Received: 13 October 2020 | Revised: 4 May 2021 | Accepted: 23 May 2021

DOI: 10.1002/mma.7604

RESEARCH ARTICLE

WILEY

Uniform convergence to equilibrium for a family of drift–diffusion models with trap-assisted recombination and self-consistent potential

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Communicated by: M. Brokate

We investigate a recombination–drift–diffusion model coupled to Poisson's equation modelling the transport of charge within certain types of semiconductors. In more detail, we study a two-level system for electrons and holes endowed with an intermediate energy level for electrons occupying trapped states. As our main result, we establish an explicit functional inequality between relative entropy and entropy production, which leads to exponential convergence to equilibrium. We stress that our approach is applied uniformly in the lifetime of electrons on the trap level assuming that this lifetime is sufficiently small.

KEYWORDS

entropy method, exponential convergence to equilibrium, PDEs in connection with semiconductor devices, reaction–diffusion equations, self-consistent potential, trapped states

MSC CLASSIFICATION

35Q81; 78A35; 35B40; 35K57

1 | INTRODUCTION AND MAIN RESULTS

We consider the following PDE–ODE recombination–drift–diffusion system coupled to Poisson's equation on a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega \in C^2$:

$$\begin{cases} \partial_t n = \nabla \cdot J_n(n, \psi) + R_n(n, n_{tr}), \\ \partial_t p = \nabla \cdot J_p(p, \psi) + R_p(p, n_{tr}), \\ \varepsilon\, \partial_t n_{tr} = R_p(p, n_{tr}) - R_n(n, n_{tr}), \\ -\lambda \Delta \psi = n - p + \varepsilon n_{tr} - D, \end{cases} \tag{1}$$

where the flux terms J_n, J_p and recombination terms R_n, R_p are defined as

$$J_n := \nabla n + n\nabla(\psi + V_n) = \mu_n \nabla \frac{n}{\mu_n} + n\nabla\psi, \mu_n := e^{-V_n},$$
$$J_p := \nabla p + p\nabla(-\psi + V_p) = \mu_p \nabla \frac{p}{\mu_p} - p\nabla\psi, \mu_p := e^{-V_p},$$
$$R_n := \frac{1}{\tau_n}\left(n_{tr} - \frac{n}{n_0\mu_n}(1 - n_{tr})\right), \qquad R_p := \frac{1}{\tau_p}\left(1 - n_{tr} - \frac{p}{p_0\mu_p}n_{tr}\right).$$

The variables n, p, and n_{tr} denote the densities of electrons in the conduction band, holes in the valence band, and electrons on the trap level (see Figure 1). Moreover, ψ represents the electrostatic potential generated by n, p, n_{tr}, and the time-independent doping profile $D \in L^\infty(\Omega)$. The constants $n_0, p_0, \tau_n, \tau_p > 0$ are positive recombination parameters, while $\varepsilon \in (0, \varepsilon_0]$ (for arbitrary but fixed $\varepsilon_0 > 0$) is a dimensionless quantity which can be interpreted, on the one hand, as the density of available trapped states and, on the other hand, as the lifetime of electrons on the trap level. Note that system (1) reduces in the limit $\varepsilon = 0$ to the famous Shockley–Read–Hall model of electron recombination[1,2] in semiconductor drift–diffusion systems; see, e.g., the corresponding works.[3,4] Finally,

$$V_n, V_p \in W^{2,\infty}(\Omega) \qquad \text{with} \qquad \hat{n} \cdot \nabla V_n = \hat{n} \cdot \nabla V_p = 0 \text{ on } \partial\Omega \tag{2}$$

represent external time-independent potentials.

The system is equipped with no-flux boundary conditions for the flux components and homogeneous Neumann boundary data for the electrostatic potential,

$$\hat{n} \cdot J_n = \hat{n} \cdot J_p = \hat{n} \cdot \nabla\psi = 0 \text{ on } \partial\Omega \tag{3}$$

where $\hat{n}$ represents the outer unit normal vector on $\partial\Omega$. As we will use a result in Goudon et al[4] for proving existence and uniqueness of global solutions, we choose the initial conditions $n(0,\cdot) = n_I$, $p(0,\cdot) = p_I$, and $n_{tr}(0,\cdot) = n_{tr,I}$ in accordance with Goudon et al[4]

$$n_I, p_I \in H^1(\Omega) \cap L^\infty(\Omega), \ \ n_I, p_I \geq 0, \ \ 0 \leq n_{tr,I} \leq 1. \tag{4}$$

For convenience, we assume that the volume of Ω is normalized, i.e., $|\Omega| = 1$, and we set

$$\overline{f} := \int_\Omega f(x)\,dx$$

for any function $f \in L^1(\Omega)$. This abbreviation is consistent with the usual definition of the average of f since $|\Omega| = 1$.

The main goal of the paper is to close the gap between the models investigated in Fellner and Kniely.[5,6] While the pure Shockley–Read–Hall model including the electrostatic potential has been considered in Fellner and Kniely,[5] the family of drift–diffusion models with trap-assisted recombination already appeared in Fellner and Kniely[6] but without coupling to Poisson's equation. Here, we focus on the exponential convergence to equilibrium for a PDE–ODE model including both trapped states dynamics and the self-consistent potential. More precisely, we obtain an explicit bound for the convergence rate by employing the so-called *entropy method* which amounts to deriving a functional inequality between an entropy functional and the associated entropy production. Concerning exponential convergence to equilibrium for reaction–diffusion systems, see, e.g., the related works.[7-11] The framework of the entropy method which we shall use here to obtain explicit bounds on the convergence rate originates from Desvillettes and Fellner,[12-14] where models from reversible chemistry have been studied. An earlier application of the entropy method, but using a non-constructive com-

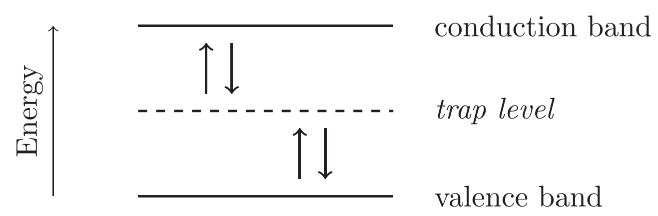

FIGURE 1 An illustration of the allowed transitions of electrons between the three energy levels

pactness argument, is presented in the works of Glitzky, Gröger, and Hünlich,,[15,16] where the authors prove exponential convergence for a model of electrically charged species taking the coupling to Poisson's equation into account.

Throughout the article, we will frequently encounter the following inhomogeneous Poisson equation with right hand side $f \in L^2(\Omega)$ subject to homogeneous Neumann boundary conditions:

$$-\lambda \Delta \psi = f \text{ in } \Omega, \qquad \hat{n} \cdot \nabla \psi = 0 \text{ on } \partial\Omega. \tag{5}$$

It is well-known that there exists a weak solution $\psi \in H^1(\Omega)$ if and only if $\overline{f} = 0$ holds true (compatibility condition with homogeneous Neumann boundary data). In this case, ψ is determined only up to an additive constant, which one can fix via the normalization $\overline{\psi} = 0$ to obtain a unique solution ψ.

Due to the previous considerations, we additionally have to demand that the initial data satisfy the charge-neutrality condition

$$\int_\Omega \left(n_I - p_I + \varepsilon n_{tr,I} - D\right) dx = 0. \tag{6}$$

As a consequence of the structure of system (1) and the no-flux boundary conditions in (3), we see that the total charge is preserved for all $t \geq 0$ in the sense that

$$\int_\Omega (n(t,\cdot) - p(t,\cdot) + \varepsilon n_{tr}(t,\cdot))\, dx = \int_\Omega \left(n_I - p_I + \varepsilon n_{tr,I}\right) dx = \int_\Omega D\, dx. \tag{7}$$

For the sake of completeness, we subsequently recall all assumptions on our model referring to the introduction above for further details and modelling issues.

Assumption 1.1. We work on a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class C^2 imposing the following constraints:

- n_0, p_0, τ_n, τ_p are positive constants, while $\varepsilon \in (0, \varepsilon_0]$ is bounded by $\varepsilon_0 > 0$,
- $D \in L^\infty(\Omega)$ and $V_n, V_p \in W^{2,\infty}(\Omega)$ with $\hat{n} \cdot \nabla V_n = \hat{n} \cdot \nabla V_p = 0$ on $\partial\Omega$ where $\hat{n}$ represents the outer unit normal vector on $\partial\Omega$,
- $\hat{n} \cdot J_n = \hat{n} \cdot J_p = \hat{n} \cdot \nabla \psi = 0$ on $\partial\Omega$,
- $n_I, p_I \in H^1(\Omega) \cap L^\infty(\Omega)$, $n_I, p_I \geq 0$, $0 \leq n_{tr,I} \leq 1$, and (6) holds.

Throughout this article, we suppose Assumption 1.1 to hold.

Definition 1.2. A global weak solution to (1)–(4) and (6) is a quadruple $(n, p, n_{tr}, \psi) : [0, \infty) \to H^1(\Omega)^2 \times L^\infty(\Omega) \times H^1(\Omega)$ such that for all $T > 0$ the following conditions are satisfied:

- $n, p \in L^2(0, T; H^1(\Omega))$, and

$$\begin{aligned}
&- \int_0^T \langle u'(t), n(t) \rangle_{H^1(\Omega)^* \times H^1(\Omega)} dt - \int_\Omega n_I u(0)\, dx \\
&= - \int_0^T \int_\Omega J_n(n, \psi) \cdot \nabla u\, dx\, dt + \int_0^T \int_\Omega R_n(n, n_{tr}) u\, dx\, dt, \\
&- \int_0^T \langle u'(t), p(t) \rangle_{H^1(\Omega)^* \times H^1(\Omega)} dt - \int_\Omega p_I u(0)\, dx \\
&= - \int_0^T \int_\Omega J_p(p, \psi) \cdot \nabla u\, dx\, dt + \int_0^T \int_\Omega R_p(p, n_{tr}) u\, dx\, dt,
\end{aligned}$$

  for all $u \in W_2(0, T) := \left\{ f \in L^2(0, T; H^1(\Omega)) \,|\, \partial_t f \in L^2(0, T; H^1(\Omega)^*) \right\}$ subject to $u(T) = 0$,
- $n_{tr}(T) = n_{tr,I} + \frac{1}{\varepsilon} \int_0^T \left(R_p(p, n_{tr}) - R_n(n, n_{tr})\right) dt$,
- $\lambda \int_\Omega \nabla \psi(T) \cdot \nabla w\, dx = \int_\Omega (n(T) - p(T) + \varepsilon n_{tr}(T) - D)\, w\, dx$ for all $w \in H^1(\Omega)$.

We further mention the embedding $W_2(0, T) \hookrightarrow C([0, T], L^2(\Omega))$ known from PDE theory (see, e.g., Chipot[17]).

