Risk Analysis in Capital Investment Appraisal with Correlated Cash Flows: Simple Analytical Methods[1]

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Abstract

Since uncertainty is the crucial point of a capital investment decision, risk analysis in capital budgeting is often applied. Usually risk analysis is carried out by a Monte Carlo simulation. The aim of this article is to present simple analytical methods which allow us to calculate the standard deviation of a project with correlated cash flows as a risk measure. These methods are compared with simulation procedures carried out with R, and it is shown that the proposed simple analytical methods are indeed a quick and efficient procedure for assessing the risk of an investment project where the cash flows are correlated.

Keywords: capital budgeting, net present value, finance, uncertainty.

1. Introduction

The net present value (NPV) is an indicator of how much value an investment or project adds to the wealth of the investor. It is defined as the discounted sum of future cash flows:

$$C_0 = -I_0 + \sum_{t=1}^{n} \frac{c_t}{q^t} = -I_0 + \sum_{t=1}^{n} \frac{c_t}{(1+i)^t}$$

with C_0: net present value (NPV), $I_0 > 0$: investment expenditure at time 0, $c_t > 0$: cash flow at time t, n>0: number of years, q>1: discount factor, i=q-1: discount rate (see, e.g., Pflaumer, 2004, 2013). The estimation of future cash flows has to be done under uncertainty. Considering future cash flows as random variables leads to a stochastic investment appraisal model, where the net present value is a random variable, too. The distribution of the net present value can be determined by Monte Carlo simulation methods (see, e.g., Hess/Quigley, 1963; Hertz, 1964; Savvides, 1994; Vose, 1997) or analytical methods (see, e.g., Hillier, 1963; Wagle, 1967; Jöckel/ Pflaumer, 1980, 1981).

2. A Simple Model with Correlated Cash Flows

In order to consider the uncertainty in stochastic investment appraisal, one has to specify a mean estimate of the cash flow, a pessimistic and an optimistic estimate. It is assumed that the cash flow is equally distributed between the lower bound a_t and the upper bound b_t. Statistically speaking, the cash flow is a random variable c_t (risk profile) with a rectangular distribution. The expected value and the variance of the cash flow are:

$$\mu_t = E(c_t) = \frac{a_t + b_t}{2}, \sigma_t^2 = Var(c_t) = \frac{(b_t - a_t)^2}{12}.$$

The correlation coefficient between the cash flow, c_j, at time *j* and the cash flow, c_k, at time *k* shall be $r_{jk}=r_{kj}$. The correlation of a time series with its own past and future values is called the autocorrelation. It is also sometimes called "lagged correlation" or "serial correlation".

The expected value and the variance of the net present value (NPV) will be

$$E(C_0) = -I_0 + \sum_{t=1}^{n} \frac{E(c_t)}{q^t} \text{ and } Var(C_0) = \sum_{t=1}^{n} \frac{Var(c_t)}{q^{2t}} + 2 \cdot \sum_{t=1}^{n-1} \sum_{s=t+1}^{n} r_{ts} \cdot \frac{\sigma_{c_t} \sigma_{c_s}}{q^{t+s}}.$$

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Positive correlation increases the variance and thus the risk of a capital budgeting project. The model is not easy to manage since a large number of correlation coefficients, exactly $\frac{n\cdot(n-1)}{2}$, have to be specified, which are, in general, difficult to estimate.
Therefore, it is reasonable to assume that the correlation depends only on the time-distance between the pair of cash flows, however not on their position in time:

$$r_{t,t+\tau}=r_{t+\tau,t}=r^{\tau}.$$

The autocorrelation decreases with increasing time lags. The influence of the cash flow of the previous year on this year´s cash flow is more important than the influence of the cash flow from, e.g., three years ago. Simplifying the investigation, we further assume that the cash flows are identical distributed with a rectangular distribution. Together with the assumptions about the autocorrelation, it is possible to derive the expectation and the variance of the NPV distribution, using statistical standard methods regarding the variance formula of the sum of discounted correlated random variables and sum formulas of geometric series.

