\mu)\rangle}{\sqrt{n}} - \frac{\lambda\tilde{m}_{\mathbf{2}}(\mu)\lambda^t}{2n} + o(\frac{1}{n})\right)\times\right. \\
&\qquad\qquad\left.\times\left(1 + \frac{i\langle\lambda,\tilde{m}_{\mathbf{1}}(\mu)\rangle}{\sqrt{n}} - \frac{\langle\lambda,\tilde{m}_{\mathbf{1}}(\mu)\rangle^2}{2n} + o(\frac{1}{n})\right)\right]^n \\
&\qquad = \left(1 - \frac{\lambda\tilde{\Sigma}(\mu)\lambda^t}{2n} + o(\frac{1}{n})\right)^n.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} E(\varphi^A_{\lambda/\sqrt{n}-i\rho^A}(T(\tilde{S}_n^{p_n}))\cdot\exp(i\langle\lambda,\tilde{m}_{\mathbf{1}}(\nu)\rangle\sqrt{n})) = \exp(-\lambda\tilde{\Sigma}(\mu)\lambda^t/2). \tag{82}$$

On the other hand, from Lemma 5.2(5) we have

$$\lim_{n\to\infty} E(\varphi^A_{\lambda/\sqrt{n}-i\rho^A}(T(\tilde{S}_n^{p_n})) - \exp(-i\langle\lambda,\tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n})\rangle/\sqrt{n}) = 0. \tag{83}$$

Eq. (82) and (83) and the fact that $|e^{i\langle\lambda,\sqrt{n}\tilde{m}_{\mathbf{1}}(\mu)\rangle}| \leqslant 1$ together yield that for all $\lambda \in \mathbb{R}^q$,

$$\lim_{n\to\infty} \exp(-i\langle\lambda,\tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\mu)\rangle/\sqrt{n}) = \exp(-\lambda\tilde{\Sigma}(\mu)\lambda^t/2).$$

Lévy's continuity theorem for the classical q-dimensional Fourier transform implies that

$$(\tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\mu))/\sqrt{n}$$

tends to the normal distribution $\mathcal{N}(0,\tilde{\Sigma}(\mu))$.
Now, Lemma 5.2(2) implies that $(T(\tilde{S}_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\nu)\rangle)/\sqrt{n}$ also converges to $\mathcal{N}(0,\tilde{\Sigma}(\nu))$.
Finally, with the same argument as in the proof of Theorem 6.4 above we get that $(\tilde{S}_n^{p_n} - n\tilde{m}_{\mathbf{1}}(\mu))/\sqrt{n} \to N(0,\tilde{\Sigma}(\mu))$ in distribution, as desired. □

For the weak LLN result the assumption of existence of first moments of $\mu \in \mathcal{M}^1(C_q^B)$ is sufficient.

THEOREM 6.5. *Let $(p_n)_{n\geq 1} \subset (2q-1,\infty)$ be an increasing sequence with $\lim_{n\to\infty} n/p_n = 0$. Let $\mu \in \mathcal{M}^1(C_q^B)$ be with $\mu \neq \delta_0$ and first moments. Consider the associated random walks $(\tilde{S}_n^p)_{n\geqslant 0}$ on C_q^B for $p > 2q-1$ and let $\varepsilon > \frac{1}{2}$. Then*

$$\frac{1}{n^\varepsilon}(\tilde{S}_n^{p_n} - n\cdot\tilde{m}_{\mathbf{1}}(\mu)) \longrightarrow 0 \textit{ in probability.}$$

This means in particular

$$\frac{\tilde{S}_n^{p_n}}{n} \longrightarrow \tilde{m}_{\mathbf{1}}(\mu) \text{ in probability.}$$

PROOF OF THEOREM 6.5. From Eq. (80), (81) and the initial assumption $O(1/\sqrt{np_n}) = o(1/n)$ it follows that