Proposition 1.3 (Global solutions). *There exists a unique global weak solution* $(n, p, n_{tr}, \psi) : [0, \infty) \to H^1(\Omega)^2 \times L^\infty(\Omega) \times H^1(\Omega)$ *of (1)–(4) and (6) with* $\overline{\psi(t, \cdot)} = 0$ *for all* $t \geq 0$. *This solution satisfies* $n, p \in L^2(0, T; H^1(\Omega))$ *for all* $T > 0$ *uniformly in* $\varepsilon \in (0, \varepsilon_0]$ *as well as (7). Moreover,* $n, p \geq 0$, $0 \leq n_{tr} \leq 1$, *and there exist positive constants* M, $K(M)$ *(again uniformly in* $\varepsilon \in (0, \varepsilon_0]$*) such that*

$$\|n(t)\|_{L^\infty(\Omega)} + \|p(t)\|_{L^\infty(\Omega)} \leq M \text{ and } \|\psi(t)\|_{H^2(\Omega)} + \|\psi(t)\|_{C(\overline{\Omega})} \leq K \tag{8}$$

for all $t \geq 0$. *In addition, there exists a positive constant* $\mu(M, K) < \frac{1}{2}$ *(uniformly in* $\varepsilon \in (0, \varepsilon_0]$*) such that*

$$n(t, x),\ p(t, x) \geq \mu \min\left\{t^2, 1\right\} \tag{9}$$

and

$$n_{tr}(t, x) \in \left[\min\left\{\frac{1}{2\varepsilon_0 \tau_p} t, \mu\right\}, 1 - \min\left\{\frac{1}{2\varepsilon_0 \tau_n} t, \mu\right\}\right] \tag{10}$$

for all $t \geq 0$ *and a.e.* $x \in \Omega$. *Finally,* $n, p \in W_2(0, T) \hookrightarrow C([0, T], L^2(\Omega))$, $n_{tr} \in C([0, T], L^\infty(\Omega))$, *and* $\psi \in C([0, T], H^2(\Omega))$, *where each inclusion holds true uniformly in* $\varepsilon \in (0, \varepsilon_0]$.

Proposition 1.4 (Equilibrium states). *The stationary system*

$$\begin{cases} \nabla \cdot J_n(n) + R_n(n, n_{tr}) = 0, \\ \nabla \cdot J_p(p) + R_p(p, n_{tr}) = 0, \\ R_n(n, n_{tr}) = R_p(p, n_{tr}), \\ -\lambda \Delta\psi = n - p + \varepsilon n_{tr} - D, \end{cases} \tag{11}$$

subject to $\hat{n} \cdot J_n = \hat{n} \cdot J_p = \hat{n} \cdot \nabla\psi = 0$ *admits a unique solution* $(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty) \in (H^1(\Omega) \cap L^\infty(\Omega))^4$ *satisfying* $\overline{\psi_\infty} = 0$. *The equilibrium potential* ψ_∞ *is continuous and there exists a positive constant* K_∞ *such that*

$$\|\psi_\infty\|_{H^2(\Omega)} + \|\psi_\infty\|_{C(\overline{\Omega})} \leq K_\infty.$$

Moreover, there exist positive constants $M_\infty(K_\infty) > 1$ *and* $\mu_\infty(K_\infty) < \frac{1}{2}$ *such that*

$$n_\infty(x),\ p_\infty(x) \in (\mu_\infty, M_\infty) \text{ and } n_{tr,\infty}(x) \in (\mu_\infty, 1 - \mu_\infty) \tag{12}$$

for a.e. $x \in \Omega$. *The constants* K_∞, M_∞, *and* μ_∞ *are independent of* $\varepsilon \in (0, \varepsilon_0]$. *In detail, the equilibrium densities* n_∞, p_∞, *and* $n_{tr,\infty}$ *read*

$$n_\infty = n_* e^{-\psi_\infty - V_n},\ p_\infty = p_* e^{\psi_\infty - V_p},\ n_{tr,\infty} = \frac{n_*}{n_* + n_0 e^{\psi_\infty}} = \frac{p_0}{p_0 + p_* e^{\psi_\infty}} \tag{13}$$

where the positive constants n_* *and* p_* *are uniquely determined in terms of* ψ_∞ *by*

$$n_* p_* = n_0 p_0 \text{ and } n_* \overline{e^{-\psi_\infty - V_n}} - p_* \overline{e^{\psi_\infty - V_p}} + \varepsilon \overline{\frac{n_*}{n_* + n_0 e^{\psi_\infty}}} - \overline{D} = 0. \tag{14}$$

Furthermore, the following relations hold true:

$$n_{tr,\infty} = \frac{n_\infty (1 - n_{tr,\infty})}{n_0 \mu_n} \text{ and } 1 - n_{tr,\infty} = \frac{p_\infty n_{tr,\infty}}{p_0 \mu_p}. \tag{15}$$

We introduce the *entropy functional* $E(n, p, n_{tr}, \psi)$ for non-negative functions $n, p, n_{tr} \in L^2(\Omega)$ satisfying $n_{tr} \leq 1$ and $\overline{n} - \overline{p} + \varepsilon\overline{n_{tr}} = \overline{D}$ where $\psi \in H^1(\Omega)$ is the unique solution of (5) with right hand side $f = n - p + \varepsilon n_{tr} - D$ and normalization $\overline{\psi} = 0$:

$$E(n,p,n_{tr},\psi) := \int_\Omega \Bigg(n \ln \frac{n}{n_0\mu_n} - (n - n_0\mu_n) \\ + p \ln \frac{p}{p_0\mu_p} - (p - p_0\mu_p) + \frac{\lambda}{2}|\nabla\psi|^2 + \varepsilon \int_{1/2}^{n_{tr}} \ln\left(\frac{s}{1-s}\right) ds \Bigg)\, dx. \tag{16}$$

The densities n and p enter via Boltzmann entropy contributions $a \ln a - (a-1) \geq 0$, whereas n_{tr} appears within the entropy functional via an integral term. We first mention that the integral $\int_{1/2}^{n_{tr}} \ln\left(\frac{s}{1-s}\right) ds$ is non-negative and finite for all $n_{tr}(x) \in [0,1]$. In more detail, we may write

$$\int_{1/2}^{n_{tr}} \ln\left(\frac{s}{1-s}\right) ds = [n_{tr} \ln n_{tr} - (n_{tr} - 1)] + [(1-n_{tr}) \ln(1-n_{tr}) - ((1-n_{tr}) - 1)] + \ln 2 - 1.$$

Consequently, both the occupied and unoccupied trapped states (n_{tr} and $1 - n_{tr}$) are described via Boltzmann statistics within the entropy functional, and the integral $\int_{1/2}^{n_{tr}} \ln\left(\frac{s}{1-s}\right) ds$ allows to combine the contributions of n_{tr} and $1 - n_{tr}$ in a compact fashion.

One can further verify that the entropy functional (16) is indeed a Lyapunov functional: By defining the *entropy production functional*

$$P(n,p,n_{tr},\psi) := -\frac{d}{dt} E(n,p,n_{tr},\psi), \tag{17}$$

we first calculate along solutions of (1) that formally

$$P(n,p,n_{tr},\psi) = \int_\Omega \left(\frac{|J_n|^2}{n} + \frac{|J_p|^2}{p} - R_n \ln\left(\frac{n(1-n_{tr})}{n_0\mu_n n_{tr}}\right) - R_p \ln\left(\frac{p n_{tr}}{p_0\mu_p(1-n_{tr})}\right)\right) dx. \tag{18}$$

The entropy production functional involves non-negative flux terms as well as recombination terms of the form $(a-1)\ln a \geq 0$. The entropy production P is, therefore, a non-negative functional, which ensures the monotone decrease in time of the entropy E along trajectories of (1). More precisely, it will be shown in Theorem 1.6 that the global weak solutions to (1) obtained in Proposition 1.3 satisfy a suitable weak version of (17), see (19) below.

The following theorem constitutes a so-called entropy–entropy production (EEP) estimate. This is a functional inequality between entropy and entropy production for arbitrary, yet admissible non-negative functions $n, p, n_{tr} \in L^\infty(\Omega)$, $n_{tr} \leq 1$; in particular, the electrostatic potential $\psi \in H^1(\Omega)$ in the following theorem must be the unique solution of (5) subject to $f = n - p + \varepsilon n_{tr} - D$ and the normalization $\overline{\psi} = 0$.

Theorem 1.5 (Entropy–entropy production estimate). *Consider all non-negative functions $n, p, n_{tr} \in L^\infty(\Omega)$ subject to $n, p \leq \mathcal{M}$, $n_{tr} \leq 1$, and $\overline{n} - \overline{p} + \varepsilon\overline{n_{tr}} = \overline{D}$ and accordingly determine $\psi \in H^1(\Omega)$ as the unique solution to (5) with $f = n - p + \varepsilon n_{tr} - D$ and $\overline{\psi} = 0$. Then, there exist explicit constants $\varepsilon_0 > 0$ and $C_{EEP} > 0$ depending on $\mathcal{M}$ and on K_∞ (as given in Proposition 1.4) such that*

$$E(n,p,n_{tr},\psi) - E(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty) \leq C_{EEP} P(n,p,n_{tr},\psi)$$

holds true for all $\varepsilon \in (0, \varepsilon_0]$.

Note that C_{EEP} is independent of $\varepsilon \in (0, \varepsilon_0]$ and that this abstract EEP inequality can be applied to the global solution to (1) by using $\mathcal{M} = M$ from Proposition 1.3. We are then able to prove the exponential decay of the entropy relative to the equilibrium by using a Gronwall argument.

Theorem 1.6 (Exponential decay of the relative entropy). *Let $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ from Theorem 1.5, and let (n, p, n_{tr}, ψ) be the unique global weak solution to (1) with non-negative initial datum $(n_I, p_I, n_{tr,I}) \in (H^1(\Omega) \cap L^\infty(\Omega))^2 \times L^\infty(\Omega)$, $n_{tr,I} \leq 1$, satisfying $\overline{n_I} - \overline{p_I} + \varepsilon\overline{n_{tr,I}} = \overline{D}$ according to Proposition 1.3. In addition, let $(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty)$ be the unique equilibrium state characterised in Proposition 1.4 as a solution to (11). Then,*

(n, p, n_{tr}, ψ) fulfils the weak entropy production law

$$E(n,p,n_{tr},\psi)(t_1) + \int_{t_0}^{t_1} P(n,p,n_{tr},\psi)(s)\,ds = E(n,p,n_{tr},\psi)(t_0) \tag{19}$$

for all $0 < t_0 \leq t_1 < \infty$. As a consequence, $E(n, p, n_{tr}, \psi)$ converges exponentially to $E(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty)$ with explicit rate and constant as a function of time $t \geq 0$. More precisely,

$$E(n,p,n_{tr},\psi) - E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty) \leq \left(E(n_I,p_I,n_{tr,I},\psi_I) - E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty)\right) e^{-C_{EEP}^{-1}t} \tag{20}$$

where $\psi_I \in H^1(\Omega)$ is the unique weak solution to (5) with $f = n_I - p_I + \varepsilon n_{tr,I} - D$ and $\overline{\psi_I} = 0$.