The mean and variance of the NPV distribution are now defined as

$$E(C_0)=\frac{a+b}{2}\cdot\frac{1}{q^n}\cdot\frac{q^n-1}{q-1}$$

and

$$Var(C_0)=\frac{(b-a)^2}{12}\cdot\frac{1}{q^{2n}}\cdot\frac{q^{2n}-1}{q^2-1}+2\cdot\frac{(b-a)^2}{12}\cdot\sum_{t=1}^{n-1}\left(\frac{q^{t-2n}}{1-q^2}+\frac{1}{q^t\left(q^2-1\right)}\right)\cdot r^t$$

or

$$Var(C_0)=\frac{(b-a)^2}{12}\cdot\left(\frac{1}{q^{2n}}\cdot\frac{q^{2n}-1}{q^2-1}+2\cdot S_n\right),$$

where

$$S_n=\frac{1}{q^{2n}\left(1-q^2\right)}\cdot\frac{(rq)^n-rq}{rq-1}+\frac{1}{q^2-1}\cdot\frac{\left(\frac{r}{q}\right)^n-\frac{r}{q}}{\frac{r}{q}-1}$$

with

$$\lim_{q\to 1}S_n=\frac{n\cdot r}{1-r}+\frac{r^{n+1}-r}{(r-1)^2}.$$

For the sake of simplicity, it may be sometimes assumed that the correlation coefficient is always $r_{t,t+\tau}=r_{t+\tau,t}=1$.
In this case we get

$$Var(C_0)=\frac{(b-a)^2}{12}\cdot\left(\frac{1}{q^n}\cdot\frac{q^n-1}{q-1}\right)^2 \text{ or } \sigma_{C_0}=\sqrt{\frac{(b-a)^2}{12}}\cdot\frac{1}{q^n}\cdot\frac{q^n-1}{q-1}.$$

If all autocorrelation coefficients are unity than the following relationship holds:

$$\frac{1}{q^{2n}}\cdot\frac{q^{2n}-1}{q^2-1}+2\cdot S_n=\left(\frac{1}{q^n}\cdot\frac{q^n-1}{q^2-1}\right)^2.$$

The general formula in the case of completely autocorrelated cash flows is $Var(C_0)=\left(\sum_{t=1}^{n}\frac{\sigma_{c_t}}{q^t}\right)^2$.

With the unrealistic assumption of complete correlation between all cash flows the variance, and thus the risk of the NPV distribution, will be largely overestimated. Therefore, one may propose the calculation of the variances in the cases of uncorrelated and completely correlated cash flows. Thus, a lower and an upper limit of the variance can easily be calculated (excluding the case of negative correlation). The real variance consequently lies somewhere in the middle.

The quotient $U = \frac{Var\left(C_0, r=1\right)}{Var\left(C_0, r=0\right)} = \frac{q+1}{q-1} \cdot \frac{q^n - 1}{q^n + 1}$ reflects the relative limits of the uncertainty interval. The quotient approaches $\hat{U} = \frac{q+1}{q-1}$, as n increases.

Comparable results can be obtained by the quotients

$\hat{V} = \frac{Var\left(C_0, r=-1\right)}{Var\left(C_0, r=0\right)} = \frac{q-1}{q+1}$ and $\hat{W} = \frac{Var\left(C_0, r=1\right)}{Var\left(C_0, r=-1\right)} = \frac{\left(q+1\right)^2}{\left(q-1\right)^2}$, as n tends to infinity.

If the number of periods (years) is large then we can find an approximation formula for the variance

$$Var\left(C_0\right) = \frac{\left(b-a\right)^2}{12} \cdot \frac{q+r}{\left(q-r\right) \cdot \left(q^2-1\right)},$$

since

$$\lim_{n \to \infty} S_n = \frac{r}{\left(q-r\right) \cdot \left(q^2-1\right)}.$$

If it is assumed that only *m* adjacent cash flows are correlated and temporally far apart cash flows are not correlated, that is

$$r_{t,t+\tau} = r_{t+\tau,t} = \begin{cases} r^\tau & if \;\; \tau \le m \\ 0 & if \;\; \tau > m \end{cases},$$

then the variance formula for the NPV changes to

$$Var\left(C_0\right) = \frac{\left(b-a\right)^2}{12} \cdot \frac{1}{q^{2n}} \cdot \frac{q^{2n}-1}{q^2-1} + 2 \cdot \frac{\left(b-a\right)^2}{12} \cdot \sum_{t=1}^{m \le n-1} \left(\frac{q^{t-2n}}{1-q^2} + \frac{1}{q^t \left(q^2-1\right)} \right) \cdot r^t$$

$$= \frac{\left(b-a\right)^2}{12} \cdot \frac{1}{q^{2n}} \cdot \frac{q^{2n}-1}{q^2-1} + 2 \cdot \frac{\left(b-a\right)^2}{12} \cdot \left(\frac{1}{q^2-1} \cdot \frac{\left(\frac{r}{q}\right)^{m+1} - \frac{r}{q}}{\frac{r}{q} - 1} - \frac{1}{q^{2n}\left(q^2-1\right)} \cdot \frac{\left(r \cdot q\right)^{m+1} - r \cdot q}{r \cdot q - 1} \right).$$

<u>Example</u>

A project investment of € 100,000 for 10 years will lead to annual autocorrelated cash flows, which will be between *a* = € 14,000 and *b* = € 26,000. Rectangular distribution of the cash flows is assumed. The autocorrelation coefficient is $r_{t,t+\tau} = r_{t+\tau,t} = 0.7^\tau$. The mean and the standard deviation of the NPV distribution shall be determined at a discount rate of 12%.

