Euler and Süßmilchs's Population Growth Model[1]

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Abstract
In 1761, the German demographer Johann Peter Süßmilch published a simple population growth model that starts with a couple, in the eighth chapter of his book *Die göttliche Ordnung*. With the help of the Swiss mathematician Leonhard Euler, he projected the population for 300 years. He demonstrated that after that time the population will be growing approximately geometrically. In this paper, the population projection of Euler and Süßmilch is reanalyzed using matrix algebra. Graphs and tables show the time series of the population and its growth rates. Age structures of selected years are presented. The solution of the projection equation is derived. It is shown that the projection model can be described by a geometric trend model which is superimposed by six cyclical components. In the long run, the population time series can be explained quite well by the sum of only two components, the trend component and one component with explosive cycles of a period of about 24 years. In the very long run, the influence of the cyclical component diminishes, and the series can be solely explained by its geometric trend component, as has been also recognized by Euler and Süßmilch.

Keywords: Population Projection, Matrix Model, Historical Demography.

1. Introduction

Johann Peter Süßmilch (1707–1767) was a German Protestant pastor in Berlin, statistician and demographer. His most important publication, *Die göttliche Ordnung* (1741), is regarded as a pioneering work in demography and the history of population statistics. He is sometimes called the "father of demography and statistics in Germany", which can be read on a Berlin memorial plaque, for example. As an apologist of the Christian doctrine, Süßmilch wanted to prove in the second edition of *Die göttliche Ordnung* (1761) that the population is descended from a couple and that high population levels in antiquity are compatible with the Christian calendar, even in view of the Flood. For this task, he needed mathematical help, and he turned to one of his colleagues at the Berlin Academy of Sciences, Leonhard Euler (1707–1783), the famous Swiss mathematician. Euler calculated the doubling time of the population and carried out a population projection which starts with a couple. Given this growth model, Euler seeked to understand the long-term behavior of the growing population superimposed by cycles. He noted that the tripling time is finally about 24 years, which corresponds to an annual growth rate of 4.7 percent. The collaboration between Süßmilch and Euler is described in detail in the essays by Girlich (2007) and Klyve (2014).

2. Population statistics in Süßmilch's chapter eight

Euler calculated for Süßmilch, in Chapter 8[2] of his revised edition of *Die göttliche Ordnung* from the year 1761, the doubling time of the population assuming geometric growth (§ 152 and § 156) and a population projection without information of the analytical methods (§ 160).
The assumptions are:

A1. The projection begins with one married couple, and each person is 20 years old.
A2. Marriage age 20 years.
A3. Each pair should give birth to one daughter and one son at the age of 22, 24 and 26 years.
A4. Everyone reaches the age of 40 and dies afterwards.

[1] Paper presented at the **XV CLAPEM** (*Congreso Latinoamericano de Probabilidad y Estadística Matemática*), 2 December 2019, Mérida, Yucatán, Mexico; a shortened version of the essay (without mathematical details) was presented at the St. Petersburg Historical Forum, St. Petersburg, Russia, 29 October -3 November 2019.

[2] The eighth chapter, "Von der Geschwindigkeit der Vermehrung und von der Zeit der Verdopplung", is heavily inspired by Euler, and the paragraphs 147 to 161 were regarded by Du Pasquier as the intellectual property of Euler in his *Opera Omnia* (Euler, Du Pasquier, Leipzig, 1923), pp. 507–534.

The projection steps are **biennial** (occurring every two years), which is important for the stability of the model (see 4.4).

Table 1: Recalculated numerical results of Euler and Süßmilch's population growth model (see also Süßmilch's original table in https://reader.digitale-sammlungen.de/de/fs1/object/display/bsb11283119_00325.html)

Year	Births	Total Births	Deaths	People alive
0	0	2	0	2
2	2	4	0	4
4	2	6	0	6
6	2	8	0	8
8	0	8	0	8
10	0	8	0	8
12	0	8	0	8
14	0	8	0	8
16	0	8	0	8
18	0	8	0	8
20	0	8	2	6
22	0	8	2	6
24	2	10	2	8
26	4	14	2	12
28	6	20	2	18
30	4	24	2	22
…				
216	13530	92444	17828	74616
218	16700	109144	18838	90306
220	17906	127050	19388	107662
…				
294	404378	3806204	534612	3271592
296	346580	4152784	555314	3597470
298	273884	4426668	589546	3837122
300	214370	4641038	646724	3994314

Notes: The number of deaths is the cumulative number of deaths. Euler undercalculated ten births in 218. (See also Euler, Du Pasquier, 1923, p. 529). Instead of the correct number of 16700 births, Euler's table lists only 16690 births. This mathematical error affects the other results. For example, in the year 300, the number of living persons in his table is 3993954 (360 persons too low). In the years 172, 190, and 298, there are obvious literal errors in the numbers of the "total births" (172, 190) and the "people alive" (298).

The number of births and deaths increases, strongly fluctuating, while the number of the living persons increases, moderately fluctuating, as can be seen in Table 1. Süßmilch interprets the results as follows (§ 161): "It can be seen from this that at any time after 24 years the number of the living persons becomes almost exactly three times greater, from which after 1000 and more years an astonishing increase must grow."

A note at the end of § 161 in smaller fonts than the rest of the chapter, probably written by Euler himself, is:

> "The great disorder that seems to prevail (in Euler's table) does not prevent the number of births from following a kind of progression that one calls recurrent series […] Whatever the initial disorder of these progressions, they turn into a geometric progression if they are not interrupted and the disorders of the beginning fade little by little and vanish almost completely." (Translation adopted from Bacaër, 2011, p. 17)

The chapter does not say more about the mathematics of this population model. However, Euler wrote a manuscript entitled "On the multiplication of the human race"[3], which stayed unpublished during his lifetime. This manuscript contains the mathematical background.