Corollary 1.7 (Exponential convergence to the equilibrium)**.** *Under the hypotheses of Theorem 1.6, the following improved convergence properties with constants $0 < c, C < \infty$ both depending on M and K_∞ but not on $\varepsilon \in (0, \varepsilon_0]$ hold true for all $t \geq 0$:*

$$\|n - n_\infty\|_{L^\infty(\Omega)} + \|p - p_\infty\|_{L^\infty(\Omega)} + \|n_{tr} - n_{tr,\infty}\|_{L^\infty(\Omega)} + \|\psi - \psi_\infty\|_{H^2(\Omega)} \leq Ce^{-ct}. \tag{21}$$

In particular, $\psi \to \psi_\infty$ in $L^\infty(\Omega)$ at an exponential rate.

The remainder of this article is devoted to the proofs of the various statements above. Section 2 collects the proofs of Propositions 1.3 and 1.4. The proof of Theorem 1.5 along with the necessary prerequisites is contained in Section 3, while the results on exponential convergence to equilibrium are proven in Section 4. A brief section providing an outlook to future research concludes the paper.

2 | GLOBAL SOLUTION AND EQUILIBRIUM STATE

Proof of Proposition 1.3. The existence of such a unique global solution (n, p, n_{tr}, ψ) as well as $(n, p) \in L^2(0, T; H^1(\Omega))$ for all $T > 0$ uniformly for $\varepsilon \in (0, \varepsilon_0]$ and $0 \leq n_{tr} \leq 1$ are a consequence of Goudon et al[4, Lemma 3.1] The uniform-in-time L^∞ bounds for n and p follow similar to Di Francesco et al,[18, Lemma 4.1] where a Nash–Moser-type iteration for L^r norms, $r \geq 1$, of n and p has been employed. But as the coupling to Poisson's equation is missing in Di Francesco et al,[18] we have to slightly modify the line of arguments.

The evolution of the L^{r+1} norm, $r \geq 1$, of n and p can be reformulated as

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left(n^{r+1} + p^{r+1}\right) dx &= (r+1)\int_\Omega \left(-rn^{r-1}\nabla n \cdot (\nabla n + n\nabla(\psi + V_n))\right. \\
&\quad \left. -rp^{r-1}\nabla p \cdot \left(\nabla p + p\nabla(-\psi + V_p)\right) + n^r R_n + p^r R_p\right) dx \\
&\leq -\frac{4r}{r+1}\int_\Omega \left|\nabla n^{\frac{r+1}{2}}\right|^2 dx - \frac{4r}{r+1}\int_\Omega \left|\nabla p^{\frac{r+1}{2}}\right|^2 dx + r\int_\Omega n^{r+1}\Delta(\psi + V_n)\,dx \\
&\quad + r\int_\Omega p^{r+1}\Delta(-\psi + V_p)\,dx + (r+1)\int_\Omega \left(n^r R_n + p^r R_p\right) dx.
\end{aligned}$$

To the last term, we apply the estimate $(r+1)n^r \leq \frac{1}{r} + 2rn^{r+1}$ which follows from Young's inequality $ab \leq \frac{1}{q}a^q + \frac{1}{s}b^s$ with $a := \left(\frac{1}{r+1}\right)^{\frac{1}{r+1}}$, $b := \left(\frac{1}{r+1}\right)^{-\frac{1}{r+1}} n^r$, $q := r+1$, and $s := \frac{r+1}{r}$. As a consequence of $-\lambda\Delta\psi = n - p + \varepsilon n_{tr} - D$, $(n^{r+1} - p^{r+1})(n-p) \geq 0$, $|R_n| \leq C(1+n)$, and $|R_p| \leq C(1+p)$, we then deduce

$$\frac{d}{dt}\int_\Omega \left(n^{r+1} + p^{r+1}\right) dx \leq -\frac{4r}{r+1}\int_\Omega \left|\nabla n^{\frac{r+1}{2}}\right|^2 dx - \frac{4r}{r+1}\int_\Omega \left|\nabla p^{\frac{r+1}{2}}\right|^2 dx + \widehat{C}r\int_\Omega \left(n^{r+1} + p^{r+1}\right) dx + \frac{\widehat{C}}{r} \tag{22}$$

with a constant $\widehat{C} > 0$ depending on ε_0 but not on r. One can now proceed as in Di Francesco et al[18, Lemma 4.1] For completeness, we briefly collect the main arguments below and refer to Di Francesco et al[18] for the details. By utilizing the Gagliardo–Nirenberg-type inequality $\|f\|_{L^2(\Omega)} \le C_{GN}\|f\|_{L^1(\Omega)}^{\frac{2}{5}}\|f\|_{H^1(\Omega)}^{\frac{3}{5}}$ for $f := n^{\frac{r+1}{2}}$ and $f := p^{\frac{r+1}{2}}$, one derives

$$\int_\Omega \left(n^{r+1} + p^{r+1}\right) dx \le \delta \int_\Omega \left(\left|\nabla n^{\frac{r+1}{2}}\right|^2 + \left|\nabla p^{\frac{r+1}{2}}\right|^2 \right) dx + \frac{\widetilde{C}}{\delta} \left(\int_\Omega \left(n^{\frac{r+1}{2}} + p^{\frac{r+1}{2}} \right) dx \right)^2 \tag{23}$$

where $\widetilde{C} > 0$ is a constant independent of r and $\delta > 0$. We now introduce $\lambda_k := 2^k - 1$ for $k \ge 1$ and set $r := \lambda_k$. Choosing a sufficiently small constant $A > 0$ and defining $\delta_k := \frac{A}{\lambda_k}$ results in

$$\delta_k \left(\widehat{C}\lambda_k + \delta_k \right) \le \frac{4\lambda_k}{\lambda_k + 1}$$

for all $k \ge 1$. By multiplying (23) with $\widehat{C}\lambda_k + \delta_k$ and by combining the result with (22), we arrive at

$$\frac{d}{dt}\int_\Omega \left(n^{\lambda_k+1} + p^{\lambda_k+1}\right) dx \le -\delta_k \int_\Omega \left(n^{\lambda_k+1} + p^{\lambda_k+1}\right) dx + B\lambda_k(\lambda_k + \delta_k) \sup_{0\le\tau\le t} \left(\int_\Omega \left(n^{\frac{r+1}{2}} + p^{\frac{r+1}{2}} \right) dx \right)^2 + \frac{\widehat{C}}{\lambda_k}$$

with the constant $B := \frac{\widehat{C}\widetilde{C}}{A}$. The uniform L^∞ bounds on n and p now follow from Di Francesco et al[18, Lemma 4.2]

As, in particular, $\|n(t) - p(t) + \varepsilon n_{tr}(t) - D\|_{L^2(\Omega)}$ is uniformly bounded in $t \ge 0$, we conclude that $\|\psi(t)\|_{H^2(\Omega)}$ is uniformly bounded in time by applying standard elliptic regularity theory (see, e.g., Mikhaılov[19, Chap. IV. §2. Theorem 4]). The announced bound on $\|\psi(t)\|_{C(\overline{\Omega})}$ follows from the embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ valid in $\mathbb{R}^3$.

The regularity $\partial_t n, \partial_t p \in L^2(0,T;H^1(\Omega)^*)$ and, hence, $n, p \in W_2(0,T) \hookrightarrow C([0,T], L^2(\Omega))$ uniformly for $\varepsilon \in (0, \varepsilon_0]$ is easily inferred from the corresponding bounds on $J_n, J_p \in L^2((0,T)\times\Omega)$ and $R_n, R_p \in L^\infty((0,T)\times\Omega)$. Likewise, $n_{tr} \in C([0,T], L^\infty(\Omega))$ and $\psi \in C([0,T], H^2(\Omega))$, where both inclusions hold true uniformly for $\varepsilon \in (0, \varepsilon_0]$.

For showing the upper and lower bound on n_{tr} in (10), we multiply the third equation in (1) with τ_p and observe that

$$\varepsilon \partial_t(\tau_p n_{tr}) \ge 1 - \rho n_{tr}$$

holds true with a constant $\rho(M) > 1$ due to $\|p\|_{L^\infty(\Omega)} \le M$. We now distinguish the following three cases for all $t \ge 0$ and a.e. $x \in \Omega$: $n_{tr}(t,x) \ge \frac{1}{\rho}$, $n_{tr}(t,x) \in [\frac{1}{2\rho}, \frac{1}{\rho})$, and $n_{tr}(t,x) < \frac{1}{2\rho}$. In the first case, $\partial_t(\tau_p n_{tr}(t,x)) \le 0$, while in the second case $\partial_t(\tau_p n_{tr}(t,x)) > 0$. And in the third case, $\partial_t(\tau_p n_{tr}(t,x)) > \frac{1}{2\varepsilon_0}$. Defining $t_0 := \frac{\varepsilon_0 \tau_p}{\rho}$, this ensures

$$\tau_p n_{tr}(t,x) \ge \frac{t}{2\varepsilon_0}, \ t \in [0, t_0], \qquad \text{and} \qquad n_{tr}(t,x) \ge \frac{1}{2\rho}, \ t \ge t_0. \tag{24}$$

The upper bound on n_{tr} follows by applying the same arguments to $\tau_n(1 - n_{tr})$.

Concerning the bounds on n and p in (9), we follow the lines in Fellner and Kniely[6] and concentrate on the arguments for n as the result for p can be derived analogously. For simplicity, we set w.l.o.g. $\tau_n = \tau_p = 1$ in the following calculations. The temporal derivative of n is then bounded from below by

$$\partial_t n \ge \nabla \cdot (\nabla n + n\nabla(\psi + V_n)) + n_{tr} - \alpha n \tag{25}$$

with a constant $\alpha > 0$. Employing the no-flux boundary conditions from (3), we first test (25) with $\left(n - \mu_1 t^2\right)_-$ for $t \in [0, t_0]$ where $\mu_1 > 0$ is a constant specified below and where we abbreviate $(\cdot)_- := \min\{\cdot, 0\}$. This entails

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega \left(n - \mu_1 t^2\right)_-^2 dx &= \int_\Omega \left(n - \mu_1 t^2\right)_- (\partial_t n - 2\mu_1 t)\, dx \\
&\leq \int_\Omega \left(n - \mu_1 t^2\right)_- (\nabla \cdot (\nabla n + n\nabla(\psi + V_n)) + n_{tr} - \alpha n - 2\mu_1 t)\, dx \\
&= -\int_\Omega \mathbb{1}_{n \leq \mu_1 t^2} \nabla n \cdot \left(\nabla n + (n - \mu_1 t^2)\nabla(\psi + V_n) + \mu_1 t^2 \nabla(\psi + V_n)\right) dx \\
&\quad + \int_\Omega \left(n - \mu_1 t^2\right)_- (n_{tr} - \alpha n - 2\mu_1 t)\, dx.
\end{aligned}
$$

Omitting the first term on the right hand side, we further derive

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega \left(n - \mu_1 t^2\right)_-^2 dx \leq &-\frac{1}{2}\int_\Omega \nabla\left[\left(n - \mu_1 t^2\right)_-^2\right] \cdot \nabla(\psi + V_n) dx \\
&- \int_\Omega \nabla\left[\left(n - \mu_1 t^2\right)_-\right] \cdot \mu_1 t^2 \nabla(\psi + V_n) dx \\
&+ \int_\Omega \left(n - \mu_1 t^2\right)_- (n_{tr} - \alpha n - 2\mu_1 t)\, dx.
\end{aligned}
$$