$$\begin{aligned}
&E(\varphi^A_{\frac{\lambda}{n^\varepsilon}-i\rho^A}(T(\tilde{S}_n^{p_n}))e^{i\langle\lambda, n^{1-\varepsilon}\tilde{m}_{\mathbf{1}}(\mu)\rangle} \\
&\quad = \mathcal{F}_A(T(\mu^{(n,p_n)}))(\lambda/n^\varepsilon - i\rho^A)\cdot e^{i\langle\lambda, n^{1-\varepsilon}\tilde{m}_{\mathbf{1}}(\nu)\rangle} \\
&\quad = \left[\left(\mathcal{F}_A(T(\mu))(\frac{\lambda}{n^\varepsilon} - i\rho^A) + O(\frac{\|\lambda\|_1}{n^\varepsilon\sqrt{p_n}})\right)^n + O(\frac{\|\lambda\|_1}{\sqrt{np_n}})\right]\cdot e^{i\langle\lambda, \frac{\tilde{m}_{\mathbf{1}}(\nu)}{n^\varepsilon}\rangle n} \\
&\quad = \left[\left(1 - \frac{i\langle\lambda, \tilde{m}_{\mathbf{1}}(\mu)\rangle}{n^\varepsilon} + O(\frac{1}{n^{\varepsilon+1/2}})\right)\left(1 + \frac{i\langle\lambda, \tilde{m}_{\mathbf{1}}(\mu)\rangle}{n^\varepsilon} + O(\frac{1}{n^{2\epsilon}})\right)\right]^n \\
&\qquad = \left(1 + o(\frac{\|\lambda\|^2}{n})\right)^n.
\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} E(\varphi^A_{\frac{\lambda}{n^\varepsilon}-i\rho^A}(T(\tilde{S}_n^{p_n}))e^{i\langle\lambda, n^{1-\varepsilon}\tilde{m}_{\mathbf{1}}(\mu)\rangle} = 1. \tag{84}$$

On the other hand, from Lemma 5.2(5) we have

$$\lim_{n\to\infty} E(\varphi^A_{\lambda/n^\varepsilon-i\rho^A}(T(\tilde{S}_n^{p_n})) - \exp(-i\langle\lambda, \tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n})\rangle/n^\varepsilon)) = 0. \tag{85}$$

Eq. (84) and (85) and the fact that $|e^{i\langle\lambda,\sqrt{n}\tilde{m}_{\mathbf{1}}(\mu)\rangle}| \leqslant 1$ together yield that for all $\lambda \in \mathbb{R}^q$,

$$\lim_{n\to\infty} \exp(-i\langle\lambda, (\tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\mu)\rangle)/n^\varepsilon) = 1.$$

Lévy's continuity theorem for the classical q-dimensional Fourier transform implies that

$$(\tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\mu))/n^\varepsilon \longrightarrow 0 \text{ in distribution.}$$

Now, Lemma 5.2(2) implies that

$$(T(\tilde{S}_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\nu)\rangle)/n^\epsilon$$

also converges to 0.
Finally, with the same argument as in the proof of Theorem 6.4 above we get

$$(\tilde{S}_n^{p_n} - n\tilde{m}_{\mathbf{1}}(\mu))/n^\varepsilon \to 0 \text{ in distribution.}$$

This implies also convergence in probability since the limit is constant.

□

Remark 6.6. *For the rank one case ($q = 1$) the preceding CLT was derived in [**Gr1**] with different techniques under weaker assumptions, namely without the restriction $n/p_n \to 0$ as $n \to \infty$. The proof in [**Gr1**] relies on the convergence of the moment functions*

$$(m_1^p(x))^2 - m_2^p(x) \to 0 \tag{86}$$

on $[0,\infty)$ for $p \to \infty$. However, for $q \geq 2$ this convergence is no longer available.

We next try to get rid of the restriction $n/p_n \to 0$. We shall achieve this by assuming the existence of fourth moments in addition.

Theorem 6.7. *Let $(p_n)_{n\geq 1} \subset (2q-1,\infty)$ be an increasing sequence with $\lim_{n\to\infty} p_n = \infty$. Let $\mu \in \mathcal{M}^1(C_q^B)$ with $\mu \neq \delta_0$ and with fourth moments. Consider the associated random walks $(\tilde S_n^p)_{n\geqslant 0}$ on C_q^B for $p \geq 2q-1$. Then*

$$\frac{\tilde S_n^{p_n} - n\cdot m_{\mathbf{1}}^{p_n}(\mu)}{\sqrt{n}})$$

converges in distribution to $\mathcal{N}(0,\tilde\Sigma(\mu))$.