Entering the values into the appropriate formulas leads to the following results:

$$E\left(C_0\right) = 13{,}004 \text{ and } \sigma_{C_0} = 12{,}872.$$

The approximation for large *n* yields:

$$\sigma_{C_0} = \sqrt{\frac{\left(b-a\right)^2}{12} \cdot \frac{q+r}{\left(q-r\right) \cdot \left(q^2-1\right)}} = \sqrt{\frac{12{,}000^2}{12} \cdot \frac{1.12+0.7}{\left(1.12-0.7\right) \cdot \left(1.12^2-1\right)}} = 14{,}297.$$

If all correlation coefficients are unity, then the upper limit of the variance will be:

$$\sigma_{C_0} = \sqrt{\frac{\left(b-a\right)^2}{12}} \cdot \frac{1}{q^n} \cdot \frac{q^n-1}{q-1} = \sqrt{\frac{12{,}000^2}{12}} \cdot \frac{1}{1.12^{10}} \cdot \frac{1.12^{10}-1}{q-1} = 19{,}573.$$

The case of annual stochastically independent cash flows leads to the lower limit of the standard deviation, which is calculated to be $\sigma_{C_0} = 6{,}502$. The relative uncertainty interval is therefore

$$U^{\frac{1}{2}} = \sqrt{\frac{q+1}{q-1} \cdot \frac{q^n - 1}{q^n + 1}} = \frac{1.12}{0.12} \cdot \frac{1.12^{10} - 1}{1.12^{10} + 1} = \frac{19{,}573}{6{,}502} \approx 3.$$

In Table 1 the standard deviations of the NPV of our example are shown under different assumptions about the correlation coefficient and the length of the autocorrelation. The influence of the length of the autocorrelation increases with the value of the correlation coefficient.

Table 1: Standard deviations of the NPV under different assumptions

	r=0.1		*r*=0.4		*r*=0.7		*r*=1	
m	σ_{C_0}	Index	σ_{C_0}	Index	σ_{C_0}	Index	σ_{C_0}	Index
0	6,502	1.00	6,502	1.00	6,502	1.00	6,502	1.00
1	7,043	1.08	8,461	1.30	9,673	1.49	10,750	1.65
2	7,088	1.09	9,037	1.39	11,154	1.72	13,360	2.05
3	7,092	1.09	9,224	1.42	11,947	1.84	15,228	2.34
4	7,092	1.09	9,286	1.43	12,387	1.90	16,627	2.56
5	7,092	1.09	9,305	1.43	12,631	1.94	17,685	2.72
6	7,092	1.09	9,312	1.43	12,763	1.96	18,473	2.84
7	7,092	1.09	9,314	1.43	12,830	1.97	19,035	2.93
8	7,092	1.09	9,314	1.43	12,862	1.98	19,396	2.98
9	7,092	1.09	9,314	1.43	12,872	1.98	19,573	3.01

Index= $\sigma_{C_0} / 6{,}052$

In the case of stochastically independent cash flows, the NPV distribution can be approximated by a normal distribution (due to the central limit theorem) from which, e.g., quantiles and the probability of a negative net present value can be calculated. In the case of autocorrelation the central limit theorem is no longer valid. The NPV distribution and its quantiles can only be determined by simulation.

3. Simulation

A simulation with R of uncorrelated and autocorrelated cash flows was carried out. The simulation size was N=10,000. The simulation procedure followed an idea of Smart (2014):

“The idea is simple. 1. Draw any number of variables from a joint normal distribution. 2. Apply the univariate normal CDF (cumulative distribution function) of variables to derive probabilities for each variable. 3. Finally apply the inverse CDF of any distribution to simulate draws from that distribution. The result is that the final variables are correlated in a similar manner to that of the original variables. This is because the rank order of the variables is maintained and thus correlations are approximately the same though not exact.”