We now integrate by parts utilizing $\hat{n} \cdot \nabla\psi = \hat{n} \cdot \nabla V_n = 0$ on $\partial\Omega$. Due to the bound $n_{tr}(t, x) \geq \frac{t}{2\varepsilon_0}$ on the considered interval $t \in [0, t_0]$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega \left(n - \mu_1 t^2\right)_-^2 dx &\leq \frac{1}{2}\int_\Omega \left(n - \mu_1 t^2\right)_-^2 \|\Delta(\psi + V_n)\|_{L^\infty(\Omega)} dx \\
&+ \int_\Omega \left(n - \mu_1 t^2\right)_- \left(\frac{1}{2\varepsilon_0} - \alpha\mu_1 t_0 - 2\mu_1 - \mu_1 t_0 \|\Delta(\psi + V_n)\|_{L^\infty(\Omega)}\right) t\, dx.
\end{aligned}
$$

Choosing $\mu_1 > 0$ according to $\mu_1 \left(2 + \alpha t_0 + t_0 \|\Delta(\psi + V_n)\|_{L^\infty(\Omega)}\right) \leq \frac{1}{2\varepsilon_0}$, we deduce

$$
\frac{d}{dt}\int_\Omega \left(n - \mu_1 t^2\right)_-^2 dx \leq \|\Delta(\psi + V_n)\|_{L^\infty(\Omega)} \int_\Omega \left(n - \mu_1 t^2\right)_-^2 dx.
$$

Because of $\int_\Omega (n(0, x))_-^2\, dx = 0$, we derive $\int_\Omega \left(n - \mu_1 t^2\right)_-^2 dx = 0$ for all $t \in [0, t_0]$ by applying a Gronwall argument. We thus arrive at $n(t, x) \geq \mu_1 t^2$ for all $t \in [0, t_0]$ and a.e. $x \in \Omega$.

In the situation $t \geq t_0$, we test (25) with $(n - \mu_2)_-$ where $\mu_2 > 0$ is another constant to be specified. As above, we calculate

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega (n - \mu_2)_-^2\, dx &= \int_\Omega (n - \mu_2)_- \partial_t n\, dx \\
&\leq \int_\Omega (n - \mu_2)_- (\nabla \cdot (\nabla n + n\nabla(\psi + V_n)) + n_{tr} - \alpha n)\, dx \\
&= -\int_\Omega \mathbb{1}_{n \leq \mu_2} \nabla n \cdot (\nabla n + (n - \mu_2)\nabla(\psi + V_n) + \mu_2 \nabla(\psi + V_n))\, dx \\
&\quad + \int_\Omega (n - \mu_2)_- (n_{tr} - \alpha n)\, dx.
\end{aligned}
$$

The same reasoning as above gives rise to

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega (n - \mu_2)_-^2\, dx \leq &-\frac{1}{2}\int_\Omega \nabla\left[(n - \mu_2)_-^2\right] \cdot \nabla(\psi + V_n) dx \\
&- \int_\Omega \nabla\left[(n - \mu_2)_-\right] \cdot \mu_2 \nabla(\psi + V_n) dx + \int_\Omega (n - \mu_2)_- (n_{tr} - \alpha n)\, dx.
\end{aligned}
$$

For $t \geq t_0$, we have the lower bound $n_{tr}(t,x) \geq \frac{1}{2\rho}$, which yields

$$\frac{d}{dt}\frac{1}{2}\int_\Omega (n-\mu_2)_-^2\,dx \leq \frac{1}{2}\int_\Omega (n-\mu_2)_-^2\,\|\Delta(\psi+V_n)\|_{L^\infty(\Omega)}\,dx$$
$$+\int_\Omega (n-\mu_2)_-\left(\frac{1}{2\rho}-\alpha\mu_2-\mu_2\|\Delta(\psi+V_n)\|_{L^\infty(\Omega)}\right)dx.$$

If we impose the conditions $\mu_2(\alpha+\|\Delta(\psi+V_n)\|_{L^\infty(\Omega)}) \leq \frac{1}{2\rho}$ and $\mu_2 \leq \mu_1 t_0^2$ on $\mu_2 > 0$, we infer

$$\frac{d}{dt}\int_\Omega (n-\mu_2)_-^2\,dx \leq \|\Delta(\psi+V_n)\|_{L^\infty(\Omega)}\int_\Omega (n-\mu_2)_-^2\,dx$$

as well as $\int_\Omega (n(t_0,x)-\mu_2)_-^2\,dx = 0$. Finally, Gronwall's lemma guarantees that $\int_\Omega (n-\mu_2)_-^2\,dx = 0$ and, hence, $n(t,x) \geq \mu_2$ for all $t \geq t_0$ and a.e. $x \in \Omega$. □

Proof of Proposition 1.4. As the entropy production vanishes at the stationary state $(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty)$, straightforward calculations show that $J_n = J_p = R_n = R_p = 0$ yields the representations for n_∞ and p_∞ as well as the two expressions for $n_{tr,\infty}$ in (13). The relation $n_* p_* = n_0 p_0$ results from a combination of the formulas arising from $R_n = 0$ and $R_p = 0$, while the fact that the conservation law is also fulfilled in the equilibrium follows from integrating Poisson's equation in (11). Note that n_* (and hence p_*) is uniquely determined from the second relation in (14) as its left hand side is strictly monotonously increasing and surjective from $(0,\infty)$ to $(-\infty,\infty)$ as a function of n_*. The identities in (15) are equivalent versions of $R_n(n_\infty, n_{tr,\infty}) = 0$ and $R_p(p_\infty, n_{tr,\infty}) = 0$.

Next, we establish the existence of the limiting potential ψ_∞. A technical difficulty stems from the fact that the constant n_* (or equally p_*) depends non-locally on ψ_∞, see (14). However, this can be avoided by substituting $\widetilde{\psi_\infty} := \psi_\infty - \ln n_*$ and rewriting $-\lambda\Delta\psi_\infty = n_\infty - p_\infty + \varepsilon n_{tr,\infty} - D$ as

$$-\lambda\Delta\widetilde{\psi_\infty} - e^{-\widetilde{\psi_\infty}-V_n} + n_0p_0e^{\widetilde{\psi_\infty}-V_p} - \frac{\varepsilon}{1+n_0e^{\widetilde{\psi_\infty}}} = -D. \tag{26}$$

We now aim to apply Tröltzsch[20, Theorem 4.8] to (26), which we further reformulate as

$$-\lambda\Delta\widetilde{\psi_\infty} + \min\{e^{-V_n}, n_0p_0e^{-V_p}\}\widetilde{\psi_\infty} + d\left(\cdot,\widetilde{\psi_\infty}\right) = -D \tag{27}$$

where

$$d(x,y) := -e^{-y-V_n} + n_0p_0e^{y-V_p} - \frac{\varepsilon}{1+n_0e^y} - \min\{e^{-V_n}, n_0p_0e^{-V_p}\}y.$$

The structure of (27) is suitable to apply Tröltzsch[20, Theorem 4.8] for the existence of a unique continuous solution provided that d is monotone increasing w.r.t. y: Indeed, direct computations show

$$e^{-y-V_n} + n_0p_0e^{y-V_p} \geq 2\sqrt{n_0p_0}e^{-\frac{V_p+V_n}{2}}, \qquad \forall y \in \mathbb{R},$$

(where the lower bound is attained at the unique minimum $e^y = e^{\frac{V_p-V_n}{2}}/\sqrt{n_0p_0}$). Hence, we estimate independently of ε

$$\partial_y d(x,y) \geq e^{-y-V_n} + n_0p_0e^{y-V_p} - \min\{e^{-V_n}, n_0p_0e^{-V_p}\}$$
$$\geq 2e^{-\frac{V_n}{2}}\sqrt{n_0p_0}e^{-\frac{V_p}{2}} - \min\{e^{-V_n}, n_0p_0e^{-V_p}\} > 0,$$

and therefore strict monotonicity of d w.r.t. y follows.

As a consequence, (27) admits a unique solution $\widetilde{\psi_\infty} \in H^1(\Omega)\cap L^\infty(\Omega)$, which is continuous on $\overline{\Omega}$ and bounded via

$$\|\widetilde{\psi_\infty}\|_{H^1(\Omega)} + \|\widetilde{\psi_\infty}\|_{C(\overline{\Omega})} \leq \widetilde{K_\infty}$$

where the constant $\widetilde{K_\infty}$ is independent of ε. Going back to ψ_∞, the constraint $\overline{\psi_\infty} = 0$ implies

$$n_* = e^{-\int_\Omega \widetilde{\psi_\infty}\, dx},$$

which in turn uniquely determines $\psi_\infty = \widetilde{\psi_\infty} - \int_\Omega \widetilde{\psi_\infty}\, dx$. The bounds on $\widetilde{\psi_\infty}$ directly transfer to ψ_∞.

As in Fellner and Kniely,[5] we verify the bounds (12) by solving the two equations in (14) for $n_* > 0$ abbreviating $V_\infty := \max\{\|V_n\|_{L^\infty(\Omega)}, \|V_p\|_{L^\infty(\Omega)}\}$:

$$n_* = \frac{\overline{D - \varepsilon n_{tr,\infty}}}{2\overline{e^{-\psi_\infty - V_n}}} + \sqrt{\frac{\overline{D - \varepsilon n_{tr,\infty}}^2}{4\overline{e^{-\psi_\infty - V_n}}^2} + n_0 p_0 \frac{\overline{e^{\psi_\infty - V_p}}}{\overline{e^{-\psi_\infty - V_n}}}} \le e^{K_\infty + V_\infty}(\sqrt{n_0 p_0} + \varepsilon_0 + |\overline{D}|).$$

We stress that the same bound is valid also for $p_* > 0$, and that the upper and lower bounds on n_∞, p_∞ and $n_{tr,\infty}$ are a consequence of the bounds on n_* and p_* as well as $n_* p_* = n_0 p_0$. Finally, the estimate

$$\|\psi_\infty\|_{H^2(\Omega)} \le C\|n_\infty - p_\infty + \varepsilon n_{tr,\infty} - D\|_{L^2(\Omega)} \le C(K_\infty)$$

ensures the higher regularity of ψ_∞. □

3 | DERIVATION OF AN EEP INEQUALITY

As an auxiliary result, we first derive a convenient expression for the entropy relative to the equilibrium.