Proof of Theorem 6.7. We first notice that by Taylor's theorem and Proposition 6.3 for all $p > 2q-1$,

(87)

$$\Big|E(\varphi^p_{\lambda/\sqrt{n}-i\rho}(\tilde S_n^p)) - \Big(1 - \frac{i\langle\lambda, m_{\mathbf{1}}^p(\mu)\rangle}{\sqrt{n}} - \frac{\lambda m_{\mathbf{2}}^p(\mu)\lambda^t}{2n}\Big)\Big|$$

$$\leq \sum_{l\in\mathbb{N}^q,|l|=3} m_l^p(\mu)\frac{\lambda_1^{l_1}\ldots\lambda_q^{l_q}}{l_1!\ldots l_q!}$$

$$\leq \frac{1}{n^{3/2}}\sum_{l\in\mathbb{N}^q,|l|=3}(\tilde m_l(\mu)+C/\sqrt{p})\frac{\lambda_1^{l_1}\ldots\lambda_q^{l_q}}{l_1!\ldots l_q!}$$

$$\leq K_1\frac{\|\lambda\|_\infty^3}{n^{3/2}} \tag{88}$$

for some constant $K_1 > 0$ which is independent of p. Analogously, for all $p \geqslant 2q-1$,

$$\Big|e^{i\langle\lambda,\sqrt{n}m_{\mathbf{1}}^p(\mu)\rangle} - \Big(1 + \frac{i\langle\lambda, m_{\mathbf{1}}^p(\mu)\rangle}{\sqrt{n}} - \frac{\langle\lambda, m_{\mathbf{1}}^p(\mu)\rangle^2}{2n}\Big)\Big| \leqslant K_2\frac{\|\lambda\|_\infty^3}{n^{3/2}} \tag{89}$$

for some $K_2 > 0$ independent of p.
Using estimates (87) and (89) we now follow similar paths as in the proof of Theorem 6.4. We however use the BC-type Fourier transform and BC-moments instead of objects of type A, and then approximate A-type moments by BC-type

moments using Proposition 6.3. Now, we have

$$
\begin{aligned}
E(\varphi^{p_n}_{\lambda/\sqrt{n}-i\rho(n)}(\tilde{S}^{p_n}_n))e^{i\langle\lambda,\sqrt{n}m^{p_n}_{\mathbf{1}}(\mu)\rangle} &= \\
&= \mathcal{F}^{p_n}_{BC}(\mu^{(n,p_n)})(\lambda/\sqrt{n}-i\rho(n))\cdot e^{i\langle\lambda,\sqrt{n}m^{p_n}_{\mathbf{1}}(\nu)\rangle} \\
&= \Bigg[\left(1-\frac{i\langle\lambda,m^{p_n}_{\mathbf{1}}(\mu)\rangle}{\sqrt{n}}-\frac{\lambda m^{p_n}_{\mathbf{2}}(\mu)\lambda^t}{2n}+o(\frac{1}{n})\right) \\
&\qquad\times\left(1+\frac{i\langle\lambda,m^{p_n}_{\mathbf{1}}(\mu)\rangle}{\sqrt{n}}-\frac{\langle\lambda,m^{p_n}_{\mathbf{1}}(\mu)\rangle^2}{2n}+o(\frac{1}{n})\right)\Bigg]^n \\
&= \left(1-\frac{\lambda\Sigma^{p_n}(\mu)\lambda^t}{2n}+o(\frac{1}{n})\right)^n
\end{aligned}
$$

From Lemma 6.3 we also obtain that

$$
|\lambda\Sigma^{p_n}(\mu)\lambda^t-\lambda\tilde{\Sigma}(\mu)\lambda^t| = O(\frac{|\lambda|^2}{\sqrt{p_n}})
$$

for $p_n\to\infty$. Therefore, we have

$$
\begin{aligned}
\lim_{n\to\infty} E(\varphi^{p_n}_{\lambda/\sqrt{n}-i\rho(n)}(\tilde{S}^{p_n}_n))e^{i\langle\lambda,\sqrt{n}m^{p_n}_{\mathbf{1}}(\mu)\rangle} &= \\
&= \lim_{n\to\infty}\left(1-\frac{\lambda\tilde{\Sigma}(\mu)\lambda^t}{2n}+\frac{\lambda(\Sigma^{p_n}(\mu)-\tilde{\Sigma(\mu)})\lambda^t}{2n}+o(\frac{1}{n})\right)^n \\
&= \exp(-\lambda\tilde{\Sigma}(\mu)\lambda^t/2)
\end{aligned}
$$

