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Infinite-horizon optimal control – Asymptotics and dissipativity

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This note discusses the interplay between dissipativity and the asymptotics of continuous-time infinite-horizon optimal control problems. We focus on the results on convergence of optimal primal solutions derived in [6]. Moreover, we present a result on the attractivity of the infinite-horizon optimal adjoint trajectories, which is closely related to transversality conditions for infinite-horizon optimal control problems. Proofs and further results can be found in [6].

1 Introduction

The analysis of infinite-horizon optimal control problems arises in different contexts ranging from economics, to design of optimal feedback strategies, and inverse optimal control approaches. Early treatments of the problem include [10], a comprehensive overview is provided by [2]. It is also well-understood that there exist close relations between stability and infinite-horizon optimal control, see [12], and between dissipativity and stability, cf. [8, 11, 14]. Moreover, one may argue that Jan Willems constructed his formal system-theoretic definition of dissipativity by leveraging infinite-horizon optimal control [15, 16]. Recently, in the context of economic model predictive control a dissipativity notion of Optimal Control Problems (OCPs), proposed by [1], has proven to be of crucial importance in the stability analysis, see e.g. [5]. Moreover, it has been shown by [7, 9] that dissipativity allows certifying turnpike properties of OCPs.[1]

Despite the crucial importance of infinite-horizon optimal control for many problems, there are also open issues related to it. Obviously—and except for special cases like LQR problems—computing the transient solution to infinite-horizon problems is intrinsically difficult since the objective functional does not need to be bounded. Moreover, it is known since the seminal insights of Hubert Halkin [10] that in the infinite-horizon case solving the adjoint/co-state dynamics is challenging. This is mainly due to the fact that the corresponding adjoint transversality condition cannot be inferred by taking the asymptotic limit of the finite-horizon one. Specifically, Halkin constructed an example of a Lagrange problem wherein for any finite horizon the adjoint at $t = T$ has to be 0 (due to the absence of a Mayer term), while for the infinite-horizon cases it is shown that the adjoint does not converge 0. This note summarizes parts of the results of a recent paper [6] in which we analyze infinite-horizon OCPs with respect to the interplay between dissipativity and stability. Moreover, in [6] we have shown that under a strict dissipativity assumption, the optimal adjoint converges to the value of the optimal steady-state Lagrange multiplier of the dynamics. Put differently, strict dissipativity imposes an asymptotic limit on the co-state trajectory which can be different from 0.

2 Problem Statement

We are interested in time-invariant OCPs in Lagrange form given by

$$V_T(x_0) \doteq \inf_{u(\cdot)\in\mathcal{L}_\infty([0,T],\mathbb{R}^{n_u})} \int_0^T \ell(x(t),u(t))\mathrm{d}t \tag{1a}$$

$$\text{subject to} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t),u(t)),\ x(0)=x_0 \quad \text{and} \quad 0 \geq g_i(x(t),u(t)),\ i=1\dots n_g, \tag{1b}$$

wherein the horizon $T \in \mathbb{R}^+ \cup \infty$ can be finite or infinite. The dynamics $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, the stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, and the mixed input-path constraints $g_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, i = 1 \dots n_g$ are at least twice continuously differentiable. Moreover, we suppose that for all initial conditions of interest, i.e. $x_0 \in \mathbb{X}_0 \subseteq \mathbb{R}^{n_x}$, an optimal solution exists, such that the optimal state response is absolutely continuous. The object of investigation is the stability of the considered dynamics under the open-loop infinite-horizon optimal control $u^\star : \mathbb{R}_0^+ \times \mathbb{X}_0 \to \mathbb{R}^{n_u}$, i.e.,

$$\dot{x} = f(x, u^\star(t,x_0)), \quad x_0 \in \mathbb{X}_0. \tag{Σ}$$

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[1] The term turnpike property was coined by [3] and has received considerable attention in economics [2, 13]. It refers to similarity properties of solutions of OCPs being parametric in the initial condition and the horizon length, see [4] for a recent overview.

Our analysis relies on the following strict integral dissipation inequality

$$S(x^\star(t_1)) - S(x_0) \leq \int_0^{t_1} -\alpha_\ell\left(\|(x^\star(t),u^\star(t)) - \bar z\|\right) + \ell(x^\star(t),u^\star(t)) - \ell(\bar z)\,\mathrm{d}t, \tag{sDI}$$

to hold along optimal pairs for all $x_0 \in \mathbb{X}_0$, where $\alpha_\ell \in \mathcal{K}_\infty$ and $\bar z = (\bar x, \bar u)^\top$ is a steady state pair, i.e., $0 = f(\bar x, \bar u)$. We remark that strict dissipativity implies that $\bar z$ is optimal in problem (2) introduced below. Moreover, let $\bar\lambda, \bar\mu$ denote the corresponding Lagrange multipliers in (2).

Theorem 2.1 (Strict dissipativity ⇒ primal attractivity [6]) *For all $x_0 \in \mathbb{X}_0$, let OCP$_T(x_0)$ be strictly dissipative with respect to $\bar z = (\bar x, \bar u)^\top$ and suppose that, for all $x_0 \in \mathbb{X}_0$, $V_\infty(x_0) < \infty$. Then, for all $x_0 \in \mathbb{X}_0$, the solutions of (Σ) satisfy $\lim_{t\to\infty} x(t, x_0, u^\star(\cdot, x_0)) = \bar x$.*

Furthermore, if there exists an optimal input $u^\star(\cdot, x_0)$ absolutely continuous on $[0,\infty)$, then $\lim_{t\to\infty} u^\star(t, x_0) = \bar u$.

The proof given in [6] relies on Barbalat's Lemma. The extension towards the asymptotics of the adjoints is given next.

Theorem 2.2 (Strict dissipativity ⇒ adjoint attractivity [6]) *For all $x_0 \in \mathbb{X}_0$, let OCP$_T(x_0)$ be strictly dissipative at $\bar z = (\bar x, \bar u) \in \mathbb{Z}$, let $V_\infty(x_0) < \infty$, suppose that*

- *the Jacobian linearization of (Σ) at $(\bar x, \bar u)$, $(A, B) \doteq (f_x, f_u)$, is stabilizable and*
- *the uniqueness of the Lagrange multipliers $\bar\lambda \in \mathbb{R}^{n_x}, \bar\mu \in \mathbb{R}^{n_g}$ in*

$$\min_{(x,u)\in\mathbb{R}^{n_x+n_u}} \ell(x,u) \quad \textit{subject to} \quad 0 = f(x,u), \quad 0 \geq g_i(x,u),\ i = 1\dots n_g. \tag{2}$$

Then, for all $x_0 \in \mathbb{X}_0$, the infinite-horizon adjoint $\lambda^\star(\cdot, x_0)$ satisfies $\lim_{t\to\infty} \lambda^\star(t, x_0) = \bar\lambda$.

Further results without uniqueness of multipliers can also be derived. For details and for the discussion of the link to Hamilton-Jacobi-Bellman-Equations we refer to [6].

3 Conclusions

This note has recapitulated on key insights on the implications of strict dissipativity in infinite-horizon optimal control. Specifically, we have commented on adjoint transversality conditions for infinite-horizon optimal control and on the convergence of states and inputs based on a strict dissipativity assumption. These results and further ones in [6] can be regarded as a nonlinear extension to the classic results of Jan Willems [15].

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