Spectral Inequalities for Schrödinger Operators and Parabolic Observability

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

Der Fakultät für Mathematik der
Technischen Universität Dortmund
vorgelegt von

Alexander Dicke

im Juli 2022

Dissertation
Spectral Inequalities for Schrödinger Operators and Parabolic Observability

Fakultät für Mathematik
Technische Universität Dortmund

Erstgutachter: Prof. Dr. Ivan Veselić
Zweitgutachterin: Prof. Dr. Birgit Jacob

Tag der mündlichen Prüfung: 31. Oktober 2022

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor Professor Ivan Veselić. This thesis would not have been possible without his excellent guidance and his mathematical insights he shared with me. Thank you Ivan, for giving me great freedom in how I organized my research while still always being there when I needed support.

Many thanks go to Albrecht Seelmann. The countless mathematical discussions we had over a coffee in the morning have contributed a great deal to our joint works!

Without my current and former colleagues the university could never have been such a welcoming place. On this occasion, I would like to extend my sincere thanks to Albrecht Seelmann, Christopher Strothmann, and Max Kämper for valuable discussions, on both mathematical and other topics, for relaxing coffee breaks and entertaining game nights, and for many shared laughs.

Special thanks go to Albrecht Seelmann, Christopher Strothmann, Max Kämper, and Matthias Täufer for proofreading parts of this thesis and for the helpful comments they provided in the progress.

I thank Lea for already more than seven wonderful years we spend together and for moral support. Last but not least, I am very grateful to my parents for their love and constant encouragement.

Preface

This thesis consists of eight chapters and two appendices. Chapter 1 grants an overview over the main results. Thereafter, in Chapter 2 the notation is fixed, the operators are introduced, and some basic results needed in the subsequent chapters are discussed. Chapter 3 proceeds with a short introduction into specific aspects of control theory and paves the way to the main results of this thesis that are presented and put into context in Chapter 4. The proofs of the main results are postponed to Chapters 5–8. Some supplementing computations and technical details are presented in Appendix A, while Appendix B is an excursus presenting results of the author which have not been included in the main body of text.

Parts of this thesis are based on and coincide with the following publications and preprints by the author that were partially obtained in collaboration with **Christian Rose**, **Albrecht Seelmann**, **Martin Tautenhahn**, and **Ivan Veselić**:

[Dic21] A. Dicke. Wegner estimate for random divergence-type operators monotone in the randomness. *Math. Phys. Anal. Geom.*, 24(3):22, 2021.

[DRST] A. Dicke, C. Rose, A. Seelmann, and M. Tautenhahn. Quantitative unique continuation for spectral subspaces of Schrödinger operators with singular potentials. Preprint: arXiv:2011.01801.

[DS22] A. Dicke and A. Seelmann. Uncertainty principles with error term in Gelfand–Shilov spaces. *Archiv der Mathematik*, 119(4):413–425, 2022.

[DSVa] A. Dicke, A. Seelmann, and I. Veselić. Control problem for quadratic differential operators with sensor sets of decaying density via partial harmonic oscillators. Preprint: arXiv:2201.02370.

[DSVb] A. Dicke, A. Seelmann, and I. Veselić. Spectral inequality with sensor sets of decaying density for Schrödinger operators with power growth potentials. Preprint: arXiv:2206.08682.

[DSVc] A. Dicke, A. Seelmann, and I. Veselić. Uncertainty principle for Hermite functions and null-controllability with sensor sets of decaying density. Preprint: arXiv:2201.11703.

[DV] A. Dicke and I. Veselić. Unique continuation for the gradient of eigenfunctions and Wegner estimates for random divergence-type operators. Preprint: arXiv:2003.09849.

It is here set out on which of the aforementioned articles the chapters that are listed below are based on or coincide with.

Chapter 2: Parts of that chapter are extracted from [DSVa], in particular, Subsection 2.2.1 is an adaptation of [DSVa, Appendix A] to more general potentials.

Chapter 4: The main results of this thesis presented in that chapter are taken from the works [DRST, DSVc, DSVa, DS22, DS22, DSVb] and parts of the chapter coincide with or are based on these articles.

Chapter 5: That chapter coincides for the most part with [DSVb, Section 2].

Chapter 6: Sections 6.2 and 6.3 are based upon [DRST, Sections 2 and 3], while Section 6.4 combines and elaborates further results from [DRST, DSVb].

Chapter 7: Section 7.2 is based upon [DSVc, DSVa], while the proof in Section 7.3 is a shortened version of the proof in [DS22].

Chapter 8: The proof given in that chapter is taken from [DSVa, Section 4].

Contents

CHAPTER 1

Introduction

Imagine the task of measuring the state of a system at a given time through observations of sensors over a time interval. For physical or economical reasons, it is usually not preferable or not even possible to observe the entire system. Instead, one wants to use as few or as small sensors as possible. However, taking fewer or smaller sensors increases the possible deviation of the measurement from the actual state or even prevents the measurement entirely. It is therefore a fundamental problem to derive conditions on the sensors such that the system can be measured adequately. In this case, one also wants to derive upper bounds for the deviation between the actual state and the measurement in terms of some conditions imposed on the sensors. If one is interested in the possibility to steer the system to a designated state in a given time, a similar problem arises: One again wants to determine under which configurations of control units one can be insured that the designated state is attainable. At the same time, it is desirable to minimize the number or the size of the control units that influence the system, but this causes the control costs to increase. Hence, it is of interest to obtain bounds for the control costs depending on the configuration at hand.

In the present thesis, the state of the system is governed by some differential equation $w'(t) = Aw(t)$ with initial value $w(0) = w_0 \in L^2(\mathbb{R}^d)$ and the sensors are represented by a *sensor set* $\omega \subset \mathbb{R}^d$. Provided that the operator A generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$, the above task of measuring the state at a given time $T > 0$ can be put into mathematical terms as the question whether there exists an *observability constant* $C_{\mathrm{obs}} > 0$ such that the observability estimate

$$\|w(T)\|^2_{L^2(\mathbb{R}^d)} \leq C^2_{\mathrm{obs}} \int_0^T \|\mathcal{T}(t)w_0\|^2_{L^2(\omega)} \,\mathrm{d}t \tag{1.1}$$

holds for all initial values w_0. In this setting, the constant C_{obs} accounts for the possibly occurring deviations of the measurement from the actual state. Any action to steer the system can then be modeled by adding an inhomogeneity $f \in L^2([0,\infty); L^2(\omega))$ the values $f(t)$ of which are constrained to the control set we again denote by ω. Hence, the system is then governed by $w'(t) = Aw(t) + f(t)$. By linearity, the task of steering the system to some state (in the range of $\mathcal{T}(T)$) is then reduced to the task of choosing the control f in such a way that $w(T) = 0$. Here, the norm of f corresponds to the cost of the control. It is well-established that for a given set ω the existence of an observability constant such that the observability estimate (1.1) holds is equivalent to the existence of a null-control f constrained to ω with norm at most C_{obs} for the dual system. Because of this duality it is not a limitation to investigate only the problem of observability.

The main objective of this thesis is to study sufficient geometric conditions on the sensor set ω that guarantee the existence of an observability constant for different types of systems characterized by the semigroup generator A. Furthermore, given a sufficient condition, we also investigate how the geometry of the set impacts the constant C_{obs}.

There is a vast amount of literature dealing with several different ways to establish the observability estimate. In this work, we rely on methods going back to Lebeau and Robbiano. These allow to conclude the existence of an observability constant by combining two ingredients, namely an *uncertainty principle* and a *dissipation estimate*. In Chapter 3, we recall different forms of Lebeau-Robbiano methods together with a short outline of basic results from control theory that are related to the present work.

The main results of this thesis are tailored towards the aforementioned methods and are presented, discussed, and put into context to earlier results in Chapter 4. They can be roughly divided into three categories: Spectral inequalities, dissipation estimates, and uncertainty principles with error term. While all of these are stated in Chapter 4, their proofs are deferred to Chapters 5–8. Let us emphasize that Sections 6.1 and 7.1 provide an overview of the origins of the approaches we use.

1.1. Main results

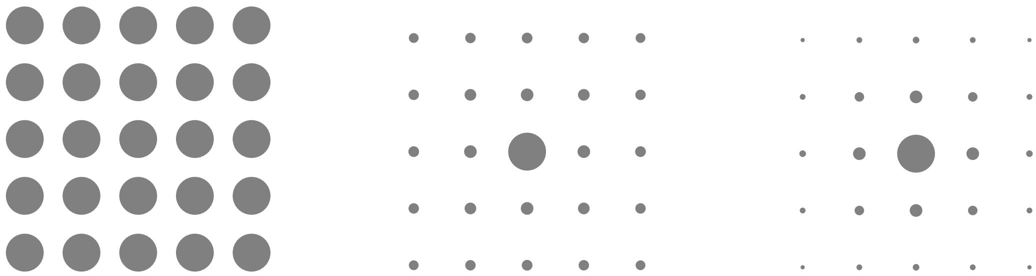
Let us now outline the main results in the order they are formulated in Chapter 4.

1.1.1. Spectral inequalities. In Section 4.1, we consider the situation where the generator $A = \Delta - V$ is a selfadjoint Schrödinger operator in $L^2(\mathbb{R}^d)$. For selfadjoint operators, establishing the observability estimate reduces by the Lebeau-Robbiano method to the proof of a *spectral inequality*. The latter has the form

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1}} \|f\|^2_{L^2(\omega)},$$

where $\omega \subset \mathbb{R}^d$ is the sensor set from the observability estimate, f is any function in the range of the spectral projection $P_\lambda(-\Delta + V)$ up to energy $\lambda \geq 1$ of the Schrödinger operator, and $d_0, d_1 > 0$, $\gamma_1 \in (0, 1)$ are constants. Proving the spectral inequality not only implies the observability estimate, but there are also explicit bounds for C_{obs} in terms of the parameters d_0, d_1, γ_1, and the time T.

Whether spectral inequalities for Schrödinger operators are available and, if so, what kind of geometric conditions on ω are required heavily depends on the potential V at hand. The spectral inequalities we prove are especially motivated by earlier research for the pure Laplacian [Kov00, Kov01, EV18, EV20], for Schrödinger operators with bounded potentials [NTTV18, NTTV20b], and for the harmonic oscillator, i.e., for the Schrödinger operator with a quadratic potential [BJPS21,



(A) $\alpha = 0$ (B) $\alpha = \frac{1}{5}$ (C) $\alpha = \frac{2}{3}$

FIGURE 1. $\omega = \bigcup_{k\in\mathbb{Z}^d} B\big(k, (\frac{1}{3})^{1+|k|^\alpha}\big)$ for different values of α.

MPS22]. We extend and improve these results in several aspects, some of which we outline in the following. Let us mention beforehand that we always compute the constants d_0, d_1, and γ_1 explicitly from geometric properties of the sensor set ω.

Singular admissible potentials. Instead of studying merely Schrödinger operators with bounded potentials, we prove the spectral inequality with potentials we call *singular admissible*. These may have mild local singularities and are therefore allowed to be unbounded. For instance, $V(x) = \mathrm{e}^{-|x|}|x|^{-q}$ with $0 \leq q < 1$ is singular admissible in all dimensions $d \geq 2$. In our spectral inequality given in Theorem 4.10, the sensor set ω is assumed to satisfy the same geometric conditions that were used in case of bounded potentials. Namely, ω is an *equidistributed set* which means that it contains a suitable union of open balls with fixed radii, a particular instance being the set in Figure 1A.

The proof is given in Chapter 6 and uses Carleman estimates and the so-called ghost dimension. We make use of the fact that singular admissible potentials can be inserted into the Carleman estimates which allows to follow a similar approach as in the case of bounded potentials. An important feature of our analysis is that we can extract two characteristic parameters of the potential in terms of certain relative bounds. These parameters are the only characteristics of the potential entering the constants d_0 and d_1 in the spectral inequality.

Quadratic potentials. In the case of the harmonic oscillator, i.e., $V(x) = |x|^2$, an orthonormal basis of eigenfunctions is given by the Hermite functions. These are known to have a fast decay at infinity. We quantify this decay and demonstrate how to exploit it in order to prove the spectral inequality with sensor sets ω that become sparse at infinity. More precisely, we show that the sensor set is allowed to have a subexponentially decaying density, a condition that even allows sets of finite Lebesgue measure, see Theorem 4.16. As an example, the subexponential decay of the density corresponds to $\alpha < 1$ for the sets depicted in Figure 1. Let us stress that in the previous works [BJPS21, MPS22], the density was not allowed to decay at all and, in particular, the sensor sets were forced to have infinite Lebesgue measure.

Besides this, we also investigate partial harmonic oscillators corresponding to $V(x) = |x_{\mathcal{I}}|^2 = \sum_{j\in\mathcal{I}} x_j^2$ for some $\mathcal{I} \subset \{1, \dots, d\}$. These have not been considered previously and we prove the spectral inequality where the density of the set ω is allowed to decay subexponentially in those directions where the potential grows. This result is formulated in Theorem 4.19.

Since (partial) quadratic potentials are analytic, we are in the position to use an approach based on complex analysis. This approach allows us to derive the spectral inequality with weaker assumptions on the sensor sets than possible by the method using Carleman estimates mentioned before. Put plainly, ω does not necessarily need to contain suitable balls, but merely suitable measurable subsets. The assumptions we make are closely related to the notion of *thick sets*. We refer to the last subsection in Appendix A, where we briefly discuss the different assumptions on ω. The proof of the spectral inequality is given in Chapter 7. There we describe how the tensor structure of the Schrödinger operator can be used to derive Bernstein inequalities for elements in the spectral subspace required for this approach. Furthermore, we also prove that these inequalities yield the aforementioned quantification of the fast decay of elements in the spectral subspace of the partial harmonic oscillators (in particular of Hermite function) in directions where the potential grows unboundedly. Thereafter, the quantified decay is used to implement the spectral inequality with sensor sets having a decaying density.

Power growth potentials. The third type of potentials we investigate are those with power growth in certain coordinate directions, see Hypothesis $(S_{\mathcal{I}})$ on page 45 for the full class of potentials we work with. For a better overview, we here restrict our attention to the potentials $V(x) = |x_{\mathcal{I}}|^\tau$ where $\tau > 0$ and $\mathcal{I} \subset \{1, \dots, d\}$. At first glance, these seem quite similar to the previous case of partial quadratic potentials. However, a major difference is that even these simple potentials are not analytic for $\tau \notin 2\mathbb{N}$, hence requiring a fundamentally different approach compared to the harmonic oscillator. This has not been considered previously, but we underline that this is a very active field of research.

As a first result in this novel territory, we use our insights gained from the study of singular admissible and quadratic potentials to establish a spectral inequality also for this class of potentials with power growth in certain coordinate directions. To this end, in Chapter 5 we establish that the growth of the potentials enforces a fast decay of eigenfunctions and, more importantly, we quantify this decay. The decay benefits us in two ways: Firstly, we are able to treat potentials with power growth by the previously mentioned approach using Carleman estimates. In fact, the proof largely parallels the one for singular admissible potentials in Chapter 6. Secondly, we can allow the sensor set ω to become sparse at infinity in directions where the potential grows. However, due to the Carleman estimate, we require that the sensor set contains a union of open balls with radii decaying at infinity, see again the sets depicted in Figure 1. The precise allowed decay rate depends on the growth of the potential at hand. Our result for potentials satisfying Hypothesis $(S_{\mathcal{I}})$ is formulated in Theorem 4.24.

1.1.2. Dissipation estimates. Section 4.2 deals with possibly non-selfadjoint operators A in $L^2(\mathbb{R}^d)$. Two situations may arise here: Either the operator is selfadjoint but a spectral inequality is not available, or it is indeed non-selfadjoint, so that it is not even possible to formulate a spectral inequality. However, for certain operators A, the Lebeau-Robbiano method allows us to use the spectral inequality of a selfadjoint Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$ in order to conclude the observability. This strategy requires a dissipation estimate of the form

$$\|\big(\mathrm{Id} - P_\lambda(-\Delta + V)\big)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq d_2 \mathrm{e}^{-d_3\lambda^{\gamma_2}t^{\gamma_3}}\|g\|_{L^2(\mathbb{R}^d)}$$

for all $\lambda \geq 1$, all $0 < t \ll 1$, and all $g \in L^2(\mathbb{R}^d)$, where $(\mathcal{T}(t))_{t\geq 0}$ is the semigroup generated by A, and $d_2, d_3, \gamma_2, \gamma_3 \geq 0$ are constants. If such a dissipation estimate holds, we say that the Schrödinger operator $-\Delta + V$ is a suitable *comparison operator* for A. Provided that we have a spectral inequality for the Schrödinger operator at our disposal, the observability estimate holds with the sensor set ω from the spectral inequality and explicit bounds on the observability constant depending on the constants in the dissipation estimate as well as in the spectral inequality. In particular, solely the spectral inequality encodes sufficient conditions on the sensor set ω.

We consider quadratic differential operators A, i.e., operators that are the Weyl quantization of a homogeneous quadratic polynomial. Inspired by the previous works [BPS18, Alp21], we prove that for these operators the suitable class of comparison operators are the partial harmonic oscillators $-\Delta + |x_{\mathcal{I}}|^2$. Furthermore, we show how the appropriate member of this class can be read off the *singular space* of A. Thereby, we unify, interpolate and generalize earlier dissipation estimates for quadratic differential operators. In combination with the spectral inequalities for partial harmonic oscillators, we establish the observability estimate with the sensor sets which we have considered there. Recall that these spectral inequalities, in particular, allow the sensor sets to have a decaying density in the directions where the potential of the comparison operator grows. Such sensor sets have not been considered before in this context. Our dissipation is formulated in Theorem 4.29 and the proof is given in Chapter 8.

In addition to the results just outlined, dissipation estimates also play a role in Section 4.4. There we consider the observability estimate for other types of operators A, namely Shubin operators, and discuss how our spectral inequalities can be applied in that setting.

1.1.3. Uncertainty principles with error term. The result we present in Section 4.3 is concerned with the situation of abstract semigroups $(\mathcal{T}(t))_{t\geq 0}$ in $L^2(\mathbb{R}^d)$ that satisfy Gelfand-Shilov smoothing properties. We describe a method that is of interest in situations where either no dissipation estimate with respect to the spectral projections of a selfadjoint operator is available, or where we have no spectral inequality for a suitable comparison operator. An approach to treat such cases goes back to the work [Mar22], where an uncertainty principle with error term for elements in the range of the semigroup allows to establish an *approximate observability estimate*. Under certain conditions on the involved constants it was shown in [Mil10] that this, in turn, leads to the above stated observability estimate (1.1), see the derivation of Corollary 3.11 in Chapter 3.

Inspired by these works, we prove an uncertainty principle with error term for functions in suitable Gelfand-Shilov spaces. This uncertainty principle is closely related to the spectral inequalities for Schrödinger operators discussed above but has an extra additive error term on the right-hand side. Postponing the precise statement to Theorem 4.35 below, we now focus only on this error term. Doing so, our result can be stated roughly as follows: For certain sensor sets ω, all functions f in the Gelfand-Shilov space, and all $\delta > 0$ we have

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \Big(\frac{1}{\delta}\Big)^{C_1} \|f\|_{L^2(\omega)} + \delta \cdot C_2,$$

where $C_1, C_2 > 0$ are constants depending on the specific Gelfand-Shilov space, the function f, and the geometry of ω, but not on δ. In contrast to the previous work [Mar22] establishing the last inequality with different techniques, the derivation we present allows to eliminate several technical assumptions on the geometry of the sensor set and streamlines the proof significantly. Since our proof uses the complex analytic approach that is also used in the proof of the spectral inequalities for the partial harmonic oscillators, it is likewise given in Chapter 7.

CHAPTER 2

Preliminaries

In this chapter we introduce the notation, give basic definitions, and provide some background material that is used in this thesis.

Some of the notation and conventions we use are set out in Table 1 below.

K, $K_\bullet$	universal constant or constant that depends only on the parameters $\bullet$ indicated in the index, may change from line to line
$\lesssim$, $\lesssim_\bullet$	short hand notation for $A \leq KB$ resp. $A \leq K_\bullet B$
C, C_j, C'_j	constants that do not change from line to line
d	dimension, unless otherwise stated a natural number ≥ 1
τ_d	volume of the unit ball in $\mathbb{R}^d$
$\lvert\cdot\rvert$	Euclidean norm or absolute value of a multi-index
$x \cdot y$	Euclidean inner product of $x, y \in \mathbb{R}^d$
$x_{\mathcal{I}}$	projection onto the coordinates indicated by $\mathcal{I} \subset \{1, \dots, d\}$
$\mathbf{1}_\omega$	characteristic function of the set ω
$B(x, r)$	ball with radius $r > 0$ centered at $x \in \mathbb{R}^d$
$\Lambda(x, a)$	rectangle with sides of length $a \in (0, \infty)^d$ centered at $x \in \mathbb{R}^d$
$\Lambda_\rho(x)$	$\Lambda(x, (\rho, \dots, \rho))$ with $\rho > 0$
$D(z, r)$	complex disc of radius $r > 0$ centered at $z \in \mathbb{C}$
D_r	$D(0, r_1) \times \dots D(0, r_d)$ for $r = (r_1, \dots, r_d) \in (0, \infty)^d$
$L^p(\Omega)$, $L^p_{\mathrm{loc}}(\Omega)$	Lebesgue spaces on the set $\Omega \subset \mathbb{R}^d$, $1 \leq p \leq \infty$
$H^k(\Omega)$, $H^k_{\mathrm{loc}}(\Omega)$	Sobolev subspace of $L^2(\Omega)$ of order $k \in \mathbb{N}$
$\lVert\cdot\rVert_X$	norm of the space X
$\mathcal{L}(X, Y)$	space of bounded operators $S \colon X \to Y$
$\operatorname{Ran} S$	range of the operator S
$\mathcal{S}(\mathbb{R}^d)$	space of Schwartz functions
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	inner product of the Hilbert space $\mathcal{H}$
$\mathcal{V}^\perp$	orthogonal complement of $\mathcal{V} \subset \mathcal{H}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$
H	lower semibounded, selfadjoint operator on some Hilbert space
$\mathbf{1}_M(H)$	spectral projection of H associated with some Borel-set $M \subset \mathbb{R}$
$P_\lambda(H)$	spectral projection $\mathbf{1}_{(-\infty, \lambda]}(H)$ for $\lambda \in \mathbb{R}$
A	generator of a contraction semigroup on some Hilbert space
$(\mathcal{T}(t))_{t \geq 0}$	strongly continuous semigroup generated by the operator A

TABLE 1. List of conventions and frequently used symbols.

2.1. Basic definitions and facts

Let $M \subset \mathbb{R}$ be any set and let $\mathcal{N} \subset \{1, \dots, d\}$. We write $x = (x_1, \dots, x_d) \in M^d$ and set $M^d_{\mathcal{N}} = \{x \in M^d : x_j = 0 \text{ for all } j \notin \mathcal{N}\}$. If $x \in M^d$, we denote by $x_{\mathcal{N}}$ the projection of x onto $M^d_{\mathcal{N}} \cong M^{|\mathcal{N}|}$ and by $L^\infty_{\text{loc}}(\mathbb{R}^d_{\mathcal{N}})$ the space of functions f defined on $\mathbb{R}^d$ for which there is $g \in L^\infty_{\text{loc}}(\mathbb{R}^{|\mathcal{N}|})$ such that $f(x) = g(x_{\mathcal{N}})$ for almost all $x \in \mathbb{R}^d$. Frequently, we use the abbreviation $\mathbb{N}^d_{0,\mathcal{N}} := (\mathbb{N}_0)^d_{\mathcal{N}}$. Denoting the j-th unit vector by $e_j \in \mathbb{R}^d$, we let $\mathrm{Id} = (e_j)_{j=1,\dots,d}$ be the identity on $\mathbb{R}^d$ and $\mathrm{Id}_{\mathcal{N}} = (v_j)_{j=1,\dots,d}$ be the matrix with columns $v_j = e_j$ if $j \in \mathcal{N}$ and $v_j = 0$ if $j \notin \mathcal{N}$.

Important subspaces of the Schwartz functions we frequently use are the so-called *Gelfand-Shilov spaces* $S^\mu_\nu(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ with $\mu, \nu > 0$ satisfying $\mu + \nu \geq 1$. These spaces were originally introduced as the spaces of functions such that both f and its Fourier transform, have a certain decay encoded by the parameters μ and ν; this is why the assumption $\mu + \nu \geq 1$ not a restriction, as the space is otherwise trivial, see, e.g., [NR10, Theorem 6.1.10]. Here, we use an equivalent characterization, see [NR10, Theorem 6.1.6 and Theorem 6.1.10], and define $S^\mu_\nu(\mathbb{R}^d)$ as the space of all functions $f \in \mathcal{S}(\mathbb{R}^d)$ for which there are constants $D_1, D_2 > 0$ such that

$$\|x^\alpha \partial^\beta f\|_{L^2(\mathbb{R}^d)} \leq D_1 D_2^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^d_0, \tag{2.1}$$

where $\partial^\beta = \partial^\beta_x$ denotes the partial derivatives. This definition shows in particular that these spaces satisfy the inclusion $S^\mu_\nu(\mathbb{R}^d) \subset S^{\mu'}_{\nu'}(\mathbb{R}^d)$ whenever $\mu \leq \mu'$ and $\nu \leq \nu'$.

2.1.1. Classic results from functional analysis.

We now give a fast paced outline of well-known results from functional analysis that are used in the present work. The reader familiar with operators on Hilbert spaces, sesquilinear forms, strongly continuous semigroups, and tensor products may skip directly to Section 2.2. The following presentation is in parts based on the textbooks [Wei80, EN00, Sch12] and we refer to these for a more detailed discussion.

Operators on Hilbert spaces. Consider a Hilbert space $\mathcal{H}$ and let $H : \mathcal{D}(H) \supset \mathcal{H} \to \mathcal{H}$ be a linear operator. If H is closed, that is if its domain $\mathcal{D}(H)$ is complete with respect to its graph norm, we define the spectrum $\sigma(H)$ of H as the set of all $\lambda \in \mathbb{C}$ such that the operator $H - \lambda\,\mathrm{Id}$ has no bounded inverse. We further decompose the spectrum into the discrete spectrum $\sigma_{\text{disc}}(H)$ consisting of isolated eigenvalues with finite multiplicity and the essential spectrum $\sigma_{\text{ess}}(H) = \sigma(H) \setminus \sigma_{\text{disc}}(H)$. Recall that if H is not closed, then the smallest closed extension $\overline{H}$ of H (if it exists) is called the closure of H. An operator core is a subset $\mathcal{D} \subset \mathcal{D}(H)$ of the domain of a closed operator H such that the operator H itself is the closure of the restriction $H|_{\mathcal{D}}$.

Most importantly, if H is a selfadjoint operators, that is if H agrees with its adjoint H^*, then H is always closed and the spectrum satisfies $\sigma(H) \subset \mathbb{R}$.

Furthermore, H is called lower semibounded, if $\sigma(H) \subset [m, \infty)$ for some $m \in \mathbb{R}$; if $\sigma(H) \subset [0, \infty)$ we simply say that H is nonnegative. The operator H is called essentially selfadjoint, if its closure is selfadjoint. Finally, let us recall the famous spectral theorem: It establishes that associated to every selfadjoint operator H there is a unique spectral measure $\mathbf{1}_M(H)$ defined on all Borel-measurable subsets M of $\mathbb{R}$ and taking values in the orthogonal projections of $\mathcal{H}$ that diagonalizes H, i.e., with $P_\lambda(H) = \mathbf{1}_{(-\infty,\lambda]}(H)$ we have $H = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_\lambda(H)$. For Borel-measurable functions f, one then defines the operators

$$f(H) = \int_{\mathbb{R}} f(\lambda) \, \mathrm{d}P_\lambda(H) \tag{2.2}$$

with domain

$$\mathcal{D}(f(H)) = \Big\{ x \in \mathcal{H} \colon \int_{\mathbb{R}} |f(\lambda)|^2 \, \mathrm{d}\langle P_\lambda(H)x, x\rangle_{\mathcal{H}} < \infty \Big\}.$$

Sesquilinear forms. A sesquilinear form $\mathfrak{h} \colon \mathcal{D}[\mathfrak{h}] \times \mathcal{D}[\mathfrak{h}] \to \mathbb{C}$ with domain $\mathcal{D}[\mathfrak{h}] \subset \mathcal{H}$ is called densely defined, if $\mathcal{D}[\mathfrak{h}]$ is dense in $\mathcal{H}$ and symmetric if $\mathfrak{h}[f, g] = \overline{\mathfrak{h}[g, f]}$ holds for all $f, g \in \mathcal{D}[\mathfrak{h}]$. A symmetric sesquilinear form is called lower semibounded if there exists an $m \in \mathbb{R}$ such that for all $f \in \mathcal{D}[\mathfrak{h}]$ we have $\mathfrak{h}[f, f] \geq m\|f\|_{\mathcal{H}}^2$. Additionally, $\mathfrak{h}$ is called closed, if $\mathfrak{h}$ is lower semibounded and the space $(\mathcal{D}[\mathfrak{h}], \|\cdot\|_{\mathfrak{h}})$ with the norm $\|f\|_{\mathfrak{h}} = (\mathfrak{h}[f, f] + (1 - m)\|f\|_{\mathcal{H}}^2)^{1/2}$ is complete. If $\mathcal{D}$ is a subspace of $\mathcal{D}[\mathfrak{h}]$ that is dense with respect to this norm, $\mathcal{D}$ is called a form core for $\mathfrak{h}$.