On the other hand from the Lemma 5.6(5) we have

$$
\lim_{n\to\infty} E(\varphi^{p_n}_{\lambda/\sqrt{n}-i\rho(n)}(\tilde{S}^{p_n}_n)-\exp(-i\langle\lambda,m^{p_n}_{\mathbf{1}}(\tilde{S}^{p_n}_n)\rangle/\sqrt{n}))=0. \tag{90}
$$

Now, Lévy's continuity theorem for the classical q-dimensional Fourier transform implies that

$$
(\tilde{m}_{\mathbf{1}}(\tilde{S}^{p_n}_n)-n\cdot m^{p_n}_{\mathbf{1}}(\mu)\rangle)/\sqrt{n}
$$

tends to the normal distribution $\mathcal{N}(0,\tilde{\Sigma}(\mu))$.
Now, Lemma 5.2(2) implies that $(T(\tilde{S}^{p_n}_n)-n\cdot m^{p_n}_{\mathbf{1}}(\mu)\rangle)/\sqrt{n}$ also converges to $\mathcal{N}(0,\tilde{\Sigma}(\mu))$.
Finally, with the same argument as in the proof of Theorem 6.4 above we get that

$$
(\tilde{S}^{p_n}_n-n\cdot m^{p_n}_{\mathbf{1}}(\mu))/\sqrt{n}\to\mathcal{N}(0,\tilde{\Sigma}(\mu)) \text{ weakly,}
$$

as desired. □

In contrast to the CLT above, in order to obtain a weak LLN the existence of second moments for the associated measure μ is sufficient.

THEOREM 6.8. *Let $(p_n)_{n\geq 1} \subset (2q-1, \infty)$ be an increasing sequence with and $\lim_{n\to\infty} p_n = \infty$. Let $\mu \in \mathcal{M}^1(C_q^B)$ with $\mu \neq \delta_0$ and with second moments. Consider the associated random walks $(\tilde{S}_n^p)_{n\geqslant 0}$ on C_q^B for $p > 2q-1$. Let $\varepsilon > \frac{1}{2}$. Then*

$$\frac{1}{n^\varepsilon}(\tilde{S}_n^{p_n} - n \cdot m_{\mathbf{1}}^{p_n}(\mu)) \longrightarrow 0 \text{ in probability.}$$

PROOF OF THEOREM 6.8. We first notice that by Taylor's theorem and Proposition 6.3 for all $p > 2q-1$

$$\begin{aligned}\Big|E(\varphi^p_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^p)) - \Big(1 - \frac{i\langle\lambda, m_{\mathbf{1}}^p(\mu)\rangle}{n^\varepsilon}\Big)\Big| &\leq \frac{1}{n^{2\varepsilon}} \sum_{l\in\mathbb{N}^q, |l|=2} m_l^p(\mu) \frac{\lambda_1^{l_1}...\lambda_q^{l_q}}{l_1!...l_q!} \\ &\leq \frac{1}{n^{2\varepsilon}} \sum_{l\in\mathbb{N}^q, |l|=2} (\tilde{m}_l(\mu) + C/\sqrt{p}) \frac{\lambda_1^{l_1}...\lambda_q^{l_q}}{l_1!...l_q!} \\ &\leq K_1 \frac{\|\lambda\|_\infty^3}{n^{2\varepsilon}} \qquad (91)\end{aligned}$$

for some constant $K_1 > 0$ which is independent of p. Analogously, for all $p > 2q-1$,

$$\Big|e^{i\langle\lambda, n^\varepsilon\cdot m_{\mathbf{1}}^p(\mu)\rangle} - \Big(1 + \frac{i\langle\lambda, m_{\mathbf{1}}^p(\mu)\rangle}{n^\varepsilon}\Big)\Big| \leqslant K_2 \frac{\|\lambda\|_\infty^3}{n^{2\varepsilon}} \qquad (92)$$

for some $K_2 > 0$ independent of p.
Using estimates (91) and (92) we now follow similar paths as in the proof of Theorem 6.7. For $\lambda \in \mathbb{R}^q$ we have