Lemma 3.1. *The entropy relative to the equilibrium equals*

$$\begin{aligned} &E(n, p, n_{tr}, \psi) - E(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty) \\ &= \int_\Omega \Bigg(n \ln \frac{n}{n_\infty} - (n - n_\infty) + p \ln \frac{p}{p_\infty} - (p - p_\infty) \\ &+ \frac{\lambda}{2} |\nabla(\psi - \psi_\infty)|^2 + \varepsilon \int_{n_{tr,\infty}}^{n_{tr}} \left(\ln \frac{s}{1-s} - \ln \frac{n_{tr,\infty}}{1 - n_{tr,\infty}} \right) ds \Bigg) dx. \end{aligned}$$

Proof. According to the definition of $E(n, p, n_{tr}, \psi)$, one has

$$\begin{aligned} &E(n, p, n_{tr}, \psi) - E(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty) \\ &= \int_\Omega \Bigg(n \ln \frac{n}{n_0 \mu_n} - n_\infty \ln \frac{n_\infty}{n_0 \mu_n} - (n - n_\infty) + p \ln \frac{p}{p_0 \mu_p} - p_\infty \ln \frac{p_\infty}{p_0 \mu_p} - (p - p_\infty) \\ &+ \frac{\lambda}{2} \left(|\nabla \psi|^2 - |\nabla \psi_\infty|^2 \right) + \varepsilon \int_{n_{tr,\infty}}^{n_{tr}} \ln \frac{s}{1-s} ds \Bigg) dx. \end{aligned}$$

We rewrite the first integrand as $n \ln \frac{n}{n_0 \mu_n} = n \ln \frac{n}{n_\infty} + n \ln \frac{n_\infty}{n_0 \mu_n}$ and use $\frac{n_\infty}{n_0 \mu_n} = \frac{n_*}{n_0} e^{-\psi_\infty}$ to find

$$\begin{aligned} &\int_\Omega \left(n \ln \frac{n}{n_0 \mu_n} - n_\infty \ln \frac{n_\infty}{n_0 \mu_n} - (n - n_\infty) \right) dx \\ &= \int_\Omega \left(n \ln \frac{n}{n_\infty} - (n - n_\infty) + (n - n_\infty) \left(\ln \frac{n_*}{n_0} - \psi_\infty \right) \right) dx. \end{aligned}$$

Together with an analogous calculation for the p-terms, we obtain

$$
\begin{aligned}
&E(n,p,n_{tr},\psi)-E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty)\\
&=\int_\Omega\Bigg(n\ln\frac{n}{n_\infty}-(n-n_\infty)+p\ln\frac{p}{p_\infty}-(p-p_\infty)\\
&\quad+(n-n_\infty)\left(\ln\frac{n_*}{n_0}-\psi_\infty\right)+(p-p_\infty)\left(\ln\frac{p_*}{p_0}+\psi_\infty\right)\\
&\quad+\frac{\lambda}{2}|\nabla\psi|^2-\frac{\lambda}{2}|\nabla\psi_\infty|^2+\varepsilon\int_{n_{tr,\infty}}^{n_{tr}}\ln\frac{s}{1-s}\,ds\Bigg)\,dx.
\end{aligned}
$$

We now employ the conservation law $\overline{p}-\overline{p_\infty}=\overline{n}-\overline{n_\infty}+\varepsilon(\overline{n_{tr}}-\overline{n_{tr,\infty}})$, the formula $n_*p_*=n_0p_0$ and the representation $\frac{p_*}{p_0}=\frac{1-n_{tr,\infty}}{n_{tr,\infty}}e^{-\psi_\infty}$ to derive

$$
\begin{aligned}
&(\overline{n}-\overline{n_\infty})\ln\frac{n_*}{n_0}+(\overline{p}-\overline{p_\infty})\ln\frac{p_*}{p_0}\\
&=(\overline{n}-\overline{n_\infty})\ln\frac{n_*p_*}{n_0p_0}+\varepsilon\int_\Omega(n_{tr}-n_{tr,\infty})\ln\frac{p_*}{p_0}\,dx\\
&=\varepsilon\int_\Omega(n_{tr}-n_{tr,\infty})\left(\ln\frac{1-n_{tr,\infty}}{n_{tr,\infty}}-\psi_\infty\right)dx\\
&=-\varepsilon\int_\Omega\left((n_{tr}-n_{tr,\infty})\psi_\infty+\int_{n_{tr,\infty}}^{n_{tr}}\ln\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\,ds\right)dx.
\end{aligned}
$$

The relative entropy now reads

$$
\begin{aligned}
&E(n,p,n_{tr},\psi)-E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty)\\
&=\int_\Omega\Bigg(n\ln\frac{n}{n_\infty}-(n-n_\infty)+p\ln\frac{p}{p_\infty}-(p-p_\infty)\\
&\quad+\frac{\lambda}{2}|\nabla\psi|^2-\frac{\lambda}{2}|\nabla\psi_\infty|^2-\big(n-n_\infty-p+p_\infty+\varepsilon(n_{tr}-n_{tr,\infty})\big)\,\psi_\infty\\
&\quad+\varepsilon\int_{n_{tr,\infty}}^{n_{tr}}\left(\ln\frac{s}{1-s}-\ln\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\right)ds\Bigg)\,dx.
\end{aligned}
$$

Poisson's equation $n-n_\infty-p+p_\infty+\varepsilon(n_{tr}-n_{tr,\infty})=-\lambda\Delta(\psi-\psi_\infty)$ and an integration by parts entail

$$
\begin{aligned}
&E(n,p,n_{tr},\psi)-E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty)\\
&=\int_\Omega\Bigg(n\ln\frac{n}{n_\infty}-(n-n_\infty)+p\ln\frac{p}{p_\infty}-(p-p_\infty)+\frac{\lambda}{2}|\nabla\psi|^2-\frac{\lambda}{2}|\nabla\psi_\infty|^2\\
&\quad-\lambda\nabla(\psi-\psi_\infty)\cdot\nabla\psi_\infty+\varepsilon\int_{n_{tr,\infty}}^{n_{tr}}\left(\ln\frac{s}{1-s}-\ln\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\right)ds\Bigg)\,dx.
\end{aligned}
$$

The claim now obviously follows from collecting the terms involving ψ and ψ_∞. □

Following ideas in Fellner and Kniely,[5] Gajewski and Gärtner,[21] and Fellner and Kniely,[6] we are able to bound the relative entropy essentially in terms of the squared L^2 distance between $(n,p,\sqrt{n_{tr}})$ and $\big(n_\infty,p_\infty,\sqrt{n_{tr,\infty}}\big)$.

Proposition 3.2. *There exists an explicit constant $c_1(K_\infty)>0$ satisfying*

$$
E(n,p,n_{tr},\psi)-E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty)\le c_1\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty}+\frac{(p-p_\infty)^2}{p_\infty}+\varepsilon\Big(\sqrt{n_{tr}}-\sqrt{n_{tr,\infty}}\Big)^2\right)dx
$$

for all $\varepsilon>0$ and all non-negative $n,p,n_{tr}\in L^2(\Omega)$, $n_{tr}\le 1$ where $\psi\in H^1(\Omega)$ is the unique solution of (5) with $f=n-p+\varepsilon n_{tr}-D$ and $\overline{\psi}=0$.

Proof. Applying the elementary inequality $\ln x \le x-1$ for $x>0$, we derive

$$n\ln\frac{n}{n_\infty}-(n-n_\infty)\le n\left(\frac{n}{n_\infty}-1\right)-n+n_\infty=\frac{(n-n_\infty)^2}{n_\infty}$$

and an analogous estimate involving p and p_∞. Integration by parts with homogeneous Neumann conditions for ψ and ψ_∞ as well as $-\lambda\Delta(\psi-\psi_\infty)=(n-n_\infty)-(p-p_\infty)+\varepsilon(n_{tr}-n_{tr,\infty})$ yield

$$\begin{aligned}\lambda\int_\Omega|\nabla(\psi-\psi_\infty)|^2dx&=\int_\Omega\big((n-n_\infty)-(p-p_\infty)+\varepsilon(n_{tr}-n_{tr,\infty})\big)(\psi-\psi_\infty)dx\\&\le\frac{1}{2}\left(\frac{1}{delta}\|(n-n_\infty)-(p-p_\infty)+\varepsilon(n_{tr}-n_{tr,\infty})\|^2+\delta\|\psi-\psi_\infty\|^2\right)\\&\le\frac{3L(\Omega)}{2\lambda}\left(\|n-n_\infty\|^2+\|p-p_\infty\|^2+\varepsilon\|n_{tr}-n_{tr,\infty}\|^2\right)+\frac{\lambda}{2}\|\nabla(\psi-\psi_\infty)\|^2.\end{aligned}$$

Here and below, we abbreviate $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and denote by $L(\Omega)>0$ a constant such that Poincaré's estimate $\|f\|^2\le L(\Omega)\,\|\nabla f\|^2$ holds true for all $f\in H^1(\Omega)$ subject to $\overline{f}=0$. The estimate in the last line is then a result of $\overline{\psi-\psi_\infty}=0$ and Poincaré's inequality together with the choice $\delta := \lambda/L(\Omega)$, whereas the previous bound follows from Hölder's inequality and Young's inequality with some constant $\delta>0$. We thus find

$$\frac{\lambda}{2}\int_\Omega|\nabla(\psi-\psi_\infty)|^2dx\le\frac{3L(\Omega)\max\{M_\infty,4\}}{2\lambda}\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty}+\frac{(p-p_\infty)^2}{p_\infty}+\varepsilon\left(\sqrt{n_{tr}}-\sqrt{n_{tr,\infty}}\right)^2\right)dx,$$

where we employed the bounds from (12) and $\left(\sqrt{n_{tr}}+\sqrt{n_{tr,\infty}}\right)^2\le 4$.

The last term within the relative entropy including n_{tr} can be controlled as in Fellner and Kniely.[6] For convenience, we briefly recall the main arguments. First, there exists for all $x\in\Omega$ some mean value

$$\theta(x)\in(\min\{n_{tr}(x),n_{tr,\infty}(x)\},\max\{n_{tr}(x),n_{tr,\infty}(x)\})$$

such that

$$\int_{n_{tr,\infty}(x)}^{n_{tr}(x)}\ln\frac{s}{1-s}ds=(n_{tr}(x)-n_{tr,\infty}(x))\ln\frac{\theta(x)}{1-\theta(x)}. \tag{28}$$

To enhance readability, we shall suppress the x-dependence of n_{tr} and $n_{tr,\infty}$ subsequently. We further use the bound $n_{tr,\infty}\in(\mu_\infty,1-\mu_\infty)$ from (12) and observe that

$$\left|\int_{n_{tr,\infty}}^{n_{tr}}\ln\frac{s}{1-s}ds\right|\le\int_0^1\left|\ln\frac{s}{1-s}\right|ds=2\ln 2$$

for all $x\in\Omega$. In combination with (28), this estimate entails

$$\left|\ln\frac{\theta(x)}{1-\theta(x)}\right|\left|n_{tr}-n_{tr,\infty}\right|\le 2\ln 2.$$

