Received: 15 July 2020 | Accepted: 23 November 2020

DOI: 10.1002/pamm.202000264

# Numerical approach for a continuum theory with higher stress gradients

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We use an extended balance of linear momentum derived from stress field analysis of higher order terms in power series expansion. Thus, the balance equation accounts for higher gradients of stress in the contiguity of continuum points. Interestingly, it does not coincide with the balance of linear momentum from strain gradient elasticity. As shown in [1], it exhibits an inverse sign for the extended term compared to strain gradient elasticity. We are interested in the mechanical interpretation of this inversed sign since it seems to inverse the stiffening effect of strain gradient elasticity. Therefore, we set up the weak form of our extended balance equation by means of Galerkin's approach. Then, we use the Finite Element Method to approximate the weak form with help of different shape functions. In this context we also use Isogeometric Analysis since it is very promising for a numerical model with higher gradients.



## 1 Introduction

A cubic subdomain of the deformed body in the actual configuration with dimension $\mathscr{L}_c$ is to be considered. The nonlinear stress tensor is approximated via power series expansion up to third order derivatives, see [1]. Lagrangian description of momentum balance of a static system without body force $\boldsymbol{\mathcal{R}}$ up to third order derivatives reads

$$\boldsymbol{\mathcal{R}} := \mathbf{R} + \mathbb{R} = \mathbf{0} \quad \text{with} \quad \mathbf{R} = \int_V \operatorname{Div} \mathbf{P} \, \mathrm{d}V \quad \text{and} \quad \mathbb{R} = \int_V \frac{\mathscr{L}_c^2}{24} \operatorname{Div} \operatorname{Grad} \operatorname{Div} \mathbf{P} \, \mathrm{d}V. \tag{1}$$

Here, $\mathbf{R}$ is the first order Lagrangian description balance of momentum, $\mathbb{R}$ is the higher order Lagrangian description balance of momentum, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor and the parameter $\mathscr{L}_c$ is defined as internal lenght scale.

## 2 Higher order discrete residual and stiffness

The concept of the isoparametric elements is used for interpolation of displacements $u_i$ and coordinates $X_i$, thus

$$u_i \approx \sum_{I=1}^{n} N^I u_i^I \quad \text{and} \quad X_i \approx \sum_{I=1}^{n} N^I X_i^I, \tag{2}$$

where $N^I$ are the shape functions, $u_i^I$ are nodal displacements and $X_i^I$ are the nodal coordinates. The total discrete residual $\mathcal{R}_i^I$ of a static system without body force is the sum of the first and higher order discrete residuals

$$\mathcal{R}_i^I = R_i^I + \mathbb{R}_i^I = 0, \tag{3}$$

where the higher order terms of the total discrete residual are given by

$$\mathbb{R}_i^I = -\int_V \frac{\mathscr{L}_c^2}{24} \frac{\partial L_k^I}{\partial X_k^J} \frac{\partial P_{ij}}{\partial X_j^J} \mathrm{d}V - \int_A \frac{\mathscr{L}_c^2}{24} \left[ \frac{\partial P_{ij}}{\partial X_j^J} L_k^I - \frac{\partial^2 P_{ij}}{\partial X_j^J \otimes \partial X_k^I} \right] N^J \mathrm{d}A_k. \tag{4}$$

The first derivative of the shape functions $N^I$ w.r.t. coordinates $X_i$ is defined as $L_i^I$. Discrete stiffness is defined as the first derivative of the total discrete residual $\mathcal{R}_i^I$ w.r.t. nodal displacements

$$\mathcal{K}_{iw}^{IW} = K_{iw}^{IW} + \mathbb{K}_{iw}^{IW}, \tag{5}$$

where the higher order terms of the total discrete stiffness are given

$$\mathbb{K}_{iw}^{IW} = -\int_V \frac{\mathscr{L}_c^2}{24} \frac{\partial L_k^I}{\partial X_k^J} \frac{\partial^2 P_{ij}}{\partial X_j^J \otimes \partial u_w^W} \mathrm{d}V - \int_A \frac{\mathscr{L}_c^2}{24} \left[ \frac{\partial^2 P_{ij}}{\partial X_j^J \otimes \partial u_w^W} L_k^I - \frac{\partial^3 P_{ij}}{\partial X_j^J \otimes \partial X_k^I \otimes \partial u_w^W} \right] N^J \mathrm{d}A_k. \tag{6}$$

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## 3 Higher order discrete derivatives