$$\begin{aligned}E(\varphi^{p_n}_{\lambda/n^\varepsilon - i\rho(n)}&(\tilde{S}_n^{p_n}))e^{i\langle\lambda, n^\varepsilon\cdot m_{\mathbf{1}}^{p_n}(\mu)\rangle} = \\ &= \mathcal{F}_{BC}^{p_n}(\mu^{(n,p_n)})(\lambda/n^\varepsilon - i\rho(n)) \cdot e^{i\langle\lambda, n^\varepsilon\cdot m_{\mathbf{1}}^{p_n}(\mu)\rangle} \\ &= \Big[\Big(1 - \frac{i\langle\lambda, m_{\mathbf{1}}^{p_n}(\mu)\rangle}{n^\varepsilon} + o(\frac{1}{n})\Big)\Big(1 + \frac{i\langle\lambda, m_{\mathbf{1}}^{p_n}(\mu)\rangle}{n^\varepsilon} + o(\frac{1}{n})\Big)\Big]^n \\ &= \Big(1 + o(\frac{1}{n})\Big)^n.\end{aligned}$$

Therefore, for all $\lambda \in \mathbb{R}^q$ we have

$$\lim_{n\to\infty} E(\varphi^{p_n}_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^{p_n}))e^{i\langle\lambda, n^\varepsilon\cdot m_{\mathbf{1}}^{p_n}(\mu)\rangle} = 1.$$

On the other hand from the Lemma 5.6(5) for all $\lambda \in \mathbb{R}^q$ we have

$$\lim_{n\to\infty} E(\varphi^{p_n}_{\lambda/n^\varepsilon - i\rho(n)}(\tilde{S}_n^{p_n}) - \exp(-i\langle\lambda, m_{\mathbf{1}}^{p_n}(\tilde{S}_n^{p_n})\rangle/n^\varepsilon)) = 0. \qquad (93)$$

Now, Lévy's continuity theorem for the classical q-dimensional Fourier transform implies that

$$(\tilde{m}_{\mathbf{1}}(\tilde{S}_n^{p_n}) - n \cdot m_{\mathbf{1}}^{p_n}(\mu))/n^\varepsilon \longrightarrow 0 \text{ in distribution.}$$

Now, Lemma 5.2(2) implies that also $(T(\tilde{S}_n^{p_n})-n\cdot m_{\mathbf{1}}^{p_n}(\mu)/n^\varepsilon \longrightarrow 0$ in distribution. Finally, with the same argument as in the proof of Theorem 6.4 above we get $(\tilde{S}_n^{p_n} - n\cdot m_{\mathbf{1}}^{p_n}(\mu))/n^\varepsilon \to 0$ in distribution. This implies convergence in probability since the limit is constant.

□

2. A law of large numbers for inner normalizations and growing parameters

We present a further limit theorem for $(S_n^{(p,n^{-1/2})})_{n\geq 1}$ when p and n go ∞ in a coupled way. It will turn out that then, under some canonical norming, the limiting distribution is a point measure, i.e., we obtain a weak law of large numbers:

THEOREM 6.9. *Let $\mu \in \mathcal{M}^1(C_q^B)$ with $\mu \neq \delta_0$ and finite second moments. Let t_0 be defined as in Theorem 5.15 and $(p_n)_{n\geq 1} \subset [2q-1,\infty)$ be increasing with $\lim_{n\to\infty} n/p_n = 0$. Then, $S_n^{(p_n,n^{-1/2})}$ tends in probability for $n \to \infty$ to the constant*

$$\ln\left(e^{t_0/2} + \sqrt{e^{t_0/4}-1}\right)\cdot(1,\ldots,1).$$

For the proof of theorem we first recapitulate the Taylor expansion for $\varphi_\lambda^A(x)$ at $x=0$ from [**G1**]:

LEMMA 6.10. *For $\|x\|_2 \to 0$,*

$$\varphi_\lambda^A(x) = 1 + \frac{1}{qd}(\lambda_1+\lambda_2+\ldots+\lambda_q)\sum_{k=1}^q x_k + R_\lambda(x)$$

with

$$R_\lambda(x) = \sum_\alpha f_\alpha(\lambda)P_\alpha(x)$$

where the $P_\alpha(x)$ are symmetric polynomials in $x_1,\ldots,x_q$ which are homogeneous of order ≥ 2.