Densely defined, lower semibounded, and closed forms are intimately related to lower semibounded selfadjoint operators by the representation theorem for semibounded forms. In fact, for every lower semibounded selfadjoint operator H there is a unique, densely defined, lower semibounded, and closed form $\mathfrak{h}$ associated to H in the sense that $\mathfrak{h}[f, g] = \langle Hf, g\rangle_{\mathcal{H}}$ for all $f \in \mathcal{D}(H)$, $g \in \mathcal{D}[\mathfrak{h}]$, where the domain of H satisfies

$$\mathcal{D}(H) = \{ f \in \mathcal{D}[\mathfrak{h}] \colon \exists h \in L^2(\mathbb{R}^d) \forall g \in \mathcal{D}[\mathfrak{h}] \colon \mathfrak{h}[f, g] = \langle h, g\rangle_{L^2(\mathbb{R}^d)} \}.$$

Conversely, for every densely defined, lower semibounded, and closed form $\mathfrak{h}$ there is a unique lower semibounded selfadjoint operator H defined by the last two identities. In this case, we say that the operator H and the form $\mathfrak{h}$ are associated to each other.

Strongly continuous semigroups. A semigroup $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(\mathcal{H})$ is called strongly continuous if for all $x \in \mathcal{H}$ we have $\lim_{t\downarrow 0} \|\mathcal{T}(t)x - x\|_{\mathcal{H}} = 0$. Given a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(\mathcal{H})$, we say that an operator $A \colon \mathcal{D}(A) \supset \mathcal{H} \to \mathcal{H}$ is the generator of $(\mathcal{T}(t))_{t\geq 0}$, if

$$\mathcal{D}(A) = \Big\{ x \in \mathcal{H} \colon \lim_{t\downarrow 0} \frac{1}{t}\big(\mathcal{T}(t)x - x\big) \text{ exists} \Big\} \quad \text{and} \quad Ax = \lim_{t\downarrow 0} \frac{1}{t}\big(\mathcal{T}(t)x - x\big).$$

It is well-known that the semigroup is uniquely determined by its generator and there are several characterizations for generators of semigroups. In this thesis, we almost exclusively encounter contraction semigroups, i.e., semigroups satisfying $\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{H})} \leq 1$ for all $t \geq 0$. The generators of these semigroups are fully characterized by the Hille-Yosida resp. the Lumer-Phillips theorem: A is the generator of a contraction semigroup if and only if A is maximal-dissipative, that is if A is dissipative (i.e., for all $\lambda > 0$ and $x \in \mathcal{D}(A)$ we have $\|(A-\lambda\,\mathrm{Id})x\|_{\mathcal{H}} \geq \lambda\|x\|_{\mathcal{H}}$) and satisfies $\mathrm{Ran}(A-\lambda\,\mathrm{Id}) = \mathcal{H}$ for some $\lambda > 0$. In particular, if A is a negative selfadjoint operator (i.e., $\sigma(A) \subset (-\infty, 0]$), then it satisfies these conditions and the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by A satisfies $\mathcal{T}(t) = \mathrm{e}^{tA}$, where the right-hand side is defined in the sense of (2.2).

Tensor products. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. For $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$ we define the conjugate-bilinear form $(f_1 \otimes f_2)(g_1, g_2) = \langle f_1, g_1\rangle_{\mathcal{H}_1}\langle f_2, g_2\rangle_{\mathcal{H}_2}$ for $g_j \in \mathcal{H}_j$, $j = 1, 2$. Moreover, given two finite sums $u = \sum_k f_{1,k} \otimes f_{2,k}$ and $v = \sum_l f'_{1,k} \otimes f'_{2,l}$ with $f_{j,k}, f'_{j,l} \in \mathcal{H}_j$, $j = 1, 2$, we set

$$\langle u, v\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = \sum_{k,l} \langle f_{1,k}, f'_{1,l}\rangle_{\mathcal{H}_1}\langle f_{2,k}, f'_{2,l}\rangle_{\mathcal{H}_2}.$$

Then, $\langle\cdot,\cdot\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}$ defines an inner product on the space $\mathcal{H}_1 \odot \mathcal{H}_2$ of finite sums $\sum_k g_{1,k} \otimes g_{2,k}$ with $g_{j,k} \in \mathcal{H}_j$, $j = 1, 2$. The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as the completion of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the norm induced by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}$. The tensor product of operators $H_j \colon \mathcal{D}(H_j) \supset \mathcal{H}_j \to \mathcal{H}_j$, $j = 1, 2$, is defined in an analogous fashion: We define

$$(H_1 \odot H_2)\Big(\sum_k g_{1,k} \otimes g_{2,k}\Big) = \sum_k H_1 g_{1,k} \otimes H_2 g_{2,k}$$

for all finite sums $\sum_k g_{1,k} \otimes g_{2,k} \in \mathcal{D}(H_1) \odot \mathcal{D}(H_2)$. If both H_1 and H_2 are densely defined and closable, then $H_1 \odot H_2$ is also densely defined and closable. We denote its closure by $H_1 \otimes H_2$.

Since tensor product are applied throughout this thesis, let us close this section with a simple example in the case of L^2-spaces.

Example 2.1 (Elementary tensor on L^2-spaces, cf. [Sch12, Example 7.9]). Let $\mathcal{H}_1 = L^2(\mathbb{R}^{m_1})$ and $\mathcal{H}_2 = L^2(\mathbb{R}^{m_2})$ for $m_1, m_2 \in \mathbb{N}$. Then for $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$, we may identify the tensor product $f_1 \otimes f_2$ with an element in $L^2(\mathbb{R}^{m_1+m_2})$ by setting $(f_1 \otimes f_2)(x) = f_1(x_1)f_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$.

2.2. Schrödinger operators

Here, we introduce two types of Schrödinger operators used in this work. In both cases, we use the well-known fact that the multiplication operator

$$V \colon L^2(\mathbb{R}^d) \supset \mathcal{D}(V) \to L^2(\mathbb{R}^d), \quad f \mapsto V \cdot f, \tag{2.3}$$

with a real-valued measurable function $V\colon \mathbb{R}^d \to \mathbb{R}$ defined on the maximal domain

$$\mathcal{D}(V) = \{f \in L^2(\mathbb{R}^d)\colon Vf \in L^2(\mathbb{R}^d)\}$$

is selfadjoint, see, for instance, [Sch12, Example 5.3].

2.2.1. Nonnegative potentials. Let us first discuss the situation where the potential V is nonnegative. In anticipation of some of the main results presented in Section 4.2 below, we also include "partial Laplacians", even though we consider mainly the situation where we have a full Laplacian. The latter can be recovered from the following for $\mathcal{J} = \{1,\dots,d\}$ and $d_3 = 0$.

Let $\mathcal{I},\mathcal{J} \subset \{1,\dots,d\}$ and consider the space of partial $H^1(\mathbb{R}^d)$-functions

$$H^1_{\mathcal{J}}(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d)\colon \partial_k f \in L^2(\mathbb{R}^d)\ \forall k \in \mathcal{J}\}$$

which is complete when equipped with the norm

$$\|f\|_{H^1_{\mathcal{J}}(\mathbb{R}^d)} := \Bigl(\|f\|^2_{L^2(\mathbb{R}^d)} + \sum_{k\in\mathcal{J}} \|\partial_k f\|^2_{L^2(\mathbb{R}^d)}\Bigr)^{1/2}.$$

Furthermore, let $V_{\mathcal{I}} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d_{\mathcal{I}})$ be nonnegative and define the forms

$$\mathfrak{a}_{\mathcal{J}}\colon \mathcal{D}[\mathfrak{a}_{\mathcal{J}}] \times \mathcal{D}[\mathfrak{a}_{\mathcal{J}}] \to \mathbb{C}, \quad (f,g) \mapsto \sum_{k\in\mathcal{J}} \langle \partial_k f, \partial_k g\rangle_{L^2(\mathbb{R}^d)}$$

and

$$\mathfrak{v}_{\mathcal{I}}\colon \mathcal{D}[\mathfrak{v}_{\mathcal{I}}] \times \mathcal{D}[\mathfrak{v}_{\mathcal{I}}] \to \mathbb{C}, \quad (f,g) \mapsto \langle V_{\mathcal{I}}^{1/2} f, V_{\mathcal{I}}^{1/2} g\rangle_{L^2(\mathbb{R}^d)}$$

with their respective domains given by

$$\mathcal{D}[\mathfrak{a}_{\mathcal{J}}] = H^1_{\mathcal{J}}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{D}[\mathfrak{v}_{\mathcal{I}}] = \{f \in L^2(\mathbb{R}^d)\colon V_{\mathcal{I}}^{1/2} f \in L^2(\mathbb{R}^d)\}.$$

It is easy to see that $\mathfrak{a}_{\mathcal{J}}$ is nonnegative and that its form norm coincides with the norm of $H^1_{\mathcal{J}}(\mathbb{R}^d)$, so that $\mathfrak{a}_{\mathcal{J}}$ is a densely defined closed form. We denote the unique nonnegative, selfadjoint operator associated to $\mathfrak{a}_{\mathcal{J}}$ by $-\Delta_{\mathcal{J}}$. The form $\mathfrak{v}_{\mathcal{I}}$ is likewise nonnegative, closed, and densely defined since it is associated to the nonnegative, selfadjoint multiplication operator with $V_{\mathcal{I}}$ defined in (2.3). Hence, the form

$$\mathfrak{h}_{\mathcal{I},\mathcal{J}} := \mathfrak{a}_{\mathcal{J}} + \mathfrak{v}_{\mathcal{I}} \quad \text{with} \quad \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}] := \mathcal{D}[\mathfrak{a}_{\mathcal{J}}] \cap \mathcal{D}[\mathfrak{v}_{\mathcal{I}}] \tag{2.4}$$

is again closed as a sum of two nonnegative closed forms, see [Sch12, Corollary 10.2]. This form is also densely defined and the associated nonnegative, selfadjoint operator is given by $H_{\mathcal{I},\mathcal{J}} = -\Delta_{\mathcal{J}} + V_{\mathcal{I}}$.

We now prove a tensor representation for the operator $H_{\mathcal{I},\mathcal{J}}$ with $\mathcal{J} \neq \emptyset$ and derive related representations for the elements of the spectral subspaces for $H_{\mathcal{I},\mathcal{J}}$. Without loss of generality, we may reorder the coordinates of $\mathbb{R}^d$ in the following way: There are $d_1, d_2, d_3 \in \{0,\dots,d\}$ with $1 \le d_1 + d_2 \le d$ and $d_3 = d - d_1 - d_2$ such that

$$\mathcal{J} = \mathcal{N}_1 \cup \mathcal{N}_2 \quad \text{and} \quad \mathcal{I} = \mathcal{N}_1 \cup \mathcal{N}_3 \tag{2.5}$$

where $\mathcal{N}_1 = \{1,\dots,d_1\}$, $\mathcal{N}_2 = \{d_1+1,\dots,d_1+d_2\}$, and $\mathcal{N}_3 = \{d_1+d_2+1,\dots,d\}$. In case $d_3 \neq 0$ we need to make a technical constraint here since otherwise the operator can not be represented as a tensor product: We always assume that $V_{\mathcal{I}}$ can be written as a sum $V_{\mathcal{I}}(x) = V_1(x_{\mathcal{N}_1}) + V_3(x_{\mathcal{N}_3})$ with some $V_j \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d_{\mathcal{N}_j})$, $j = 1,3$. However, let us emphasize that this is satisfied in all our applications. Now, analogously to $H_{\mathcal{I},\mathcal{J}}$ above, we introduce the selfadjoint nonnegative operators H_1, H_2, and H_3 corresponding to the expressions

$$-\Delta + V_1 \text{ in } L^2(\mathbb{R}^{d_1}), \quad -\Delta \text{ in } L^2(\mathbb{R}^{d_2}), \quad V_3 \text{ in } L^2(\mathbb{R}^{d_3}),$$

respectively.

LEMMA 2.2. *With $\mathcal{I}$ and $\mathcal{J}$ as in* (2.5)*, the operator $H = H_{\mathcal{I},\mathcal{J}}$ admits the tensor representation*

$$H = H_1 \otimes I_2 \otimes I_3 + I_1 \otimes H_2 \otimes I_3 + I_1 \otimes I_2 \otimes H_3, \tag{2.6}$$

where I_j denotes the identity operator in $L^2(\mathbb{R}^{d_j})$, $j = 1,2,3$, respectively.

PROOF. Denote the operator corresponding to the right-hand side of (2.6) by $\tilde{H}$. Following [Sch12, Theorem 7.23 and Exercise 7.17.a], $\tilde{H}$ is nonnegative and selfadjoint with operator core $\mathcal{D} := \operatorname{span}_{\mathbb{C}}\{f_1 \otimes f_2 \otimes f_3 \colon f_j \in \mathcal{D}(H_j)\}$. Moreover, using the form domains of H_j, $j = 1,2,3$, we have $\mathcal{D} \subset \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}]$. We now proceed similarly as in [See21, Section 3]: Consider $f = f_1 \otimes f_2 \otimes f_3 \in \mathcal{D}$ and $g \in \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}]$. By Fubini's theorem, we then have that $g(\cdot,y,z) \in \mathcal{D}[\mathfrak{h}_1]$ (the form domain of H_1) for almost every $(y,z) \in \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}$. Using this, we see that

$$\begin{aligned}
\langle (H_1 \otimes I_2 \otimes I_3) f, g\rangle_{L^2(\mathbb{R}^d)} &= \langle (H_1 f_1 \otimes f_2 \otimes f_3), g\rangle_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^{d_2}\times\mathbb{R}^{d_3}} f_2(y) f_3(z) \langle H_1 f_1, g(\cdot,y,z)\rangle_{L^2(\mathbb{R}^{d_1})} \,\mathrm{d}(y,z) \\
&= \int_{\mathbb{R}^{d_2}\times\mathbb{R}^{d_3}} f_2(y) f_3(z) \mathfrak{h}_1[f_1, g(\cdot,y,z)] \,\mathrm{d}(y,z) \\
&= \mathfrak{h}_{\mathcal{N}_1,\mathcal{N}_1}[f,g].
\end{aligned}$$

In a completely analogous way, we establish

$$\langle (I_1 \otimes H_2 \otimes I_3) f, g\rangle_{L^2(\mathbb{R}^d)} = \mathfrak{h}_{\emptyset,\mathcal{N}_2}[f,g]$$

and

$$\langle (I_1 \otimes I_2 \otimes H_3) f, g\rangle_{L^2(\mathbb{R}^d)} = \mathfrak{h}_{\mathcal{N}_3,\emptyset}[f,g].$$

Summing up gives

$$\langle \tilde{H} f, g\rangle_{L^2(\mathbb{R}^d)} = \mathfrak{h}_{\mathcal{N}_1,\mathcal{N}_1}[f,g] + \mathfrak{h}_{\emptyset,\mathcal{N}_2}[f,g] + \mathfrak{h}_{\mathcal{N}_3,\emptyset}[f,g] = \mathfrak{h}_{\mathcal{I},\mathcal{J}}[f,g].$$

By sesquilinearity, the latter extends to all $f \in \mathcal{D}$, so that $\tilde{H}|_{\mathcal{D}} \subset H$. Since $\mathcal{D}$ is an operator core for $\tilde{H}$ and both H and $\tilde{H}$ are selfadjoint, we conclude that $\tilde{H} = \overline{\tilde{H}|_{\mathcal{D}}} = H$, which proves the claim. □

Corollary 2.3. *In the situation of Lemma 2.2, we have*

$$\sigma(H) = \sigma(H_1) + \sigma(H_2) + \sigma(H_3),$$

and the restriction of H to the Schwartz functions on $\mathbb{R}^d$ is essentially selfadjoint.

Proof. The first part follows from [Sch12, Corollary 7.25]; cf. also [Sch12, Exercise 7.18.a]. For the second part, we observe that H_2 and H_3 are essentially selfadjoint on $\mathcal{S}(\mathbb{R}^{d_2})$ and $\mathcal{S}(\mathbb{R}^{d_3})$, respectively. Moreover, [Kat72] implies that the operator H_1 is essentially selfadjoint on the smooth and compactly supported functions $C_c^\infty(\mathbb{R}^{d_1}) \subset \mathcal{S}(\mathbb{R}^{d_1})$. Hence, H is essentially selfadjoint on the space $\operatorname{span}_\mathbb{C}\{f_1 \otimes f_2 \otimes f_3 \colon f_j \in \mathcal{S}(\mathbb{R}^{d_j})\} \subset \mathcal{S}(\mathbb{R}^d)$, see, e.g., [Wei80, Theorem 8.33]. □

We are mostly concerned with the situation where $d_3 = 0$. In this situation we simply write $V(x) = V_1(x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Thereby, the previous calculations simplify and the third tensor factor in (2.6) can be dropped, that is, we have $H = H_1 \otimes I_2 + I_1 \otimes H_2$ and $\sigma(H) = \sigma(H_1) + \sigma(H_2)$. If, in addition, V_1 satisfies $V_1(x) \to \infty$ as $|x| \to \infty$ (where $x \in \mathbb{R}^{d_1}$), [Sch12, Proposition 12.7] implies that H_1 has purely discrete spectrum. In this case, we obtain the following result.

Corollary 2.4. *Suppose $V_1(x) \to \infty$ as $|x| \to \infty$ and that $d_3 = 0$ holds in addition to the assumptions of Lemma 2.2. Then:*

(a) Every $f \in \operatorname{Ran} P_\lambda(H)$, $\lambda \geq 0$, can be represented as a finite sum

$$f = \sum_k \phi_k \otimes \psi_k$$

with suitable $\phi_k \in \operatorname{Ran} P_\lambda(H_1)$ and $\psi_k \in \operatorname{Ran} P_\lambda(H_2)$. Moreover, $(\partial^\alpha f)(\cdot, y)$ belongs to $\operatorname{Ran} P_\lambda(H_1)$ for all $y \in \mathbb{R}^{d_2}$ and all multi-indices $\alpha \in \mathbb{N}^d_{0,\mathcal{I}^c}$, while $(\partial^\beta f)(x, \cdot)$ belongs to $\operatorname{Ran} P_\lambda(H_2)$ for all $x \in \mathbb{R}^{d_1}$ and all $\beta \in \mathbb{N}^d_{0,\mathcal{I}}$.

(b) If all elements of $\operatorname{Ran} P_\lambda(H_1)$, $\lambda \geq 0$, can be extended to analytic functions on $\mathbb{C}^{d_1}$, then f can be extended to an analytic function on $\mathbb{C}^d$.

Proof. We proceed similarly as in the proof of [ES21, Lemma 2.3]. Let $f \in \operatorname{Ran} P_\lambda(H)$ for some $\lambda \geq 0$, and let $(\phi_k)_k$ be an orthonormal basis of eigenfunctions of H_1 with corresponding eigenvalues μ_k. Write

$$f(x, y) = \sum_k \langle f(\cdot, y), \phi_k \rangle_{L^2(\mathbb{R}^{d_1})} \phi_k(x) = \sum_k \phi_k(x) g_k(y),$$

where $g_k \in L^2(\mathbb{R}^{d_2})$ is given by $g_k(y) = \langle f(\cdot, y), \phi_k \rangle_{L^2(\mathbb{R}^{d_1})}$.

By [Wei80, Theorem 8.34 and Exercise 8.21], the spectral family $P_\lambda(H)$ for H admits the representation

$$P_\lambda(H) = \sum_{\mu \leq \lambda} \mathbf{1}_{\{\mu\}}(H_1) \otimes P_{\lambda - \mu}(H_2).$$

This implies that

$$f = P_\lambda(H)f = \sum_{k:\, \mu_k \le \lambda} \phi_k \otimes \psi_k$$

with $\psi_k = P_{\lambda-\mu_k}(H_2)g_k \in \operatorname{Ran} P_{\lambda-\mu_k}(H_2) \subset \operatorname{Ran} P_\lambda(H_2)$, which shows part (a) of the statement.

In order to prove part (b), recall that each element of $\operatorname{Ran} P_\lambda(H_2)$ can be extended to an analytic function on $\mathbb{C}^{d_2}$ by the Paley–Wiener theorem. Hence, the statement follows by using the corresponding properties of ϕ_k and ψ_k and Hartogs' theorem on separate analyticity. □

It is also easy to show that the smooth and compactly supported functions are a form core for H.

LEMMA 2.5 (cf. [CFKS87, Theorem 1.13]). *Let $d_3 = 0$. Then a form core for H as in Lemma 2.2 is given by the test functions $C_c^\infty(\mathbb{R}^d)$.*

PROOF. We give a simple proof for the current situation with $V_1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{d_1})$.

Let $\mathfrak{h} = \mathfrak{a} + \mathfrak{v}$ be the form associated to H as in the proof of Lemma 2.2 and let $f \in \mathcal{D}[\mathfrak{h}]$. First, suppose that f has compact support, and choose a sequence of test functions (φ_n) with $\varphi_n \to f$ in $H^1(\mathbb{R}^d)$ such that the supports of f and each φ_n are contained in a common compact set. Since $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{d_1})$, it is then clear that also $\mathfrak{v}[f - \varphi_n, f - \varphi_n] \to 0$, that is, $\mathfrak{h}[f - \varphi_n, f - \varphi_n] \to 0$, as $n \to \infty$.

If f does not have compact support, we approximate f in the form norm by functions in $\mathcal{D}[\mathfrak{h}]$ with compact support. To this end, we follow the proof of [GY12, Lemma 2.2] and introduce $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$ and define $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ for $\varepsilon > 0$ by $\chi_\varepsilon(x) := \chi(\varepsilon x)$. Crucially, each $\chi_\varepsilon f$ belongs to $\mathcal{D}[\mathfrak{h}]$ and has compact support. Also observe that $1 - \chi_\varepsilon$ is uniformly bounded in ε with $1 - \chi_\varepsilon \to 0$ pointwise as $\varepsilon \to 0$. Moreover, $\||\nabla \chi_\varepsilon|\|_{L^\infty(\mathbb{R}^d)} \le \varepsilon \||\nabla \chi|\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. With this, we conclude by the dominated convergence theorem that $\chi_\varepsilon f \to f$ in $H^1(\mathbb{R}^d)$ and that $\mathfrak{v}[(1-\chi_\varepsilon)f, (1-\chi_\varepsilon)f] \to 0$. In particular, $\mathfrak{h}[(1-\chi_\varepsilon)f, (1-\chi_\varepsilon)f] \to 0$, as $\varepsilon \to 0$, which together with the first part completes the proof. □

Let us recall the following classic statement of elliptic regularity which we use in the proof given in Chapter 5.

LEMMA 2.6. *Let $d_3 = 0$ and let H be as in Lemma 2.2. We have $\mathcal{D}(H) \subset H^2_{\mathrm{loc}}(\mathbb{R}^d)$, and every $f \in \mathcal{D}(H)$ satisfies $Hf = -\Delta f + Vf$ almost everywhere on $\mathbb{R}^d$. Moreover, if $Hf \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ for some $f \in \mathcal{D}(H)$, then $f \in H^3_{\mathrm{loc}}(\mathbb{R}^d)$.*

PROOF. Let $f \in \mathcal{D}(H)$, let $\mathfrak{h} = \mathfrak{a} + \mathfrak{v}$ be the form associated to H, and let $\Omega \subset \mathbb{R}^d$ be open and bounded. Abbreviate $g := (Hf)|_\Omega \in L^2(\Omega)$ and $h := (Vf)|_\Omega \in H^1(\Omega)$. For all test functions $\varphi \in C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^d)$ we then have $\mathfrak{a}[f, \varphi] = \mathfrak{h}[f, \varphi] - \mathfrak{v}[f, \varphi] = \langle g - h, \varphi \rangle_{L^2(\Omega)}$. Classical elliptic regularity results, see, e.g., [BS91, Theorem S2.2.1], imply that $f \in H^2_{\mathrm{loc}}(\Omega)$. In particular, we have

$-\Delta f = g - h$ almost everywhere on Ω. Since Ω was chosen arbitrarily, this proves the first claim.

If, in addition, $Hf \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, then the above choices of g and h satisfy even $g - h \in H^1(\Omega)$, so that the same procedure with the regularity result from [BS91, Theorem S2.2.1] yields $f \in H^3_{\mathrm{loc}}(\mathbb{R}^d)$. $\square$

The next lemma is presented here for the sake of completeness, since we cannot specify the domain of the operator H explicitly.

LEMMA 2.7. *Let $d_3 = 0$ and let H be as in Lemma 2.2. If $f \in \mathcal{D}(H)$ satisfies $Vf \in L^2(\mathbb{R}^d)$, then $f \in H^2(\mathbb{R}^d)$ and $Hf = -\Delta f + Vf$ pointwise almost everywhere.*

PROOF. Let $f \in \mathcal{D}(H)$ with $Vf \in L^2(\mathbb{R}^d)$, and let $g \in C_c^\infty(\mathbb{R}^d)$. Then,

$$\mathfrak{a}[f,g] = \mathfrak{h}[f,g] - \mathfrak{v}[f,g] = \langle Hf, g\rangle_{L^2(\mathbb{R}^d)} - \langle Vf, g\rangle_{L^2(\mathbb{R}^d)} = \langle Hf - Vf, g\rangle_{L^2(\mathbb{R}^d)}$$

with the to H associated form $\mathfrak{h} = \mathfrak{a} + \mathfrak{v}$. By approximation, we conclude that $\mathfrak{a}[f,g] = \langle Hf - Vf, g\rangle_{L^2(\mathbb{R}^d)}$ for all $g \in H^1(\mathbb{R}^d)$, which proves that $f \in \mathcal{D}(\Delta) = H^2(\mathbb{R}^d)$ and $-\Delta f = Hf - Vf$. $\square$

Partial harmonic oscillators. Let us point out some specifics of the situation where the potential is of the form $V_{\mathcal{I}}(x) = |x_{\mathcal{I}}|^2$ and, accordingly, $H = H_{\mathcal{I},\mathcal{J}}$ is a so-called partial harmonic oscillator. First, we consider the set

$$\mathcal{G}_{\mathcal{I},\mathcal{J}} = \{f \in L^2(\mathbb{R}^d) \colon x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d) \ \forall \alpha \in \mathbb{N}^d_{0,\mathcal{I}},\ \beta \in \mathbb{N}^d_{0,\mathcal{J}}\} \subset \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}]$$

that is used in the proof we give in Chapter 8.

LEMMA 2.8. *We have $\mathcal{G}_{\mathcal{I},\mathcal{J}} \subset \mathcal{D}(H_{\mathcal{I},\mathcal{J}})$ and $H_{\mathcal{I},\mathcal{J}} f = -\Delta_{\mathcal{J}} f + |x_{\mathcal{I}}|^2 f$ for all $f \in \mathcal{G}_{\mathcal{I},\mathcal{J}}$. In particular, $\mathcal{G}_{\mathcal{I},\mathcal{J}}$ is invariant for $H_{\mathcal{I},\mathcal{J}}$.*

PROOF. Let $f \in \mathcal{G}_{\mathcal{I},\mathcal{J}}$ and $g \in \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}] \subset H^1_{\mathcal{J}}(\mathbb{R}^d)$. Then, using Fubini's theorem and that $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, integration by parts in each coordinate of $\mathcal{J}$ yields

$$\begin{aligned}
\mathfrak{h}_{\mathcal{I},\mathcal{J}}[f,g] &= \sum_{k\in\mathcal{J}} \langle \partial_k f, \partial_k g\rangle_{L^2(\mathbb{R}^d)} + \langle |x_{\mathcal{I}}| f, |x_{\mathcal{I}}| g\rangle_{L^2(\mathbb{R}^d)} \\
&= \langle -\Delta_{\mathcal{J}} f, g\rangle_{L^2(\mathbb{R}^d)} + \langle |x_{\mathcal{I}}|^2 f, g\rangle_{L^2(\mathbb{R}^d)} \\
&= \langle -\Delta_{\mathcal{J}} f + |x_{\mathcal{I}}|^2 f, g\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Since $g \in \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}]$ was arbitrary, this proves the claim. $\square$

Next, we show that for general $\mathcal{I}, \mathcal{J} \subset \{1, \dots, d\}$ we can trade the parts of the potential corresponding to elements in $\mathcal{I} \setminus \mathcal{J}$ for additional derivatives via an appropriate partial Fourier transform. Let $m = \#(\mathcal{I} \setminus \mathcal{J})$, and decompose

$$x = (x^{(1)}, x^{(2)}) \quad \text{with} \quad x^{(1)} \in \mathbb{R}^d_{\mathcal{I}\setminus\mathcal{J}},\ x^{(2)} \in \mathbb{R}^d_{(\mathcal{I}\setminus\mathcal{J})^c}.$$

We consider the partial Fourier transform

$$(\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}f)(x) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(\eta, x^{(2)}) \mathrm{e}^{-i\eta\cdot x^{(1)}}\,\mathrm{d}\eta \quad \text{for} \quad f \in L^2(\mathbb{R}^d), \tag{2.7}$$

which, by Plancherel's and Fubini's theorems, defines a unitary operator on $L^2(\mathbb{R}^d)$. We utilize this transformation to show that $H_{\mathcal{I},\mathcal{J}}$ is unitary equivalent to a partial harmonic oscillator $H_{\mathcal{I}',\mathcal{J}'}$ with $\mathcal{J}' \supset \mathcal{I}'$.