By an elementary argumentation, one can now conclude that $\theta(x)\in(\xi,1-\xi)$ where $\xi\in\left(0,\frac{1}{2}\right)$ only depends on μ_∞. Therefore, we obtain

$$\begin{aligned}\varepsilon\int_\Omega\int_{n_{tr,\infty}}^{n_{tr}}\left(\ln\frac{s}{1-s}-\ln\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\right)ds\,dx&=\varepsilon\int_\Omega\left(\ln\frac{\theta(x)}{1-\theta(x)}-\ln\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\right)(n_{tr}-n_{tr,\infty})dx\\&=\varepsilon\int_\Omega\frac{1}{\sigma(x)(1-\sigma(x))}(\theta(x)-n_{tr,\infty})(n_{tr}-n_{tr,\infty})dx\end{aligned}$$

with some $\sigma(x) \in (\min\{\theta(x), n_{tr,\infty}(x)\}, \max\{\theta(x), n_{tr,\infty}(x)\}) \subset [\xi, 1-\xi]$ employing the mean-value theorem and taking into account that

$$\frac{d}{ds}\ln\frac{s}{1-s} = \frac{1}{s(1-s)}.$$

As $(\sigma(x)(1-\sigma(x)))^{-1}$ is uniformly bounded in Ω in terms of $\xi(\mu_\infty)$, there exists some $c > 0$ only depending on μ_∞ such that

$$\varepsilon\int_\Omega\int_{n_{tr,\infty}}^{n_{tr}}\left(\ln\frac{s}{1-s} - \ln\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\right)ds\,dx \le c\varepsilon\int_\Omega |\theta(x) - n_{tr,\infty}||n_{tr} - n_{tr,\infty}|\,dx \le 4c\varepsilon\int_\Omega\left(\sqrt{n_{tr}} - \sqrt{n_{tr,\infty}}\right)^2 dx$$

where the last line results from estimating $\left(\sqrt{n_{tr}} + \sqrt{n_{tr,\infty}}\right)^2 \le 4$. This proves the claim. □

The subsequent lemma contains rather non-intuitive estimates for bilinear terms like $(n - n_\infty)(p - p_\infty)$. These expressions will appear in the proof of Proposition 3.4 below. Admissible functions are typically assumed to belong to the set

$$\mathcal{N} := \left\{(n, p, n_{tr}) \in L^2_+(\Omega)^3 \ : \ n, p \le M,\ n_{tr} \le 1 \text{a.e. in } \Omega\right\}. \tag{29}$$

Lemma 3.3. *The following estimates hold true for all $(n, p, n_{tr}) \in \mathcal{N}$ with explicit constants $\Gamma_1(M) > 0$ and $\Gamma_2 > 0$:*

$$\begin{aligned}
(n - n_\infty)(p - p_\infty) &\le \Gamma_1\left(-R_n\ln\frac{n(1-n_{tr})}{n_0\mu_n n_{tr}} - R_p\ln\frac{pn_{tr}}{p_0\mu_p(1-n_{tr})}\right),\\
(n - n_\infty)(-n_{tr} + n_{tr,\infty}) &\le \Gamma_2\left(-R_n\ln\frac{n(1-n_{tr})}{n_0\mu_n n_{tr}} + \left(\sqrt{n_{tr}} - \sqrt{n_{tr,\infty}}\right)^2\right),\\
(p - p_\infty)(n_{tr} - n_{tr,\infty})&\\
&\le \Gamma_2\left(-R_p\ln\frac{pn_{tr}}{p_0\mu_p(1-n_{tr})} + \left(\sqrt{1-n_{tr}} - \sqrt{1-n_{tr,\infty}}\right)^2\right).
\end{aligned}$$

Proof. As in Gajewski and Gärtner[21] and Fellner and Kniely,[5] we first recall the elementary inequalities $(a - a_0)(b - b_0) \le \left(\sqrt{ab} - \sqrt{a_0b_0}\right)^2$ for all $a, a_0, b, b_0 \ge 0$ and $4\left(\sqrt{x} - \sqrt{y}\right)^2 \le (x - y)\ln\frac{x}{y}$ for all $x \ge 0$ and $y > 0$.

Concerning the first inequality we write

$$(n - n_\infty)(p - p_\infty) \le \left(\sqrt{np} - \sqrt{n_\infty p_\infty}\right)^2 = n_\infty p_\infty\left(\sqrt{\frac{np}{n_0\mu_n p_0\mu_p}} - 1\right)^2$$

and distinguish the two cases $n_{tr} > \frac{1}{2}$ and $n_{tr} \le \frac{1}{2}$. In the case $n_{tr} > \frac{1}{2}$, we infer

$$\begin{aligned}
(n - n_\infty)(p - p_\infty) &\le n_0p_0\mu_n\mu_p\left(\sqrt{\frac{n}{n_0\mu_n n_{tr}}}\left(\sqrt{\frac{p}{p_0\mu_p}n_{tr}} - \sqrt{1-n_{tr}}\right)\right.\\
&\quad\left.+\sqrt{\frac{1}{n_{tr}}}\left(\sqrt{\frac{n}{n_0\mu_n}(1-n_{tr})} - \sqrt{n_{tr}}\right)\right)^2\\
&\le \Gamma_1(M)\left(\left(\frac{n}{n_0\mu_n}(1-n_{tr}) - n_{tr}\right)\ln\frac{n(1-n_{tr})}{n_0\mu_n n_{tr}}\right.\\
&\quad\left.+\left(\frac{p}{p_0\mu_p}n_{tr} - (1-n_{tr})\right)\ln\frac{pn_{tr}}{p_0\mu_p(1-n_{tr})}\right)
\end{aligned}$$

employing the L^∞ bound on n. Using analogous arguments we derive the same result also in the case $n_{tr} \le \frac{1}{2}$. The second inequality arises from

$$(n - n_\infty)(-n_{tr} + n_{tr,\infty}) \le \left(\sqrt{n(1-n_{tr})} - \sqrt{n_\infty(1-n_{tr,\infty})}\right)^2 = n_0\mu_n\left(\sqrt{\frac{n}{n_0\mu_n}(1-n_{tr})} - \sqrt{n_{tr,\infty}}\right)^2$$

where we used the relation $n_\infty(1-n_{tr,\infty}) = n_0\mu_n n_{tr,\infty}$ from Proposition 1.4. The claim is now a consequence of

$$n_0\mu_n\left(\left(\sqrt{\frac{n}{n_0\mu_n}(1-n_{tr})}-\sqrt{n_{tr}}\right)+\left(\sqrt{n_{tr}}-\sqrt{n_{tr,\infty}}\right)\right)^2$$
$$\leq \Gamma_2\left(\left(\frac{n(1-n_{tr})}{n_0\mu_n}-n_{tr}\right)\ln\frac{n(1-n_{tr})}{n_0\mu_n n_{tr}}+\left(\sqrt{n_{tr}}-\sqrt{n_{tr,\infty}}\right)^2\right).$$

Similarly, one can also verify the third inequality stated above. □

The next result establishes an upper bound for the L^2 distance between (n,p) and (n_∞,p_∞) basically in terms of the entropy production P and $\|\sqrt{n_{tr}}-\sqrt{n_{tr,\infty}}\|^2_{L^2(\Omega)}$. Similar arguments already appeared in Gajewski and Gärtner[21] and Fellner and Kniely.[5]

Proposition 3.4. *There exists an explicit constant $c_2(M,K_\infty)>0$ satisfying*

$$\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty}+\frac{(p-p_\infty)^2}{p_\infty}\right)dx \leq c_2 P(n,p,n_{tr},\psi)$$
$$+c_2\,\varepsilon\int_\Omega\left(\left(\sqrt{n_{tr}}-\sqrt{n_{tr,\infty}}\right)^2+\left(\sqrt{1-n_{tr}}-\sqrt{1-n_{tr,\infty}}\right)^2\right)dx$$

for all $\varepsilon>0$ and all $(n,p,n_{tr})\in\mathcal{N}$ where additionally $n,p\in H^1(\Omega)$ and $\psi\in H^1(\Omega)$ is the unique solution of (5) with $f=n-p+\varepsilon n_{tr}-D$ and $\overline{\psi}=0$.

Proof. We start by defining intermediate equilibria $N := n_* e^{-\psi-V_n}$ and $P := p_* e^{\psi-V_p}$ which fulfil $J_n(N,\psi) = J_p(P,\psi) = 0$. Due to $J_n(n,\psi) = N\nabla(\frac{n}{N})$ and $J_p(p,\psi) = P\nabla\left(\frac{p}{P}\right)$, we derive the following lower bounds for the flux terms involving J_n and J_p:

$$\frac{|J_n|^2}{n_\infty n} = \frac{N^2}{n_\infty n}\left|\nabla\left(\frac{n}{N}\right)\right|^2 = \frac{N^2}{n_\infty n}\left|\frac{n_\infty}{N}\nabla\left(\frac{n}{n_\infty}\right)+\frac{n}{n_\infty}\nabla\left(\frac{n_\infty}{N}\right)\right|^2$$
$$=\frac{N^2}{n_\infty n}\left|e^{\psi-\psi_\infty}\nabla\left(\frac{n}{n_\infty}\right)+\frac{n}{n_\infty}e^{\psi-\psi_\infty}\nabla(\psi-\psi_\infty)\right|^2$$
$$\geq 2\frac{N^{2\,2}}{n_{\ \infty}}e^{2(\psi-\psi_\infty)}\nabla\left(\frac{n}{n_\infty}\right)\cdot\nabla(\psi-\psi_\infty) = 2\nabla(\psi-\psi_\infty)\cdot\nabla\left(\frac{n-n_\infty}{n_\infty}\right).$$

In the same way, we obtain $\frac{|J_p|^2}{p_\infty p}\geq -2\nabla(\psi-\psi_\infty)\cdot\nabla\left(\frac{p-p_\infty}{p_\infty}\right)$ and, therefore,

$$\frac{\lambda}{2}\int_\Omega\left(\frac{|J_n|^2}{n_\infty n}+\frac{|J_p|^2}{p_\infty p}\right)dx \geq \lambda\int_\Omega\nabla(\psi-\psi_\infty)\cdot\nabla\left(\frac{n-n_\infty}{n_\infty}-\frac{p-p_\infty}{p_\infty}\right)dx$$
$$=\int_\Omega\left((n-n_\infty)-(p-p_\infty)+\varepsilon(n_{tr}-n_{tr,\infty})\right)\left(\frac{n-n_\infty}{n_\infty}-\frac{p-p_\infty}{p_\infty}\right)dx$$

via integration by parts and Poisson's equation. Rearranging this inequality now yields

$$\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty}+\frac{(p-p_\infty)^2}{p_\infty}\right)dx$$
$$\leq\frac{\lambda}{2}\int_\Omega\left(\frac{|J_n|^2}{n_\infty n}+\frac{|J_p|^2}{p_\infty p}\right)dx+\int_\Omega\left(\left(\frac{1}{n_\infty}+\frac{1}{p_\infty}\right)(n-n_\infty)(p-p_\infty)\right.$$
$$\left.+\varepsilon(-n_{tr}+n_{tr,\infty})\frac{n-n_\infty}{n_\infty}+\varepsilon(n_{tr}-n_{tr,\infty})\frac{p-p_\infty}{p_\infty}\right)dx.$$

Together with the bounds from (12) and Lemma 3.3, we arrive at the desired result. □

We are now in a position to prove the EEP inequality from Theorem 1.5, where the main task is to provide an appropriate bound on $\left(\sqrt{n_{tr}} - \sqrt{n_{tr,\infty}}\right)^2$.