We also need the following fact:

LEMMA 6.11. *For $p \geq 2q-1$, the half sum $\rho = \rho^{BC}(p)$ satisfies the condition $\rho^A - \rho \in co(W_q^B\cdot\rho)$, where W_q^B is the Weyl group of type B_q.*

PROOF. Denote $\hat{\rho} := (\rho_q, \rho_{q-1}\ldots,\rho_1)$. Then, obviously $, -\rho, -\hat{\rho} \in W_q^B\cdot\rho$. On the other hand we have

$$\rho^A - \rho = \left(\frac{d}{2}(p+1)-1\right)(1,\ldots.,1) = \frac{1}{2}(-\rho-\hat{\rho}).$$

This proves the result. □

PROPOSITION 6.12. *Let μ, t_0 and $(p_n)_{n\geq 1}$ be defined as in Theorem 6.9. Let $\rho(n) := \rho^{BC}(p_n)$ be the half sum of positive roots of type BC associated with the parameters p_n. Then, for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda = \rho^A$,*

$$\int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}})d\mu(x) = 1 + \frac{t_0}{4n}\cdot\sum_{k=1}^{q}(\lambda_k - i\rho_k^A) + o(1/n) \text{ as } n \to \infty. \tag{94}$$

PROOF. Lemma 6.10 and the Taylor expansion $\ln\cosh x = x^2 + O(x^4)$ show that for all $\lambda \in \mathbb{C}^q$ with such that $\Im\lambda \in co(W_q^A \cdot \rho^A)$

$$\varphi_\lambda^A(\ln\cosh\frac{x}{\sqrt{n}}) = 1 + \sum_{i=1}^{q}\lambda_i\frac{\|x\|_2^2}{2nqd} + R_\lambda(\frac{\|x\|^2}{n}) \tag{95}$$

for $n \to \infty$. On the other hand, Theorem 4.2(2) in [**RV1**] states that

$$|\varphi^p_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}}) - \varphi^A_{\lambda - i\rho_A}(\ln\cosh\frac{x}{\sqrt{n}})| \leq C\cdot\frac{\|\lambda\|_1\cdot\min(1, x_1/\sqrt{n})}{\sqrt{p}} \tag{96}$$

for all $\lambda \in \mathbb{C}^q$ such that $\Im\lambda - \rho(n) \in co(W_q^B\cdot\rho(n))$. Notice that the analysis of the proof of Theorem 4.2(2) in [**RV1**] shows that (96) is in fact precisely valid for

$$\lambda \in \{\lambda \in \mathbb{C}^q : \Im\lambda - \rho(n) \in co(W_q^B\cdot\rho(n)) \text{ and } \Im\lambda - \rho^A \in co(W_q^A\cdot\rho^A)\}.$$

If we combine (95) and (96) and use the Lemma 6.11 we see that as $p_n/n \to \infty$

$$\left|\varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}}) - 1 - \sum_{k=1}^{q}(\lambda_k - i\rho_k^A)\frac{\|x\|_2^2}{2qnd}\right| = o(\frac{\|x\|_2^2}{n}) \quad \text{for all } \lambda\in\mathbb{C}^q \text{ with } \Im\lambda = \rho^A \tag{97}$$

which, by integrating w.r.t ν yields the result.

□

PROOF OF THE THEOREM 6.9. Let $\mu^{(n,p_n)}$ be the n-fold $*_{p_n}$ convolution power of μ. The Proposition 6.12 shows that for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda = \rho^A$

$$\begin{aligned}\lim_{n\to\infty}\int_{C_q^B}\varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}})d\mu^{(n,p_n)}(x) &= \lim_{n\to\infty}\left(\int_{C_q^B}\varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}})d\mu(x)\right)^n \\ &= \lim_{n\to\infty}\left(1 + \frac{t_0}{4n}\cdot\sum_{k=1}^{q}(\lambda_k - i\rho_k^A) + o(1/n)\right)^n \\ &= e^{\frac{t_0}{4}\cdot\sum_{k=1}^{q}(\lambda_k - i\rho_k^A)}.\end{aligned}$$