Euler has assumed the following recursion equation, which is a variant of his high fertility model II (10 children) in Section 2 of his manuscript (see also Girlich, p. 13 or Chapter 4.2 of this paper),

$$B_n = B_{n-11} + B_{n-12} + B_{n-13}$$

with the charcteristic equation

$$\lambda^{13} - \lambda^2 - \lambda - 1 = 0.$$

The (dominant) solution is $\lambda \approx 1.0961$ with a tripling time of 23.94 years and a doubling time of 15.1 years. The annual growth rate is $r = \left(\sqrt{1.0961} - 1\right) \cdot 100\% \approx 4.7\%$.

Gumbel (1917) has analytically shown, for Euler and Süßmilch's model, that in the long run the population will grow geometrically with the factor $\lambda = 1.0961$. He remarks also that the model is similar to that of Fibonacci from the year 1202.

Bacaër (2011) shows that the total population is about ten times the number of births

$$P_t^{40} = B_t \cdot \left(1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{19}}\right) = B_t \cdot \frac{1}{q^{19}} \cdot \frac{q^{20} - 1}{q - 1} = 9.59$$

However, if the age at death tends to infinity, then the factor is only 19% higher:

$$P_t^{\infty} = B_t \cdot \frac{q}{q-1} = 11.41$$

Age at death does not have an important influence on population size, if the population growth rate is so high.

3. Matrix population model of Euler and Süßmilch

3.1 Presentation and results

The matrix representation of the female part of the model is given by

$$n_t = A \cdot n_{t-1},\ t = 0,1,2,3,\ldots$$

or

$$n_t = A^t \cdot n_0,\ t = 0,1,2,3,\ldots$$

with the population vector

$$n_0^T = (0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0)$$

and the projection matrix

[3] "Sur la multiplication du genre humain", first published in: Euler, L; Du Pasquier, L. G.: *Leonhardi Euleri opera omnia*, Serie I, Vol. 7, Leipzig 1923, pp. 545–552. The manuscript is in notebook H6, probably written between 1750 and 1755 (see comments in Euler, Du Pasquier, 1923, p. 534).

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & & & & \vdots \\ & & & & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & 1 & 0 \end{pmatrix}.$$

We describe the biennial projection step with t. The population vector of the total population is given by

$p_t = 2 \cdot n_t$. $p_t = 2 \cdot n_t$.

The 20×20 projection matrix L is a special case of the Leslie matrix, which is well known in demography. Now, we can apply methods of matrix algebra, projecting the population and analyzing the ergodic characteristics of the growth model. The results of Table 1 have been calculated with the above matrix population model.

The projection results in Figs. 1 to 3 show that the initial population is approaching a stable population. In fact, the oscillations decrease very slowly. Finally, the typically stable age structure of a growing population results, whereby the biennial growth rate roughly tends to 9.6%. The population size increases very rapidly after a while, because of the high growth rate. Population sizes are: 456 (after 100 years), 3,994,314 (after 300 years), 35,161,956,600 (after 500 years), $3.4 \cdot 10^{20}$ or 340 quintillion (after 1000 years) . Even after 500 years, the stable state has not yet been achieved, where the population size is nearly five times higher than today's world population (see Fig. 3). Probably, Euler's projection ended already after 300 years to avoid publishing unbelievably high population numbers. That the model leads to implausible and impossible populations sizes in the long run is evident, if one calculates world population densities (people per sq. km of land area) which are, in year 500, 235 and, in year 1000, $2.2753 \cdot 10^{12}$ (2.3 trillion) or 2,275,313 people per sq. m.