Proof of Theorem 1.5. **Step 1.** Due to (15) we easily calculate

$$\begin{aligned}
\sqrt{n_{tr}} - \sqrt{\frac{n}{n_0\mu_n}(1-n_{tr})} &= \sqrt{n_{tr}} - \sqrt{n_{tr,\infty}} - \sqrt{\frac{n}{n_0\mu_n}(1-n_{tr})} \\
&\quad + \sqrt{\frac{n_\infty}{n_0\mu_n}(1-n_{tr})} - \sqrt{\frac{n_\infty}{n_0\mu_n}(1-n_{tr})} + \sqrt{\frac{n_\infty}{n_0\mu_n}(1-n_{tr,\infty})} \\
&= \sqrt{n_{tr}} - \sqrt{n_{tr,\infty}} - \sqrt{\frac{1-n_{tr}}{n_0\mu_n}}\left(\sqrt{n} - \sqrt{n_\infty}\right) + \sqrt{\frac{n_\infty}{n_0\mu_n}}\left(\sqrt{1-n_{tr,\infty}} - \sqrt{1-n_{tr}}\right).
\end{aligned}$$

Observing that $\sqrt{n_{tr}} - \sqrt{n_{tr,\infty}}$ and $\sqrt{1-n_{tr,\infty}} - \sqrt{1-n_{tr}}$ have the same sign and using the inequality $4\left(\sqrt{x} - \sqrt{y}\right)^2 \le (x-y)\ln\frac{x}{y}$ for $x \ge 0$ and $y > 0$, we reformulate the previous identity to find

$$\begin{aligned}
\left(\sqrt{n_{tr}} - \sqrt{n_{tr,\infty}}\right)^2 &\le \left(\sqrt{n_{tr}} - \sqrt{n_{tr,\infty}} + \sqrt{\frac{n_\infty}{n_0\mu_n}}\left(\sqrt{1-n_{tr,\infty}} - \sqrt{1-n_{tr}}\right)\right)^2 \\
&\le 2\left(\sqrt{n_{tr}} - \sqrt{\frac{n}{n_0\mu_n}(1-n_{tr})}\right)^2 + \frac{2}{n_0\mu_n}\left(\sqrt{n} - \sqrt{n_\infty}\right)^2 \\
&\le \frac{1}{2}\left(\frac{n(1-n_{tr})}{n_0\mu_n} - n_{tr}\right)\ln\frac{n(1-n_{tr})}{n_0\mu_n n_{tr}} + \frac{2}{n_0\mu_n}\frac{(n-n_\infty)^2}{n_\infty}.
\end{aligned}$$

Along the same lines, we also deduce

$$\begin{aligned}
&\left(\sqrt{1-n_{tr}} - \sqrt{1-n_{tr,\infty}}\right)^2 \\
&\le \frac{1}{2}\left(\frac{pn_{tr}}{p_0\mu_p} - (1-n_{tr})\right)\ln\frac{pn_{tr}}{p_0\mu_p(1-n_{tr})} + \frac{2}{p_0\mu_p}\frac{(p-p_\infty)^2}{p_\infty}.
\end{aligned}$$

Step 2. We can now improve the claim of Proposition 3.2 in the sense that there exists a constant $c_1(K_\infty) > 0$ satisfying

$$E(n,p,n_{tr},\psi) - E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty) \le c_1 P(n,p,n_{tr},\psi) + c_1\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty} + \frac{(p-p_\infty)^2}{p_\infty}\right)dx$$

for all $(n,p,n_{tr}) \in \mathcal{N}$ where $\psi \in H^1(\Omega)$ is the unique solution of (5) with $f = n - p + \varepsilon n_{tr} - D$ and $\overline{\psi} = 0$ for any $\varepsilon > 0$. Furthermore, we notice that Proposition 3.4 now gives rise to a constant $c_2(M, K_\infty) > 0$ such that

$$\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty} + \frac{(p-p_\infty)^2}{p_\infty}\right)dx \le c_2 P(n,p,n_{tr},\psi) + c_2\,\varepsilon\int_\Omega\left(\frac{(n-n_\infty)^2}{n_\infty} + \frac{(p-p_\infty)^2}{p_\infty}\right)dx$$

holds true for all $\varepsilon > 0$ and all $(n,p,n_{tr}) \in \mathcal{N}$ with $n, p \in H^1(\Omega)$ and $\psi \in H^1(\Omega)$ being the unique solution of (5) with $f = n - p + \varepsilon n_{tr} - D$ and $\overline{\psi} = 0$.

Step 3. If we restrict ε to the interval $\left(0, \frac{1}{2c_2}\right)$, we finally arrive at

$$E(n,p,n_{tr},\psi) - E(n_\infty,p_\infty,n_{tr,\infty},\psi_\infty) \le (c_1 + 2c_1c_2)P(n,p,n_{tr},\psi)$$

employing the notation from Step 2. □

4 | PROOF OF THE EXPONENTIAL CONVERGENCE

As soon as the weak entropy production law (19) is settled, the exponential decay of the relative entropy arises from a Gronwall argument as carried out in Fellner and Kniely[6] (see also the references[22,23] on Gronwall inequalities).

Proof of Theorem 1.6. The weak entropy production law (19) readily follows from (16) and (18) for $0 < t_0 \leq t_1 < \infty$ utilizing the regularity and bounds on n, p, n_{tr}, and ψ from Proposition 1.3. The statement of the theorem is then a consequence of Theorem 1.5 applied to the global solution (n, p, n_{tr}, ψ). Note, however, that the weak entropy production law (19) only allows to derive

$$E(n,p,n_{tr},\psi)(t) - E(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty) \leq (E(n,p,n_{tr},\psi)(t_0) - E(n_\infty, p_\infty, n_{tr,\infty}, \psi_\infty))e^{-C_{EEP}^{-1}(t-t_0)}$$

for all $t_0 \in (0, t]$. The assertion in (20) is then a consequence of the fact that the entropy $E(n, p, n_{tr}, \psi)(t_0)$ continuously extends to $t_0 \to 0$ since $n, p \in C([0, T), L^2(\Omega))$, $n_{tr} \in C([0, T), L^\infty(\Omega))$, and $\psi \in C([0, T), H^2(\Omega))$ for all $T > 0$ by Proposition 1.3. □

Exponential convergence in L^∞ and H^2 for (n, p, n_{tr}) and ψ, respectively, is a consequence of standard regularity techniques, which have been partially employed already in Fellner and Kniely.[6] As a prerequisite, we formulate a Csiszár–Kullback–Pinsker-type inequality, which we believe to be well-known. But as we were not able to find a precise reference, we provide a proof in the subsequent lemma.

Lemma 4.1 (A Csiszár–Kullback–Pinsker-type inequality). *Let $f, g : \Omega \to \mathbb{R}$ be non-negative and measurable functions and g be strictly positive. Then,*

$$\int_\Omega \left(f \ln \frac{f}{g} - f + g \right) dx \geq \frac{3}{2\overline{f} + 4\overline{g}} \|f - g\|_{L^1(\Omega)}^2.$$

Proof. Going back to an idea of Pinsker, we first prove the elementary inequality $h(u) := (2u + 4)(u \ln u - u + 1) - 3(u-1)^2 \geq 0$ for scalar $u \geq 0$. The claim follows from the identities $h(1) = h'(1) = h''(1) = 0$ and the sign of $h'''(u) = \frac{4}{u} - \frac{4}{u^2}$ for $u > 1$ and $u < 1$, respectively. As a consequence, we obtain

$$\begin{aligned}
\|f - g\|_{L^1(\Omega)} = \int_\Omega \left| \frac{f}{g} - 1 \right| g\,dx &\leq \int_\Omega \frac{g}{\sqrt{3}} \sqrt{\frac{2f}{g} + 4} \sqrt{\frac{f}{g} \ln \frac{f}{g} - \frac{f}{g} + 1}\,dx \\
&\leq \frac{1}{\sqrt{3}} \sqrt{\int_\Omega (2f + 4g)\,dx} \sqrt{\int_\Omega \left(f \ln \frac{f}{g} - f + g \right) dx},
\end{aligned}$$

where we employed Hölder's inequality in the last step. □

Proof of Corollary 1.7. An immediate consequence of the exponential decay of the relative entropy as stated in (20) is the exponential convergence to the equilibrium of $n(t)$ and $p(t)$ in $L^1(\Omega)$, and of $n_{tr}(t)$ in $L^2(\Omega)$. To see this, we first recall the explicit representation of the relative entropy from Lemma 3.1. Lemma 4.1 allows us to control

$$\int_\Omega \left(n \ln \frac{n}{n_\infty} - (n - n_\infty) \right) dx \geq \frac{3}{2\overline{n} + 4\overline{n_\infty}} \|n - n_\infty\|_{L^1(\Omega)}^2 \geq c\|n - n_\infty\|_{L^1(\Omega)}^2$$

and analogously $\|p - p_\infty\|_{L^1(\Omega)}^2$ in terms of a (rough) constant $c(M, K_\infty) > 0$. Next, we notice that

$$\frac{d}{ds} \ln\left(\frac{s}{1-s}\right) = \frac{1}{s(1-s)} \geq 4$$

is valid for all $s \in (0,1)$, which enables us to estimate

$$\varepsilon \int_\Omega \int_{n_{tr,\infty}}^{n_{tr}} \left(\ln\left(\frac{s}{1-s}\right) - \ln\left(\frac{n_{tr,\infty}}{1-n_{tr,\infty}}\right)\right) ds\, dx$$
$$= \varepsilon \int_\Omega \int_{n_{tr,\infty}}^{n_{tr}} \frac{1}{\sigma(s)(1-\sigma(s))}(s - n_{tr,\infty})\, ds\, dx \geq 2\varepsilon \| n_{tr} - n_{tr,\infty}\|^2_{L^2(\Omega)}$$

where $\sigma(s)$ serves as an intermediate point between $n_{tr,\infty}$ and s.