Thus, using (96) we have that

$$\begin{aligned}\lim_{n\to\infty}\mathcal{F}^A(\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})})(\lambda-i\rho^A) &= \lim_{n\to\infty}\int_{C_q^B}\varphi^A_{\lambda-i\rho^A}(\ln\cosh\frac{x}{\sqrt{n}})d\mu^{(n,p_n)}(x)\\ &= \lim_{n\to\infty}\int_{C_q^B}\varphi^{p_n}_{\lambda-i\rho(n)}(\frac{x}{\sqrt{n}})d\mu^{(n,p_n)}(x)\\ &= e^{\frac{t_0}{4}\cdot\sum_{k=1}^q(\lambda_k-i\rho_k^A)}\end{aligned}$$

for all $\lambda\in\mathbb{C}^q$ with $\Im\lambda=\rho^A$. By making substitution $\lambda\mapsto\lambda+i\rho^A$ above, we get

$$\lim_{n\to\infty}\mathcal{F}^A(\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})})(\lambda)=e^{\frac{t_0}{4}\cdot\sum_{k=1}^q\lambda_k} \tag{98}$$

for all $\lambda\in\mathbb{R}^q$. On the other hand from (49) we can easily see that

$$\begin{aligned}e^{\frac{t_0}{4}\cdot\sum_{k=1}^q\lambda_k} &= \varphi^A_\lambda(\frac{t_0}{4}(1,...,1))\\ &= \mathcal{F}^A(\delta_{\frac{t_0}{4}(1,....,1)})(\lambda)\end{aligned}$$

for all $\lambda\in\mathbb{C}^q$ with $\Im\lambda\in co(W_q^A\cdot\rho^A)$. Since the equality (98) is satisfied on $\mathbb{R}^q$, i.e. the support of Plancherel measure, from Theorem 2.12(iv) it follows that $\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})}$ converges vaguely to the Dirac point measure $\delta_{t_0(1,...,1)}$. Moreover, as the $\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})}$ and $\delta_{\frac{t_0}{4}(1,...,1)}$ are probability measures, the sequence $(\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})})_n$ is tight and the convergence becomes weak. Now, since T^{-1} is a continuous function, from continuous mapping theorem we conclude that $\mathbb{P}_{S_n^{(p_n,n^{-1/2})}}$ converges weakly to

$$T^{-1}(\delta_{\frac{t_0}{4}\cdot(e_1,...,e_q)})=\delta_{\ln\left(e^{\frac{t_0}{4}}+\sqrt{e^{\frac{t_0}{2}}-1}\right)\cdot(1,...,1)}$$

as desired. $\square$

List of Symbols

Symbol	Meaning
$\mathfrak{a}$	Euclidean space with dimension q and scalar product $\langle\cdot,\cdot\rangle$
$\mathcal{B}(X)$	Borel sigma algebra of X
$\mathfrak{B}_b(X)$, $\mathfrak{B}_b(X)$	Space of (bounded) Borel measures on X
$\mathcal{C}(X)$, $\mathcal{C}_b(X)$	Space of (bounded) continuous measures on X
δ_x	Dirac measure at x
Δ	Euclidean Laplace operator on $\mathbb{R}^q$
Δ_m	Heckman-Opdam Laplacian, see (33)
C_q^A	Weyl chamber of type A
C_q^B	Weyl chamber of type B
$d(\underline{x},\underline{y},u,w)$	see (22)
F_λ	hypergeometric function
m_α	multiplicity: W-invariant map $m: R \to \mathbb{C}$
$d\mathbf{m}_p(w)$	see (20)
$\mathcal{M}(X), \mathcal{M}_b(X)$	Space of (bounded) Borel measures on X
$\mathcal{M}^1(X)$	Space of probability measures on X
$\mathcal{M}_b(G\vert K), \mathcal{M}_b(G\vert\vert K)$	K-(bi)invariant measures in $\mathcal{M}_b(G)$, see (**??**), (3)
$\mathcal{M}_{b,K}(M)$	space of K-invariant (invariant under action of K) measure on M
$\mathfrak{M}_k^1(X)$	space of probability measures with moments up to order k, see Definition 3.8
$\mathcal{P}, \mathcal{P}^W$	space of (W-invariant) polynomials
$\mathcal{S}(\mathfrak{a})$	symmetric algebra on $\mathfrak{a}$
$\rho(m)$	half sum of roots, see (31)
R	root system, see Definition 4.4
R^+	a positive subsystem root system R
$\chi(X), \chi_b(X)$	(semi)characters, see Definition 2.6
$\hat{X}$	dual space of X, see Definition 2.6
$\Delta_\lambda(x)$	see (45)
$\mathcal{F}_{BC}^p$	Fourier transform of type BC
$\mathcal{F}_A$	Fourier transform of type A
$co(\cdot)$	convex hull of a set
$int(\cdot)$	interior of a set
$\Re x$	real part of x
$\Im x$	imaginary part of x

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