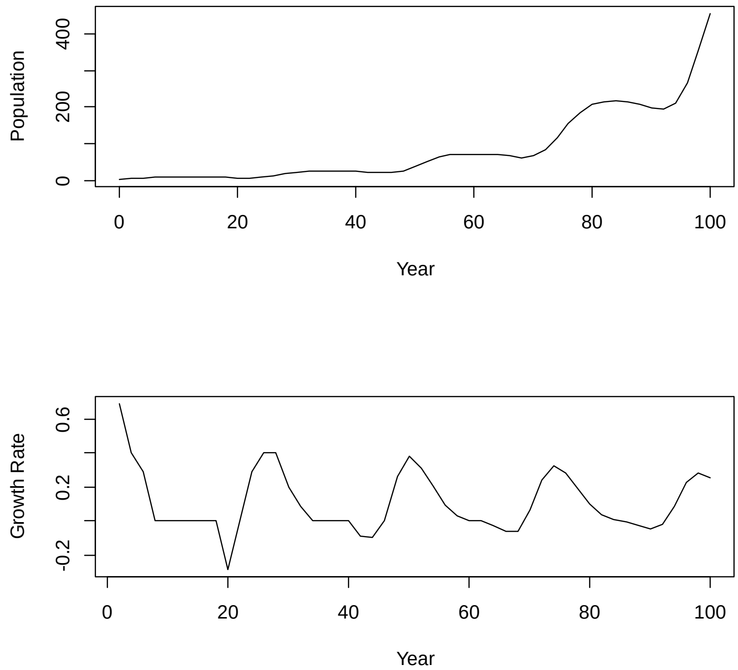

Fig. 1: Total population sizes and biennial growth rates up to the year 100

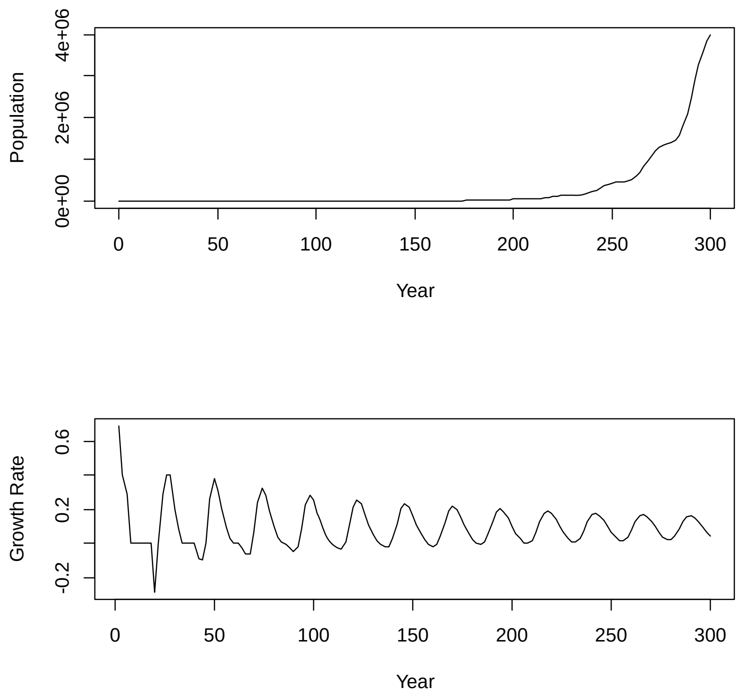

Fig. 2: Total population sizes and biennial growth rates up to the year 300

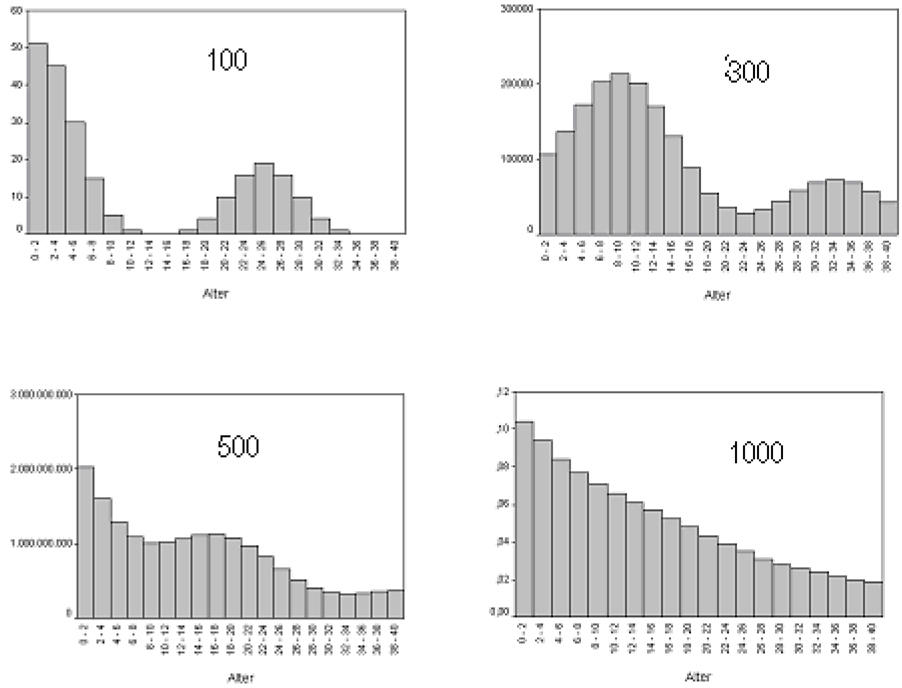

Fig. 3: Age structures after 100, 300, 500 und 1000 years

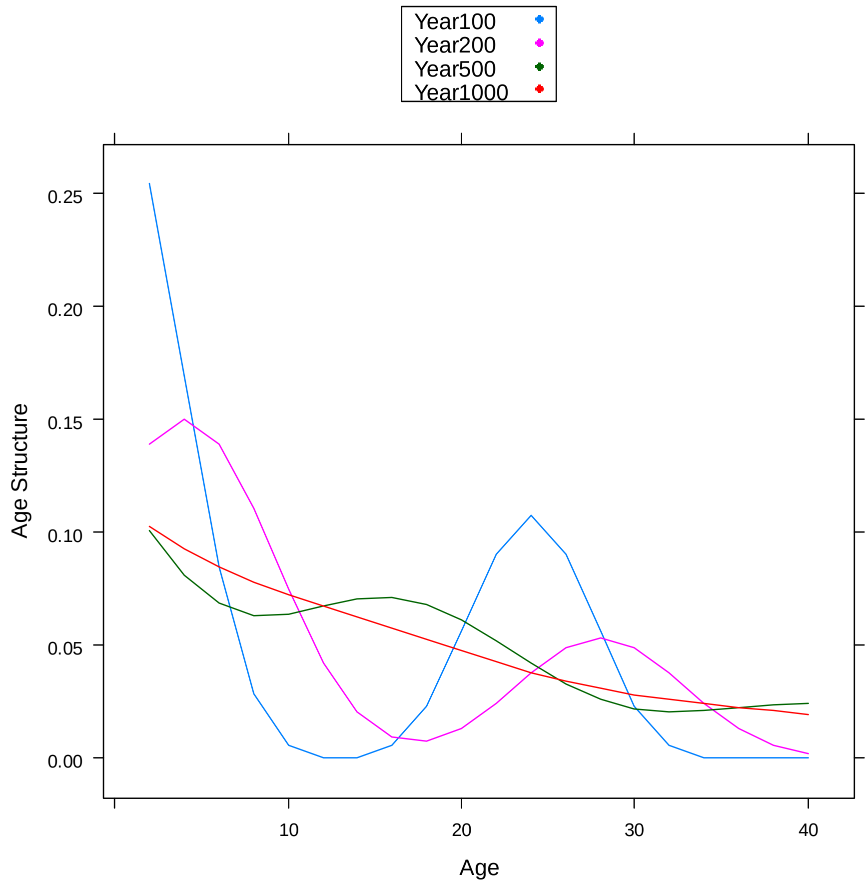

Fig. 4: Comparison of the age structures

3.2 Solution of the projection equation

Since the properties of the projection matrix A do not depend on the post-reproductive age classes, let us confine ourselves to the irreducible part of the matrix. This Leslie matrix L consists of the first 13 age classes. The 13×13 matrix L and the population vector n_0 are:

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$n_0^T = (0,0,0,0,0,0,0,0,0,0,1,0,0)$$

We regard only the female population. The projection results up to the year 300 are given in the appendix.

The eigenvalues λ_i of L are the solution of the characteristic equation

$\det(L - I\lambda) = 0$ or $\lambda^{13} - \lambda^2 - \lambda - 1 = 0$,

where I is the identity matrix.

Since L is an irreducible, primitive matrix, there exists one positive eigenvalue that is greater than any of the others in magnitude. This eigenvalue λ_1 is called the dominant eigenvalue of L. The dominant eigenvalue is the growth factor of the population per projection step. The right eigenvector belonging to λ_1 contains only positive elements and reflects the age structure of the stable population.

The solution of the matrix equation $\qquad n_t = L^t \cdot n_0$

is (see, e.g., Caswell, 2001, p. 76)

$$n_t = \sum \lambda_i^t \cdot w_i^{-1} \cdot n_0 \cdot w_i = \sum \lambda_i^t \cdot c_i \cdot w_i \,.$$

w_i *and* w_i^{-1} are column and row vectors of W and W^{-1}, where W is the matrix of the eigenvalues of L and c_i is a row vector of $W^{-1}n_0$; n_t is the population vector at time t.

The dominant eigenvalue[4] determines the ergodic properties of the population:
if $\lambda_1 > 1$, then $n_t \to \lambda_1^t \cdot \left(w_1^{-1} \cdot n_0\right) \cdot w_1$, $w_1^{-1} \cdot n_0$ is a scalar, and $\ddot{a} = \left(w_1^{-1} \cdot n_0\right) \cdot w_1$ is the stable equivalent population.