The calculated and simulated means and standard deviations are similar. If one regards the formulas, the results in Table 2 and the graphs in Fig. 1, one clearly recognizes the influence of (positive) correlated cash flows:

1. Standard deviation and, thus, risk increases.
2. The probability of loss (probability of a negative NPV) increases considerably, from about 2% to about 18%.
3. The mean and median are not affected.
4. The histogram of the correlated cash flows significantly deviates from the density of the normal distribution. The histogram density is much flatter (kurtosis = -0656), an impressive demonstration that the central limit theorem applies only for independent random variables.
5. With autocorrelated cash flows, the probability of a low or a high NPV increases. The risk of a low outcome is high; however, the chance of a high outcome is also high.
6. The expected value and the standard deviation of the net present value are decreasing functions of the discount factor.

7. The relative risk, expressed as coefficient of variation is an increasing function of the discount factor as long as the net present value is positive. The relative risk rises dramatically as soon as the discount factor approaches the factor of the internal rate of return, since the net present value tends to zero, whereas the standard deviation decreases only slightly.

Table 2: Results of the simulation

mean	stdev	loss prob.	IQR	skewness	kurtosis	min	1. Q.	median	3. Q.	max	N
Autocorrelation r=0.7											
13,000	12,793	0.1762	18,974	0.017	-0.656	-19,726	3,403	13,026	22,378	46,239	10,000
No correlation											
13,054	6,510	0.0237	8,797	-0.024	-0.078	-10,919	8,672	13,061	17,469	35,955	10,000

Q=Quartile, IQR=Interquartile Range

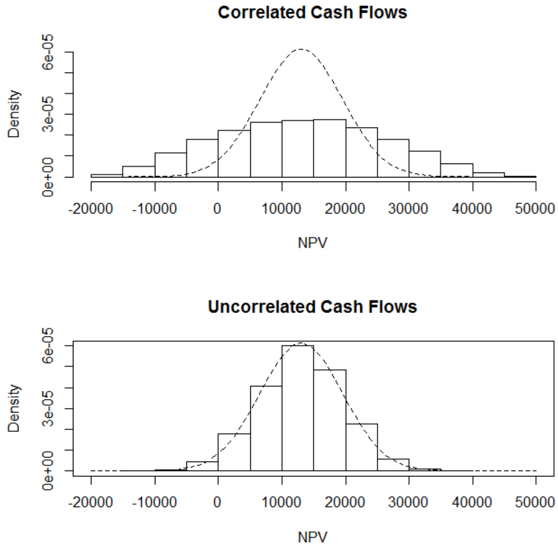

Fig. 1: Simulation results with R: NPV histograms with overlapping normal densities

Sometimes the beta distribution (PERT distribution) is considered as an adequate distribution for specifying the cash flows. The estimates of the mean and variance of the constant cash flows are

$$E(c_t) = \frac{a + 2m + b}{6} \text{ and } Var(c_t) = \frac{(b-a)^2}{36}$$

if using the traditional PERT approximation for the variance (see, e.g., Malcolm et al., 1959 or Williams, 2005). The parameters a, m, and b are the pessimistic (minimum), the most likely (modus) and the optimistic (maximum) estimates. With this specification it is possible to model symmetric and skewed distributions of the cash flow. The mean and variance of the NPV distribution are in this case

$$E(C_0)_{PERT} = \frac{a+2m+b}{6} \cdot \frac{1}{q^n} \cdot \frac{q^n - 1}{q-1} \text{ and } Var(C_0)_{PERT} = \frac{(b-a)^2}{36} \cdot \left(\frac{1}{q^{2n}} \cdot \frac{q^{2n}-1}{q^2-1} + 2 \cdot S_n \right).$$

The standard deviation of the NPV distribution is here around 42% smaller than in the case of a rectangular distribution, since the quotient of the standard deviations is $\sqrt{3}/3 \approx 0.5774$.

4. Conclusions

Risk analysis is a useful tool for identifying and assessing the risks of investment decisions. Failure to take the risk of a project into account can lead to wrong decisions. An average in statistics without the standard deviation is as useless as a net present value without a risk measure. The proposed simple methods are a first, and only a rough, indicator of the risk associated with an investment project. The standard deviations can be calculated easily and quickly. The method is particularly suitable for educational purposes, with the object being to make students and practitioners familiar with the concept of uncertainty in capital budgeting. In practice, managers, experts, or specialists often have a good imagination about the lower and upper limit of future cash flows. Thus, these parameters can be estimated subjectively. In general, no expert can subjectively specify the autocorrelation coefficients. Therefore, the correlation coefficient has to be estimated from past observations by time series or regression methods (see, e.g., Jöckel/Pflaumer, 1980).

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