LEMMA 2.9. *With* $\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$ *as in* (2.7) *we have*

$$H_{\mathcal{I},\mathcal{J}} = \mathcal{F}^{-1}_{\mathcal{I}\setminus\mathcal{J}} H_{\mathcal{I}\cap\mathcal{J},\mathcal{I}\cup\mathcal{J}} \mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}.$$

In particular, for all $\lambda \geq 0$ *we have*

$$\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}} \operatorname{Ran} P_\lambda(H_{\mathcal{I},\mathcal{J}}) = \operatorname{Ran} P_\lambda(H_{\mathcal{I}\cap\mathcal{J},\mathcal{I}\cup\mathcal{J}}).$$

PROOF. We first observe that for $j \in \mathcal{J}$, the partial derivative ∂_{x_j} in direction e_j commutes with $\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$, while for $j \in \mathcal{I}\cap\mathcal{J}$, the multiplication by x_j commutes with $\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$. Moreover, for $f \in L^2(\mathbb{R}^d)$ and $k \in \mathcal{I}\setminus\mathcal{J}$, we have $x_k f \in L^2(\mathbb{R}^d)$ if and only if $\partial_{x_k}\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}} f \in L^2(\mathbb{R}^d)$ and $\partial_{x_k}\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}} f = -\mathrm{i}\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}(x_k f)$. With this, we see that $f \in \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}]$ if and only if $\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}} f \in \mathcal{D}[\mathfrak{h}_{\mathcal{I}\cap\mathcal{J},\mathcal{I}\cup\mathcal{J}}]$ and that

$$\mathfrak{h}_{\mathcal{I}\cap\mathcal{J},\mathcal{I}\cup\mathcal{J}}[\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}} f, \mathcal{F}_{\mathcal{I}\setminus\mathcal{J}} g] = \mathfrak{h}_{\mathcal{I},\mathcal{J}}[f,g] \quad \text{for all} \quad f,g \in \mathcal{D}[\mathfrak{h}_{\mathcal{I},\mathcal{J}}],$$

which proves the first part of the lemma. Immediately, the unitary equivalence implies the second part. □

2.2.2. Operator bounded potentials. For $\mathcal{J} = \{1,\dots,d\}$ the space of partial H^1-functions $H_{\mathcal{J}}(\mathbb{R}^d)$ introduced in the previous subsection coincides with the Sobolev space $H^1(\mathbb{R}^d)$. Moreover, the unique selfadjoint operator associated to the form $\mathfrak{a} = \mathfrak{a}_{\mathcal{J}}$ is in this case the usual Laplacian

$$-\Delta\colon L^2(\mathbb{R}^d) \supseteq \mathcal{D}(\Delta) \to L^2(\mathbb{R}^d) \quad \text{with} \quad \mathcal{D}(\Delta) = H^2(\mathbb{R}^d).$$

If $V\colon \mathbb{R}^d \to \mathbb{R}$ is a measurable function, the associated multiplication operator is said to be *infinitesimally* Δ*-bounded* if for every $\varepsilon > 0$ there exists some $b = b(\varepsilon) \geq 0$ such that

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq \varepsilon\|\Delta f\|_{L^2(\mathbb{R}^d)} + b\|f\|_{L^2(\mathbb{R}^d)} \quad \text{for all} \quad f \in \mathcal{D}(\Delta).$$

For such V the well-known Kato-Rellich theorem implies that the Schrödinger operator H defined as the operator sum

$$H = -\Delta + V\colon L^2(\mathbb{R}^d) \supset \mathcal{D}(\Delta) \to L^2(\mathbb{R}^d), \tag{2.8}$$

is selfadjoint and lower semibounded.

REMARK 2.10. If $\Omega \subsetneq \mathbb{R}^d$, then the choice of the form-domain $\mathcal{D}[\mathfrak{a}]$ of

$$\mathfrak{a}[f,g] = \sum_{j=1}^{d} \langle \partial_k f, \partial_k g\rangle_{L^2(\Omega)} \quad \text{with} \quad f,g \in \mathcal{D}[\mathfrak{a}]$$

encodes the boundary conditions of the Laplacian. For instance,

- if $\mathcal{D}[\mathfrak{a}] = H^1(\Omega)$, the associated selfadjoint Laplacian $-\Delta_\Omega^N$ has Neumann boundary conditions.
- if $\mathcal{D}[\mathfrak{a}] = H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm of $H^1(\Omega)$, the associated selfadjoint Laplacian $-\Delta_\Omega^D$ has Dirichlet boundary conditions.
- if Ω is a bounded and rectangular shaped domain and $\mathcal{D}[\mathfrak{a}] = H^1_{\mathrm{per}}(\Omega)$ is the closure of the periodic $C^\infty(\Omega)$-functions with respect to the norm of $H^1(\Omega)$, the associated selfadjoint Laplacian $-\Delta_\Omega^{\mathrm{per}}$ has periodic boundary conditions.

If $V\colon \Omega \to \mathbb{R}$ is a measurable function that is infinitesimally $\Delta_\Omega^\bullet$-bounded, where $\bullet \in \{N, D, \mathrm{per}, \dots\}$ denotes the chosen boundary conditions, the corresponding Schrödinger operator $H_\Omega^\bullet$ with these boundary conditions can be defined analogously to (2.8) via the Kato-Rellich theorem.

2.3. Quadratic differential operators

Here we recall some results from the theory of quadratic differential operators that are used in the formulation and proof of our main result in Section 4.2. For a broader overview on the general theory of pseudo-differential operators, we refer, among others, to [Hör07, NR10]. To begin with, we consider the complex quadratic polynomial

$$q\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}, \quad q(x,\xi) = \sum_{\substack{|\alpha+\beta|=2\\ \alpha,\beta\in\mathbb{N}_0^d}} c_{\alpha,\beta} x^\alpha \xi^\beta, \quad c_{\alpha,\beta} \in \mathbb{C}. \tag{2.9}$$

It is well-known that the distribution kernel

$$\mathcal{K}(x,y) = (2\pi)^{-d/2}\mathcal{F}^{-1}\big(\mathbb{R}^d \ni \xi \mapsto q((x+y)/2,\xi)\big)(x-y)$$

defines a continuous operator $q^{\mathrm{w}}\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle q^{\mathrm{w}} u, v\rangle = \langle \mathcal{K}, v \otimes u\rangle \quad \text{for} \quad u,v \in \mathcal{S}(\mathbb{R}^d).$$

Here $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions, $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denotes the Fourier transform, and $\langle\cdot,\cdot\rangle$ is the pairing between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ and between $\mathcal{S}'(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^{2d})$, respectively. Moreover, the thus defined operator q^{w} extends to a continuous operator on $\mathcal{S}'(\mathbb{R}^d)$, see [NR10, Proposition 1.2.13], and we may therefore define

$$A\colon L^2(\mathbb{R}^d) \supset \mathcal{D}(A) \to L^2(\mathbb{R}^d), \quad f \mapsto q^w f \tag{2.10}$$

on

$$\mathcal{D}(A) = \{f \in L^2(\mathbb{R}^d)\colon q^{\mathrm{w}} f \in L^2(\mathbb{R}^d)\}.$$

We call A the *quadratic differential operator* associated to q and q its *symbol*.

The next lemma due to Hörmander shows that quadratic differential operators with symbol A are generators of contraction semigroups if the real part of q is non-positive.

PROPOSITION 2.11 ([Hör95]). *Let q be as in* (2.9). *Then the operator A defined by* (2.10) *is closed, densely defined, and agrees with the closure of the restriction of q^{w} to the space $\mathcal{S}(\mathbb{R}^d)$. If $\operatorname{Re} q \leq 0$, then A generates a contraction semigroup.*

We point out that a symbol q may also be complex valued in which case the generator in the previous proposition is non-selfadjoint.

The operator $A = q^{\mathrm{w}}$ introduced via a distribution in (2.10), is explicitly given by

$$(Af)(x) = (q^{\mathrm{w}} f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}y\cdot\xi} q\Big(\frac{x+y}{2}, \xi\Big) f(y) \,\mathrm{d}y \,\mathrm{d}\xi \tag{2.11}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, see [NR10, Propositon 1.2.3]. With this explicit expression at hand, one easily verifies that A is a simple differential operator.

LEMMA 2.12. *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $k, j \in \{1, \ldots, d\}$ we have:*
(a) If $q(x,\xi) = x_k^2$ then $(q^{\mathrm{w}} f)(x) = x_k^2 f(x)$.
(b) If $q(x,\xi) = \xi_j^2$ then $(q^{\mathrm{w}} f)(x) = (\partial_{x_j}^2 f)(x)$.
(c) If $q(x,\xi) = x_k \xi_j$ then $(q^{\mathrm{w}} f)(x) = \frac{1}{2}\big(\mathrm{i}x_k(\partial_{x_j} f)(x) + \mathrm{i}(\partial_{x_j}(x_k f))(x)\big)$.

Let now $\mathcal{R} \colon \mathbb{R}^d \to \mathbb{R}^d$ be any unitary map and let $\tilde{q}$ be the quadratic polynomial with $\tilde{q}(x,\xi) = q(\mathcal{R}x, \mathcal{R}\xi)$. The following lemma relates the Weyl quantizations of q and $\tilde{q}$.

LEMMA 2.13. *Let $(\mathcal{U}_{\mathcal{R}} f)(x) = f(\mathcal{R}x)$ for $f \in L^2(\mathbb{R}^d)$. Then $\mathcal{U}_{\mathcal{R}} q^{\mathrm{w}} \mathcal{U}_{\mathcal{R}}^{-1} = \tilde{q}^{\mathrm{w}}$ on $\mathcal{S}(\mathbb{R}^d)$. Moreover, if $\tilde{A}$ is the quadratic differential operator with symbol $\tilde{q}$ we have $\mathcal{U}_{\mathcal{R}} A \mathcal{U}_{\mathcal{R}}^{-1} = \tilde{A}$.*

PROOF. Let $f \in \mathcal{S}(\mathbb{R}^d)$. We have

$$(\tilde{q}^{\mathrm{w}} f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}y\cdot\xi} q\Big(\frac{\mathcal{R}x + \mathcal{R}y}{2}, \mathcal{R}\xi\Big) f(y) \,\mathrm{d}y \,\mathrm{d}\xi$$

and substituting $y' = \mathcal{R}y$ and $\xi' = \mathcal{R}\xi$ yields

$$\begin{aligned}
&(\tilde{q}^{\mathrm{w}} f)(x) \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\mathcal{R}^{-1}\xi'} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\mathcal{R}^{-1}y'\cdot\mathcal{R}^{-1}\xi'} q\Big(\frac{\mathcal{R}x + y'}{2}, \xi'\Big) f(\mathcal{R}^{-1}y') \,\mathrm{d}y' \,\mathrm{d}\xi' \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\mathcal{R}x\cdot\xi'} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}y'\cdot\xi'} q\Big(\frac{\mathcal{R}x + y'}{2}, \xi'\Big) f(\mathcal{R}^{-1}y') \,\mathrm{d}y' \,\mathrm{d}\xi' \\
&\quad = (q^{\mathrm{w}}(f \circ \mathcal{R}^{-1}))(\mathcal{R}x).
\end{aligned}$$

Hence $\mathcal{U}_{\mathcal{R}}\big(q^{\mathrm{w}}(\mathcal{U}_{\mathcal{R}}^{-1}f)\big)(x) = \tilde{q}^{\mathrm{w}}f(x)$. Finally, since the operator A is the closure of $q^{\mathrm{w}}|_{\mathcal{S}(\mathbb{R}^d)}$ by Proposition 2.11, the statement of extends to the operators A and $\tilde{A}$ corresponding to q and $\tilde{q}$, respectively. □

REMARK 2.14. It is also possible to give the proof of Lemma 2.13 using Lemma 2.12 and establishing the identity $\mathcal{U}_{\mathcal{R}}q^{\mathrm{w}}\mathcal{U}_{\mathcal{R}}^{-1} = \tilde{q}^{\mathrm{w}}$ for the elementary symbols $q(x,\xi) = x^{\alpha}\xi^{\beta}$ with $|\alpha+\beta| = 2$ via the chain rule.

2.3.1. The singular space. It is a result of Hörmander [Hör95] that the semigroup generated by a quadratic differential operator can be identified with the Hamilton flow of the corresponding quadratic symbol.[1] Interpreting q as a quadratic form on $\mathbb{R}^{2d}$ and using the same letter to denote the polarized form, the Hamilton flow associated with q is the solution of the system of $2d$-ordinary differential equations $Z' = -\mathrm{i}J\frac{\partial}{\partial Z}q(Z)$, where J is the matrix corresponding to the standard symplectic form σ on $\mathbb{R}^{2d}$. Since q is a quadratic form on $\mathbb{R}^{2d}$, there is a matrix $Q \in \mathbb{R}^{2d\times 2d}$ such that for all $X, Y \in \mathbb{R}^{2d}$ we have $q(X,Y) = X\cdot QY$. Setting $F = JQ$ we therefore have $q(X,Y) = \sigma(X, FY)$ for all $X, Y \in \mathbb{R}^{2d}$ and with this matrix the Hamilton flow can be written as $\mathrm{e}^{-\mathrm{i}tF}$. It is easy to compute that

$$F = \frac{1}{2}\begin{pmatrix} (\partial_{\xi_j}\partial_{x_k}q(x,\xi))_{j,k=1}^d & (\partial_{\xi_j}\partial_{\xi_k}q(x,\xi))_{j,k=1}^d \\ -(\partial_{x_j}\partial_{x_k}q(x,\xi))_{j,k=1}^d & -(\partial_{x_j}\partial_{\xi_k}q(x,\xi))_{j,k=1}^d \end{pmatrix}$$

and F is called the *fundamental matrix*. With the fundamental matrix at hand, the *singular space* of the quadratic form q (or the operator $A = q^{\mathrm{w}}$) has been introduced in [HPS09] as

$$S = S(A) = S(q) = \left(\bigcap_{j=0}^{2d-1} \ker\big[\operatorname{Re}F(\operatorname{Im}F)^j\big]\right) \cap \mathbb{R}^{2d}, \tag{2.12}$$

where the real $\operatorname{Re}F$ and imaginary part $\operatorname{Im}F$ are taken entrywise. We denote by $k_0 \in \{0,\dots 2d-1\}$ the smallest number such that

$$S = \left(\bigcap_{j=0}^{k_0} \ker\big[\operatorname{Re}F(\operatorname{Im}F)^j\big]\right) \cap \mathbb{R}^{2d}, \tag{2.13}$$

which exists due to the Cayley-Hamilton theorem. Throughout this work, we call k_0 the *rotation exponent* of q (resp. A). One important case are the real-valued symbols where $k_0 = 0$ since $\operatorname{Im}F = 0$.

The singular space is strongly connected to so-called smoothing properties of the semigroup generated by $A = q^{\mathrm{w}}$ which play a major role in the result we state in Section 4.2. More precisely, it encodes directions in the phase space in

[1] In [AB] it is stated i.a. that the so-called *exact classical-quantum correspondance* due to Hörmander “allows to identify a semigroup generated by the Weyl quantization of a quadratic form with the Hamiltonian flow of this quadratic form (i.e. the exponential of a matrix).”

The author is grateful to Paul Alphonse for pointing out this correspondence in a personal conversation in Dortmund.

which elements in the range of the semigroup associated to the operator A do not behave like a Schwartz function. We refer the reader to Chapter 8 below for more information and references concerning the connection between the singular space and the smoothing properties of the semigroup.

The singular space can be easily computed for the partial harmonic oscillators $H_{\mathcal{I},\mathcal{J}}$ introduced via the form method in Section 2.2 above.

LEMMA 2.15. *Let $\mathcal{I},\mathcal{J} \subset \{1,\dots,d\}$ and define $q(x,\xi) = -|\xi_{\mathcal{J}}|^2 - |x_{\mathcal{I}}|^2$ for $x,\xi \in \mathbb{R}^d$. Then, for the quadratic differential operator A with symbol q, we have $A = -H_{\mathcal{I},\mathcal{J}}$ and the singular space is given by $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d_{\mathcal{J}}$.*

PROOF. The operators A and $-H_{\mathcal{I},\mathcal{J}}$ agree on the Schwartz space and therefore the desired equality follows from Corollary 2.3 and Proposition 2.11. In order to calculate the singular space, we note that the rotation exponent k_0 of q is zero since q is real-valued. A direct calculation yields $S(q) = \ker F = \mathbb{R}^d_{\mathcal{I}^c} \times \mathbb{R}^d_{\mathcal{J}^c}$. □

In order to have explicit examples at hand where our main result from Section 4.2 is applicable, we present the particular models of a *Kolmogorov* and *Kramers-Fokker-Planck operators* here. These operators are special examples of so-called *generalized Ornstein-Uhlenbeck operators* and our result in Section 4.2 can also be applied for this larger class of operators. However, the following examples already allow us to distinguish our results sufficiently from previous results. Therefore, for simplicity we restrict ourselves to these.

In the next examples we let $d = 2m$ with $m \in \mathbb{N}$ and write $y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^m$. The singular space in both examples can be computed explicitly as demonstrated for instance in [Alp21, Proof of Theorem 5.2].

EXAMPLE 2.16 (Kolmogorov equation). Let $q(x,\xi) = -|\xi^{(1)}|^2 + \mathrm{i}x^{(2)}\cdot\xi^{(1)}$. Then the quadratic differential operator corresponding to this quadratic polynomial is the *Kolmogorov operator* $A = \Delta_{x^{(1)}} - x^{(2)}\cdot\nabla_{x^{(1)}}$. For this operator the singular space is given by $S(A)^\perp = \{0\} \times \mathbb{R}^d$.

EXAMPLE 2.17 (Kramers-Fokker-Planck operator). Let $\mathcal{I}_1 \subset \{1,\dots,m\}$ and define

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 2\,\mathrm{Id}_m \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mathrm{Id}_m \\ -\mathrm{Id}_{\mathcal{I}_1} & 0 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\,\mathrm{Id}_m \end{pmatrix}.$$

The quadratic differential operator A corresponding to the quadratic polynomial

$$q(x,\xi) = -\frac{1}{2}Q\xi\cdot\xi - \frac{1}{2}Rx\cdot x - \mathrm{i}Bx\cdot\xi \tag{2.14}$$

is given by

$$A = \Delta_{x^{(2)}} - \frac{1}{4}|x^{(2)}|^2 - x^{(2)}\cdot\nabla_{x^{(1)}} + \nabla_{x^{(1)}}V(x^{(1)})\cdot\nabla_{x^{(2)}},$$

where $V(x^{(1)}) = |(x^{(1)})_{\mathcal{I}_1}|^2$ is the so-called *external potential.* In this setting, the singular space satisfies $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d$ with $\mathcal{I} = \mathcal{I}_1 \cup \{m+1,\dots,d\}$.

CHAPTER 3

Control theory

In this chapter, we give a brief introduction into control theory for evolution equations on Hilbert spaces. The abstract concepts of (null-)controllability and (final-state) observability are introduced in Section 3.1, where also the duality of these notions is discussed. As a consequence of this duality, we limit ourselves in the rest of this thesis to the study of observability. In Section 3.2, we present criteria to conclude observability of a system solely from properties of the generator. Our study of observability in Chapter 4 mainly relies on these criteria.

The setting we introduce here is contained in several works dealing with control theory and this chapter is oriented in part on the presentations in [EN00, TW09, ENS$^+$20].

3.1. Observability and controllability

The time evolution of the state of an autonomous linear system $w'(t) = Aw(t)$ with initial value $w(0) = w_0 \in \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is described by the strongly continuous operator semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $A\colon \mathcal{H} \supset \mathcal{D}(A) \to \mathcal{H}$. These semigroups were and are studied extensively, see, e.g., the monographs [HP57, Paz83, EN00]. When modeling manipulations or measurements of the system from the outside one enters the field of *control theory.* Let us start by formulating the problem in more detail: We deal with the *controlled abstract Cauchy problem*

$$
\begin{cases} w'(t) = Aw(t) + \mathcal{B}u(t), & t \in (0, T], \\ v(t) = \mathcal{C}w(t), & t \in [0, T], \\ w(0) = w_0, & w_0 \in \mathcal{H}, \end{cases} \tag{3.1}
$$

where $T > 0$, $\mathcal{U}$ and $\mathcal{V}$ are the control and observation Hilbert spaces, respectively, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ is the *control operator*, $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{V})$ is the *observation operator*, and $u \in L^2((0, T); \mathcal{U})$ is the *control function.* We denote the abstract control system associated to (3.1) by $\Sigma(A, \mathcal{B}, \mathcal{C})$ and write $\Sigma_c(A, \mathcal{B})$ or $\Sigma_o(A, \mathcal{C})$ if there is no observation or no control operator, respectively.

The mild solution of the system $\Sigma_c(A, \mathcal{B})$ is given by Duhamel's formula as

$$
w(t) = \mathcal{T}(t)w_0 + \int_0^t \mathcal{T}(t - s)\mathcal{B}u(s)\,\mathrm{d}s, \quad t \in (0, T]. \tag{3.2}
$$

Hence, the free evolution $t \mapsto \mathcal{T}(t)w_0$ is influenced only by the so-called *controllability map*

$$\mathcal{B}_t \colon L^2((0,T),\mathcal{U}) \to \mathcal{H},\ \mathcal{B}_t u = \int_0^t \mathcal{T}(t-s)\mathcal{B}u(s)\,\mathrm{d}s \quad \text{for} \quad t \in (0,T]. \tag{3.3}$$

This motivates the labeling of $\mathcal{B}$ as a control operator, since $\mathcal{B}$ restricts how the control function can interact with the system. The question of interest in this setting is whether we can find a control function for a given control operator such that the mild solution vanishes in time T. Note that this implies by linearity that one can steer the system to every state in the range of $\mathcal{T}(T)$.

DEFINITION 3.1 (Null-controllability). The system $\Sigma_c(A,\mathcal{B})$ is said to be null-controllable in time $T > 0$, if for every initial value $w_0 \in \mathcal{H}$ there exists a control function $u = u(w_0) \in L^2((0,T);\mathcal{U})$ such that the mild solution satisfies $w(T) = 0$. If $\Sigma_c(A,\mathcal{B})$ is null-controllable, the control costs in time T of the system $\Sigma_c(A,\mathcal{B})$ are

$$C_T = \sup_{\substack{w_0\in\mathcal{H}\\ \|w_0\|_{\mathcal{H}}=1}} \min\bigl\{\|u\|_{L^2((0,T);\mathcal{U})} \colon \mathcal{T}(T)w_0 + \mathcal{B}_T u = 0\bigr\}. \tag{3.4}$$

EXAMPLE 3.2. Suppose that $\mathcal{B} \in \mathcal{L}(\mathcal{U},\mathcal{H})$ is surjective. Then for all $s \in [0,T]$ there is $u_s \in \mathcal{U}$ such that $\mathcal{B}u_s = \mathcal{T}(s)w_0$ and setting $u(s) = -u_s/T$ for $s \in [0,T]$ we have $\mathcal{B}_T u = -\mathcal{T}(T)w_0$. Hence, $w(T) = 0$ and $\Sigma_c(A,\mathcal{B})$ is null-controllable.

It clearly depends on the control operator $\mathcal{B}$ (and the controllability map) whether there is any chance that the system $\Sigma_c(A,\mathcal{B})$ is null-controllable. In fact, if the system $\Sigma_c(A,\mathcal{B})$ is null-controllable in time $T > 0$, then in view of (3.2) we must have

$$\operatorname{Ran}\mathcal{T}(T) \subset \operatorname{Ran}\mathcal{B}_T \tag{3.5}$$

and this inclusion may serve as an equivalent definition of null-controllability.

When working with the system $\Sigma_o(A,\mathcal{C})$, we can observe (or measure) the state $w(t)$ of the system at time t only through the observation $v(t)$. This motivates the following question: Is it is possible to recover the final state $w(T) = \mathcal{T}(T)w_0$ only from the observations $v(\cdot) = \mathcal{C}\mathcal{T}(\cdot)w_0$ on the interval $[0,T]$?

DEFINITION 3.3 (Observability). The system $\Sigma_o(A,\mathcal{C})$ is called (final-state) observable in time $T > 0$ if there is a constant $C_{\mathrm{obs}} = C_{\mathrm{obs}}(T) > 0$ such that

$$\|\mathcal{T}(T)w_0\|_{\mathcal{H}} \leq C_{\mathrm{obs}}\biggl(\int_0^T \|\mathcal{C}\mathcal{T}(t)w_0\|_{\mathcal{V}}^2\,\mathrm{d}t\biggr)^{1/2} \quad \text{for all} \quad w_0 \in \mathcal{H}. \tag{3.6}$$

As for null-controllability, it heavily depends on the observation operator if the system is observable.

REMARK 3.4. The observability estimate (3.6) only depends on the semigroup. Therefore, in order to avoid unnecessarily complicated formulations if we work with a semigroup without specifying its generator, we also say that the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is observable through $\mathcal{C}$ (or from some set ω if $\mathcal{C} = \mathbf{1}_\omega$) in time T if (3.6) holds.

The null-controllability of the system $\Sigma_c(A, \mathcal{B})$ and the observability of the system $\Sigma_o(A, \mathcal{C})$ are intimately related since the adjoint of the controllability map satisfies

$$(\mathcal{B}_T^* w)(s) = \mathcal{B}^* \mathcal{T}(T-s)^* w \quad \text{for} \quad w \in \mathcal{H},\ s \in [0,T]. \tag{3.7}$$

Since by [Paz83, Corollary 10.6] the adjoint semigroup $(\mathcal{T}(t)^*)_{t\geq 0}$ is again a strongly continuous semigroup with generator A^*, the integral of the norm of the adjoint of the controllability map over the interval $[0,T]$ corresponds to the right-hand side of the observability estimate (3.6) for the adjoint semigroup, cf. inequality (3.8) below. Hence, we obtain that null-controllability and observability are dual concepts.

THEOREM 3.5. *The system $\Sigma_o(A, \mathcal{C})$ is observable in time $T > 0$, if and only if the system $\Sigma_c(A^*, \mathcal{C}^*)$ is null-controllable in time T. Furthermore, the control costs in time T of the system $\Sigma_c(A^*, \mathcal{C}^*)$ satisfy*

$$C_T = \min\{C_{\mathrm{obs}} \colon \text{(3.6) holds for all } w_0 \in \mathcal{H}\}.$$

Since in almost all of the results presented in this work we consider classes of generators that are closed under taking adjoints, it is always possible to switch between observability and null-controllability by Theorem 3.5. In fact, since $\mathcal{B}$ is a bounded operator, we have $\mathcal{B} = \mathcal{B}^{**}$ and therefore it always suffices to study the observability of the system $\Sigma_o(A^*, \mathcal{B}^*)$ in order to prove null-controllability of the system $\Sigma_o(A, \mathcal{B})$ and vice versa. For this reason, in the most parts of this thesis we only treat the observability of the system $\Sigma_o(A, \mathcal{C})$. One exception is Section 4.3 below, where we cannot guarantee that the class is closed under taking adjoints, cf. Remark 4.37.

The proof of Theorem 3.5 is a direct consequence of the identity (3.7) and an abstract functional analytic lemma, the so-called Douglas' lemma [Dou66]. We give here the version of the latter from [ENS+20], that also includes a statement about the control costs. For further references, we refer the reader to [DR77, Zab20, TW09].

LEMMA 3.6. *Let X, Y, and Z be Hilbert spaces and let $P \in \mathcal{L}(X,Z)$ and $Q \in \mathcal{L}(Y,Z)$. Then the following conditions are equivalent:*

(i) $\operatorname{Ran} P \subset \operatorname{Ran} Q$,
(ii) There is some $C > 0$ such that $\|P^* z\|_X \leq C\|Q^* z\|_Y$ for all $z \in Z$,
(iii) There is some $R \in \mathcal{L}(X,Y)$ such that $P = QR$.

Moreover,

$$\min\{C \colon \|P^* z\|_X \leq C\|Q^* z\|_Y \forall z \in Z\} = \min\{\|R\|_{\mathcal{L}(X,Y)} \colon P = QR\}.$$

PROOF OF THEOREM 3.5. We apply Lemma 3.6 with $X = Z = \mathcal{H}$, $Y = L^2((0,T);\mathcal{U})$, $P = \mathcal{T}(T)$ and with $Q = \mathcal{B}_T$. In this setting, (i) takes the form of the alternative definition of null-controllability in (3.5), while (ii) reads as

$$\|\mathcal{T}^*(T)z\|_{\mathcal{H}} \leq C\|\mathcal{B}_T^*(\cdot)z\|_{L^2((0,T);\mathcal{U})} = C \cdot \left(\int_0^T \|\mathcal{B}_T^*(s)z\|_{\mathcal{V}} \, \mathrm{d}s \right)^{1/2}, \quad z \in \mathcal{H}. \tag{3.8}$$

Substituting (3.7) into this expression then proves the equivalence. Moreover, if one of the two equivalent statements holds, then the operator R whose existence is guaranteed by (iii) maps an initial value to a suitable control function such that the mild solution vanishes at T and the norm of this operator agrees with the control costs C_T defined in (3.4). Thereby, we get the desired identity for the control costs. □

3.2. Criteria for observability and the observability constant

Given a generator A of a strongly continuous semigroup, we are now interested in conditions on $\mathcal{C}$ guaranteeing observability. In addition, given any sufficient condition for $\mathcal{C}$, it is also of great interest to track the observability constant (or an upper bound for it) in terms of this condition.

The possibly most studied situation is that of $\mathcal{H} = L^2(\Omega)$ for an open set $\Omega \subset \mathbb{R}^d$ where the observation operator is a multiplication operator by the characteristic function of some measurable set $\omega \subset \Omega$, i.e., $\mathcal{C} = \mathbf{1}_\omega \colon L^2(\Omega) \to L^2(\omega)$. In this case, the set ω is often referred to as a *sensor set*, which is guided by the interpretation that we measure the state of the system with sensors placed throughout the domain Ω. Consider, for example, the Dirichlet Laplacian $A = \Delta = \Delta_\Omega^D$ in $L^2(\Omega)$. Then a sufficient condition for observability of the heat system $\Sigma_o(\Delta, \mathbf{1}_\omega)$ was established by Lebeau and Robbiano in the pioneering paper [LR95], see also [LZ98, JL99]. There it is shown, that the system $\Sigma_o(\Delta, \mathbf{1}_\omega)$ is observable, if there are constants $d_0, d_1 > 0$ such that the so-called *spectral inequality*

$$\|P_\lambda(-\Delta)g\|_{L^2(\Omega)}^2 \leq d_0 \mathrm{e}^{d_1\lambda^{1/2}} \|\mathbf{1}_\omega P_\lambda(-\Delta)g\|_{L^2(\omega)}^2 \tag{3.9}$$

holds for all $g \in L^2(\Omega)$. Hence, this condition eliminates the dependence on the time parameter and reduces the investigation of the observability of the parabolic system $\Sigma_o(\Delta, \mathbf{1}_\omega)$ to the study of properties of elements in the spectral subspace of the generator.