Remark: The stable equivalent population for a population with m (13) fertile age classes and $k > m$ (20) total age classes with the growth factor λ is given by:

$$\ddot{A} = \left(1 + \frac{1}{\lambda^m} \cdot \frac{\left(1 - \frac{1}{\lambda}\right)^{k-m}}{1 - \left(\frac{1}{\lambda}\right)^m} \right) \cdot \ddot{a}$$

The numerical solution of the projection equation for the total population is

$$\mathrm{P}_t = 1.6662 \cdot 1.0961^t + \sum_{i=2}^{13} C_i \cdot \lambda_i^t \text{ with } C_i = c_i \cdot \sum w_i \text{ for } t = 0, 1, 2, \ldots \text{(see Table 2)}$$

The index t denotes the (biennial) projection steps. The stable equivalent of the total population is 1.6662.

Table 2: Numerical solutions of the characteristic equation

	C_i		Eigenvalues λ_i			
i	Re(C_i)	Im(C_i)	Re(λ_i)	Im(λ_i)	Modulus r_i	Eigenvector w_1
1	1.66624094	0	1.096129	0		0.126
2	−0.0914849	−0.2673070	0.9404209	0.5461789	1.0875214	0.115
3	−0.0914849	0.2673070	0.9404209	−0.5461789	1.0875214	0.105
4	−0.1075009	−0.0831605	0.5258241	0.9196097	1.0593267	0.096
5	−0.1075009	0.0831605	0.5258241	−0.9196097	1.0593267	0.087
6	−0.08108108	−0.0135135	0	1	1.0000000	0.080
7	−0.08108108	0.0135135	0	−1	1.0000000	0.073
8	−0.00210702	0.0107986	−0.9603462	0.2570448	0.9941513	0.066
9	−0.00210702	−0.0107986	−0.9603462	−0.2570448	0.9941513	0.060
10	−0.01414979	0.0296763	−0.6729737	0.6502474	0.9357966	0.055
11	−0.01414979	−0.0296763	−0.6729737	−0.6502474	0.9357966	0.050
12	−0.03679678	0.0125678	−0.3809896	0.8056402	0.8911842	0.046
13	−0.03679678	−0.0125678	−0.3809896	−0.8056402	0.8911842	0.042
					sum	1

In the long term, the population is growing with declining oscillations around a geometric trend with a biennial growth rate of 9.61%. The stable equivalent population is 1.6662. The final age structure is given by the eigenvector w_1 of the dominant eigenvalue.

The following table (Table 3) shows the stable age structure for all 20 age classes and the reproductive values. The reproductive value is given by the left eigenvector v_1 corresponding to λ_1. The reproductive value is the total number of female offspring, discounted with the population growth rate, who can be expected to be born to an x-year-old woman. The reproductive value has its maximum in the age class 20–22, the beginning of the reproductive phase.

[4] Eigenvalues and eigenvectors are calculated with R: eigen$val und eigen$vec.