In Eq. (4) and Eq. (6), higher order stress gradients occur. For instance, a third order gradient of the first Piola-Kirchhoff stress tensor $P_{ij}$ is demanded and reads

$$\begin{aligned}\frac{\partial^3 P_{ij}}{\partial X_j^J \otimes \partial X_k^I \otimes \partial u_w^W} &= \frac{\partial^3 P_{ij}}{\partial F_{mn} \otimes \partial X_k^I \otimes \partial u_w^W}\frac{\partial F_{mn}}{\partial X_j^J} + \frac{\partial P_{ij}}{\partial F_{mn}}\frac{\partial^3 F_{mn}}{\partial X_j^J \otimes \partial X_k^I \otimes \partial u_w^W} \\ &+ \frac{\partial^2 P_{ij}}{\partial F_{mn} \otimes \partial X_k^I}\frac{\partial^2 F_{mn}}{\partial X_j^J \otimes \partial u_w^W} + \frac{\partial^2 P_{ij}}{\partial F_{mn} \otimes \partial u_w^W}\frac{\partial^2 F_{mn}}{\partial X_j^J \otimes \partial X_k^I}.\end{aligned} \tag{7}$$

Here, the deformation gradient $F_{ij}$ is a function of nodal displacements and the discrete gradient of the shape functions $L_i^I$

$$F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij} = L_j^I u_i^I + \delta_{ij}. \tag{8}$$

From [2], we know the partial variation of the deformation gradient w.r.t. geometry. The discrete version reads

$$\frac{\partial F_{ij}}{\partial X_k}\mathrm{d}X_k = -\frac{\partial u_i}{\partial X_k}\frac{\partial \mathrm{d}X_k}{\partial X_j} \quad \text{and hence,} \quad \frac{\partial L_i^I}{\partial X_j^J} = (-1)^1 \left(L_i^J L_j^I\right). \tag{9}$$

Some parts of Eq. (7) require the computation of even higher order gradients of the shape functions. Following Eq. (9), these can easily be identified as

$$\frac{\partial^2 L_i^I}{\partial X_j^J \otimes \partial X_k^K} = (-1)^2 \left(L_i^J L_j^K L_k^I + L_i^K L_j^I L_k^J\right). \tag{10}$$

In the same manner it is possible to compute gradients of shape functions of arbitrary order, presumed sufficient differentiability. In this context, isogeometric analysis seems a promising technique, as NURBS are chosen as shape functions, cf. e.g. [3], which can be easily constructed of higher order. Additionally, the continuity on element boundaries is not necessarily $C^0$.

## 4 Numerical example

For comparison of the standard solution with the extended model, a shear test is considered and analyesd using IGA. The number of control points are 9 in x- and 9 in y-direction. The elastic material parameters for the Neo-Hookean material are Young's modulus $E = 0.05$ and Poisson's ratio $\nu = 0.3$. Fig. (1) illustrates the different deformations and stress distribution in x-direction for different values of the internal lenght scale.

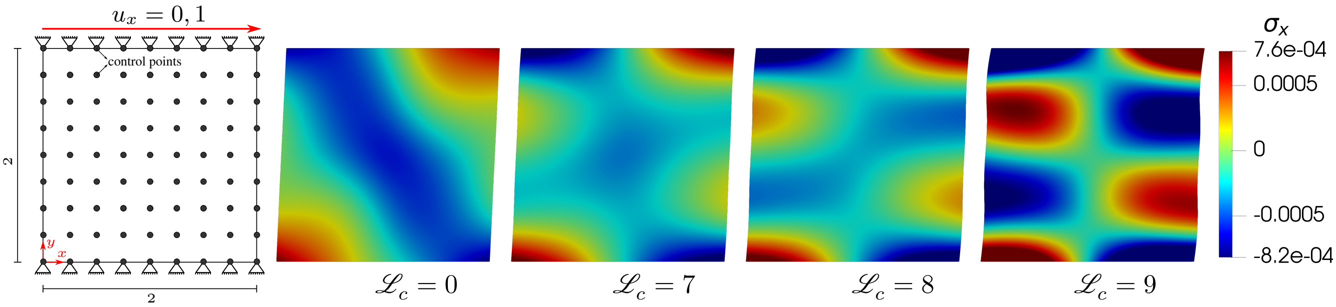


**Fig. 1:** Boundary conditions and Cauchy stress in x-direction presented on the Eulerian Configuration (displacements are not scaled.)

## 5 Conclusion

In addition to the classical standard solutions it is capable to evolve meaningful microstructural solutions from smooth and symmetric sets of boundary conditions without any imperfections. The model gives the homogeneous solution without specifying imperfections. The higher gradients of the shape functions can be determined by summation of the products of the first gradient of the shape functions.

**Acknowledgements** Open access funding enabled and organized by Projekt DEAL.

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