Similar results, often called a *Lebeau-Robbiano method* in the literature, were subsequently obtained in various situations. For expository reasons, we first consider selfadjoint operators on general Hilbert spaces with abstract observation operators which were studied explicitly in [TT11] (for operators with purely discrete spectrum) and in [NTTV20a]. However, let us emphasize that the papers [Mil10, BPS18], which are only mentioned further below, have already obtained similar results before and have served as an important inspiration for [NTTV20a]. The papers [TT11, NTTV20a] improve on the original formulation of [LR95] in two significant ways: Firstly, they give estimates on the observability constant C_{obs} in terms of the parameters d_0 and d_1 in the uncertainty relation (3.9). Secondly, they allow a dependence of the constant on the spectral parameter of the form λ^{γ_1} for some $\gamma_1 \in (0,1)$. We here recall the main result from [NTTV20a], as it gives the best upper bound for the observability constant in terms of d_0, d_1, and γ_1. Note that this theorem contains the statement from [LR95] as a special case.

THEOREM 3.7 ([NTTV20a, Theorems 2.8 and 2.12]). *Let $-A$ be a lower semi-bounded selfadjoint operator and let $\kappa = \inf \sigma(-A)$. Suppose that there are $d_0 > 0$, $d_1 \geq 0$, and $\gamma_1 \in (0,1)$ such that we have the spectral inequality*

$$\|P_\lambda(-A)g\|_{\mathcal{H}}^2 \leq d_0 \mathrm{e}^{d_1(\lambda-\kappa_-)^{\gamma_1}} \|\mathcal{C}P_\lambda(-A)g\|_{\mathcal{V}}^2 \quad \text{for all} \quad \lambda > \kappa,\ g \in \mathcal{H}. \tag{3.10}$$

Then the system $\Sigma_o(A, \mathcal{C})$ is observable in all times $T > 0$ and the observability constant satisfies

$$C_{\mathrm{obs}}^2 \leq \frac{C_1 d_0}{T}\bigl(2d_0\|\mathcal{C}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} + 1\bigr)^{C_2} \exp\Bigl(C_3 \frac{d_1^{1/(1-\gamma_1)}}{T^{\gamma_1/(1-\gamma_1)}} - \kappa_- T\Bigr),$$

where $C_1, C_2, C_3 > 0$ are constants depending only on γ_1 and $\kappa_- = \min\{0, \kappa\}$.

It is worth pointing out that if we have established the spectral inequality only for, say, $\lambda \geq 1$, then we still get (3.10) by slightly adapting the constants d_0 and d_1.

When dealing with non-selfadjoint generators A, it is not possible to formulate the spectral inequality (3.10), since there is no natural replacement for the spectral projections $P_\lambda(-A)$. However, there are similar results where the spectral projections are replaced by some family of operators $(P_\lambda)_\lambda$ satisfying, in addition to an analog of (3.10), a so-called *dissipation estimate* (see (3.12) below) which guarantees an exponential decay of the semigroup on $(\operatorname{Ran} P_\lambda)^\perp$. Results in this direction were proven in [Mil10, WZ17, BPS18]. Moreover, [BEPS20] consider the situation of contraction semigroups with sensor sets varying in time and with a certain blow-up of the dissipation estimate in small times (that is, $d_2 = d_2(t)$ in (3.12) below with a polynomial blow-up as $t \to 0$), while general semigroups on Banach spaces were treated in [GST20].

Here we formulate the result from the last mentioned reference in the special case of Hilbert spaces.

THEOREM 3.8 ([GST20, Theorem 2.1]). *Let A be the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ such that for some $M \geq 1$ and $\varkappa \in \mathbb{R}$ we have $\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{H})} \leq M\mathrm{e}^{\varkappa t}$ for all $t \geq 0$. Suppose that for some $\lambda^* \geq 0$ and a family $(P_\lambda)_{\lambda \geq \lambda^*} \subset \mathcal{L}(\mathcal{H})$ of operators there are*

(i) $d_0, d_1, \gamma_1 > 0$ with

$$\|P_\lambda g\|_{\mathcal{H}}^2 \leq d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1}} \|CP_\lambda g\|_{\mathcal{H}}^2 \quad \text{for all} \quad \lambda > \lambda^*,\ g \in \mathcal{H}, \tag{3.11}$$

and

(ii) $d_2 \geq 1$, $d_3, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$ such that for all $t \in (0, T/2]$ we have

$$\|(\mathrm{Id} - P_\lambda)\mathcal{T}(t)g\|_{\mathcal{H}} \leq d_2 \mathrm{e}^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}} \|g\|_{\mathcal{H}} \quad \text{for all} \quad \lambda > \lambda^*,\ g \in \mathcal{H}. \tag{3.12}$$

Then the system $\Sigma_o(A,\mathcal{C})$ *is observable in time* T *and*

$$C_{\mathrm{obs}}^2 \leq \frac{C_1^2}{T} \exp\Big(\frac{C_2}{T^{\gamma_1\gamma_3/(\gamma_2-\gamma_1)}} + C_3 T\Big),$$

where the constants are explicitly given by

$$C_1 = (4Md_0^{1/2}) \max\{(4d_2M^2)(d_0^{1/2}\|\mathcal{C}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} + 1)^{8/(\mathrm{e}\log 2)}, \mathrm{e}^{2d_1(2\lambda^*)^{\gamma_1}}\},$$
$$C_2 = 8(2^{\gamma_1-\gamma_2}(2\cdot 4^{\gamma_3})^{\gamma_1\gamma_3/(\gamma_2-\gamma_1)} d_1^{\gamma_2}/d_3^{\gamma_1})^{1/(\gamma_2-\gamma_1)},$$
$$C_3 = \max\{\varkappa, 0\}(1 + 10/(\mathrm{e}\log 2)).$$

Oftentimes, in applications Theorem 3.8 can be applied with $\gamma_2 = 1$ and therefore we are particularly interested in (3.11) with $\gamma_1 < \gamma_2 = 1$. In fact, this is the situation we encounter for selfadjoint generators: By spectral calculus, a negative selfadjoint operator A generates a contraction semigroup $(\mathcal{T}(t))_{t\geq 0}$ with $\mathcal{T}(t) = \mathrm{e}^{tA}$, so that we can set $M = 1$ and $\varkappa = 0$ in Theorem 3.8. Furthermore, setting $P_\lambda = P_\lambda(-A)$ for $\lambda \geq \lambda^* = \kappa = 0$ we have by spectral calculus

$$\|(\mathrm{Id} - P_\lambda)\mathrm{e}^{tA} f\|_{\mathcal{H}} \leq \mathrm{e}^{-t\lambda}\|f\|_{\mathcal{H}} \quad \text{for all} \quad t > 0,$$

which indeed shows the dissipation estimate (3.12) with $\gamma_2 = 1$ and with $\gamma_3 = d_2 = d_3 = 1$. Since also the (spectral) inequalities (3.10) and (3.11) are equivalent in this case, Theorem 3.7 and Theorem 3.8 coincide for selfadjoint generators.

Let us now discuss the dependence of the observability constant on the time $T > 0$. To this end, we suppress for now all constants depending on parameters other then T. Theorem 3.7 establishes $C_{\mathrm{obs}} \lesssim 1/T^{1/2}$ as $T \to \infty$. This is can also be seen in Theorem 3.8 for the case of contraction semigroups, i.e., if $\varkappa \leq 0$ and therefore $C_3 = 0$. It is known that this behavior is optimal, in fact [NTTV20a, Theorem 2.13] shows that also $C_{\mathrm{obs}} \gtrsim 1/T^{1/2}$ as $T \to \infty$. However, if the semigroup is not a contraction semigroup, it is expected to experience an exponential blow-up in the large time regime.

For small times the observability constant in both theorems blows up at most exponentially, e.g., the observability constant of the heat system $\Sigma_o(\Delta, \mathbf{1}_\omega)$ satisfies $C_{\mathrm{obs}} \lesssim \mathrm{e}^{c/T}$ as $T \to 0$. That this exponential blow-up is actually the worst case scenario for heat systems was shown in [Sei84] (in dimension one) and in [FI96, Mil10] (in higher dimensions). Complementing lower bounds of the form $C_{\mathrm{obs}} \gtrsim \mathrm{e}^{c/T}$ for $T \to 0$ were given in [Güi85, Proposition 3] (in one dimension) and in [Mil04, Theorem 2.1] (in higher dimensions). Hence, the estimates in both theorems above are optimal in the small as well as in the large time regime for these systems.

Besides the already mentioned references, there is a vast amount of literature on the dependence of the observability constant on the time $T > 0$ which we cannot reproduce here in its entirety. We therefore refer the reader to the discussion in the papers [NTTV20a, GST20]. Since in all our applications we prove observability using either Theorem 3.7 or a corollary of Theorem 3.8, see Corollary 3.9 below, we do not discuss the time dependence in more detail. Instead, the dependency of most interest to us is that of the geometry of ω on the observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$, where $\omega \subset \mathbb{R}^d$ and A is the generator of a contraction semigroup on $L^2(\mathbb{R}^d)$. Moreover, if the system is indeed observable, we also briefly investigate the influence of the geometry on the observability constant of the system. There are also previous works studying this dependence in several situations, the ones most important to our results being [EV18, NTTV18, Egi21, NTTV20a, BPS18, BJPS21, MPS22, Alp21]. We address these in the discussion in Chapter 4.

So far we have only treated the case where the dissipation estimate (3.12) holds for all times $t \in (0, T/2]$. However, in all our applications below where the generator of the semigroup is non-selfadjoint, we can only prove the dissipation estimate (3.12) for small times $0 < t \ll 1$. In order to still achieve observability for all times $T > 0$ we formulate the following simple corollary to Theorem 3.8. It extends the statement from [BPS18, Theorem 2.1] which agrees with the corollary for families $(P_\lambda)_\lambda$ of projections on some L^2-space. Essentially, this corollary mirrors the fact that observability of a contraction semigroup is transitive in time, i.e., if the system is observable in some time T_0, then it is also observable in all times $T \geq T_0$ and the observability constant does not get worse. Since in the next proof we only use the contractivity of the semigroup it is apparent that we no longer obtain the decay for large times T.

Corollary 3.9. *Let A be the generator of a contraction semigroup and suppose that part (i) of Theorem 3.8 holds. Then, if part (ii) of Theorem 3.8 holds only for $t \in (0, T_0/2]$ with some $T_0 > 0$, the system $\Sigma_o(A, \mathcal{C})$ is observable in all times $T > 0$ and we have*

$$C_{\mathrm{obs}}^2 \leq C_1' \exp\biggl(\frac{C_2'}{T^{\gamma_1\gamma_3/(\gamma_2-\gamma_1)}}\biggr), \tag{3.13}$$

where $C_1', C_2' > 0$ are constants depending only on $\gamma_1\gamma_3/(\gamma_2 - \gamma_1)$, C_1, C_2, and T_0.

Proof. Let $C_2' = C_2 + CC_1^{2\gamma_1\gamma_3/(\gamma_2-\gamma_1)}$ and $C_1' = \exp\bigl(C_1 T_0^{-\gamma_1\gamma_3/(\gamma_2-\gamma_1)}\bigr)$, where $C > 0$ is a constant that depends only on $\gamma_1\gamma_3/(\gamma_2 - \gamma_1)$ and that is chosen in such a way that $\log x \leq Cx^{\gamma_1\gamma_3/(\gamma_2-\gamma_1)}$ for all $x > 0$. Then the statement follows from Theorem 3.8 if $T \leq T_0$ and from the contractivity of the semigroup if $T > T_0$. ☐

We point out that the constant in the previous corollary was only given in this specific form to avoid distinguishing between the two cases $T \leq T_0$ and $T > T_0$ in inequality (3.13).

The last method to prove observability we present in this chapter goes back to the works [Mil10, Mar22]. In contrast to the criteria we considered up to now it does not combine a spectral inequality with a dissipation estimate but instead supposes an uncertainty principle with an error term for elements in the range of the contraction semigroup $(\mathcal{T}(t))_{t\geq 0}$. For some set $\omega \subset \mathbb{R}^d$, this reads as

$$\|\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)}^2 \leq U_t\|\mathcal{T}(t)g\|_{L^2(\omega)}^2 + E_t\|g\|_{L^2(\mathbb{R}^d)}^2, \quad g \in L^2(\mathbb{R}^d), \tag{3.14}$$

for all $t \in (0, t_0)$, where $t_0 > 0$, and where U_t, E_t are suitable constants depending, amongst others, on the parameter t and are both monotone decreasing in t. This way of establishing observability is of interest whenever we are not able to use Theorem 3.7 or Theorem 3.8 (respectively Corollary 3.9), but we can use properties of elements in the range of the semigroup to prove (3.14). We pursue this approach in Section 4.3 below.

Since the result we use in Section 4.3, see Corollary 3.11 below, was only stated implicitly in [Mar22, Proof of Theorem 2.11], we give a short derivation here. To this end, we first recall the following lemma which is a slightly customized version of [Mil10, Lemma 2.1].

LEMMA 3.10. *Let $(\mathcal{T}(t))_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^2(\mathbb{R}^d)$ and suppose that there is a monotone increasing function $h\colon [0,\infty) \to [0,1)$ with $h(\tau) \to 0$ as $\tau \to 0$ such that for some $q \in (0,1)$ and some $\tau_0 > 0$ we have the approximate observability estimate*

$$h(\tau)\|\mathcal{T}(t)g\|^2_{L^2(\mathbb{R}^d)} \leq \int_{q\tau}^{\tau} \|\mathcal{T}(s)g\|^2_{L^2(\omega)}\,\mathrm{d}s + h(q\tau)\|g\|^2_{L^2(\mathbb{R}^d)} \tag{3.15}$$

for all $g \in L^2(\mathbb{R}^d)$ and all $\tau \in (0, \tau_0)$. Then, for all $T > 0$ we have

$$\|\mathcal{T}(T)g\|^2_{L^2(\mathbb{R}^d)} \leq \frac{C_1}{h\big((1-q)T\big)} \int_0^T \|\mathcal{T}(t)g\|^2_{L^2(\omega)}\,\mathrm{d}t,$$

where $C_1 = 1/h\big((1-q)\tau_0/2\big)$.

The idea introduced in [Mar22] is then to build the proof of the observability estimate upon the uncertainty principle with error term (3.14) and Lemma 3.10. Indeed, using that $(\mathcal{T}(t))_{t\geq 0}$ is a contraction semigroup, we see that we can interpret the approximate observability estimate (3.15) as an integrated version of (3.14): Since the semigroup is a contraction, for all $t > 0$ and $q \in (0,1)$ we have

$$\|\mathcal{T}(t)g\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{(1-q)t}\int_{qt}^{t} \|\mathcal{T}(t)g\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \leq \frac{1}{(1-q)t}\int_{qt}^{t} \|\mathcal{T}(s)g\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s,$$

so that plugging in (3.14) we get

$$\|\mathcal{T}(t)g\|^2_{L^2(\mathbb{R}^d)} \leq \frac{U_{qt}}{(1-q)t}\int_{qt}^{t} \|\mathcal{T}(s)g\|^2_{L^2(\omega)}\,\mathrm{d}s + E_{qt}\|g\|^2_{L^2(\mathbb{R}^d)}$$

for all $t \in (0, t_0)$, where we used that $E_{(\cdot)}$ is monotone decreasing. Thus, if we can find a monotone increasing function h as in Lemma 3.10 such that for some $\tau_0 \leq t_0$ and some appropriately chosen and fixed q we have

$$\frac{U_{q\tau}}{(1-q)\tau} \leq \frac{1}{h(\tau)} \quad \text{and} \quad E_{q\tau} \leq \frac{h(q\tau)}{h(\tau)} \quad \text{for} \quad \tau \in (0, \tau_0), \tag{3.16}$$

then (3.14) implies the approximate observability estimate (3.15) with this function h. In particular, we get the following corollary to Lemma 3.10.

COROLLARY 3.11. *Let $(\mathcal{T}(t))_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^2(\mathbb{R}^d)$ and suppose that* (3.14) *holds for all $\tau \in (0,\tau_0)$ with constants U_τ and E_τ that satisfy* (3.16) *with some monotone increasing function $h\colon [0,\infty) \to [0,1)$ satisfying $h(\tau) \to 0$ as $\tau \to 0$. Then, with the constant C_1 from Lemma 3.10 we have*

$$\|\mathcal{T}(T)g\|^2_{L^2(\mathbb{R}^d)} \leq \frac{C_1}{h\bigl((1-q)T\bigr)} \int_0^T \|\mathcal{T}(t)g\|^2_{L^2(\omega)} \,\mathrm{d}t \quad \textit{for all} \quad T > 0.$$

CHAPTER 4

Observability

In this chapter the main results of this thesis are presented and discussed. We first state spectral inequalities for two types of Schrödinger operators in Section 4.1: Those with *admissible singular potentials* and with *partial power growth potentials*. In the case of partial power growth potentials, we further distinguish between the *partial harmonic oscillator* (i.e., the case of quadratic potentials) and Schrödinger operators with more general partial power growth potentials including, amongst others, *(partial) anisotropic Shubin-type operators*. After that, in Section 4.2 we present a class of accretive operators that are *comparable* to the partial harmonic oscillators in the sense of a dissipation estimate. Using the Lebeau-Robbiano method and falling back to the spectral inequality of the partial harmonic oscillators, we present observability estimates also for the systems associated with these possibly non-selfadjoint generators. Finally, Section 4.3 discusses an observability result that can be obtained from *smoothing effects* of the semigroup, while in Section 4.4 we discuss supplementary results that can be derived using the spectral inequalities from Section 4.1.

Throughout the chapter we use the notation introduced in Chapter 2, see for instance Table 1. In particular, K always denotes a universal constant, whereas K_d denotes a constant depending only on the dimension d. Recall that both may change from line to line. Furthermore, we remind the reader of the notation $\Sigma_o(A, \mathbf{1}_\omega)$ from Section 3.1.

4.1. Spectral inequalities for selfadjoint Schrödinger operators

Let $\omega \subset \mathbb{R}^d$ be some measurable sensor set and let $H = -A\colon L^2(\mathbb{R}^d) \supset \mathcal{D}(A) \to L^2(\mathbb{R}^d)$ be some lower semibounded selfadjoint Schrödinger operator. In this case, the spectral inequality (3.10) (with $\kappa_- = 0$) is often formulated as

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1}} \|f\|^2_{L^2(\omega)} \tag{4.1}$$

for all $\lambda \geq 1$ and all $f \in \operatorname{Ran} P_\lambda(H)$. Such estimates bear various names depending on the area of analysis where they appear. For instance, while in the context of control theory one often sticks to the term spectral inequality, in, e.g., the context of random Schrödinger operators an estimate of the form (4.1) is also referred to as a *quantitative unique continuation estimate*. We here adopt the following way of speaking.

Definition 4.1. We say that (4.1) is a *spectral inequality* if and only if the inequality holds for all $f \in \operatorname{Ran} P_\lambda(H)$, $\lambda \geq 1$, and with $\gamma_1 < 1$. If it holds merely for all f in a subspace of $\operatorname{Ran} P_\lambda(H)$ (e.g., for individual eigenfunctions) or if $\gamma_1 \geq 1$, we call it a *quantitative unique continuation estimate.*

Let us now turn to the observability of the systems $\Sigma_o(A, \mathbf{1}_\omega)$ in time $T > 0$. In view of Theorem 3.7, it suffices to study spectral inequalities for the operator $H = -A$ in order to extract sufficient conditions on the sensor set that guarantee observability in all times. In the observability estimate, the geometry of the set ω is then expressed in the constants d_0, d_1, and γ_1 of the spectral inequality (4.1) and, therefore, we are especially interested in the dependence of these constants on properties of the sensor set. The parameter γ_1 plays an important role here since the proof of observability via the Lebeau-Robbiano strategy in Theorem 3.7 fails if $\gamma_1 \geq 1$. This is why our sensor sets need to be chosen in a way that guarantees $\gamma_1 < 1$.

In the case where $A = \Delta = \Delta_\Omega^D$ is the Dirichlet Laplacian on a bounded domain Ω, it is well-known that the system $\Sigma_o(\Delta, \mathbf{1}_\omega)$ is observable in all times $T > 0$ if the sensor set is any open set [LR95] or even any measurable set with positive Lebesgue measure if the domain is additionally Lipschitz and locally star-shaped (e.g., a convex set) [AE13, AEWZ14]. Both results do not give explicit estimates on the observability constant in terms of the geometry of the sensor set. If the domain is a cube $\Omega = (0, 2\pi L)^d$ with $L > 0$, such estimates were given in [EV18, NTTV20a] if A is the selfadjoint realization of the Laplacian with periodic, Dirichlet, or Neumann boundary conditions, see Corollary 4.5 below.

The situation is remarkably different if the domain is unbounded: On the whole of $\mathbb{R}^d$, it was shown independently in [EV18] and [WWZZ19] that the system $\Sigma_o(\Delta, \mathbf{1}_\omega)$ is observable in time $T > 0$ if and only if the sensor set is a *thick set* in the sense of the following definition.

Definition 4.2 (Thick set). Let $\gamma \in (0, 1]$ and $\rho > 0$. The measurable set $\omega \subset \mathbb{R}^d$ is said to be (γ, ρ)-thick if

$$\frac{|B(x, \rho) \cap \omega|}{|B(x, \rho)|} \geq \gamma \quad \text{for all} \quad x \in \mathbb{R}^d. \tag{4.2}$$

We say that a set ω is thick if there are γ and ρ such that ω is (γ, ρ)-thick and, in this case, we call γ and ρ the thickness parameters of ω.

Similar conditions have also been used for heat systems on infinite strips [Egi21], halfspaces, orthants, sectors of certain angle [ES22] (where the Laplacian may sometimes be replaced by a constant coefficient divergence-type operator), and also for null-controllability of systems in $L^p(\mathbb{R}^d)$ $(1 < p < \infty)$ corresponding to an elliptic operator associated with a homogeneous strongly elliptic polynomial of degree at least two [GST20].

The necessity of thick sensor sets ω for the observability of the heat system $\Sigma_o(A, \mathbf{1}_\omega)$ on $\mathbb{R}^d$ is shown in [EV18, Section 4] by considering a sequence of Gaussians and exploiting the translation invariance of the Laplacian, while the proof of the sufficiency and the estimate for the observability constant therein follows from Corollary 3.9 and a suitable spectral inequality for the Laplacian. With the definition of thick sets at hand, we are in the position to state the results from [EV18] for bounded and unbounded domains simultaneously. To this end, we first formulate a spectral inequality for the Laplacian. On the whole of $\mathbb{R}^d$, it goes back to an estimate for functions with compactly supported Fourier transform [Kov01, Kov00], which has then been adapted to functions on a torus having a finite Fourier series [EV20]. We refer the reader to Section 7.1 below for the theorems from the last mentioned papers and for more substantial background.

These results were turned into spectral inequalities in [EV18] by the simple observation that functions in the spectral subspace of the Laplacian have a compactly supported Fourier transform or a finite Fourier series, respectively, see Lemma 7.5 below.

THEOREM 4.3 ([Kov01, Kov00, EV20, EV18]). *Let ω be a (γ,ρ)-thick set. If either*

(i) $\Omega = \mathbb{R}^d$ and Δ is the Laplacian on $\mathbb{R}^d$, or

(ii) $\Omega = (0, 2\pi L)^d$ for some $L > 0$, $\Delta = \Delta_\Omega^\bullet$, $\bullet \in \{D, N, \mathrm{per}\}$, is the Laplacian on Ω with Dirichlet, Neumann, or periodic boundary conditions, and $\rho \leq 2\pi L$,

then

$$\|f\|_{L^2(\Omega)}^2 \leq \Big(\frac{K^d}{\gamma}\Big)^{K(d+\rho\lambda^{1/2})} \|f\|_{L^2(\Omega\cap\omega)}^2 \quad \text{for all} \quad \lambda \geq 1,\ f \in \operatorname{Ran} P_\lambda(-\Delta).$$

In particular, we have the spectral inequality (4.1) *with $\gamma_1 = 1/2$, $d_0 = (K^d/\gamma)^{Kd}$, and $d_1 = K\rho\log(K^d/\gamma)$.*

REMARK 4.4. The constants d_0 and d_1 are uniform in the sidelength L in the situation of Theorem 4.3 (ii). For this reason, the spectral inequality for the Laplacian on a cube is called *scale-free*. Furthermore, it is also possible to formulate the above theorem if thick sets are defined with respect to hyperrectangles instead of balls, see Theorem 7.4 below. However, for reasons of comparability to our main results below we refrain from doing so here.

The exponent $\gamma_1 = 1/2$ one encounters in the previous theorem is known to be sharp. For bounded domains, this can be inferred from [LL12, Proposition 5.5], see also [JL99, Proposition 14.9], for all sets $\omega \subset \Omega$ with $\overline{\omega} \neq \Omega$, while for thick sets on the whole of $\mathbb{R}^d$ this is shown in [Kov01]. This exponent is also the benchmark when comparing our main results for Schrödinger operators to the case of the pure Laplacian.

Combining the spectral inequality with Theorem 3.7 we conclude observability of the system $\Sigma_o(\Delta, \mathbf{1}_{\Omega\cap\omega})$ with the following bound for the observability constant.

We point out that Theorem 3.7 was not yet available when [EV18] was published and therefore the estimate in the latter paper lacks the decay in large times.

COROLLARY 4.5 (cf. [ENS+20, Theorem 5.3], [NTTV20a, Theorem 4.9]). *Let ω be a (γ, ρ)-thick set and suppose (i) or (ii) of Theorem 4.3. Then the system $\Sigma_o(\Delta, \mathbf{1}_{\Omega\cap\omega})$ is observable in time $T > 0$ with*

$$C_{\mathrm{obs}}^2 \leq \frac{K^{d^2}}{T\gamma^{Kd}} \cdot \exp\left(\frac{Kd^2\rho^2 \log\bigl(\frac{1}{\gamma}\bigr)^2}{T}\right).$$

When the pure Laplacian is perturbed by some potential, the situation might change drastically depending on the properties of the potential (also called the *heat generation term* in the situation of heat systems). In the present work, we are concerned with two types of potentials and corresponding Schrödinger operators $H = -A = -\Delta + V$ in $L^2(\mathbb{R}^d)$: The first one corresponds to the case where V is some real-valued potential with mild local singularities, a so-called *singular admissible potential*. Such potentials are, amongst others, infinitesimally Δ-bounded in $L^2(\mathbb{R}^d)$. The second type describes potentials that grow unboundedly in some or all coordinate directions. The prime example for this case is a quadratic potential $V(x) = |x|^2$, but we also consider more general so-called *confinement potentials*, i.e., real-valued $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying $V(x) \to \infty$ as $|x| \to \infty$.

The different nature of these two types of potentials manifests itself in various ways. Let us mention the most important difference for our analysis here. As operators in $L^2(\mathbb{R}^d)$, Schrödinger operators with confinement potentials have purely discrete spectrum with exponentially decaying eigenfunctions. On the other hand, Schrödinger operators with singular admissible potentials generically have continuous spectrum and elements in spectral subspaces do not enjoy a quantifiable decay. Hence, one expects that sensor sets for the former operators may become sparse near infinity while sensor sets for the latter operators need to be equidistributed throughout the whole domain.

4.1.1. Schrödinger operators with singular potentials. Observability for heat systems $\Sigma_o(\Delta - V, \mathbf{1}_\omega)$ with bounded heat generation term $V \in L^\infty$ was studied in [FI96, Fur00] by means of global Carleman estimates. Using methods from [FI96], it was established in [FZ00] that on bounded domains Ω with Dirichlet boundary conditions the above system (with $\Delta = \Delta_\Omega^D$) is observable in time $T > 0$ from any nonempty open sensor set $\omega \subset \Omega$ with

$$C_{\mathrm{obs}} \leq \exp\bigl(C(\|V\|_\infty^{2/3} + 1/T)\bigr) \quad \text{as} \quad T \to 0,$$

where $C > 0$ is a constant that depends on the domain Ω and the sensor set ω. Actually, this result is best possible in even space dimensions as was shown in [DZZ08], cf. also [Zua07, Proposition 5.1 and Theorem 5.2].

Although these results already establish observability on bounded domains, the dependence of the constant C_{obs} on the geometry of the domain and the sensor set ω is not explicit. These explicit dependencies on geometric properties require a more accurate analysis which was first implemented for the above heat system (again, based on a spectral inequality) in [NTTV18, Theorem 2.15] if Ω is a hypercube. Afterwards, this approach was extended in [NTTV20b] to operators on so-called *generalized rectangles* Ω, i.e., sets of the form $\Omega = \bigtimes_{j=1}^{d}(a_j, b_j)$ with $-\infty \leq a_j < b_j \leq \infty$ for $j = 1, \ldots, d$, giving, besides that, also the first result for such Schrödinger operators on unbounded domains. These approaches are restricted to rectangular shaped domains due to certain extension arguments for elements in spectral subspaces that are required in the proof, cf. Remark 6.17 below. Moreover, in contrast to Theorem 4.3 above, the proof of the spectral inequality in [NTTV18, NTTV20b] is based on Carleman estimates, and, as a consequence, the approach in these papers only allows to consider sensors sets which contain some union of open balls.