Table 3: Stable age structure of all age classes and reproductive values

Age	Age Structure in the year 1000	Stable Age Structure	Reproductive Values	**Age**	Age Structure in the year 1000	Stable Age Structure	Reproductive Values
0–2	0.102	0.104	1	**20–22**	0.042	0.042	2.504
2–4	0.093	0.095	1.096	**22–24**	0.038	0.038	1.745
4–6	0.084	0.087	1.201	**24–26**	0.034	0.035	0.912
6–8	0.078	0.079	1.317	**26–28**	0.031	0.032	0
8–10	0.072	0.072	1.444	**28–30**	0.028	0.029	0
10–12	0.067	0.066	1.582	**30–32**	0.026	0.026	0
12–14	0.062	0.060	1.734	**32–34**	0.024	0.024	0
14–16	0.057	0.055	1.901	**34–36**	0.022	0.022	0
16–18	0.052	0.050	2.084	**36–38**	0.021	0.020	0
18–20	0.047	0.046	2.284	**38–40**	0.019	0.018	0
				sum	1	1	

In the long term, the projection results for the model with all 20 age classes can be approximated by a geometric trend model, which is in this case

$$P_t^T = 2 \cdot 2.010533467 \cdot 1.096129^t$$

with the total female stable equivalent population $\ddot{A}_{tot} = 2.010533467$ (see Table 4).

Table 4: Geometric and matrix population projection

Year	Projection step t	Geometric Trend Pop. P_t^T	Projected Pop. P_t	Ratio P_t^T / P_t	log(Ratio)
0	0	4.02	2	2.011	0.6984
100	50	395.75	456	0.868	−0.1417
200	100	38950.06	46280	0.842	−0.1724
300	150	3833464.89	3994314	0.960	−0.0411
400	200	377289645.09	353041300	1.069	0.0664
500	250	3.71328E+10	3.51620E+10	1.056	0.0545
600	300	3.65462E+12	3.66017E+12	0.998	−0.0015
700	350	3.59687E+14	3.68555E+14	0.976	−0.0244
800	400	3.54004E+16	3.58201E+16	0.988	−0.0118
900	450	3.48411E+18	3.46563E+18	1.005	0.0053
1000	500	3.42906E+20	3.40028E+20	1.008	0.0084

The dampening ratios in Figs. 5a and 5b demonstrate the tendency of achieving stability.