As a preparation for the exponential convergence of n and p in $L^\infty(\Omega)$, we adapt an argument from Fellner and Kniely[6] to establish a polynomially growing $W^{1,q}(\Omega)$ bound on n for $q \geq 4$. (In fact, we shall only require such a bound for $q = 6$ below.) The same technique is also applicable to p. As in the proof of Proposition 1.3, we set w.l.o.g. $\tau_n = \tau_p = 1$ leading to

$$\partial_t n = \Delta n + \nabla n \cdot \nabla(\psi + V_n) + n\Delta(\psi + V_n) + n_{tr} - \frac{n}{n_0\mu_n}(1 - n_{tr}).$$

Using $-|\nabla n|^{q-2}\Delta n$ as a test function and recalling $\hat{n} \cdot \nabla n = 0$ on $\partial\Omega$ entails

$$\frac{1}{q(q-1)}\frac{d}{dt}\int_\Omega |\nabla n|^q\, dx = \frac{1}{q-1}\int_\Omega |\nabla n|^{q-2}\nabla n \cdot \nabla \partial_t n\, dx = -\int_\Omega |\nabla n|^{q-2}\Delta n\, \partial_t n\, dx$$
$$= -\int_\Omega |\nabla n|^{q-2}|\Delta n|^2\, dx - \int_\Omega |\nabla n|^{q-2}\Delta n \nabla n \cdot \nabla(\psi + V_n)\, dx$$
$$- \int_\Omega |\nabla n|^{q-2}\Delta n\, n\Delta(\psi + V_n)\, dx - \int_\Omega |\nabla n|^{q-2}\Delta n \left(n_{tr} - \frac{n}{n_0\mu_n}(1-n_{tr})\right) dx.$$

By estimating the third line with Young's inequality via

$$\left| \Delta n\, n\Delta(\psi + V_n) + \Delta n \left(n_{tr} - \frac{n}{n_0\mu_n}(1 - n_{tr})\right)\right| \leq \frac{1}{2}|\Delta n|^2 + \frac{1}{2}C_2^2$$

with a constant $C_2(M) > 0$, and by observing that

$$\left|\int_\Omega |\nabla n|^{q-2}\Delta n \nabla n \cdot \nabla(\psi + V_n)\, dx\right| = \left|\int_\Omega \frac{1}{q}\nabla\left(|\nabla n|^q\right) \cdot \nabla(\psi + V_n)\, dx\right|$$
$$\leq \frac{1}{q}\int_\Omega |\nabla n|^q\, |\Delta(\psi + V_n)|\, dx \leq \frac{1}{q}\int_\Omega |\nabla n|^q C_1\, dx$$

with another constant $C_1(M) > 0$, we calculate

$$\frac{1}{q(q-1)}\frac{d}{dt}\int_\Omega |\nabla n|^q\, dx \leq -\int_\Omega |\nabla n|^{q-2}|\Delta n|^2\, dx + \frac{1}{q}\int_\Omega |\nabla n|^q C_1\, dx + \int_\Omega |\nabla n|^{q-2}\left(\frac{1}{2}|\Delta n|^2 + \frac{1}{2}C_2^2\right) dx.$$

We rewrite the first term in the second line by another integration by parts and Young's inequality, which leads us to

$$\frac{1}{q}\int_\Omega |\nabla n|^q\, dx = \frac{1}{q}\int_\Omega |\nabla n|^{q-2}\nabla n \cdot \nabla n\, dx = -\frac{q-1}{q}\int_\Omega |\nabla n|^{q-2}\Delta n\, n\, dx$$
$$\leq \frac{1}{2C_1}\int_\Omega |\nabla n|^{q-2}|\Delta n|^2\, dx + \frac{C_1 M^2}{2}\int_\Omega |\nabla n|^{q-2}\, dx.$$

The previous estimates now guarantee that

$$\frac{d}{dt}\int_\Omega |\nabla n|^q\, dx \leq C_3 \int_\Omega |\nabla n|^{q-2}\, dx,$$

with a constant $C_3(M, q) > 0$. Choosing $t_0 > 0$ and $t \geq t_0$ arbitrarily and utilizing $|\Omega| = 1$, one has

$$\|\nabla n(t)\|^q_{L^q(\Omega)} \leq \|\nabla n(t_0)\|^q_{L^q(\Omega)} + C_3 \int_{t_0}^{t} \|\nabla n(s)\|^{q-2}_{L^q(\Omega)}\, ds.$$

The polynomial growth of $\|\nabla n\|_{L^q(\Omega)}$ is then obtained by an elementary Gronwall lemma (see, e.g., Beesak[23]). In detail,

$$\|\nabla n(t)\|_{L^q(\Omega)} \leq \left(\|\nabla n(t_0)\|^2_{L^q(\Omega)} + C_3(t - t_0) \right)^{\frac{1}{2}}. \tag{30}$$

Exponential convergence to the equilibrium for n in $L^q(\Omega)$, $1 < q < \infty$, is easily deduced from the exponential convergence of n in $L^1(\Omega)$ as settled above and the $L^\infty(\Omega)$ bounds on n and n_∞ in (8) and (12), respectively, by writing

$$\|n - n_\infty\|^q_{L^q(\Omega)} \leq \|n - n_\infty\|^{q-1}_{L^\infty(\Omega)} \|n - n_\infty\|_{L^1(\Omega)} \leq Ce^{-ct}$$

where the constants $c(M, K_\infty, q), C(M, K_\infty, q) > 0$ are independent of ε for $\varepsilon \in (0, \varepsilon_0]$. For $q = 2$ and together with the same bound on p and the estimate

$$\|\psi - \psi_\infty\|_{H^2(\Omega)} \leq C \left(\|n - n_\infty\|_{L^2(\Omega)} + \|p - p_\infty\|_{L^2(\Omega)} + \varepsilon \|n_{tr} - n_{tr,\infty}\|_{L^2(\Omega)} \right),$$

this directly implies the exponential convergence of ψ in $H^2(\Omega)$. The Gagliardo–Nirenberg–Moser interpolation inequality now allows us to infer exponential convergence of n and p in $L^\infty(\Omega)$. In fact, the bound on $\|\nabla n\|_{L^6(\Omega)}$ in (30) entails

$$\|n - n_\infty\|_{L^\infty(\Omega)} \leq C \|n - n_\infty\|^{\frac{1}{2}}_{W^{1,6}(\Omega)} \|n - n_\infty\|^{\frac{1}{2}}_{L^6(\Omega)} \leq Ce^{-ct}, \tag{31}$$

with constants $c(M, K_\infty), C(M, K_\infty) > 0$.

The exponential convergence of n_{tr} in $L^\infty(\Omega)$ can be verified essentially along the same lines as in Fellner and Kniely.[6] We, therefore, omit some technical details and set w.l.o.g. $\tau_n = \tau_p = 1$. By defining $u := n_{tr} - n_{tr,\infty}$, one derives the following pointwise relation by inserting $\pm n_{tr,\infty}$ several times and by applying the identities from (15):

$$\begin{aligned}
\varepsilon\, \partial_t u = R_p - R_n &= \left(1 - n_{tr} - \frac{p}{p_0 \mu_p} n_{tr}\right) - \left(n_{tr} - \frac{n}{n_0 \mu_n}(1 - n_{tr})\right) \\
&= -u \left(2 + \frac{p_\infty}{p_0 \mu_p} + \frac{n_\infty}{n_0 \mu_n}\right) - \frac{n_{tr}}{p_0 \mu_p}(p - p_\infty) + \frac{(1 - n_{tr})}{n_0 \mu_n}(n - n_\infty).
\end{aligned}$$

By recalling $0 \leq n_{tr} \leq 1$, $\mu_n = e^{-V_n}$, and $\mu_p = e^{-V_p}$ with $V_n, V_p \in L^\infty(\Omega)$, and by employing (31), we find

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq -\frac{2}{\varepsilon} \|u(t, \cdot)\|_{L^\infty(\Omega)} + \frac{C}{\varepsilon} e^{-ct}$$

where $c, C > 0$ depend on M and K_∞ but not on ε for $\varepsilon \in (0, \varepsilon_0]$. If we choose $c > 0$ sufficiently small satisfying $\varepsilon_0 c \leq 1$, we arrive at

$$\begin{aligned}
\|n_{tr}(t, \cdot) - n_{tr,\infty}\|_{L^\infty(\Omega)} &\leq e^{-2t/\varepsilon} + \frac{C}{\varepsilon} \int_0^t e^{-2(t-s)/\varepsilon - cs}\, ds \\
&\leq e^{-2t/\varepsilon} + e^{-2t/\varepsilon} \frac{C}{2 - \varepsilon c} \left(e^{(2/\varepsilon - c)t} - 1\right) \leq e^{-2t/\varepsilon_0} + Ce^{-ct}.
\end{aligned}$$

Finally, estimate (21) is proven. □

5 | CONCLUSION AND OUTLOOK

We have derived a so-called entropy–entropy production (EEP) inequality for a recombination–drift–diffusion–Poisson system modelling the dynamics of electrons and holes on separate energy levels in semiconductor materials. We have then employed this EEP inequality to establish exponential convergence to the equilibrium for the densities of the involved charge carriers and the self-consistent electrostatic potential. However, several simplifying hypotheses have been imposed (cf. Assumption 1.1), which allowed for a transparent presentation of the main ideas of the proof but which prevent one from directly applying the results to real-world semiconductor devices.

We conclude the article with a couple of comments on possible future research.

- Instead of considering trapped states allowing for a limited number of electrons, one can—at least from a mathematical perspective—also consider trapped states attracting holes. This could be achieved by replacing n_{tr} by the density of *trapped holes* p_{tr}, and by appropriately reformulating model (1). As the structure of the resulting system remains essentially unchanged, we believe that our results also transfer to this situation.
- Concerning the presence of multiple trap levels or even a continuous distribution of trap levels within the bandgap of the semiconductor as in Goudon et al,[4] we stress that this requires a different definition of the entropy functional involving the density n_{tr}^{η} of occupied trapped states with energy $\eta \in [E_{\min}, E_{\max}]$. Assuming all constants to be independent of η (which also enforces the equilibria $n_{tr,\infty}^{\eta}$ to coincide), one can redefine the entropy functional as

$$E(n, p, \{n_{tr}^{\eta}\}_{\eta}, \psi) := \int_{\Omega} \Bigg(n \ln \frac{n}{n_0 \mu_n} - (n - n_0 \mu_n) + p \ln \frac{p}{p_0 \mu_p} - (p - p_0 \mu_p) + \frac{\lambda}{2} |\nabla \psi|^2 + \varepsilon \int_{E_{\min}}^{E_{\max}} \int_{1/2}^{n_{tr}^{\eta}} \ln\left(\frac{s}{1-s}\right) ds\, d\eta \Bigg) dx. \tag{32}$$

  In this context, our strategy leads to the same results by following the line of arguments with some minor adaptions. But as soon as n_0 and p_0 depend on η, it already seems to be infeasible to derive an expression for the entropy production P similar to (18) (at least if the integral over η is placed in front of the right hand side of (32)). Further studies in this direction shall be carried out in a subsequent project.
- In order to generalise our framework to the setting of semiconductor devices subject to non-trivial boundary conditions and space-dependent material parameters, the following questions have to be addressed: How to prove the existence of global solutions (observing that heterostructures are not covered by Goudon et al[4])? How to incorporate boundary conditions into the entropy functional preserving the non-negativity of E and P along with $P = -\frac{d}{dt}E$? Moreover, the fluxes J_n, J_p and the reactions R_n, R_p are now typically non-zero in the equilibrium. Is it, therefore, reasonable to expect relative fluxes and relative reactions to appear in the entropy production? To our knowledge, questions of equilibration of such models are largely open.

ACKNOWLEDGEMENT

There are no funders to report for this submission.
Open access funding enabled and organized by Projekt DEAL.

CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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How to cite this article: Fellner K, Kniely M. Uniform convergence to equilibrium for a family of drift–diffusion models with trap-assisted recombination and self-consistent potential. *Math Meth Appl Sci*. 2021;44(17):13040-13059. https://doi.org/10.1002/mma.7604