DEFINITION 4.6. Let $G > 0$ and $\delta \in (0, G/2)$. A set $\omega \subset \mathbb{R}^d$ is said to be (G, δ)-equidistributed, if for each $k \in (G\mathbb{Z})^d$ the intersection of ω with the hypercube $\Lambda_G(k)$ centered at k with sides of length G contains a ball of radius δ.

We now formulate the spectral inequality from [NTTV20b] for Schrödinger operators on the whole of $\mathbb{R}^d$. Here the spectral parameter λ is allowed to be any real number since this allows to optimize the observability constant when H has negative spectrum.

THEOREM 4.7 ([NTTV20b, Theorem 2.1]). *Let $H = -\Delta + V$ for some bounded $V \in L^\infty(\mathbb{R}^d)$. Then for all $G > 0$, $\delta \in (0, G/2)$, all (G, δ)-equidistributed sets ω, all $\lambda \in \mathbb{R}$, and all $f \in \operatorname{Ran} P_\lambda(H)$ we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \Big(\frac{G}{\delta}\Big)^{K_d\cdot(1+G^{4/3}\|V\|_\infty^{2/3}+G\lambda_+^{1/2})} \|f\|^2_{L^2(\omega)}, \tag{4.3}$$

where $\lambda_+ = \max\{0, \lambda\}$.

This result implies by Theorem 3.7 observability of the system $\Sigma_o(\Delta - V, \mathbf{1}_\omega)$ in $L^2(\mathbb{R}^d)$. We refrain from formulating the observability estimate at this point and instead refer the reader to Corollary 4.11 below.

REMARK 4.8. A spectral inequality with thick sensor sets for Schrödinger operators on $\mathbb{R}^d$ has been obtained in [LM, Theorem 1.2]. However, this requires analytic potentials $V\colon \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity that can be extended holomorphically to some strip $U_r = \{z \in \mathbb{C}^d \colon |\operatorname{Im} z| < r\}$ with $r > 0$ and satisfy $|\partial^\alpha V(z)| \lesssim_r (1 + |z|)^{-|\alpha|-\varepsilon}$ for some $\varepsilon \in (0, 1)$ and all $|\alpha| \leq 2$ and $z \in U_r$. Furthermore, the dependence of the constants on the thickness parameters is again not explicit.

Estimates of the form (4.3) have already been obtained previously to [NTTV18, NTTV20b] in [CHK03, RMV13, GK13, BK13, Kle13], cf. the overview in Section 6.1 below. However, these papers show (4.3) only for individual eigenfunctions

[RMV13] or for elements of a spectral subspace $\operatorname{Ran}\mathbf{1}_I(H_\Omega^\bullet)$ with some (rather short) energy interval $I \subset \mathbb{R}$ [Kle13], where Ω is an appropriate hyperrectangle in $\mathbb{R}^d$ and the operator has suitable boundary conditions. In particular, these results were no spectral inequalities in the sense of Definition 4.1 and therefore not sufficient for applications in control theory. Furthermore, even these quantitative unique continuation estimates have almost exclusively been considered for bounded potentials. Only [KT16] considers some singular unbounded potentials but is still restricted to short energy intervals, cf. Theorem 6.3 below.

We now present a class of singular potentials for whose associated Schrödinger operators we prove a spectral inequality.

Definition 4.9 (Singular admissible potentials). Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable function. If the domain of the associated selfadjoint multiplication operator in $L^2(\mathbb{R}^d)$ contains $H^1(\mathbb{R}^d)$, i.e., if $\mathcal{D}(V) \supset H^1(\mathbb{R}^d)$, then the potential V is called *(singular) admissible* (on $\mathbb{R}^d$).

As shown in Lemma 6.5 below, an admissible potential is infinitesimally Δ-bounded in $L^2(\mathbb{R}^d)$ and satisfies

$$\|Vf\|^2_{L^2(\mathbb{R}^d)} \leq \lambda_1 \|\nabla f\|^2_{L^2(\mathbb{R}^d)} + \lambda_2 \|f\|^2_{L^2(\mathbb{R}^d)} \quad \text{for all} \quad f \in H^1(\mathbb{R}^d) \tag{4.4}$$

with some $\lambda_1, \lambda_2 \geq 0$. Therefore, the Schrödinger operator $H = -\Delta + V$ is again selfadjoint and lower semibounded by the Kato-Rellich theorem, cf. Subsection 2.2.2 above. Since any $V \in L^\infty(\mathbb{R}^d)$ defines a bounded operator in $L^2(\mathbb{R}^d)$ and therefore $\mathcal{D}(V) = L^2(\mathbb{R}^d)$, such V are clearly admissible with $\lambda_1 = 0$ and $\lambda_2 = \|V\|^2_\infty$. However, for $d \geq 3$ also potentials $V \in L^p(\mathbb{R}^d)$ with $p \geq d$ are singular admissible, see Example 6.7 below. In particular, the class of admissible potentials covers for $d \geq 2$ the potentials considered in [KT16].

The next theorem is our first main result which was first formulated in the joint work [DRST] of the author with Christian Rose, Albrecht Seelmann, and Martin Tautenhahn. It not only encompasses Theorem 4.7 for bounded potentials, but also extends [KT16].

Theorem 4.10. *Let $H = -\Delta + V$ for some singular admissible potential V. Then for all $G > 0$, $\delta \in (0, G/2)$, all (G,δ)-equidistributed sets $\omega \subset \mathbb{R}^d$, all $\lambda \in \mathbb{R}$, and all $f \in \operatorname{Ran} P_\lambda(H)$ we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \Big(\frac{G}{\delta}\Big)^{K_d\cdot(1+G^2\lambda_1 + G^{4/3}\lambda_2^{1/3} + G\lambda_+^{1/2})} \|f\|^2_{L^2(\omega)}, \tag{4.5}$$

where $\lambda_1, \lambda_2 \geq 0$ are as in (4.4).

We give the proof of this theorem in Chapter 6 below. Let us emphasize that only the characteristic parameters λ_1 and λ_2 of the potential enter the constant on the right-hand side of (4.5). Furthermore, the theorem is again formulated for $\lambda \in \mathbb{R}$ instead of just $\lambda \geq 1$ in order to ensure comparability with Theorem 4.7.

Theorem 4.10 implies observability of heat systems with singular admissible heat generation term V from every (G, δ)-equidistributed sensor set ω. More precisely, Theorem 3.7 is applicable with $\gamma_1 = 1/2$, $d_0 = (G/\delta)^{K_d \cdot (1+G^2\lambda_1 + G^{4/3}\lambda_2^{1/3})}$ and $d_1 = K_d \cdot G \log(G/\delta)$. This gives the following result.

COROLLARY 4.11. *Let $\omega \subset \mathbb{R}^d$ be a (G, δ)-equidistributed set. Then for all admissible V the system $\Sigma_o(\Delta - V, \mathbf{1}_\omega)$ is observable in time $T > 0$ with*

$$C_{\mathrm{obs}}^2 \leq \frac{1}{T}\Big(\frac{G}{\delta}\Big)^{K_d \cdot (1+G^2\lambda_1 + G^{4/3}\lambda_2^{1/3})} \cdot \exp\Big(\frac{K_d \cdot G^2 \log^2(G/\delta)}{T} - \kappa_- T\Big),$$

where $\kappa = \inf \sigma(-\Delta + V)$ and $\kappa_- = \min\{0, \kappa\}$.

REMARK 4.12. If V is a bounded potential and accordingly $\lambda_1 = 0$ and $\lambda_2 = \|V\|_\infty^2$, then the observability constant can be slightly optimized. In fact, in this situation one gets

$$C_{\mathrm{obs}}^2 \leq \frac{1}{T}\Big(\frac{G}{\delta}\Big)^{K_d \cdot (1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \cdot \exp\Big(\frac{K_d \cdot G^2 \log^2(G/\delta)}{T} - \kappa_- T\Big),$$

see [NTTV20a, Theorem 4.11]. Here the term $\|V - \kappa\|_\infty^{2/3}$ in the exponent results from applying the spectral inequality to the operator $H - \kappa = -\Delta + (V - \kappa)$.

4.1.2. Schrödinger operators with confinement potentials. When the Laplacian is perturbed by a confinement potential V, the operator $H = -A = -\Delta + V$ has purely discrete spectrum, cf. the discussion before Corollary 2.4 above, although it is defined on the whole of $\mathbb{R}^d$. Hence, from the spectral point of view, A has the same properties as the Laplacian on a bounded domain, where any set ω of positive Lebesgue measure serves as a sensor set for observability of the system $\Sigma_o(\Delta, \mathbf{1}_\omega)$. For the same system on $\mathbb{R}^d$ this is no longer the case and observability requires at least thick sensor sets, which excludes, for instance, sets of finite Lebesgue measure. One may ask how this spectral dichotomy influences the criteria for observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$.

Another motivation stems from the observation that the eigenfunctions of A vanish exponentially at infinity. More precisely, in Chapter 5 as well as in Lemma 7.9 below we establish weighted inequalities of the form

$$\|\mathrm{e}^{C|x|^p} f\|_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{C'\lambda^q} \|f\|_{L^2(\mathbb{R}^d)} \tag{4.6}$$

with suitable constants $C, C', p, q > 0$ for $f \in \operatorname{Ran} P_\lambda(H)$. This weighted inequality implies that half of the L^2-mass of such f is contained in a ball of radius proportional to $\lambda^{q/p}$, which can be seen as a quantification of the fast decay. In particular, this entails that the L^2-mass on areas far away from the origin contributes only a negligible amount to the whole L^2-mass. One therefore wonders whether it is possible to establish the spectral inequality with a set $\omega \subset \mathbb{R}^d$ that is sparse near infinity. This would also directly yield the observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ from such sensor sets ω.

In the literature, observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ has so far been studied mostly in the case where $V(x) = |x|^{2l}$ for some $l \in \mathbb{N}$, in which case the Schrödinger operator is called an *(anisotropic) Shubin operator* or, if $l = 1$, the *harmonic oscillator*. For arbitrary $l \in \mathbb{N}$, [Mil] established that the system $\Sigma_o(\Delta - |x|^{2l}, \mathbf{1}_\omega)$ is observable in all times $T > 0$ with the sensor set being any open cone of the form

$$\omega = \{x \in \mathbb{R}^d \colon |x| \geq r_0 \text{ and } x/|x| \in \Omega_0\}$$

for some open $\Omega_0 \subset \mathbb{S}^{d-1}$ and some $r_0 \geq 0$ if and only if $l > 1$. In this case, the observability constant satisfies the bound $C_{\mathrm{obs}} \leq \exp\bigl(C/T^{(l+1)/(l-1)}\bigr)$.[1] Apart from these results, there are approaches that show observability of the system $\Sigma_o(\Delta - |x|^{2l}, \mathbf{1}_\omega)$ with $l > 1$ using different techniques but not a spectral inequality for the generator itself; these paths are discussed in Sections 4.3 and 4.4 below. However, for the harmonic oscillator (i.e., $l = 1$), the situation is notably different and spectral inequalities have been studied extensively using that the eigenfunctions of this operator are the well-known Hermite functions, see, e.g., Theorem 4.13 and 4.15 below.

In this subsection, we formulate a spectral inequality for a class of Schrödinger operators with (partial) power growth potentials. Amongst others, this includes the operators $A = \Delta - |x_{\mathcal{I}}|^\tau$ with arbitrary $\tau > 0$ and nonempty $\mathcal{I} \subset \{1, \dots, d\}$; recall from Chapter 2 that $x_{\mathcal{I}}$ denotes the projection of x onto the coordinates indicated by $\mathcal{I}$. However, our results are still bipartite: If $A = \Delta - |x_{\mathcal{I}}|^2$ is the negative partial harmonic oscillator, we use a technique based on complex analysis to prove the spectral inequality. This uses so-called *Bernstein inequalities* for elements in the spectral subspace of A. Such inequalities heavily rely on the fact that the Hermite functions satisfy some very specific recursion formulas (see (7.8) below). Furthermore, since Bernstein inequalities imply that the eigenfunctions are analytic, it is not possible to establish such inequalities for arbitrary potentials with power growth. Therefore, the same approach is not applicable for these potentials and we instead use an approach based on Carleman estimates similar to the proof of the above Theorem 4.10 which is much more flexible with respect to the potential. However, we pay for this flexibility by having to accept stronger assumptions on the sensor set and by losing the dependence on the energy that is expected to be optimal. In particular, this explains why the results for the harmonic oscillator we present now are considerably stronger than those for other types of potentials, see the discussion after Theorem 4.23 below.

Partial harmonic oscillators. Let $H = -\Delta + |x|^2$ be the harmonic oscillator. As already pointed out above, the eigenfunctions of H are the well-known *Hermite*

[1] Note added: Shortly before completion of this work, it has been shown in [Mar, Theorem 2.5] that the system $\Sigma_o(\Delta - |x|^{2l}, \mathbf{1}_\omega)$ with $l > 1$ is observable as soon as ω has strictly positive Lebesgue measure.

functions

$$\Phi_\alpha(x) = \prod_{j=1}^{d} \phi_{\alpha_j}(x_j), \quad \alpha \in \mathbb{N}_0^d, \tag{4.7}$$

where

$$\phi_k(t) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \mathrm{e}^{t^2/2} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathrm{e}^{-t^2}, \quad k \in \mathbb{N}_0,$$

denotes the k-th standard Hermite function; note that $\phi_k(t) = p_k(t)\mathrm{e}^{-t^2/2}$ for some polynomial p_k of degree k. More precisely, Φ_α is an eigenfunction of H corresponding to the eigenvalue $2|\alpha| + d$ for all $\alpha \in \mathbb{N}_0^d$ and therefore the spectral subspace of the harmonic oscillator can be expressed as

$$\operatorname{Ran} P_\lambda(H) = \mathcal{E}_N \quad \text{for} \quad 2N + d \leq \lambda < 2(N+1) + d, \tag{4.8}$$

were $\mathcal{E}_N = \operatorname{span}\{\Phi_\alpha \colon |\alpha| \leq N\}$, $N \in \mathbb{N}_0$, denotes the space spanned by the Hermite functions Φ_α of degree $|\alpha| \leq N$. Using this representation for the spectral subspaces, spectral inequalities for the harmonic oscillator can be reformulated as uncertainty principles for Hermite functions. Indeed, if for some measurable set $\omega \subset \mathbb{R}^d$ and some constants d_0', d_1', and γ_1 we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq d_0' \mathrm{e}^{d_1' N^{\gamma_1}} \|f\|_{L^2(\omega)}^2 \quad \text{for all} \quad N \in \mathbb{N},\ f \in \mathcal{E}_N, \tag{4.9}$$

then by (4.8) this implies the spectral inequality (3.10) (in the form (4.1)) with $d_0 = d_0'$ and $d_1 = d_1'/2^{\gamma_1}$.

The first result that deals with uncertainty principles for Hermite functions of the form (4.9) was given in [BPS18] using Carleman estimates, where ω must contain a suitable union of open balls. Subsequently, in [BJPS21] it was shown that thickness of ω is sufficient for (4.9) to hold. In view of the discussion following Theorem 4.16 below, we also recall a result for arbitrary open ω from the last reference, although it has an unfavorable dependence on N, i.e., the spectral parameter.

THEOREM 4.13 ([BJPS21, Theorem 2.1]). *Let $\omega \subset \mathbb{R}^d$ be measurable.*

(i) If ω is (γ, ρ)-thick for some $\rho > 0$ and some $\gamma \in (0, 1]$, then there is a constant $C > 0$ depending on the thickness parameters and the dimension d such that

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq C \left(\frac{K_d}{\gamma}\right)^{K_d \rho N^{1/2}} \|f\|_{L^2(\omega)}^2 \quad \textit{for all} \quad N \in \mathbb{N},\ f \in \mathcal{E}_N.$$

(ii) If ω is any nonempty and open set, then there is a constant $C > 0$ such that

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq C \mathrm{e}^{CN \log N} \|f\|_{L^2(\omega)}^2 \quad \textit{for all} \quad N \in \mathbb{N},\ f \in \mathcal{E}_N.$$

Let us emphasize that part (i) of this result has been reproduced in [ES21, Corollary 1.9] with more explicit constants.

The condition on ω in part (i) of the above theorem is the same as for the spectral inequality for the Laplacian. In particular, it does not reflect that the Hermite functions decay (super-)exponentially. On the other hand, part (ii) of the theorem establishes an inequality with a superlinear dependence on the spectral parameter. In fact, this result is sharp and therefore it is not possible to prove a spectral inequality for bounded sensor sets ω. This was first observed in [Mil] and can also be seen by means of the following simple example.

EXAMPLE 4.14. Let $a > 0$ and $\omega = [-a, a]$. Suppose that for all $N \in \mathbb{N}$ the uncertainty relation

$$\|f\|^2_{L^2(\mathbb{R})} \leq \mathrm{e}^{c\cdot h(N)}\|f\|^2_{L^2(\omega)}, \quad N \in \mathbb{N},\ f \in \mathcal{E}_N, \tag{4.10}$$

holds with some nonnegative function h and a constant $c > 0$ independent from N. Consider $f_N(x) = x^N \mathrm{e}^{-|x|^2/2}$. Then $f_N \in \mathcal{E}_N$, and a simple computation shows that

$$\|f_N\|^2_{L^2(\mathbb{R})} = \int_0^\infty y^{N-\frac{1}{2}}\mathrm{e}^{-y}\,\mathrm{d}y = \Gamma\Big(N + \frac{1}{2}\Big) \quad \text{and} \quad \|f_N\|^2_{L^2(\omega)} \leq \sqrt{\pi}a^{2N},$$

where Γ denotes the Gamma function. Plugging these into (4.10), we derive

$$\Gamma\Big(N + \frac{1}{2}\Big) \lesssim \mathrm{e}^{c\cdot h(N)+2N\log a} \quad \text{for all} \quad N \in \mathbb{N}. \tag{4.11}$$

On the other hand, Stirling's formula for the Gamma function shows that

$$\Gamma\Big(N + \frac{1}{2}\Big) \geq \sqrt{\frac{2\pi}{N+\frac{1}{2}}}\Big(\frac{N+\frac{1}{2}}{\mathrm{e}}\Big)^{N+\frac{1}{2}} \gtrsim \mathrm{e}^{\frac{1}{2}\cdot N\log N} \quad \text{for large} \quad N.$$

Consequently, (4.11) can only hold if $N\log N \lesssim h(N)$ for large N. In particular, h can not be of the form $h(N) = N^\beta$ with $\beta < 1$.

In view of the previous example we are interested in unbounded sensor sets that take into account that elements in the spectral subspace are concentrated near the origin. The first result in this direction was given in [MPS22], where the authors generalized the spectral inequality from Theorem 4.13 (i) to sensor sets that are merely thick with respect to an unbounded, sublinear scale. This allows, amongst others, sensor sets which have holes of unboundedly growing diameter which are clearly not thick, see, e.g., Example 4.18 below.

THEOREM 4.15 ([MPS22, Theorem 2.1]). *Let $R > \varsigma > 0$, $\gamma \in (0, 1]$, and let $\rho\colon \mathbb{R}^d \to (0,\infty)$ be a $\frac{1}{2}$-Lipschitz continuous function satisfying*

$$\varsigma \leq \rho(x) \leq R(1 + |x|^2)^{(1-\varepsilon)/2} \quad \textit{for all} \quad x \in \mathbb{R}^d \quad \textit{and some} \quad \varepsilon \in (0, 1].$$

Then there is a constant $C_1 > 0$ depending on ς, R, γ, d, and ε as well as a constant $C_2 > 0$ depending on R, d, and ε such that for all measurable sets $\omega \subset \mathbb{R}^d$ satisfying

$$\frac{|B(x, \rho(x)) \cap \omega|}{|B(x, \rho(x))|} \geq \gamma \quad \textit{for all} \quad x \in \mathbb{R}^d \tag{4.12}$$

we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq C_1 \Big(\frac{K_d}{\gamma}\Big)^{C_2 N^{1-\varepsilon/2}} \|f\|_{L^2(\omega)}^2 \quad \textit{for all} \quad N \in \mathbb{N},\ f \in \mathcal{E}_N.$$

The drawback of the assumptions in the last theorem is that they still rule out sets of finite Lebesgue measure. Indeed, if ω is any set with finite Lebesgue measure and ρ is any unbounded scale satisfying $\rho(x) \to \infty$ as $|x| \to \infty$, then

$$\frac{|B(x,\rho(x)) \cap \omega|}{|B(x,\rho(x))|} \leq \frac{|\omega|}{\tau_d \rho(x)^d} \to 0 \quad \text{as} \quad |x| \to \infty.$$

In particular, (4.12) can not be satisfied and Theorem 4.15 is not applicable.

The first main result of this subsection is taken from the joint work [DSVc] of the author with Albrecht Seelmann and Ivan Veselić. It is formulated for sensor sets that are thick merely with respect to a variable sublinear scale and a subexponentially decaying density. Moreover, even in the case of a constant density our result removes the technical conditions from Theorem 4.15 and its constant is explicit in all model parameters.

THEOREM 4.16. *Let* $0 \leq \alpha < \varepsilon \leq 1$, $R > 0$, $\gamma \in (0,1]$, *and suppose that* $\rho\colon \mathbb{R}^d \to (0,\infty)$ *satisfies*

$$\rho(x) \leq R(1+|x|^2)^{(1-\varepsilon)/2} \quad \textit{for all} \quad x \in \mathbb{R}^d.$$

Then, for all measurable sets ω *satisfying*

$$\frac{|B(x,\rho(x)) \cap \omega|}{|B(x,\rho(x))|} \geq \gamma^{1+|x|^\alpha} \quad \textit{for all} \quad x \in \mathbb{R}^d \tag{4.13}$$

we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{3}\Big(\frac{K^d}{\gamma}\Big)^{K^{1+\alpha} d^{3+\alpha/2}(1+R)^2 N^{1-\frac{\varepsilon-\alpha}{2}}} \|f\|_{L^2(\omega)}^2 \quad \textit{for all} \quad N \in \mathbb{N},\ f \in \mathcal{E}_N.$$

This theorem allows sensor sets ω with finite Lebesgue measure which are not in scope of Theorem 4.15.

EXAMPLE 4.17. Let

$$\omega = \bigcup_{k \in \mathbb{Z}^d} B(k, d^{1/2} \cdot 2^{-1-|k|^\alpha}) \quad \text{for some} \quad 0 < \alpha < 1. \tag{4.14}$$

Then ω has finite Lebesgue measure and satisfies (4.13) with $\rho \equiv d^{1/2}$ and $\gamma = 2^{-d}$. Here the factor $d^{1/2}$ in the definition of ω is merely used to avoid an additional prefactor on the right-hand side of (4.13) since $|B(x,d^{1/2})| = \tau_d d^{d/2}$.

Let us also note here that there are sets which are thick with respect to an unbounded scale while they are not thick with respect to a fixed scale with a decaying density. Together with the preceding example this shows for the first time that the variable scale and the decaying density are two notions that can not be compared.

EXAMPLE 4.18. Let $d = 1$ and $\rho(x) = (1 + x^2)^{1/4}$. Then

$$\omega = \mathbb{R} \setminus B(100n^3, n) \quad \text{satisfies} \quad \frac{|B(x, \rho(x)) \cap \omega|}{|B(x, \rho(x))|} \geq \frac{1}{4} \quad \text{for all} \quad x \in \mathbb{R}^d,$$

see Subsection A.2.2 in Appendix A for the proof. However, for all fixed $\rho > 0$ we have $B(100n^3, \rho) \cap \omega = \emptyset$ if $n \geq \rho$ and therefore ω can not satisfy (4.13) with a constant scale ρ and any α.

A more general result can be formulated for *partial harmonic oscillators* where the potential $V(x) = |x_{\mathcal{I}}|^2$ with $\mathcal{I} \subset \{1, \dots, d\}$ is growing unboundedly only in some coordinate directions while it is constant in others. Here, naturally, the anisotropy of the potential is also reflected in the criteria for a sensor set. For reasons of readability, we postpone the full result in this case to Corollary 7.21 in Chapter 7 below and formulate here the result for partial harmonic oscillators only for sensor sets that satisfy the corresponding thickness condition with respect to a constant scale and with respect to hypercubes instead of balls. The latter allows for a simpler proof since we can cover the whole of $\mathbb{R}^d$ by cubes without any overlap, see the discussion at the beginning of Subsection 7.2.5 below. However, Subsection A.2.1 shows that it is not essential whether we take balls or hypercubes.

The next result was obtained by the author in the joint work [DSVa] with Albrecht Seelmann and Ivan Veselić. Since the partial harmonic oscillators do not have discrete spectrum, it is again formulated for elements in the spectral subspace $\operatorname{Ran} P_\lambda(H)$.

THEOREM 4.19. *Let $H = -\Delta + |x_{\mathcal{I}}|^2$ with some $\mathcal{I} \subset \{1, \dots, d\}$. Then for all measurable sets $\omega \subset \mathbb{R}^d$ satisfying*

$$\frac{|\Lambda_\rho(k) \cap \omega|}{|\Lambda_\rho(k)|} \geq \gamma^{1+|k_{\mathcal{I}}|^\alpha} \quad \textit{for all} \quad k \in (\rho\mathbb{Z})^d \tag{4.15}$$

with some $\alpha \geq 0$, $\rho > 0$, and $\gamma \in (0, 1]$, we have

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{1}{3}\Big(\frac{K^d}{\gamma}\Big)^{K^{1+\alpha} d \cdot (1+\rho)^2 \lambda^{(1+\alpha)/2}} \|f\|^2_{L^2(\omega)} \quad \textit{for all} \quad \lambda \geq 1, \ f \in \operatorname{Ran} P_\lambda(H).$$

The proof of Theorem 4.19 is given in Section 7.2 below. In that section we also formulate Corollary 7.21 which contains Theorem 4.16 as a special case.

From the spectral inequality in Theorem 4.19 we now get an observability estimate using again Theorem 3.7.

COROLLARY 4.20. *Let $\mathcal{I} \subset \{1, \dots, d\}$, and suppose that the sensor set ω satisfies (4.15) with $\alpha \in [0, 1)$. Then system $\Sigma_o(\Delta - |x_{\mathcal{I}}|^2, \mathbf{1}_\omega)$ is observable and*

$$C^2_{\mathrm{obs}} \leq \frac{C}{T} \exp\Big(\frac{C}{T^{1+2\alpha/(1-\alpha)}}\Big)$$

for a constant $C > 0$ that depends on the model parameters α, γ, and ρ. In particular, there are sets ω with finite Lebesgue measure from which the system $\Sigma_o(\Delta - |x|^2, \mathbf{1}_\omega)$ is observable.

REMARK 4.21. Actually, the observability constant can be computed more explicitly: Indeed, using Theorem 3.7 with $\kappa_- = 0$, $\gamma_1 = (1+\alpha)/2$,

$$d_0 = \frac{1}{3}\Big(\frac{K^d}{\gamma}\Big)^{K^{1+\alpha}d(1+\rho)^2} \quad \text{and} \quad d_1 = K^{1+\alpha}d(1+\rho)^2\log(K^d/\gamma),$$

we have

$$C_{\mathrm{obs}}^2 \leq \frac{C_1}{T}\Big(\frac{K^d}{\gamma}\Big)^{C_2\cdot K^{1+\alpha}d(1+\rho)^2} \exp\bigg(C_3\Big(\frac{C_4|\log\gamma|^2}{T^{1+\alpha}}\Big)^{1/(1-\alpha)}\bigg)$$

with the constants C_1, C_2, and C_3 from Theorem 3.7 and with $C_4 = K^{1+\alpha}d^4(1+\rho)^4$. Note that C_1, C_3, and C_3 depend on γ_1 and therefore also on α.

Power growth potentials. We now move on to a more general class of (partial) power growth potentials, the prime examples of which are the Shubin-type potentials $V(x) = |x|^\tau$ with some $\tau > 0$.

HYPOTHESIS (S). Let $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, that is, V and its weak derivatives of first order are contained in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose that V is such that

(i) for some $c_1, c_2 > 0$ and some $0 < \tau_1 \leq \tau_2$ we have

$$c_1|x|^{\tau_1} \leq V(x) \leq c_2|x|^{\tau_2} \quad \text{for all} \quad x \in \mathbb{R}^d;$$

(ii) for some $\nu > 0$ we have

$$M_\nu := \|\mathrm{e}^{-\nu|x|}|\nabla V|\|_{L^\infty(\mathbb{R}^d\setminus B(0,1))} < \infty. \tag{4.16}$$

Let us first consider simple examples of potentials satisfying this hypothesis.

EXAMPLE 4.22. Let $V(x) = |x|^\tau$ for some $\tau > 0$. Then (i) is trivially satisfied, and since $|\nabla V| = \tau|x|^{\tau-1}$ for $|x| \geq 1$, it is not hard to verify that (ii) holds with $\nu = \tau$ and with $M_\tau \leq \tau\mathrm{e}^{1-\tau}$. Now let $V(x) = |x|^\tau + W(x)$ with $W\colon \mathbb{R}^d \to [0,\infty)$ being any differentiable function with bounded derivatives satisfying $W(x) \leq |x|^\tau$. Then (ii) is still satisfied with the same choice for ν and a simple calculation shows (i) with $c_1 = 1$, $c_2 = 2$, and $\tau_1 = \tau_2 = \tau$.