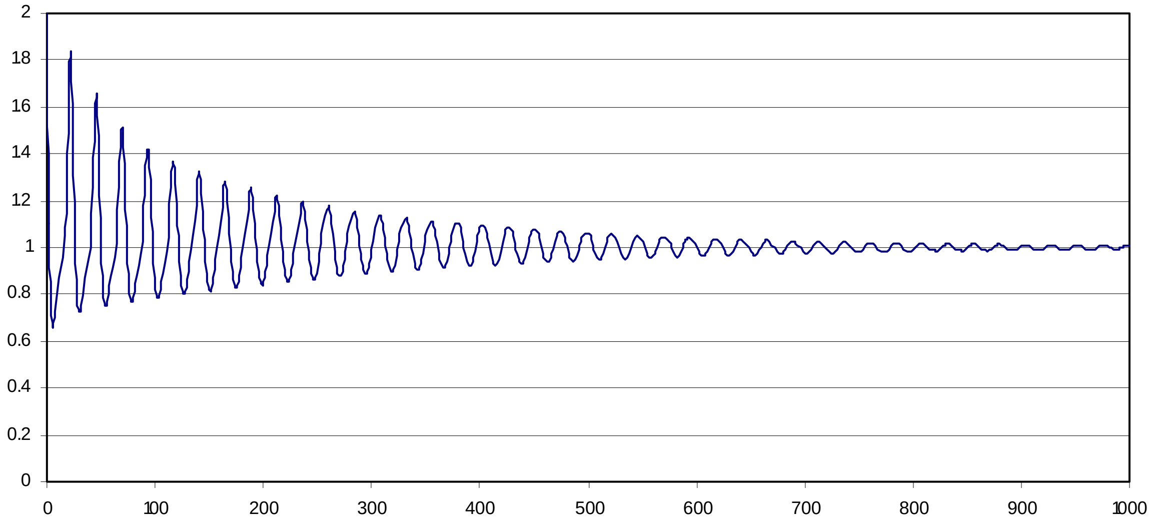

Fig. 5a: Ratio P_t^T / P_t

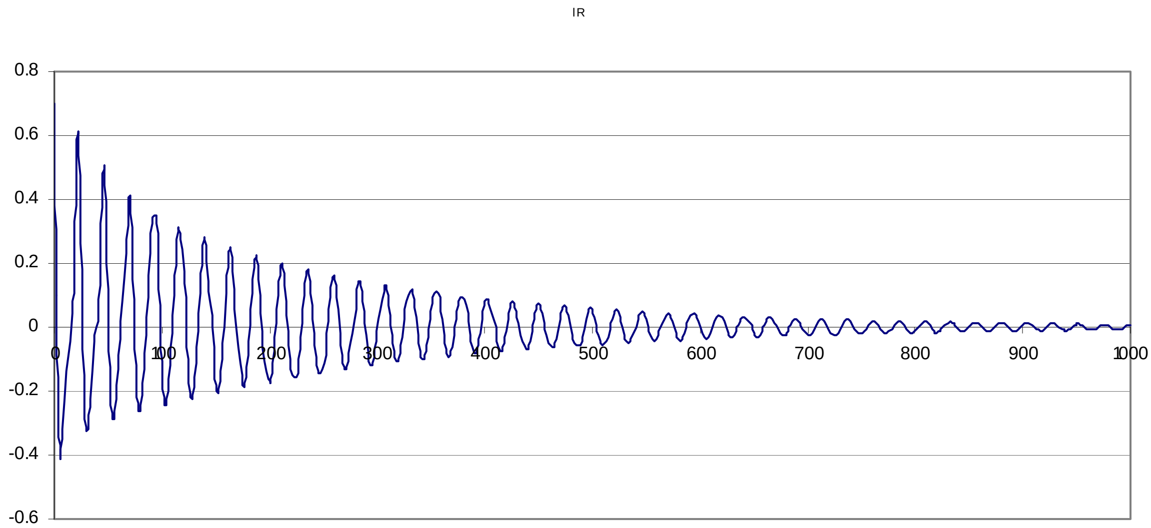

Fig. 5b: Logarithmic ratio $\ln\left(P_t^T / P_t\right)$

3.3 Influence of the cyclical components

The trigonometric form of the solution is[5]

$$P_t = 1.6662 \cdot 1.0961^t + \sum_{i=1}^{6} S_i \quad \text{for } t = 0,\ 1,\ 2,\ \ldots$$

with

$$S_i = r_i^t \cdot \left(A_i \cdot cos(phi[i] \cdot t) - B_1 \cdot sin(phi[i] \cdot t)\right) \quad t = 0,\ 1,\ 2,\ 3,\ \ldots \text{ (see Table 5).}$$

There are three possibilities for the (positive) modulus r_i of the cyclical components:

$r_i > 1$ increases the amplitudes of the population cycles over time (explosive cycles);
$r_i < 1$ decreases the amplitudes of the population cycles over time (damped cycles);
$r_i = 1$ leads to constant amplitudes and constant population cycles.

[5] The trigonometric form of a complex number $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$, where $r = |a + bi|$ is the modulus of z, and $\tan\theta = b/a$. θ is called the argument of z. Normally, we will require $0 \le \theta < 2\pi$.

Table 5: Cyclical components of the population growth model

i	Compo nent	modulus r	phi(i)	A_i	B_i	Period (double year)
	Trend	1.0961290				
1	S1	1.0875214	0.5261682	−0.1829698	−0.5346139	11.94
2	S2	1.0593267	1.0513774	−0.2150018	−0.166321	5.98
3	S3	1.0000000	1.5707963	−0.1621622	−0.02702703	4.00
4	S4	0.9941513	2.8800645	−0.00421404	0.02159716	2.18
5	S5	0.9357966	2.3733678	−0.02829957	0.05935263	2.65
6	S6	0.8911842	2.0125323	−0.07359357	0.02513567	3.12

As seen in Table 5, we have a growing geometric trend model which is superimposed by explosive (S1, S2), constant (S3), and damped cycles (S5, S6). The longest period is around 24 years (S1), whereas the shortest period is around 4 years (S4). The graphical representations of the six cycles are seen in Fig. 6 and Fig. 7.