Let us briefly comment on the assumptions in Hypothesis (S): The lower bound in part (i) on the one hand allows to bound the eigenvalue counting function for $H = -\Delta + V$ with V as in Hypothesis (S), cf. (5.11) below, and, on the other hand, is needed to control the growth of the potential. Thereby, we are able to establish a suitable L^2-decay for eigenfunctions of H, see Proposition 5.5 below. The bound in part (ii) allows to obtain a similar decay for partial derivatives of eigenfunctions by differentiating the eigenvalue equation $Hf = \lambda f$, which introduces partial derivatives of the potential to the equation, see Proposition 5.6 below. Together with the bound on the eigenvalue counting function, this amounts to the fact that even the H^1-mass of $f \in \operatorname{Ran} P_\lambda(H)$ is strongly localized, see Theorem 5.1 below, so that by a "cut-off procedure" (cf. inequality (6.24) below) the considerations can essentially be reduced to a suitable bounded subset of $\mathbb{R}^d$. Finally, the upper bound in (i) provides a corresponding bound on the potential on this bounded subset.

Through the method just discussed, we are able to use essentially the same approach as for Theorem 4.10 above to prove a spectral inequality for Schrödinger operators with potentials as in the above Hypothesis (S). However, this approach is the reason why we need to localize the H^1-mass and therefore why part (ii) in Hypothesis (S) is needed.

The next main result was established by the author in collaboration with Albrecht Seelmann and Ivan Veselić [DSVb].

THEOREM 4.23. *Let $H = -\Delta + V$ with V as in Hypothesis (S). Then there is a constant $C > 0$ depending only on $\tau_1, \tau_2, c_1, c_2, \nu, M_\nu$, and the dimension d, such that for all measurable sets $\omega \subset \mathbb{R}^d$ for which for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $(k + (-1/2, 1/2)^d) \cap \omega$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k|^\alpha}$, we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \Big(\frac{1}{\delta}\Big)^{C^{1+\alpha}\cdot\lambda^{(\alpha+2\tau_2/3)/\tau_1}} \|f\|^2_{L^2(\omega)} \quad \textit{for all} \quad \lambda \geq 1,\ f \in \operatorname{Ran} P_\lambda(H). \tag{4.17}$$

The dependence on λ in the exponent is sublinear if $\alpha + 2\tau_2/3 < \tau_1$ and, therefore, in this case the above theorem constitutes a spectral inequality. In the particular case of $V(x) = |x|^\tau$ as in the first part of Example 4.22 above, the constant in the theorem depends only on τ and the dimension d, and we must have $\alpha < \tau/3$ in order for the dependence on λ in the exponent to be sublinear.

Every $(1, \delta)$-equidistributed set satisfies the assumptions of the theorem with $\alpha = 0$. On the other hand, every set ω as in Theorem 4.23 satisfies the geometric assumptions (4.15) of Theorem 4.19 (for $\mathcal{I} = \{1, \dots, d\}$) with $\rho = 1$, $\gamma = K_d\delta^d$, and the same choice for α, see the calculations in Subsection A.2.3 in Appendix A.

Thus, in the case of the harmonic oscillator (i.e., $\tau = 2$), we must have $\alpha < 2/3$ in Theorem 4.23 while Theorem 4.19 allows $\alpha < 1$ in this situation. This difference is due to the different approach we use here. In fact, even for $\alpha = 0$ and arbitrary potentials as in Hypothesis (S) the dependence on λ in the exponent of the constant in (4.17) is of order $\gamma_1 = 2\tau_2/(3\tau_1) \geq 2/3$, while it is expected that the optimal dependence is of order $1/2$ as obtained for operators of Schrödinger type in all the aforementioned results; recall that this behavior is known to be sharp for the pure Laplacian. The slightly worse behavior in our Theorem 4.23 above is due to the mentioned cut-off procedure, which is needed in order to conduct the proof using Carleman estimates. It is also due to the Carleman approach that we are only able to consider sensor sets containing suitable open balls and not just measurable sets satisfying (4.15) with $\mathcal{I} = \{1, \dots, d\}$.

If $\tau_1 - 2\tau_2/3 > \alpha > 0$, then ω as in (4.14) above has finite measure and satisfies the assumptions of the theorem. Thus, Theorem 4.23 gives a spectral inequality for general V as in Hypothesis (S) also for some sensor sets with finite Lebesgue measure if $\tau_1 - 2\tau_2/3 > 0$.

Exploiting the tensor structure, we get as a simple generalization of Theorem 4.23 a spectral inequality for *partial power growth potentials*. In order to formulate it, we adapt Hypothesis (S) to include these potentials.

HYPOTHESIS $(S_{\mathcal{I}})$. Let $\mathcal{I} \subset \{1, \dots, d\}$. Suppose that $V(x) = V_{\mathcal{I}}(x_{\mathcal{I}})$, where $V_{\mathcal{I}} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{|\mathcal{I}|})$ satisfies the assumptions of Hypothesis (S) with $\mathbb{R}^d$ replaced by $\mathbb{R}^{|\mathcal{I}|}$.

Prototypical examples of potentials as in Hypothesis $(S_{\mathcal{I}})$ are $V(x) = |x_{\mathcal{I}}|^\tau$ for some $\tau > 0$ and some $\mathcal{I} \subset \{1, \dots, d\}$ corresponding to the *partial Shubin-type operators*.

The next theorem was also stated in [DSVb] and relates for $\tau = 2$ to Theorem 4.19 with the same choice for $\mathcal{I}$. In particular, it contains Theorem 4.23 as a special case.

THEOREM 4.24. *Let $\mathcal{I} \subset \{1, \dots, d\}$, and let $H = -\Delta + V$ with V as in Hypothesis $(S_{\mathcal{I}})$. Then there is a constant $C > 0$ depending only on the parameters $\tau_1, \tau_2, c_1, c_2, \nu, M_\nu$ connected to $V_{\mathcal{I}}$ and the dimension d, such that for all measurable sets $\omega \subset \mathbb{R}^d$ for which for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $\Lambda_1(k) \cap \omega$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k_{\mathcal{I}}|^\alpha}$, we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \Big(\frac{1}{\delta}\Big)^{C^{1+\alpha}\cdot\lambda^{(\alpha+2\tau_2/3)/\tau_1}} \|f\|^2_{L^2(\omega)} \quad \textit{for all} \quad \lambda \geq 1,\ f \in \operatorname{Ran} P_\lambda(H).$$

We derive this theorem in Section 6.4 below parallel to Theorems 4.10 and 4.23. The already mentioned H^1-localization for elements in $\operatorname{Ran} P_\lambda(H)$ we utilize is established in Chapter 5.

We close this section by formulating the observability result that is a direct consequence of Theorem 4.24 and Theorem 3.7.

COROLLARY 4.25. *Let $\mathcal{I} \subset \{1, \dots, d\}$, let V be as in Hypothesis $(S_{\mathcal{I}})$, and let $0 \leq \alpha < \tau_1 - 2\tau_2/3$. Then, if ω is a measurable set such that each intersection $\Lambda_1(k) \cap \omega$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k_{\mathcal{I}}|^\alpha}$, the system $\Sigma_o(\Delta - V, \mathbf{1}_\omega)$ is observable in time $T > 0$.*

More specifically, if $V(x) = |x_{\mathcal{I}}|^\tau$ for some $\tau > 0$, then we have

$$C^2_{\mathrm{obs}} \leq \frac{C'}{T} \exp\left(\frac{C'}{T^{1+\frac{2\alpha+\tau/3}{\tau/3-\alpha}}}\right)$$

with a constant $C' > 0$ depending on τ, α, and δ.

REMARK 4.26. The proof of the upper bound for the observability constant for potentials $V(x) = |x_{\mathcal{I}}|^\tau$ uses Theorem 3.7 with $\kappa_- = 0$, $\gamma_1 = \alpha/\tau + 2/3$, $d_0 = \delta^{-C^{1+\alpha}}$ and $d_1 = C^{1+\alpha} \log \frac{1}{\delta}$. With this, it is not hard to obtain the slightly more explicit bound

$$C^2_{\mathrm{obs}} \leq \frac{C_1}{T}\Big(\frac{1}{\delta}\Big)^{KC_2C^{1+\alpha}} \exp\left(C_3\Big(\frac{C^{(1+\alpha)\tau}|\log\delta|^\tau}{T^{\alpha+2\tau/3}}\Big)^{1/(\tau/3-\alpha)}\right).$$

Here C_1, C_2, and C_3 are the constants from Theorem 3.7 that depend on γ_1 and thus also on τ and α.

4.2. Quadratic differential operators

We now turn to the study of observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ with possibly nonselfadjoint generators A in $L^2(\mathbb{R}^d)$ and continue with our investigation of sufficient conditions on $\omega \subset \mathbb{R}^d$ that guarantee observability. The difficulties that may arise here are twofold: On the one hand, even if A is selfadjoint, a corresponding spectral inequality might not be available. Even worse, on the other hand, the generator might not be selfadjoint and therefore it makes no sense to talk about spectral projections. In both cases, it is natural to search for a suitable nonnegative selfadjoint operator H such that with its spectral projections $P_\lambda = P_\lambda(H)$ we can prove a dissipation estimate of the form (3.12) at least for small times $0 < t < t_0$, that is,

$$\|(\mathrm{Id} - P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq d_2 \mathrm{e}^{-d_3\lambda^{\gamma_2}t^{\gamma_3}} \|g\|_{L^2(\mathbb{R}^d)}$$

for all $\lambda \geq 1$, and all $g \in L^2(\mathbb{R}^d)$, where $(\mathcal{T}(t))_{t\geq 0}$ is the semigroup generated by A. If we can arrange this, we say that the operator H is a suitable *comparison operator* for A or that A is *comparable* to H. We see further below that the comparison operator is usually not unique. If we have already established a spectral inequality for H, the Lebeau-Robbiano method set out in Corollary 3.9 directly implies observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ from the same sensor sets from which the system $\Sigma_o(-H, \mathbf{1}_\omega)$ is observable. In particular, the picked comparison operator directly determines the possible sensor sets. Hence, if one is interested in the most general sensors sets possible, the comparison operator needs to fit the generator, see the discussion after Proposition 4.28 below for a more precise description of this.

In this section, the choice of the comparison is derived from properties of the operator A, which is here always a quadratic differential operator $A = q^{\mathrm{w}}$ as defined in Section 2.3, where q is a complex quadratic form satisfying $\operatorname{Re} q \leq 0$. Recall from Proposition 2.11 that A is then the generator of a contraction semigroup in $L^2(\mathbb{R}^d)$ that we always denote by $(\mathcal{T}(t))_{t\geq 0}$.

4.2.1. Dissipation estimates with a comparison operator. To the best of the authors knowledge, the first proof of observability using a comparison operator was given in [BPS18] for quadratic differential operators A with singular space $S(A) = \{0\}$ and with the harmonic oscillator $H = -\Delta + |x|^2$ as the selfadjoint comparison operator. Recall from Lemma 2.15 that the singular space of the latter operator likewise satisfies $S(H) = \{0\}$.

PROPOSITION 4.27 ([BPS18, Proposition 4.1]). *Let $S(A) = \{0\}$, let k_0 be the rotation exponent of A from* (2.13)*, and let $P_\lambda = P_\lambda(-\Delta + |x|^2)$. Then there are*

$C_0, t_0 > 0$ such that

$$\|(1 - P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq C_0 \mathrm{e}^{-C_0 t^{2k_0+1}\lambda} \|g\|_{L^2(\mathbb{R}^d)} \tag{4.18}$$

for all $0 < t < t_0$, all $\lambda \geq 1$, and all $g \in L^2(\mathbb{R}^d)$.

A similar result is also available with the pure Laplacian as the selfadjoint comparison operator: In [AB, Remark 2.9] the authors state that the technique developed in [Alp21, Section 4.2] implies that all quadratic differential operators A with $S(A) \subset \mathbb{R}^d \times \{0\}$ satisfy a dissipation estimate similar to (4.18), but with $P_\lambda = P_\lambda(-\Delta)$ the projection onto a spectral subspace of the Laplacian; recall again from Lemma 2.15 that $S(\Delta) = \mathbb{R}^d \times \{0\}$.

PROPOSITION 4.28 (see [Alp21, Section 4.2], [AB, Remark 2.9]). *Assume $S(A) = U \times \{0\}$ for some subspace $U \subset \mathbb{R}^d$, let k_0 be the rotation exponent of A, and let $P_\lambda = P_\lambda(-\Delta)$. Then there are $C_0, t_0 > 0$ such that*

$$\|(1 - P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq C_0 \mathrm{e}^{-C_0 t^{2k_0+1}\lambda} \|g\|_{L^2(\mathbb{R}^d)}$$

for all $0 < t < t_0$, all $\lambda \geq 1$, and all $g \in L^2(\mathbb{R}^d)$.

Both dissipation estimates spelled out in Propositions 4.27 and 4.28 cover the case $S(A) = \{0\}$. Hence, it is natural to compare the two complementing spectral inequalities in this case. It turns out that Proposition 4.28 is *strictly* weaker than Proposition 4.27. Indeed, recall from Section 4.1 that spectral inequalities for the Laplacian require thick sensors sets, while Theorem 4.16 shows that thickness is not necessary for spectral inequalities for the harmonic oscillator. In this sense, if $S(A) = \{0\}$, using the harmonic oscillator as a comparison operator for A and applying Corollary 3.9 allows for more general sensors sets than using the pure Laplacian.

The result we formulate next was obtained jointly by the author with Albrecht Seelmann and Ivan Veselić [DSVb]. It shows that it is reasonable to classify comparison operators for quadratic differential operators by the form of their singular space and that the partial harmonic oscillators constitute a class of comparison operators that interpolate, in some sense, between the Laplacian and the harmonic oscillator. More generally, our result allows also to use the operators $H_{\mathcal{I},\mathcal{J}} = -\Delta_{\mathcal{J}} + |x_{\mathcal{I}}|^2$ with $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$ as selfadjoint comparison operators. By Lemma 2.15, the singular space of such operators is given by $S(H_{\mathcal{I},\mathcal{J}}) = \mathbb{R}^d_{\mathcal{I}^c} \times \mathbb{R}^d_{\mathcal{J}^c}$ and, accordingly, we get a dissipation estimate with the spectral projections of the operators $H_{\mathcal{I},\mathcal{J}}$. For $\mathcal{J} = \{1, \ldots, d\}$, the next theorem covers and extends the previously mentioned dissipation estimates obtained for quadratic differential operators. We postpone the proof to Chapter 8 below. The constant C_0 from the next theorem is given in Theorem 8.4 below.

THEOREM 4.29. *Let $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d_{\mathcal{J}}$ for some sets $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$, let k_0 be the rotation exponent from (2.13), and let $P_\lambda = P_\lambda(H_{\mathcal{I},\mathcal{J}})$. Then, there are* *constants* $C_0 > 0$ *and* $t_0 \in (0,1)$ *such that*

$$\|(1-P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq 2\mathrm{e}^{-C_0 t^{2k_0+1}\lambda}\|g\|_{L^2(\mathbb{R}^d)}$$

for all $g \in L^2(\mathbb{R}^d)$*, all* $0 < t < t_0$*, and all* $\lambda \geq 1$.

Applying suitable rotations, the previous theorem can be generalized to singular spaces of the form $S(A)^\perp = V \times W$, where $V, W \subset \mathbb{R}^d$ are vector spaces of dimensions $d_1 = \dim V$ and $d_2 = \dim W$. Indeed, in this case, there is an orthogonal transformation $\mathcal{R}\colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\mathcal{R}V = \mathbb{R}^d_{\mathcal{I}} \quad \text{and} \quad \mathcal{R}W = \mathbb{R}^d_{\mathcal{J}} \tag{4.19}$$

with $\mathcal{I} = \{1, \ldots, d_1\}$ and with $\mathcal{J} = \{d_1 - l + 1, \ldots, d_1 + d_2 - l\}$, where $l = \dim(V \cap W)$. Letting q be the quadratic form corresponding to A, a simple calculation using the chain rule shows that the singular space of the form $\tilde{q}$ given by $\tilde{q}(x,\xi) = q(\mathcal{R}^{-1}x, \mathcal{R}^{-1}\xi)$ for all $x, \xi \in \mathbb{R}^d$ is characterized by $S(\tilde{q})^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d_{\mathcal{J}}$. Furthermore, the accretive operator $\tilde{A} = \tilde{q}^{\mathrm{w}}$ associated with $\tilde{q}$ by the Weyl quantization satisfies $A = \mathcal{U}_{\mathcal{R}}\tilde{A}\mathcal{U}_{\mathcal{R}}^{-1}$ where $\mathcal{U}_{\mathcal{R}}f = f \circ \mathcal{R}$, see Lemma 2.13. Thus, we have $\tilde{\mathcal{T}}(t) = \mathcal{U}_{\mathcal{R}}^{-1}\mathcal{T}(t)\mathcal{U}_{\mathcal{R}}$ for the semigroup $(\tilde{\mathcal{T}}(t))_{t\geq 0}$ generated $\tilde{A}$, so that applying Theorem 4.29 to $\tilde{A}$ proves the following result.

Corollary 4.30. *Let* $S(A)^\perp = V \times W$*, and let* $\mathcal{R}$ *be as in* (4.19)*. Then, with* $P_\lambda = \mathcal{U}_{\mathcal{R}} P_\lambda(H_{\mathcal{I},\mathcal{J}})\mathcal{U}_{\mathcal{R}}^{-1}$*, we have*

$$\|(1-P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq 2\mathrm{e}^{-ct^{2k_0+1}\lambda}\|g\|_{L^2(\mathbb{R}^d)}$$

for all $g \in L^2(\mathbb{R}^d)$, $0 < t < t_0$*, and* $\lambda \geq 1$.

Remark 4.31. There are also quadratic forms whose singular space does not satisfy $S(q)^\perp = V \times W$. Consider, e.g., the form $q(x,\xi) = -(x+\xi)^2$ on $\mathbb{R}^2$ with singular space $S(q) = \{r \cdot (1,-1)^\top : r \in \mathbb{R}\}$. The operators corresponding to such forms are not covered by Corollary 4.30.

4.2.2. Observability and applications. We now combine the dissipation estimate from Theorem 4.29 with the corresponding spectral inequality for a partial harmonic oscillator from Subsection 4.1.2 above and discuss applications of our results. Accordingly, in all these applications we have $\mathcal{J} = \{1, \ldots, d\}$. Note, however, that in Corollary 7.26 below we also prove a spectral inequality for the operator $H_{\mathcal{I},\mathcal{J}}$ with $\mathcal{J} \neq \{1, \ldots, d\}$ where the observation operator is not the multiplication operator by a characteristic function of a sensor set.

For simplicity, in the rest of this section we refrain from giving explicit estimates for the observability constant.

If A is any quadratic differential operator with singular space $S(A) = \{0\}$, then Theorem 4.29 holds for the projections $P_\lambda = P_\lambda(-\Delta + |x|^2)$ onto the spectral subspace of the harmonic oscillator. This situation has already been considered in [BJPS21, MPS22] based on Proposition 4.27 and the spectral inequalities established in the last mentioned papers, see Theorems 4.13 and 4.15 above. The important point to note in this case is that replacing the last mentioned theorems by our spectral inequality from Theorem 4.16 we conclude observability from sensor sets satisfying the geometric assumptions of Theorem 4.16.

COROLLARY 4.32. *Suppose that* $S(A) = \{0\}$. *Then, for all measurable sets* ω *satisfying*

$$\frac{|B(x,\rho(x))\cap\omega|}{|B(x,\rho(x))|} \geq \gamma^{1+|x|^\alpha} \quad \textit{for all} \quad x \in \mathbb{R}^d,$$

where $0 \leq \alpha < \varepsilon \leq 1$, $R > 0$, $\gamma \in (0,1]$, *and* $\rho\colon \mathbb{R}^d \to (0,\infty)$ *is a function satisfying* $\rho(x) \leq R(1+|x|^2)^{(1-\varepsilon)/2}$ *for all* $x \in \mathbb{R}^d$, *the system* $\Sigma_o(A, \mathbf{1}_\omega)$ *is observable in time* $T > 0$.

Sensor sets as in the above corollary were not accessible before in this context. In particular, recall that there are sets of finite Lebesgue measure satisfying the assumptions of this lemma, e.g., the set given in Example 4.17.

If the singular space is not zero, the dissipation estimate from Theorem 4.29 shows that the spectral inequalities for the partial harmonic oscillator formulated in Theorem 4.19 are applicable towards observability. Thereby, we establish observability from sensor sets with a decaying density in directions encoded by the singular space. This corollary covers also situations where the sensor set is not thick and is therefore new.

COROLLARY 4.33. *Suppose that* $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d$ *for some* $\mathcal{I} \subset \{1,\dots,d\}$. *Then, for all measurable sets* ω *satisfying*

$$\frac{|\Lambda_\rho(k)\cap\omega|}{|\Lambda_\rho(k)|} \geq \gamma^{1+|k_{\mathcal{I}}|^\alpha} \quad \textit{for all} \quad k \in (\rho\mathbb{Z})^d$$

and some $0 \leq \alpha < 1$, $\gamma \in (0,1]$, *the system* $\Sigma_o(A, \mathbf{1}_\omega)$ *is observable in time* $T > 0$.

Corollaries 4.32 and 4.33 can be applied for instance in the context of Examples 2.16 and 2.17:

The Kolmogorov operator A from Example 2.16 satisfies $S(A)^\perp = \{0\} \times \mathbb{R}^d$. For this case, the corresponding dissipation estimate has already been established in [BPS18] and in combination with the spectral inequality for the Laplacian this already establishes observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ from thick sensor sets. This is covered by Corollary 4.33. Let us point out that observability of the Kolmogorov equation has already been shown previously to [BPS18], e.g., using Carleman estimates [LM16, Zha16] if the sensor set contains a suitable union of open balls.

In all situations where $S(A)^\perp \supsetneq \{0\} \times \mathbb{R}^d$, the above corollaries improve earlier results. Previously, dissipation estimates for such operators A were only established with the Laplacian as the selfadjoint comparison operator, see [Alp21]. With this method it is only possible to establish the observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$

with thick sensor sets. Corollary 4.33 removes this conditions. This is best seen for the Kramers-Fokker-Planck operator introduced in Example 2.17: Without an external potential the singular space of this operator is given by $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d$, where $d = 2m$ for $m \in \mathbb{N}$ and $\mathcal{I} = \{m+1, \dots, d\}$. Hence, $S(A)^\perp \supsetneq \{0\} \times \mathbb{R}^d$ and Corollary 4.33 yields that $\Sigma_o(A, \mathbf{1}_\omega)$ is observable from sensor sets ω which are not thick, whereas [Alp21] only establishes observability from thick sets in this setting.

4.3. Semigroups with smoothing effects

In the preceding sections we discussed observability of systems $\Sigma_o(A, \mathbf{1}_\omega)$ by inspecting the generator A. More precisely, properties of A were used as an input for our results: While in Section 4.1 we proved a spectral inequality for the negative $H = -A$ of the selfadjoint generator, in Section 4.2 we used the singular space of A to determine suitable comparison operators. This section takes a different point of view and examines the observability solely based on properties of the semigroup. There are results in this direction that use the so-called smoothing properties of the semigroup to choose an appropriate comparison operator, see the discussion in Section 4.4 below. However, the approach we present in this section does not use comparison operators. Instead, we follow the approach presented in Corollary 3.11 and derive observability estimates from *uncertainty principles with error term* established for functions in the range of the semigroup associated to the abstract Cauchy problem. In our setting, the semigroup $(\mathcal{T}(t))_{t\geq 0}$ in $L^2(\mathbb{R}^d)$ is always a contraction semigroup that is smoothing in some Gelfand-Shilov space in the following sense.

Definition 4.34 (Smoothing properties). Let $(\mathcal{T}(t))_{t\geq 0}$ be a strongly continuous contraction semigroup in $L^2(\mathbb{R}^d)$, and let $\nu, \mu > 0$. We say that $(\mathcal{T}(t))_{t\geq 0}$ is smoothing in the Gelfand-Shilov space $S^\mu_\nu(\mathbb{R}^d)$ if there are constants $r_1, r_2 \geq 0$, $C_1, C_2 > 0$, and $t_0 \in (0,1)$ such that for all $t \in (0, t_0)$, all $g \in L^2(\mathbb{R}^d)$, and all $\alpha, \beta \in \mathbb{N}_0^d$ we have

$$\|x^\alpha \partial^\beta \mathcal{T}(t) g\|_{L^2(\mathbb{R}^d)} \leq \frac{C_1 C_2^{|\alpha|+|\beta|}}{t^{r_1 + r_2(|\alpha|+|\beta|)}} (\alpha!)^\nu (\beta!)^\mu \|g\|_{L^2(\mathbb{R}^d)}. \tag{4.20}$$

In what follows, we always assume that $\nu > 0$ and $\mu \in (0,1]$ are such that $\mu + \nu \geq 1$. Here, the assumption on μ guarantees that $\mathcal{T}(t)g$ is analytic if $t \in (0, t_0)$ and $g \in L^2(\mathbb{R}^d)$, see Lemma A.8 for a proof of this statement. Recall that the assumption $\mu + \nu \geq 1$ rules out that the Gelfand-Shilov spaces are trivial.

From the smoothing properties (4.20) it is easy to conclude an L^2-inequality with a strictly positive weight function for elements in the range of the semigroup. In fact, expanding the weight function by the multinomial theorem and using the smoothing properties for each summand we obtain

$$\|(1+|x|^2)^{n/2} \partial^\beta \mathcal{T}(t) g\|_{L^2(\mathbb{R}^d)} \leq \frac{C_1 (d+1)^{n/2} C_2^{1+n+|\beta|}}{t^{r_1 + r_2(n+|\beta|)}} (n!)^\nu (|\beta|!)^\mu \|g\|_{L^2(\mathbb{R}^d)} \tag{4.21}$$

for all $n \in \mathbb{N}$, all $\beta \in \mathbb{N}_0^d$, all $t \in (0, t_0)$, and all $g \in L^2(\mathbb{R}^d)$ if $(\mathcal{T}(t))_{t\geq 0}$ is smoothing in the Gelfand-Shilov space $S_\nu^\mu(\mathbb{R}^d)$, cf. [Mar22, Lemma 5.2]. This inequality enters the main result of this section as an essential assumption. It replaces, in some sense, the assumption of the spectral inequalities that the element for which we prove an uncertainty principle lies in the spectral subspace of the operator. Setting $f = \mathcal{T}(t)g$ with $g \in L^2(\mathbb{R}^d)$ and $t \in (0, t_0)$, inequality (4.21) can be reformulated as

$$\|(1+|x|^2)^{n/2}\partial^\beta f\|_{L^2(\mathbb{R}^d)} \leq D_1 D_2^{n+|\beta|}(n!)^\nu(|\beta|!)^\mu \quad \text{for all} \quad n \in \mathbb{N}_0,\ \beta \in \mathbb{N}_0^d, \tag{4.22}$$

with

$$D_1 = \frac{C_1}{t^{r_1}}\|g\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad D_2 = \frac{(d+1)^{1/2}C_2}{t^{r_2}}. \tag{4.23}$$

Motivated by this inequality, the main result of this section reads as follows. It was first obtained in this form in the authors work [DS22] with Albrecht Seelmann. Its proof is given in Section 7.3 below.

THEOREM 4.35. *Suppose that $f \in C^\infty(\mathbb{R}^d)$ satisfies (4.22) with some $D_1 > 0$, $D_2 \geq 1$, $\nu \geq 0$, and $0 \leq \mu < 1$ satisfying $\mu + \nu \geq 1$. Moreover, let $\varepsilon \in [0, 1)$ with $s = \varepsilon\nu + \mu < 1$, let $\gamma \in (0, 1]$, and let $\rho\colon \mathbb{R}^d \to (0, \infty)$ be a measurable function satisfying*

$$\rho(x) \leq R(1+|x|^2)^{\varepsilon/2} \quad \textit{for all} \quad x \in \mathbb{R}^d$$

with some $R \geq 1$.

Then, there is $C > 0$ depending on γ, R, r_0, ν, s, and the dimension d, such that for all $\delta \in (0, 1]$ and all measurable sets $\omega \subset \mathbb{R}^d$ satisfying

$$\frac{|B(x, \rho(x)) \cap \omega|}{|B(x, \rho(x))|} \geq \gamma \quad \textit{for all} \quad x \in \mathbb{R}^d \tag{4.24}$$

we have

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{C\cdot(1+\log\frac{1}{\delta}+D_2^{2/(1-s)})}\|f\|^2_{L^2(\omega)} + \delta D_1^2. \tag{4.25}$$

The estimate (4.25) differs from the usual form of an uncertainty principle (e.g., those presented in Section 4.1 above) by the appearance of the term δD_1^2. We call this the *error term* since it can be chosen arbitrarily small. However, for small values of δ, which corresponds to a small error term, the constant in the uncertainty term gets (polynomially) large.

In [Mar22], the same result is proved under more technical assumptions, namely that ρ is a Lipschitz contraction with a uniform positive lower bound. On the other hand, the case $s = 1$, which also allows $\mu = 1$, is treated in [Mar22] but is not in the scope of the method we discuss here. Nevertheless, [Mar22] does not present any applications in terms of observability in this case.

Applying Theorem 4.35 to $f = \mathcal{T}(t)g$ with D_1 and D_2 as in (4.23) shows that a semigroup that is smoothing in a Gelfand-Shilov space satisfies the assumptions of Corollary 3.11. This, in turn, proves observability of the semigroup from sensor sets satisfying (4.24) and reproduces [Mar22, Theorem 2.11] with less restrictive conditions on the sensor set. The technical proof of the following corollary is postponed to Subsection A.1.2 in Appendix A.