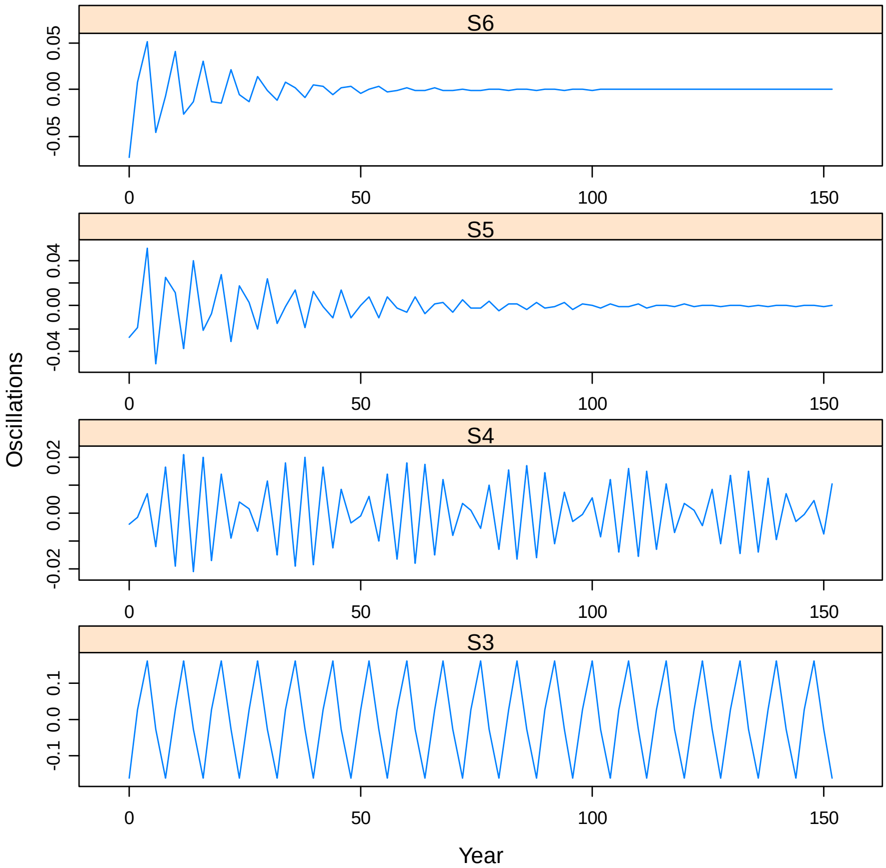

Fig. 6: Damped and constant cycles

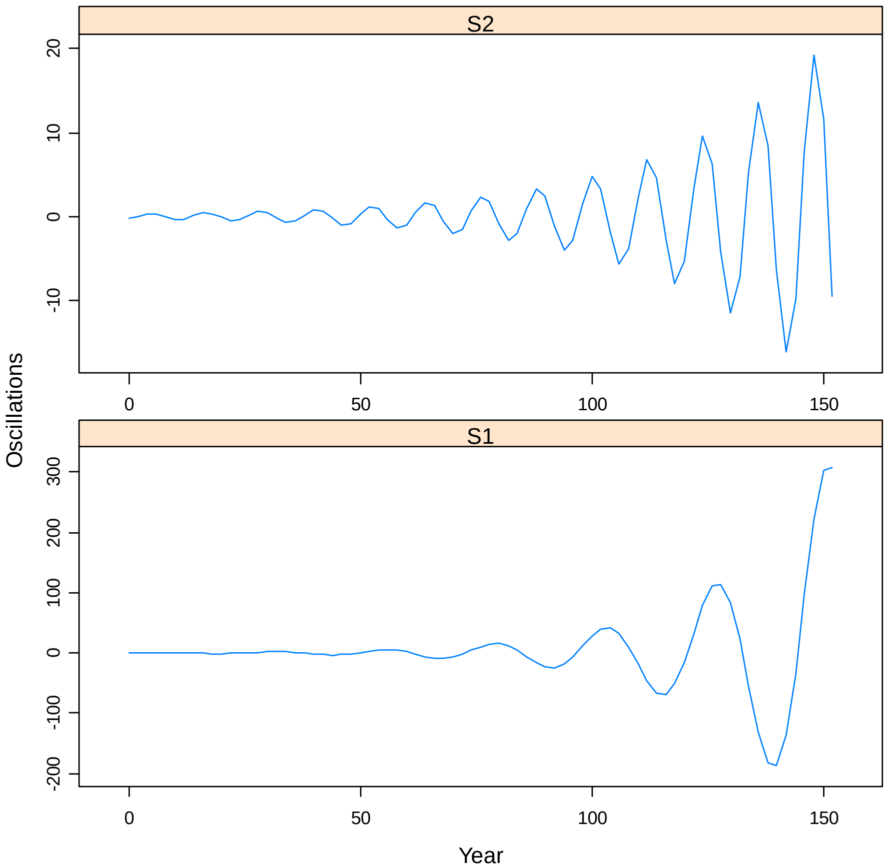

Fig. 7: Explosive cycles

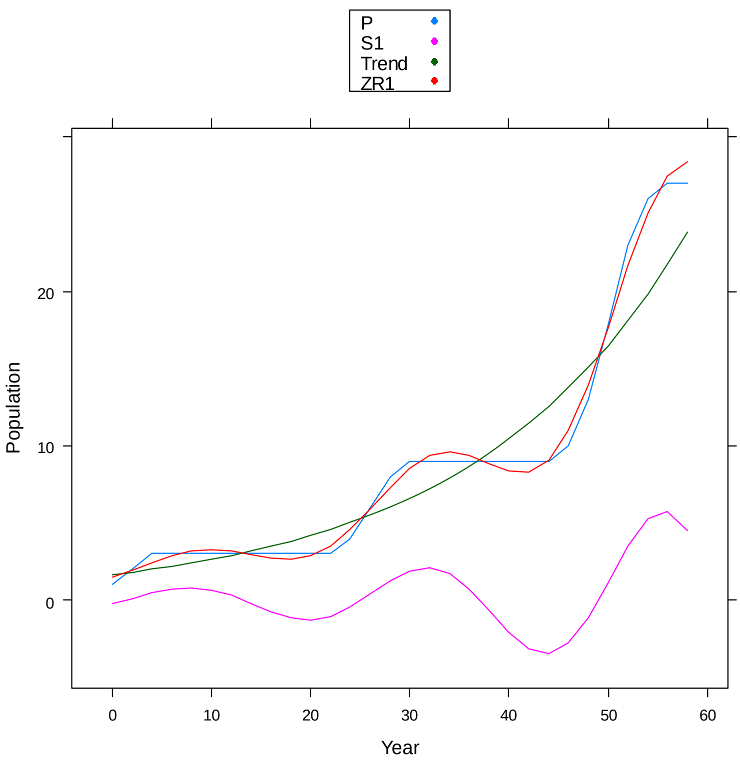

Fig. 8: Influence of the trend and cyclical component S1 on the population size at time t (ZR1 = Trend + S1)

In the long run, the population time series can be explained quite well by the sum of two components: ZR1 = Trend + S1. The short term fluctuations are only important at the beginning of the time series. In the very long run, the influence of the cycle S1 diminishes, although S1 is an explosive cycle, because the ratio S1/Trend tends to zero. The series can be explained solely by its trend component (see Fig. 8).

4. Questions

4.1 How realistic are the assumptions on fertility and mortality?

The fertility assumptions concerning the gross reproduction rate of 3 are more or less acceptable for that period. Completely unrealistic are the assumptions on mortality, which would imply a net reproduction of 3, too. Euler probably made these simple assumptions in order to carry out the calculations by hand at all. With today's computing options, we can assume more realistic assumptions. Assuming Süßmilch's life table, we obtain the following projection matrix: biennial survivor rates have been calculated from the lx-column of Süßmilch's life table (see Moser, 1839, pp. 77–78)

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0.661 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.897 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.956 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.965 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.973 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.985 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.984 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.984 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.984 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.980 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.979 & 0 \end{pmatrix}$$

The dominant eigenvalue of this matrix is 1.0311, which corresponds to a yearly growth rate of 1.54%. The net reproduction rate decreases from 3 to 1.45.