COROLLARY 4.36. *Suppose that* $(\mathcal{T}(t))_{t\geq 0}$ *is a strongly continuous contraction semigroup that is smoothing in the Gelfand-Shilov space* $S^{\mu}_{\nu}(\mathbb{R}^d)$ *with* $0 \leq \mu < 1$ *and* $\nu > 0$ *satisfying* $\mu + \nu \geq 1$. *Let* $\varepsilon \in [0,1)$ *be such that* $\varepsilon\nu + \mu < 1$ *and let* $\rho\colon \mathbb{R}^d \to (0,\infty)$ *be as in Theorem 4.35. Then* $(\mathcal{T}(t))_{t\geq 0}$ *is observable from every measurable set* $\omega \subset \mathbb{R}^d$ *satisfying* (4.24) *with some* $\gamma \in (0,1]$. *More precisely, in this case we have*

$$\|\mathcal{T}(T)g\|^2_{L^2(\mathbb{R}^d)} \leq C \exp\biggl(\frac{C}{T^{\frac{2r_2}{1-(\varepsilon\nu+\mu)}}}\biggr) \int_0^T \|\mathcal{T}(t)g\|^2_{L^2(\omega)}\,\mathrm{d}t$$

for all $g \in L^2(\mathbb{R}^d)$ *and all* $T > 0$, *where* $C \geq 1$ *is a constant depending on* $\gamma, R, r_0, \nu, \varepsilon, \mu, \nu, C_1, C_2, r_2$, *and the dimension* d.

This result can be considered as a "backup result" since it is applicable for all semigroups that are smoothing in a Gelfand-Shilov space. However, it does not allow, e.g., the sensor set to have a decaying density. In particular, stronger result are available in certain situations, see Theorem 4.42 and the discussion following Theorem 4.43 below.

REMARK 4.37. It is not clear if the class of operators that generate semigroups as in the previous corollary is closed under taking adjoints. For this reason, if one is interested in the null-controllability of the system $\Sigma_c(A, \mathbf{1}_\omega)$, one needs to check whether the semigroup generated by A^* is a strongly continuous contraction semigroup that is smoothing in the Gelfand-Shilov space $S^{\mu}_{\nu}(\mathbb{R}^d)$ with $0 \leq \mu < 1$ and $\nu > 0$. If this is the case, Corollary 4.36 and the duality in Theorem 3.5 establish null-controllability.

4.4. Supplementary results and discussion

We close this chapter by discussing supplementary results which are either simple corollaries or closely related to the ones presented so far. To this end, for brevity, we restrict our considerations to the case of Shubin-type operators $H = -\Delta + |x|^\tau$ with $\tau > 0$ and, in the case of the harmonic oscillator, we only consider sets that are thick with respect to a decaying density and a fixed scale. Furthermore, we track the constants less explicit than we did in Theorem 4.16 above and we do not formulate estimates for the observability constant here.

We first recall that for a nonnegative selfadjoint operator H we have $P_\lambda(H^\theta) = P_{\lambda^{1/\theta}}(H)$ for $\theta > 0$ by the transformation formula for spectral measures. Combining this identity with the spectral inequalities from Theorem 4.16 and Theorem 4.23, respectively, we get the following spectral inequalities for the fractional operators H^θ which are needed in the subsequent discussion.

COROLLARY 4.38. *Let $\theta, \rho > 0$, $\alpha \geq 0$, and $H = -\Delta + |x|^2$. Then there is a constant $C > 0$ such that for all measurable sets $\omega \subset \mathbb{R}^d$ satisfying*

$$\frac{|B(x,\rho) \cap \omega|}{|B(x,\rho)|} \geq \gamma^{1+|x|^\alpha} \quad \text{for all} \quad x \in \mathbb{R}^d \tag{4.26}$$

and some $\gamma \in (0,1]$, we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq \Bigl(\frac{1}{\gamma}\Bigr)^{C\lambda^{(1/2+\alpha/2)/\theta}} \|f\|_{L^2(\omega)}^2 \quad \text{for all} \quad \lambda \geq 1,\ f \in \operatorname{Ran} P_\lambda(H^\theta). \tag{4.27}$$

The exponent in (4.27) satisfies $(1/2 + \alpha/2)/\theta < 1$ if $0 \leq \alpha < 2(\theta - 1/2)$, and in this case we have a spectral inequality.

COROLLARY 4.39. *Let $\theta, \tau > 0$, $\alpha \geq 0$, and $H = -\Delta + |x|^\tau$. Then there is a constant $C > 0$ such that for all $\delta \in (0, 1/2)$, all measurable sets ω satisfying that each intersection $\Lambda_1(k) \cap \omega$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k|^\alpha}$, we have*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \Bigl(\frac{1}{\delta}\Bigr)^{C\cdot\lambda^{(2/3+\alpha/\tau)/\theta}} \|f\|_{L^2(\omega)} \quad \text{for all} \quad \lambda \geq 1,\ f \in \operatorname{Ran} P_\lambda(H^\theta). \tag{4.28}$$

In contrast to the previous corollary, we here need $0 \leq \alpha < \tau(\theta - 2/3)$ in order for the exponent to satisfy $(2/3 + \alpha/\tau)/\theta < 1$, which is again a manifestation of the cut-off procedure used in the proof, cf. the discussion following Theorem 4.23 above.

These corollaries are of particular interest for (anisotropic) Shubin operators $H = -\Delta + |x|^{2l}$ for $l \in \mathbb{N}$. This is motivated by the following characterization of Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^d)$ with

$$\mu = \frac{l\theta}{l+1}, \quad \nu = \frac{\theta}{l+1} \quad \text{for some} \quad \theta \geq 1. \tag{4.29}$$

PROPOSITION 4.40 ([CGPR19, Theorem 1.4]). *Let $H = -\Delta + |x|^{2l}$ and let μ and ν be as in (4.29). Then $f \in S^\mu_\nu(\mathbb{R}^d)$ if and only if there is $s > 0$ such that*

$$\|\mathrm{e}^{sH^{(l+1)/(2l\theta)}} f\|_{L^2(\mathbb{R}^d)} < \infty. \tag{4.30}$$

For $\theta = l = 1$ we have $\mu = \nu = 1/2$ while the usual harmonic oscillator appears in the exponent in (4.30). This suggests that for semigroups that are smoothing in the symmetric Gelfand-Shilov space $S^{1/2}_{1/2}(\mathbb{R}^d)$ (in the sense of Definition 4.34 above), the harmonic oscillator may serve as a suitable selfadjoint comparison operator. In fact, using only this smoothing property, it is possible to adapt the reasoning from the proof of Theorem 4.29 (see Chapter 8 below) to show the following theorem. This can also be extracted from [MPS22, Eq. (4.26)].

THEOREM 4.41. *Let $\theta \in [1, 2)$, $P_\lambda = P_\lambda((-\Delta + |x|^2)^{1/\theta})$, and suppose that the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is smoothing in the symmetric Gelfand-Shilov space $S^{\theta/2}_{\theta/2}(\mathbb{R}^d)$*

in the sense of Definition 4.34. Then there are $C, t_1 > 0$ such that for all $\lambda \geq 1$, all $g \in L^2(\mathbb{R}^d)$, and all $t \in (0, t_1)$ we have

$$\|(1 - P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{t^{r_1+2r_2}} \mathrm{e}^{-t^{2r_2/\theta+1}\lambda} \|g\|_{L^2(\mathbb{R}^d)}, \tag{4.31}$$

where r_1, r_2 are the constants from Definition 4.34.

In combination with the spectral inequality from Corollary 4.38 and the Lebeau-Robbiano method formulated in Corollary 3.9 above, this proves observability for semigroups satisfying the smoothing properties in Theorem 4.41. In light of the stronger spectral inequality for the harmonic oscillator we proved above this improves upon the statement in [MPS22].

THEOREM 4.42. *Let $\theta \in [1, 2)$ and suppose that A is the generator of a semigroup $(\mathcal{T}(t))_{t\geq 0}$ that is smoothing in the Gelfand-Shilov space $S^{\theta/2}_{\theta/2}(\mathbb{R}^d)$. Then the system $\Sigma_o(A, \mathbf{1}_\omega)$ is observable if there are $\rho > 0$, $\gamma \in (0, 1]$, and $0 \leq \alpha < 2(\theta - 1/2)$ such that the sensor set ω satisfies* (4.26).

In the discussion following Proposition 8.1 below we point out that quadratic differential operators A with singular space $S = \{0\}$ generate semigroups that satisfy the assumptions of the previous theorem with $\theta = 1$. Other examples of such semigroups are those generated by general *(an-)isotropic Shubin operators* $A = -((-\Delta)^m + |x|^{2l})^\kappa$ for $m, l \in \mathbb{N}$ and $\kappa > 0$.

THEOREM 4.43 ([Alp, Eq. (2.3)]). *Let $A = -((-\Delta)^m + |x|^{2l})^\kappa$ for $m, l \in \mathbb{N}$ and $\kappa > 0$. Then A generates a semigroup $(\mathcal{T}(t))_{t\geq 0}$ that is smoothing in the Gelfand-Shilov space $S^\mu_\nu(\mathbb{R}^d)$ with*

$$\mu = \max\Big\{\frac{1}{2\kappa m}, \frac{l}{l+m}\Big\} \quad \textit{and} \quad \nu = \max\Big\{\frac{1}{2\kappa l}, \frac{m}{l+m}\Big\}.$$

More precisely, we have (4.20) *with $r_1 = d(m + l)/(2\kappa ml)$ and $r_2 = \min\{\mu, \nu\}$.*

In the isotropic case $l = m$, the semigroup in the above theorem is smoothing in the symmetric Gelfand-Shilov space $S^\mu_\mu(\mathbb{R}^d)$ with $\mu = \max\{1/2, 1/(2\kappa l)\}$. Thus, if $\kappa > 1/(2l)$, then $\mu \in [1/2, 1)$ and we may apply Theorem 4.42 with $\theta = 2\mu \in [1, 2)$ to conclude observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ with ω as in this theorem.

For anisotropic Shubin operators $H = -A = -\Delta + |x|^{2l}$, $l \in \mathbb{N}$, observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ from thick sensor sets ω was shown in [Alp, Theorem 2.6] by establishing a dissipation estimate with respect to the spectral projections of the Laplacian and combining it with the spectral inequality formulated in Theorem 4.3 (i). However, Theorems 4.42 and 4.43 allow also to study observability of the system corresponding to anisotropic Shubin operators using so-called *forced symmetrization*, cf. [Alp]. This argument simply takes into account that every semigroup that is smoothing in the Gelfand-Shilov space $S^\mu_\nu(\mathbb{R}^d)$ is trivially smoothing in the symmetric Gelfand-Shilov space $S^{\max\{\mu,\nu\}}_{\max\{\mu,\nu\}}(\mathbb{R}^d) \supset S^\mu_\nu(\mathbb{R}^d)$. In particular, by Theorem 4.43, the semigroup generated by $A = \Delta - |x|^{2l}$ is smoothing in the Gelfand-Shilov space $S^{l/(l+1)}_{1/(l+1)}(\mathbb{R}^d) \subset S^{l/(l+1)}_{l/(l+1)}(\mathbb{R}^d)$, so that we may apply Theorem 4.42 with $\theta = 2l/(l+1)$. This yields observability of the system $\Sigma_o(A, \mathbf{1}_\omega)$ if ω satisfies (4.26) with $\alpha < 1/l$. However, here larger l requires smaller α which is counterintuitive and stands in strong contrast to the situation we encountered in Corollary 4.39 with $\theta = 1$, where larger l allows for larger α.

In view of the preceding discussion, it appears to be of interest to characterize spaces $\mathcal{G}_l$, $l \in \mathbb{N}$, such that we have a dissipation estimate with respect to the spectral projections of the operator $H = -\Delta + |x|^{2l}$ if the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is smoothing in $\mathcal{G}_l$. Such dissipation estimate in combination with the spectral inequality from Corollary 4.39 would imply observability for these semigroups from the same sensor sets from which the system $\Sigma_o(-H, \mathbf{1}_\omega)$ is observable. However, Proposition 4.40 suggests that for $l \neq 1$ the Gelfand-Shilov spaces are not the right spaces for this task since the power of the operator in (4.30) satisfies $(l+1)/(2l\theta) < 1$ if $l > 1$ and $\theta \geq 1$.

CHAPTER 5

Decay of linear combinations of eigenfunctions

In this chapter we quantify decay properties of linear combinations of eigenfunctions of Schrödinger operators with potentials as in Hypothesis (S). Although there are several results available for eigenfunctions establishing a fast decay in L^2-sense, see, e.g., [Agm82, Dav82, BS91], we need an explicit weighted L^2-estimate also for the partial derivatives of first order. The approach in [GY12] seems to be the most convenient one for this task. However, since it is essential for us to have the dependence of the decay on the spectral parameter explicitly quantified, we have to revisit the reasoning from [GY12] and extract the statements we need.

The main objective of the present chapter is to prove the following result.

THEOREM 5.1. *Let $H = -\Delta + V$ with V as in Hypothesis (S). Then there is a constant $C_0 > 0$, depending only on τ_1, c_1, ν, M_ν, and the dimension d, such that*

$$\|f\|^2_{H^1(\mathbb{R}^d \setminus B(0, C_0\lambda^{1/\tau_1}))} \leq \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)} \tag{5.1}$$

for every $f \in \operatorname{Ran} P_\lambda(H)$, $\lambda \geq 1$.

REMARK 5.2. If desired, the dependence of C_0 in Theorem 5.1 on τ_1, c_1, ν, M_ν can be traced explicitly from the proof. We refrain from doing so here for simplicity and brevity.

Throughout this chapter, let $H = -\Delta + V$ where V is as in Hypothesis (S). We use the parameters c_1, c_2, τ_1, and M_ν introduced in the hypothesis and we denote by $\mathfrak{h} = \mathfrak{a} + \mathfrak{v}$ the form associated to H.

5.1. Weighted inequalities

We prove Theorem 5.1 by establishing weighted L^2-estimates for functions and gradients of functions in the spectral subspace of H with an exponential weight. As a preparation, we need the following lemma which, together with its proof, is essentially taken from [GY12, Lemma 2.1].

LEMMA 5.3. *Suppose that for some $\phi \in L^2(\mathbb{R}^d)$ and $\lambda \geq 0$ the function $f \in H^2_{\mathrm{loc}}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ satisfies $-\Delta f + Vf - \lambda f = \phi$ almost everywhere. Then, $f \in \mathcal{D}[\mathfrak{h}]$ and for all $g \in \mathcal{D}[\mathfrak{h}]$ we have $\mathfrak{h}[f, g] - \lambda\langle f, g\rangle_{L^2(\mathbb{R}^d)} = \langle \phi, g\rangle_{L^2(\mathbb{R}^d)}$.*

PROOF. Consider the function $\chi\colon \mathbb{R}^d \to [0, 1]$ with $\chi(x) = (1 - |x|^2)^2$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| \geq 1$, and define $\chi_\varepsilon\colon \mathbb{R}^d \to [0, 1]$ for $0 < \varepsilon < 1/2$

by $\chi_\varepsilon(x) = \chi(\varepsilon x)$. Observe that $\chi_\varepsilon \to 1$ pointwise and monotonically as $\varepsilon \to 0$. Moreover, χ_ε vanishes outside of $B(0, 1/\varepsilon)$, in particular on the boundary $\partial B(0, 1/\varepsilon)$. With $\nabla(\chi_\varepsilon f) = \chi_\varepsilon \nabla f + f \nabla \chi_\varepsilon$, integration by parts therefore gives

$$\langle -\Delta f, \chi_\varepsilon f \rangle_{L^2(B(0,1/\varepsilon))} = \langle |\nabla f|, \chi_\varepsilon |\nabla f| \rangle_{L^2(B(0,1/\varepsilon))} + \langle \nabla f \cdot \nabla \chi_\varepsilon, f \rangle_{L^2(B(0,1/\varepsilon))},$$

so that

$$\begin{aligned} &\langle |\nabla f|, \chi_\varepsilon |\nabla f| \rangle_{L^2(B(0,1/\varepsilon))} + \langle V f, \chi_\varepsilon f \rangle_{L^2(B(0,1/\varepsilon))} \\ &\qquad = \langle -\Delta f + V f, \chi_\varepsilon f \rangle_{L^2(B(0,1/\varepsilon))} - \langle \nabla f \cdot \nabla \chi_\varepsilon, f \rangle_{L^2(B(0,1/\varepsilon))} \\ &\qquad = \langle \lambda f + \phi, \chi_\varepsilon f \rangle_{L^2(B(0,1/\varepsilon))} - \langle \nabla f \cdot \nabla \chi_\varepsilon, f \rangle_{L^2(B(0,1/\varepsilon))}. \end{aligned} \tag{5.2}$$

Now, for $x \in B(0, 1/\varepsilon)$ and $j = 1, \dots, d$ we have $\partial_j \chi_\varepsilon(x) = -4\varepsilon^2 x_j \chi_\varepsilon(x)^{1/2}$, that is,

$$|\nabla \chi_\varepsilon(x)| = 4\varepsilon^2 |x| \chi_\varepsilon(x)^{1/2} \leq 4\varepsilon \chi_\varepsilon(x)^{1/2}.$$

Thus,

$$\begin{aligned} |\langle \nabla f \cdot \nabla \chi_\varepsilon, f \rangle_{L^2(B(0,1/\varepsilon))}| &\leq \|f\|_{L^2(B(0,1/\varepsilon))} \| |\nabla f| |\nabla \chi_\varepsilon| \|_{L^2(B(0,1/\varepsilon))} \\ &\leq 4\varepsilon \|f\|_{L^2(B(0,1/\varepsilon))} \| |\nabla f| \chi_\varepsilon^{1/2} \|_{L^2(B(0,1/\varepsilon))} \\ &\leq 2\varepsilon \big(\|f\|^2_{L^2(B(0,1/\varepsilon))} + \| |\nabla f| \chi_\varepsilon^{1/2} \|^2_{L^2(B(0,1/\varepsilon))} \big) \\ &= 2\varepsilon \big(\|f\|^2_{L^2(B(0,1/\varepsilon))} + \langle |\nabla f|, \chi_\varepsilon |\nabla f| \rangle_{L^2(B(0,1/\varepsilon))} \big). \end{aligned}$$

Plugging the latter into (5.2) implies

$$\begin{aligned} &(1 - 2\varepsilon) \Big(\int_{\mathbb{R}^d} \chi_\varepsilon |\nabla f|^2 + \int_{\mathbb{R}^d} \chi_\varepsilon V |f|^2 \Big) \\ &\qquad \leq (1 - 2\varepsilon) \langle |\nabla f|, \chi_\varepsilon |\nabla f| \rangle_{L^2(B(0,1/\varepsilon))} + \langle V f, \chi_\varepsilon f \rangle_{L^2(B(0,1/\varepsilon))} \\ &\qquad \leq |\langle \lambda f + \phi, \chi_\varepsilon f \rangle_{L^2(B(0,1/\varepsilon))}| + 2\varepsilon \|f\|^2_{L^2(B(0,1/\varepsilon))} \\ &\qquad \leq \|\phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} + (\lambda + 2\varepsilon) \|f\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

that is,

$$\int_{\mathbb{R}^d} \chi_\varepsilon |\nabla f|^2 + \int_{\mathbb{R}^d} \chi_\varepsilon V |f|^2 \leq \frac{1}{1 - 2\varepsilon} \Big(\|\phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} + (\lambda + 2\varepsilon) \|f\|^2_{L^2(\mathbb{R}^d)} \Big)$$

for all $\varepsilon > 0$. Since the right-hand side of the last inequality is uniformly bounded as $\varepsilon \to 0$, it follows from the monotone convergence theorem that indeed $f \in \mathcal{D}[\mathfrak{h}]$ with

$$\mathfrak{h}[f, f] \leq \|\phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} + \lambda \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Finally, for $g \in C_c^\infty(\mathbb{R}^d)$ integration by parts shows

$$\begin{aligned} \mathfrak{h}[f, g] = \mathfrak{a}[f, g] + \mathfrak{v}[f, g] &= \int_{\mathbb{R}^d} (-\Delta f + V f) \overline{g} = \int_{\mathbb{R}^d} (\phi + \lambda f) \overline{g} \\ &= \langle \phi + \lambda f, g \rangle_{L^2(\mathbb{R}^d)}, \end{aligned}$$

and the latter extends to all $g \in \mathcal{D}[\mathfrak{h}]$ by approximation since $C_c^\infty(\mathbb{R}^d)$ is a form core for H by Lemma 2.5. □

The next result is now at the core of our proof of Theorem 5.1 and is a quantitative version of the statement in [GY12, Lemma 2.3]. Its proof is also extracted from that reference.

LEMMA 5.4. *Let $\lambda \geq 0$, $\mu > 0$, and $R \geq 1$ be such that $V(x) \geq \mu^2 + \lambda + 1$ whenever $|x| \geq R$. Moreover, suppose that the function $f \in H^2_{\mathrm{loc}}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ satisfies $-\Delta f + Vf - \lambda f = \phi$ almost everywhere with some $\phi \in L^2(\mathbb{R}^d)$. Then, if $\mathrm{e}^{2\mu|x|}\phi \in L^2(\mathbb{R}^d)$, we have*

$$\|\mathrm{e}^{\mu|x|} f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{1}{2}\|\mathrm{e}^{2\mu|x|}\phi\|^2_{L^2(\mathbb{R}^d\setminus B(0,R))} + (4\mu+6)\mathrm{e}^{2\mu(R+1)}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{5.3}$$

PROOF. According to Lemma 5.3, we have $f \in \mathcal{D}[\mathfrak{h}]$. Suppose first that f is real-valued and choose an infinitely differentiable function $\chi\colon \mathbb{R}^d \to [0,1]$ with $\chi(x) = 0$ for $|x| \leq R$ and $\chi(x) = 1$ for $|x| \geq R+1$ such that $\||\nabla\chi|\|_{L^\infty(\mathbb{R}^d)} \leq 2$. For $\varepsilon > 0$ let $w(x) = w_\varepsilon(x) = \mu|x|/(1+\varepsilon|x|)$. Then w is bounded and infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$. Accordingly, the same is true for $\chi\mathrm{e}^w$ and $\chi\mathrm{e}^{2w}$. Therefore, $\chi\mathrm{e}^{2w}f$, $\chi^2\mathrm{e}^{2w}f$, and $g := \chi\mathrm{e}^w f$ are all real-valued, belong to $\mathcal{D}[\mathfrak{h}]$, and vanish in the ball $B(0,R)$. In particular, the choice of R implies that $\mathfrak{v}[g,g] \geq (\mu^2+\lambda+1)\|g\|^2_{L^2(\mathbb{R}^d)}$. Moreover, with the relation $\nabla(\mathrm{e}^{\pm w}g) = \mathrm{e}^{\pm w}\nabla g \pm g\mathrm{e}^{\pm w}\nabla w$ and the identity $\||\nabla w|\|_{L^\infty(\mathbb{R}^d\setminus\{0\})} = \mu$ we obtain

$$\nabla(\mathrm{e}^{-w}g)\cdot\nabla(\mathrm{e}^w g) = |\nabla g|^2 - |g|^2|\nabla w|^2 \geq -\mu^2|g|^2,$$

so that

$$\mathfrak{h}[\chi f, \chi\mathrm{e}^{2w}f] = \mathfrak{h}[\mathrm{e}^{-w}g, \mathrm{e}^w g] = \mathfrak{a}[\mathrm{e}^{-w}g, \mathrm{e}^w g] + \mathfrak{v}[g,g] \geq (\lambda+1)\|g\|^2_{L^2(\mathbb{R}^d)},$$

or, in other words,

$$\|\chi\mathrm{e}^w f\|^2_{L^2(\mathbb{R}^d)} \leq \mathfrak{h}[\chi f, \chi\mathrm{e}^{2w}f] - \lambda\langle f, \chi^2\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)}. \tag{5.4}$$

Clearly, $\mathfrak{v}[\chi f, \chi\mathrm{e}^{2w}f] = \mathfrak{v}[f, \chi^2\mathrm{e}^{2w}f]$. Furthermore, a straightforward computation shows $\nabla(\chi f)\cdot\nabla(\chi\mathrm{e}^{2w}f) = \nabla f\cdot\nabla(\chi^2\mathrm{e}^{2w}f) + \eta\mathrm{e}^{2w}|f|^2$ with

$$\eta := 2\chi\nabla\chi\cdot\nabla w + |\nabla\chi|^2. \tag{5.5}$$

Taking into account Lemma 5.3 with $g = \chi^2\mathrm{e}^{2w}f$, we therefore have

$$\begin{aligned}\mathfrak{h}[\chi f, \chi\mathrm{e}^{2w}f] &= \mathfrak{h}[f, \chi^2\mathrm{e}^{2w}f] + \langle f, \eta\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)}\\ &= \langle \phi + \lambda f, \chi^2\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)} + \langle f, \eta\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Plugging the latter into (5.4) gives

$$\begin{aligned}\|\chi\mathrm{e}^w f\|^2_{L^2(\mathbb{R}^d)} &\leq \langle \phi, \chi^2\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)} + \langle f, \eta\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)}\\ &= \langle \chi^2\mathrm{e}^{2w}\phi, f\rangle_{L^2(\mathbb{R}^d)} + \langle f, \eta\mathrm{e}^{2w}f\rangle_{L^2(\mathbb{R}^d)}\\ &\leq \|\chi^2\mathrm{e}^{2w}\phi\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)} + \|\eta\mathrm{e}^{2w}\|_{L^\infty(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{5.6}$$

The function η in (5.5) vanishes outside of the annulus $R < |x| < R+1$ and satisfies $|\eta| \leq 2|\nabla\chi||\nabla w| + |\nabla\chi|^2 \leq 4(\mu+1)$. Hence,

$$\|\eta \mathrm{e}^{2w}\|_{L^\infty(\mathbb{R}^d)} \leq 4(\mu+1)\mathrm{e}^{2\mu(R+1)}.$$

We thus conclude from (5.6) that

$$\begin{aligned}\|\mathrm{e}^w f\|_{L^2(\mathbb{R}^d)}^2 &= \|\mathrm{e}^w f\|_{L^2(B(0,R+1))}^2 + \|\mathrm{e}^w f\|_{L^2(\mathbb{R}^d\setminus B(0,R+1))}^2\\ &\leq \mathrm{e}^{2\mu(R+1)}\|f\|_{L^2(\mathbb{R}^d)}^2 + \|\chi \mathrm{e}^w f\|_{L^2(\mathbb{R}^d)}^2\\ &\leq \|\chi^2\mathrm{e}^{2w}\phi\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)} + (4\mu+5)\mathrm{e}^{2\mu(R+1)}\|f\|_{L^2(\mathbb{R}^d)}^2\\ &\leq \|\mathrm{e}^{2w}\phi\|_{L^2(\mathbb{R}^d\setminus B(0,R))}\|f\|_{L^2(\mathbb{R}^d)} + (4\mu+5)\mathrm{e}^{2\mu(R+1)}\|f\|_{L^2(\mathbb{R}^d)}^2\\ &\leq \frac{1}{2}\|\mathrm{e}^{2w}\phi\|_{L^2(\mathbb{R}^d\setminus B(0,R))}^2 + (4\mu+6)\mathrm{e}^{2\mu(R+1)}\|f\|_{L^2(\mathbb{R}^d)}^2,\end{aligned}$$

where we used Young's inequality for products for the last estimate. Since $w(x) = w_\varepsilon(x) \to \mu|x|$ as $\varepsilon \to 0$ pointwise and monotonically, (5.3) now follows by the monotone convergence theorem.

If f is not real-valued, we proceed analogously for $\operatorname{Re} f$ and $\operatorname{Im} f$ separately and combine the obtained inequalities to arrive again at (5.3). ☐

Applying Lemma 5.4 with $\phi = 0$ allows us to obtain the desired weighted L^2-estimates for eigenfunctions of H, where R can be computed from λ and the constants in part (i) of Hypothesis (S).

PROPOSITION 5.5. *Suppose that $f \in \mathcal{D}(H)$ with $Hf = \lambda f$ for some $\lambda \geq 0$, and choose $R \geq 1$ such that $R^{\tau_1} \geq (\lambda+2)/c_1$. Then*

$$\|\mathrm{e}^{|x|/2} f\|_{L^2(\mathbb{R}^d)}^2 \leq 7\mathrm{e}^{R+1}\|f\|_{L^2(\mathbb{R}^d)}^2.$$

PROOF. According to Lemma 2.6, we have $f \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $-\Delta f + Vf - \lambda f = 0$ almost everywhere. Applying Lemma 5.4 with $\mu = 1/2$ and $\phi = 0$ therefore proves the claim. ☐

In order to obtain by means of Lemma 5.4 an analogous result for the partial derivatives of an eigenfunction, we follow the approach of [GY12] and differentiate the eigenvalue equation $Hf = \lambda f$. Indeed, since $Hf \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$, we know that, in fact, $f \in H^3_{\mathrm{loc}}(\mathbb{R}^d)$ by elliptic regularity, see Lemma 2.6, and it follows that each $\partial_j f \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ with $j = 1, \dots, d$ satisfies

$$-\Delta\partial_j f + V\partial_j f - \lambda\partial_j f = -f\partial_j V \tag{5.7}$$

almost everywhere. This allows to apply Lemma 5.4 to $\partial_j f$ with a corresponding right-hand side and, thus, leads to the following result.