4.2 Similar models of Euler?

In "Sur la multiplication du genre humain" (pp. 548–552), Euler proposes a projection with the following assumptions (see also Girlich, 2007)

A1. The projection begins with one married couple, and each person is 20 years old.
A2. Marriage age 20 years
A3a. Each pair should give birth to one daughter and one son at the age of 22, 24, 26, 28 and 30 years.
A4a. Everyone reaches the age of 50 and dies afterwards.

Euler calculates the dominant eigenvalue as $\lambda = 1.13315$. This result corresponds to an annual growth rate of 6.4%. He considered this growth rate as probablytoo high, and as a consequence he changed the assumptions 3 and 4 for the calculations which he finally made for Süßmilch.

4.3 What is the effect of maximum age?

With another projection, Euler assumed a maximum age of 50 (see 4.2). If we do not change the irreducible part of the projection matrix, then the maximum age has no effect on the eigenvalue or the biennial growth rate. The maximum age, however, affects the population size and the stable equivalent population. We can model immortality in the Euler–Süßmilch model with the following matrix (Lefkovitch matrix)

$$L^{\text{inf}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & & & & & \vdots \\ & & & & & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & & 1 & 1 \end{pmatrix}$$

The ratio of the stable equivalent populations of L^{inf} and L is 2.391441/2.010051 ≈ 1.1898 (see 3.2). Thus, in the long run, the population in which immortality occurs is only about 19% higher than the population in which all people die at the age of 40.

4.4 Annual instead of biennial projection steps?

Smith; Keyfitz (1977, p. 79) and Pflaumer (1988, p. 12) assume one-year age groups of the Euler–Süßmilch model and thus implicitly an irreducible 26×26 matrix. The recursion equation is $B_n = B_{n-22} + B_{n-24} + B_{n-26}$ with the characteristic equation $\lambda^{26} - \lambda^4 - \lambda^2 - 1 = 0$. But the resulting eigenvalue $\lambda_1 = 1.04696$ is not dominant. Their statement that the population grows by about 4.7% each year in the long term is therefore only correct on average. Over the long term, consecutive annual growth rates of 0% and 9.61% are occurring with unstable age structures that fluctuate with constant cycles. Although the 26×26 matrix is irreducible, it is not primitive. Two non-complex eigenvalues, $\lambda_1 = 104696$ and $\lambda_2 = -1.04696$, with same magnitude exist. If one wants to project with annual steps, then the first row of the projection matrix must be modified: at least two "fertility rates" must be next to each other (see Pollard, 1973). This change would have a small effect on the dominant eigenvalue and thus on the growth rate

5. Concluding remarks

Although the projection model of Euler and Süßmilch is not realistic, it impressively shows that constant fertility and mortality rates lead in the long term to geometric growth and a stable age structure of the population. In this view, it is the first (numerical) demonstration of the strong ergodic theorem of demography which assumes fixed age-specific birth and death rates: in the long run, the population will geometrically grow with a constant growth rate. These findings were important for the further development of scientific theories, because until that time it was not obvious that population will grow geometrically. Famous researchers adopted the thesis of geometric population growth rates. Thomas Malthus's ideas about geometric population growth in his *Essay on the Principle of Population* (1798) came from the work of Euler and Süßmilch. Darwin, on the other hand, was inspired to formulate his theory of natural selection by reading Thomas Malthus's *Essay* (Klyve, 2014).

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Appendix

Numerical results of the female population up to the age of 26 (13×13 matrix)

Year	Population	Year	Population	Year	Population	Year	Population
0	1	76	67	152	2081	228	58555
2	2	78	76	154	2152	230	59950
4	3	80	80	156	2180	232	62218
6	3	82	81	158	2195	234	66574
8	3	84	81	160	2231	236	74174
10	3	86	81	162	2336	238	85680
12	3	88	81	164	2574	240	100795
14	3	90	82	166	3001	242	118098
16	3	92	87	168	3624	244	135402
18	3	94	101	170	4374	246	150529
20	3	96	127	172	5124	248	162111
22	3	98	162	174	5747	250	170042
24	4	100	197	176	6174	252	175499
26	6	102	223	178	6413	254	180723
28	8	104	237	180	6527	256	188742
30	9	106	242	182	6606	258	202966
32	9	108	243	184	6762	260	226428
34	9	110	243	186	7141	262	260649
36	9	112	244	188	7911	264	304573
38	9	114	250	190	9199	266	354295
40	9	116	270	192	10999	268	404029
42	9	118	315	194	13122	270	448042
44	9	120	390	196	15245	272	482682
46	10	122	486	198	17045	274	507652
48	13	124	582	200	18334	276	526264
50	18	126	657	202	19114	278	544964
52	23	128	702	204	19546	280	572431
54	26	130	722	206	19895	282	618136
56	27	132	728	208	20509	284	690043
58	27	134	730	210	21814	286	791650
60	27	136	737	212	24251	288	919517
62	27	138	764	214	28109	290	1062897
64	27	140	835	216	33320	292	1206366
66	27	142	975	218	39366	294	1334753
68	28	144	1191	220	45412	296	1438376
70	32	146	1458	222	50624	298	1516598
72	41	148	1725	224	54493	300	1578880
74	54	150	1941	226	56994		

Geometric trend model:

$P_t = 1.6662 \cdot 1.0961^t$ for $t = 0, 1, 2, 3, \ldots$.

Approximation by the geometric trend model for large t, e.g.,

$P_{year=300} = P_{150} = 1.6662 \cdot 1.0961^{150} = 1582172$.