PROPOSITION 5.6. *Let $f \in \mathcal{D}(H)$ with $Hf = \lambda f$ for some $\lambda \geq 0$, and choose $R \geq 1$ such that $R^{\tau_1} \geq ((\nu+1)^2 + \lambda + 1)/c_1$. Then, we have*

$$\|\mathrm{e}^{|x|/2}|\nabla f|\|_{L^2(\mathbb{R}^d)}^2 \leq \big(8\lambda + (2\nu+5)M_\nu^2\big)\mathrm{e}^{2(1+\nu)(R+1)}\|f\|_{L^2(\mathbb{R}^d)}^2.$$

PROOF. Denote by $\phi_j := -f\partial_j V$ the right-hand side of (5.7). Then, in light of the hypothesis on R, we may first apply Lemma 5.4 to f with $\mu = \nu + 1$ and $\phi = 0$ to obtain

$$\|\mathrm{e}^{(1+\nu)|x|} f\|_{L^2(\mathbb{R}^d)}^2 \leq (4\nu + 9)\mathrm{e}^{2(1+\nu)(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2. \tag{5.8}$$

Since $|\phi_j(x)| \leq M_\nu \mathrm{e}^{\nu|x|}|f|$ on $\mathbb{R}^d \setminus B(0,1)$ by part (ii) of Hypothesis (S), we conclude that $\mathrm{e}^{|x|}\phi_j \in L^2(\mathbb{R}^d \setminus B(0,1))$. In view of (5.7), we may then again apply Lemma 5.4, this time to $\partial_j f$ with $\mu = 1/2$ and $\phi = \phi_j = -f\partial_j V$, which gives

$$\|\mathrm{e}^{|x|/2}\partial_j f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{2}\|\mathrm{e}^{|x|}\phi_j\|_{L^2(\mathbb{R}^d \setminus B(0,1))}^2 + 8\mathrm{e}^{R+1}\|\partial_j f\|_{L^2(\mathbb{R}^d)}^2. \tag{5.9}$$

Taking into account (5.8) and that

$$\||\nabla f|\|_{L^2(\mathbb{R}^d)}^2 = \mathfrak{a}[f,f] \leq \mathfrak{h}[f,f] = \langle Hf, f\rangle_{L^2(\mathbb{R}^d)} = \lambda\|f\|_{L^2(\mathbb{R}^d)}^2,$$

summing over $j = 1, \dots, d$ then yields

$$\begin{aligned}
\|\mathrm{e}^{|x|/2}|\nabla f|\|_{L^2(\mathbb{R}^d)}^2 &\leq \frac{1}{2}\|\mathrm{e}^{|x|}f|\nabla V|\|_{L^2(\mathbb{R}^d\setminus B(0,1))}^2 + 8\mathrm{e}^{R+1}\||\nabla f|\|_{L^2(\mathbb{R}^d)}^2 \\
&\leq \frac{M_\nu^2}{2}\|\mathrm{e}^{(1+\nu)|x|}f\|_{L^2(\mathbb{R}^d\setminus B(0,1))}^2 + 8\lambda\mathrm{e}^{R+1}\|f\|_{L^2(\mathbb{R}^d)}^2 \\
&\leq \big(8\lambda + (2\nu+5)M_\nu^2\big)\mathrm{e}^{2(1+\nu)(R+1)}\|f\|_{L^2(\mathbb{R}^d)}^2,
\end{aligned}$$

which proves the claim. ☐

5.2. Localization of linear combinations of eigenfunctions

Recall from the discussion before Corollary 2.4 that H has purely discrete spectrum, and let $(\lambda_k)_{k\in\mathbb{N}}$ be an enumeration of its spectrum $\sigma(H)$ in nondecreasing order (without multiplicities). With

$$N(\lambda) := \#(\sigma(H) \cap (-\infty, \lambda]),$$

we may then expand every $f \in \operatorname{Ran} P_H(\lambda)$ as

$$f = \sum_{k=1}^{N(\lambda)} f_k \tag{5.10}$$

where $f_k = P_H(\{\lambda_k\})f$ for $k \in \{1, \dots, N(\lambda)\}$. We have the simple bound

$$N(\lambda) \leq \#\{k\colon \lambda_k \leq \lambda\} \leq \sum_{k\colon \lambda_k \leq \lambda} (\lambda + 1 - \lambda_k) \leq \sum_{k\colon \lambda_k \leq \lambda+1} (\lambda + 1 - \lambda_k)$$

and in light of the lower bound $V(x) \geq c_1|x|^{\tau_1}$ on the potential in part (i) of Hypothesis (S), the right-hand side can be estimated explicitly by means of the classic Lieb-Thirring bound from [LT91, Theorem 1]. More precisely, for $\lambda \geq 1$ this theorem yields

$$\begin{aligned}\sum_{k\colon \lambda_k \leq \lambda+1} (\lambda + 1 - \lambda_k) &\lesssim_d \int_{\mathbb{R}^d} \max\{\lambda + 1 - V(x), 0\}^{d/2+1} \, \mathrm{d}x \\ &\leq \int_{B(0,((\lambda+1)/c_1)^{1/\tau_1})} (\lambda+1)^{d/2+1} \, \mathrm{d}x \\ &\lesssim_d \left(\frac{2}{c_1}\right)^{d/\tau_1} \lambda^{1+d(1/2+1/\tau_1)},\end{aligned}$$

and, therefore,

$$N(\lambda) \leq K_d \Big(\frac{2}{c_1}\Big)^{d/\tau_1} \lambda^{1+d(1/2+1/\tau_1)} \leq \mathrm{e}^{C_1 \lambda^{1/\tau_1}} \tag{5.11}$$

for some constant $C_1 > 0$ depending only on c_1, τ_1, and the dimension d.

Remark 5.7. The Lieb-Thirring bound actually also takes into account multiplicities. It is worth to mention that for $d \geq 3$ the classic Cwikel-Lieb-Rozenblum bound provides a sharper estimate for $N(\lambda)$, but the above is more than sufficient for our purposes.

We are now in position to prove the main result of this chapter.

Proof of Theorem 5.1. For every $r > 0$, we have

$$\begin{aligned}\|f\|^2_{H^1(\mathbb{R}^d \setminus B(0,r))} &\leq \|\mathrm{e}^{-|x|/2}\|^2_{L^\infty(\mathbb{R}^d\setminus B(0,r))} \cdot \big(\|\mathrm{e}^{|x|/2} f\|^2_{L^2(\mathbb{R}^d)} + \|\mathrm{e}^{|x|/2}|\nabla f|\|^2_{L^2(\mathbb{R}^d)}\big) \\ &\leq \mathrm{e}^{-r}\big(\|\mathrm{e}^{|x|/2} f\|^2_{L^2(\mathbb{R}^d)} + \|\mathrm{e}^{|x|/2}|\nabla f|\|^2_{L^2(\mathbb{R}^d)}\big).\end{aligned}$$

Moreover, using the expansion (5.10) and Hölder's inequality, we may estimate

$$\|\mathrm{e}^{|x|/2} f\|^2_{L^2(\mathbb{R}^d)} \leq \left(\sum_{k=1}^{N(\lambda)} \|\mathrm{e}^{|x|/2} f_k\|_{L^2(\mathbb{R}^d)}\right)^2 \leq N(\lambda) \sum_{k=1}^{N(\lambda)} \|\mathrm{e}^{|x|/2} f_k\|^2_{L^2(\mathbb{R}^d)}$$

and similarly, taking into account $|\nabla f| \leq \sum_{k=1}^{N(\lambda)} |\nabla f_k|$,

$$\|\mathrm{e}^{|x|/2}|\nabla f|\|^2_{L^2(\mathbb{R}^d)} \leq N(\lambda) \sum_{k=1}^{N(\lambda)} \|\mathrm{e}^{|x|/2}|\nabla f_k|\|^2_{L^2(\mathbb{R}^d)}.$$

We choose

$$R := ((\nu+1)^2 + \lambda + 1)^{1/\tau_1} \leq 3^{1/\tau_1}(1+\nu)^{2/\tau_1}\lambda^{1/\tau_1}/c_1 \lesssim_{\nu,\tau_1,c_1} \lambda^{1/\tau_1},$$

which meets the requirement on R in both Propositions 5.5 and 5.6 for all eigenfunctions corresponding to eigenvalues not exceeding λ. In particular, this is the case for the functions f_k in the expansion (5.10). Since $\sum_{k=1}^{N(\lambda)} \|f_k\|^2_{L^2(\mathbb{R}^d)} = \|f\|^2_{L^2(\mathbb{R}^d)}$

and in light of the bound for $N(\lambda)$ in (5.11), applying Propositions 5.5 and 5.6 for each f_k separately therefore implies that

$$\|\mathrm{e}^{|x|/2}f\|_{L^2(\mathbb{R}^d)}^2 + \|\mathrm{e}^{|x|/2}|\nabla f|\|_{L^2(\mathbb{R}^d)}^2 \leq \mathrm{e}^{C_2\lambda^{1/\tau_1}}\|f\|_{L^2(\mathbb{R}^d)}^2$$

for some constant $C_2 > 0$ depending only on c_1, τ_1, ν, M_ν, and d. Hence,

$$\|f\|_{H^1(\mathbb{R}^d\setminus B(0,r))}^2 \leq \mathrm{e}^{-r}\mathrm{e}^{C_2\lambda^{1/\tau_1}}\|f\|_{L^2(\mathbb{R}^d)}^2$$

and choosing $r := \log 2 + C_2\lambda^{1/\tau_1} \leq (C_2 + \log 2)\lambda^{1/\tau_1}$ then proves the claim with the constant $C_0 = C_2 + \log 2$. □

We end this chapter by stating a simple qualitative result which we also obtain from Lemma 5.4 with the same technique as exercised above. This is needed in certain situations in Subsection 6.4.2 below only because we have no explicit representation for the domain $\mathcal{D}(H)$ at our disposal.

PROPOSITION 5.8. *If $\|\mathrm{e}^{-\mu|x|}V\|_{L^\infty(\mathbb{R}^d)} < \infty$ for some $\mu > 0$, then there is a constant $C_3 > 0$ depending on V, μ, λ, and the dimension d, such that*

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq C_3\|f\|_{L^2(\mathbb{R}^d)} \tag{5.12}$$

for all $f \in \operatorname{Ran} P_\lambda(H)$, $\lambda \geq 0$. In particular, for each such f we have $Vf \in L^2(\mathbb{R}^d)$ and $f \in H^2(\mathbb{R}^d)$ with $Hf = -\Delta f + Vf$.

PROOF. By Lemma 2.7 it suffices to show (5.12). However, applying Lemma 5.4 we easily see

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq \|\mathrm{e}^{-\mu|x|}V\|_{L^\infty(\mathbb{R}^d)}\|\mathrm{e}^{\mu|x|}f\|_{L^2(\mathbb{R}^d)} \leq C_3\|f\|_{L^2(\mathbb{R}^d)} < \infty. \qquad \square$$

CHAPTER 6

Spectral inequalities based on Carleman estimates

The goal of this chapter is to give the proofs of Theorems 4.10 and 4.23 which are both based on Carleman estimates. In the first Section 6.1, we give a short presentation of certain quantitative unique continuation estimates that are based on Carleman estimates and can be seen as predecessor results to the aforementioned theorems. We discuss properties of the potentials we work with in Section 6.2 and, thereafter, in Section 6.3, we deduce two Carleman estimates valid for Schrödinger operators with singular admissible potentials by perturbing existing Carleman estimates for the pure Laplacian. The main results are then proven in Section 6.4: In Subsection 6.4.1 we first verify covering inequalities that follow from the Carleman estimates proven in Section 6.3 for functions satisfying some differential equation almost everywhere. From these we conclude an interpolation inequality that is applied to specifically constructed eigenfunctions of a Schrödinger operator on $\mathbb{R}^{d+1}$, which are related to elements in the spectral subspace by the so-called *ghost dimension* framework we recall in Subsection 6.4.2. Using this framework we finally conclude the proofs of the main results in Subsection 6.4.3.

6.1. The Carleman approach to spectral inequalities

By unique continuation properties of differential operators one refers to the fact that solutions of certain differential equations must vanish identically if they vanish on a nonempty open subset. For operators with analytic coefficients, such results were first obtained already in [Hol01]. The case of operators with non-analytic coefficients was studied in two dimensions by Carleman [Car45] using weighted inequalities that nowadays are known as *Carleman estimates*. This method was subsequently generalized by several authors, see the introduction in [Hör63, Chapter VIII] for further references. Carleman estimates have found a wide variety of applications, e.g., in inverse problems [Yam09, Isa17], unique continuation [Ken87], control theory [LR95, FI96], uniqueness of Cauchy problems [Zui83], and in the theory of random Schrödinger operators [BK05], see also [BK13, RMV13, GK13]. We refer to the book [Ler19] for an exhaustive discussion of Carleman estimates and their applications.

Quantitative forms of unique continuation properties using Carleman estimates go back at least to the works [LR95, LZ98, JL99], wherein the following spectral inequality for the Laplacian is proven. Here one has no control over the constant C that encodes the geometric properties of the set ω.

THEOREM 6.1 ([LZ98, JL99], see also [LR95]). *Let* $\Omega \subset \mathbb{R}^d$ *be bounded and let* $\omega \subset \Omega$ *be open. Then, there is a constant* $C > 0$ *such that for all* $\lambda \geq 0$ *and all* $f \in \operatorname{Ran} P_\lambda(-\Delta_\Omega^D)$ *we have*

$$\|f\|_{L^2(\Omega)}^2 \leq C\mathrm{e}^{C\lambda^{1/2}} \|f\|_{L^2(\omega)}^2.$$

A lot of progress in quantitative unique continuation based on Carleman estimates has been made due to its applications in the theory of random Schrödinger operators, where such results were first used in the highly influential paper [BK05]. In this setting, the sensor set is usually an equidistributed set (since this models the position or influence of ions, cf., e.g., the exposition in [Sto01]) and also quantitative unique continuation estimates for individual eigenfunctions are of interest since these already imply a so-called *eigenvalue lifting*, see, for instance, [NTTV18, Corollary 2.6] and the excursus in Appendix B below.

In order to present results preceding our Theorem 4.10, we now recall the main result from [RMV13].

THEOREM 6.2. *Let* $\delta \in (0, 1/2)$ *and let* $\omega \subset \mathbb{R}^d$ *be* $(1, \delta)$*-equidistributed as in Definition 4.6. Then for all* $L \in \mathbb{N}$*, all real valued* $V \in L^\infty(\mathbb{R}^d)$*, and all* $f \in \mathcal{D}(\Delta_{\Lambda_L}^D) \cup \mathcal{D}(\Delta_{\Lambda_L}^{\mathrm{per}})$ *satisfying the differential inequality* $|\Delta f| \leq |Vf|$ *almost everywhere on* Λ_L*, we have*

$$\|f\|_{L^2(\Lambda_L)}^2 \leq \Big(\frac{1}{\delta}\Big)^{K_d \cdot (1 + \|V\|_\infty^{2/3})} \|f\|_{L^2(\Lambda_L \cap \omega)}^2.$$

While this theorem holds for eigenfunctions f of the Schrödinger operator $H = -\Delta + V$, it is not applicable for all functions in a spectral subspace and therefore no spectral inequality. Furthermore, even for eigenfunctions f corresponding to an eigenvalue $\lambda \geq 0$, the exponent takes the form $K_d \cdot (1 + \|V\|_\infty^{2/3} + \lambda^{2/3})$ and therefore the power $2/3$ of the eigenvalue differs from the expected $1/2$ we get for the pure Laplacian ($V \equiv 0$) and which is known to be sharp in this case. This is due to the fact that instead of working with the parameter λ separately, one here considers f as an eigenfunction (with eigenvalue 0) of the Schrödinger operator $\tilde{H} = -\Delta + (V - \lambda)$, so that λ inherits the exponent $2/3$ from the potential-term. Actually, a similar phenomenon occurs in the proof of Theorem 4.23 below.

A first approach to generalize Theorem 6.2 to elements in the spectral subspace was given in [Kle13, Theorem 1.1] and subsequently generalized in [KT16, Theorem 1.2] to certain singular potentials. For brevity, we only recall the result from the last mentioned reference since it agrees with [Kle13, Theorem 1.1] in the case of bounded potentials.

THEOREM 6.3 ([KT16, Theorem 1.2]). *Let* $p \geq d$ *for* $d \geq 3$*,* $p > 2$ *for* $d = 2$*, and* $p \geq 2$ *for* $d = 1$*. Suppose that* $V = V_1 + V_2$ *with some* $V_1 \in L^\infty(\mathbb{R}^d)$*,* $V_2 \in L^p(\mathbb{R}^d)$*, and consider the Schrödinger operator* $H = -\Delta + V$ *on* $L^2(\mathbb{R}^d)$*. Let* $q = 4p^2/((3p - d)(2p - d))$ *for* $d \geq 2$ *and let* $q = 2p^2/((3p - 4)(p - 1))$ *for* $d = 1$*.*

Then, there is a constant $C > 0$, depending only on the dimension d, such that if for $\delta \in (0, 1/2)$ and $\lambda > 0$ we set

$$\eta^2 = \frac{1}{2}\delta^{C\cdot(1+(\|V_1\|_\infty+\|V_2\|_p+\lambda)^q)},$$

then for all finite rectangles $\Lambda = \Lambda_a(x_0)$ with $a \in (114\sqrt{d}, \infty)^d$ and $x_0 \in \mathbb{R}^d$ and all $(1, \delta)$-equidistributed sets $\omega \subset \mathbb{R}^d$ we have

$$\|f\|^2_{L^2(\Lambda)} \leq 4 \cdot \Big(\frac{1}{\delta}\Big)^{2C\cdot\big(1+(\|V_1\|_\infty+\|V_2\|_p+\lambda)^q\big)} \|f\|^2_{L^2(\Lambda\cap\omega)}$$

for all intervals $I \subset (-\infty, \lambda]$ with $|I| \leq 2\eta$ and all $f \in \operatorname{Ran} \mathbf{1}_I(H^\bullet_\Lambda)$, $\bullet \in \{D, \mathrm{per}\}$.

REMARK 6.4. Since $q \to 2/3$ as $p \to \infty$, setting $q = 2/3$ if $p = \infty$ we recover [Kle13, Theorem 1.1].

The reason why this theorem merely allows short energy intervals I is that the proof is based on an unique continuation estimate with an additional error term (which, in case of bounded potentials, essentially follows from [RMV13] with minor adjustments) and the observation that for $I = [\mu - \eta, \mu + \eta]$ this error term $\|(H^\bullet_\Lambda - \mu)f\|_{L^2(\Lambda)}$ is bounded by $\eta\|f\|_{L^2(\Lambda)}$ and can therefore be subsumed into the norm on the left-hand side.

The result from [Kle13] (for bounded potentials) was generalized to arbitrary linear combinations of eigenfunctions, that is, to a spectral inequality, in [NTTV18] for Schrödinger operators on finite cubes and in [NTTV20b] for Schrödinger operators on generalized rectangles, see Theorem 4.7 above for the most general of these results in the case of operators on the whole of $\mathbb{R}^d$. Here, while the proofs in [RMV13, Kle13, KT16] apply only a single Carleman estimate to the eigenfunction itself, [NTTV18, NTTV20b] is based on two Carleman estimates (with and without a boundary term) applied to a function that is constructed in a way introduced in [JL99] from an element in the spectral subspace of the operator. The Carleman estimate with the boundary term allows to come back to the original function whilst the Carleman estimate without the boundary term is used to perform the actual step from the equidistributed sensor set to the whole generalized rectangle.

We follow the same path in the proofs of Theorems 4.10 and 4.23 below. For the former, we prove generalizations of the two aforementioned Carleman estimates that feature singular admissible singular potentials and we also generalize the resulting interpolation inequality using these Carleman estimates. For the latter, we use Theorem 5.1 to essentially reduce our considerations to suitable regions where the a priori unbounded potential behaves like a bounded one.

6.2. Basic properties of the Schrödinger operators

In this section we investigate properties of the potentials and their associated Schrödinger operators we deal with in this chapter. First, we provide some properties of singular admissible potentials that are at the core of our considerations.

LEMMA 6.5. *Let* $V\colon \mathbb{R}^d \to \mathbb{R}$ *be singular admissible in the sense of Definition 4.9. Then:*

(a) V *is infinitesimally operator bounded with respect to the Laplacian* Δ *in* $L^2(\mathbb{R}^d)$*. More precisely, there are constants* $a, b \geq 0$ *such that*

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq a\varepsilon\|\Delta f\|_{L^2(\mathbb{R}^d)} + \Big(\frac{a}{\varepsilon} + b\Big)\|f\|_{L^2(\mathbb{R}^d)} \tag{6.1}$$

for all $f \in \mathcal{D}(\Delta)$ *and all* $\varepsilon > 0$*.*

(b) There are constants $\lambda_1, \lambda_2 \geq 0$ *such that*

$$\|Vf\|^2_{L^2(\mathbb{R}^d)} \leq \lambda_1\|\nabla f\|^2_{L^2(\mathbb{R}^d)} + \lambda_2\|f\|^2_{L^2(\mathbb{R}^d)} \tag{6.2}$$

for all $f \in H^1(\mathbb{R}^d)$*.*

PROOF. (a) We clearly have $\mathcal{D}((-\Delta)^{1/2}) \subset H^1(\mathbb{R}^d) \subset \mathcal{D}(V)$ which already implies that V is infinitesimally operator bounded with respect to the Laplacian by [Tre08, Corollary 2.1.20]. However, in order to get the more precise bound (6.1), we observe that by [Sch12, Lemma 8.4] the multiplication operator V is $(-\Delta)^{1/2}$-bounded and there are $a, b \geq 0$ such that

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq a\varepsilon\|(-\Delta)^{1/2}f\|_{L^2(\mathbb{R}^d)} + b\|f\|_{L^2(\mathbb{R}^d)}$$

for all $f \in \mathcal{D}((-\Delta)^{1/2})$. Now, as in the proof of [Tre08, Corollary 2.1.20], it is easy to calculate that for all $f \in \mathcal{D}(\Delta)$ we have

$$\|(-\Delta)^{1/2}f\|_{L^2(\mathbb{R}^d)} \leq \|\Delta f\|^{1/2}_{L^2(\mathbb{R}^d)}\|f\|^{1/2}_{L^2(\mathbb{R}^d)} \leq \varepsilon\|\Delta f\|_{L^2(\mathbb{R}^d)} + \frac{1}{\varepsilon}\|f\|_{L^2(\mathbb{R}^d)}.$$

In particular, $f \in \mathcal{D}((-\Delta)^{1/2})$, and combining the last two inequalities we obtain the desired inequality (6.1).

(b) We have $\mathcal{D}(\nabla) = H^1(\mathbb{R}^d) \subset \mathcal{D}(V)$, where ∇ denotes the gradient as a closed operator in $H^1(\mathbb{R}^d)$. Using again [Sch12, Lemma 8.4] this implies that V is relatively bounded with respect to the gradient which agrees with the claim. □

Recall that Lemma 6.5 (a) ensures that the operator sum $H = -\Delta + V$ with a singular admissible potential V is selfadjoint in $L^2(\mathbb{R}^d)$ and lower semibounded, cf. Subsection 2.2.2 above.

REMARK 6.6. (a) V is singular admissible if V^2 is form bounded with respect to the Laplacian $-\Delta$. Indeed, an inequality like (6.2) holds for all $f \in \mathcal{D}(\Delta) \subset \mathcal{D}(V^2)$, which extends to (6.2) by taking the closure of ∇. Hence, for $f \in H^1(\mathbb{R}^d)$ we have $\|Vf\|_{L^2(\mathbb{R}^d)} < \infty$ and therefore $f \in \mathcal{D}(V)$, which shows that V is singular admissible.

(b) If $V\colon \mathbb{R}^d \to \mathbb{R}$ is measurable with $\mathcal{D}(\Delta) \subset \mathcal{D}(V^2)$, then V is singular admissible. Indeed, in this case, V^2 is Δ-bounded, see, e.g., [Kat80, Remark IV.1.5]. The claim follows from the fact that V^2 is also form bounded with respect to $-\Delta$, see, e.g., [Kat80, Theorem VI.1.38], and part (i).

Based on the preceding discussion, we now construct examples of singular admissible potentials.

EXAMPLE 6.7. (a) Every real-valued $V \in L^\infty(\mathbb{R}^d)$ is singular admissible.

(b) It follows from [KM82, Lemma 2.1] that V^2 is form bounded with respect to Δ if $V \in L^p(\mathbb{R}^d)$ with $p \geq d$ for $d \geq 3$, $p > 2$ for $d = 2$ and $p = 2$ for $d = 1$. Therefore, by Remark 6.6 (a), such potentials are singular admissible.

(c) If $V : \mathbb{R}^d \to \mathbb{R}$ is measurable such that V^2 belongs to the Kato class in $\mathbb{R}^d$ (see, e.g., [AS82] or [CFKS87, Section 1.2] for a discussion), then V^2 is infinitesimally form bounded with respect to the Laplacian. Hence, V is singular admissible on $\mathbb{R}^d$ by Remark 6.6 (a). In particular, this is the case if V belongs to

$$L^p_{\mathrm{loc,unif}}(\mathbb{R}^d) = \bigl\{h \in L^p(\mathbb{R}^d) \colon \sup_{x\in\mathbb{R}^d} \|h\|_{L^p(B(x,1))} < \infty\bigr\}$$

with $p > d$ for $d \geq 2$ and $p = 4$ for $d = 1$.

Since the singular admissible potentials clearly form a vector space, also sums of potentials from (a), (b), and (c) are singular admissible. Therefore, in dimension $d \geq 2$, this class essentially covers the singular potentials considered in Theorem 6.3. However, potentials of the type (c) have not been in scope of this theorem.

We close this section with a lemma that allows to bound the Laplacian of an element of the spectral subspace $\operatorname{Ran} P_\lambda(H)$. These bounds are used in the proofs of the spectral inequalities in Section 6.4 below and are formulated here simultaneously for both types of potentials, since this avoids redundancies later on.

LEMMA 6.8. *Let $H = -\Delta + V$ where V is either*

(i) as in Hypothesis ($S_\mathcal{I}$), or *(ii) singular admissible.*

Then, in both cases, there are constants $N_1, N_2 \geq 0$ such that for all $\lambda \geq 0$ and all $f \in \operatorname{Ran} P_\lambda(H)$ we have

$$\|\Delta f\|^2_{L^2(\mathbb{R}^d)} \leq 4(\lambda^2 + N_1)\|f\|^2_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \langle -\Delta f, f\rangle_{L^2(\mathbb{R}^d)} \leq 2(\lambda + N_2)\|f\|^2_{L^2(\mathbb{R}^d)}.$$

More precisely, we have $N_2 = 0$ in case (i) and $N_2 = 1 + \lambda_1 + \lambda_2$ in case (ii).

The constant N_1 enters only on an abstract level in the proof of Lemma 6.16 below, and the exact knowledge of it is therefore not necessary.

PROOF. For $f \in \operatorname{Ran} P_\lambda(H) \subset \mathcal{D}(H)$ we have $-\Delta f = Hf - Vf$ and therefore

$$\|\Delta f\|^2_{L^2(\mathbb{R}^d)} \leq 2 \cdot \bigl(\|Hf\|^2_{L^2(\mathbb{R}^d)} + \|Vf\|^2_{L^2(\mathbb{R}^d)}\bigr) \tag{6.3}$$

by Young's inequality. Now we distinguish between the two cases for V.

(i) Suppose that V is as in Hypothesis ($S_\mathcal{I}$). According to the discussion in Subsection 2.2.1 we may assume without loss of generality that $\mathcal{I} = \{1, \dots, d_1\}$ for some $d_1 \leq d$. Then, using the tensor representation of H proven in Lemma 2.2 we have $H = H_1 \otimes I_2 + I_1 \otimes H_2$, where H_1 is a Schrödinger operator with a potential as in Hypothesis (S) in $L^2(\mathbb{R}^{d_1})$ and where H_2 is the Laplacian in $L^2(\mathbb{R}^{d-d_1})$. Thus, by Corollary 2.4 we have $f(\cdot, x_2) \in \operatorname{Ran} P_\lambda(H_1)$ for almost all $x_2 \in \mathbb{R}^{d-d_1}$ and we may apply Proposition 5.8 to $f(\cdot, x_2)$ so that by Fubini's theorem we get $\|Vf\|^2_{L^2(\mathbb{R}^d)} \leq C_3^2 \|f\|^2_{L^2(\mathbb{R}^d)}$, where C_3 is the corresponding constant from Proposition 5.8. Inequality (6.3) now easily implies

$$\|\Delta f\|^2_{L^2(\mathbb{R}^d)} \leq 2(\lambda^2 + C_3^2)\|f\|^2_{L^2(\mathbb{R}^d)}$$

and since the potential is nonnegative, we have

$$\langle -\Delta f, f\rangle_{L^2(\mathbb{R}^d)} \leq \langle Hf, f\rangle^2_{L^2(\mathbb{R}^d)} \leq \lambda \|f\|^2_{L^2(\mathbb{R}^d)}.$$

(ii) Suppose that V is singular admissible. Using that V is infinitesimally operator bounded with respect to $-\Delta$ according to Lemma 6.5, the bound (6.3) yields

$$\|\Delta f\|^2_{L^2(\mathbb{R}^d)} \leq 2\|Hf\|^2_{L^2(\mathbb{R}^d)} + 2 \cdot \Big(2a\varepsilon \|\Delta f\|^2_{L^2(\mathbb{R}^d)} + 2\Big(\frac{a}{\varepsilon} + b\Big)\|f\|^2_{L^2(\mathbb{R}^d)}\Big)$$

for some constants $a, b \geq 0$ and all $\varepsilon > 0$. Hence, choosing $\varepsilon = 1/(8a)$ we have

$$\|\Delta f\|^2_{L^2(\mathbb{R}^d)} \leq 4\|Hf\|^2_{L^2(\mathbb{R}^d)} + N_1\|f\|^2_{L^2(\mathbb{R}^d)} \leq (4\lambda^2 + N_1)\|f\|^2_{L^2(\mathbb{R}^d)}$$

for some constant $N_1 \geq 0$. For the second inequality we estimate

$$\langle -\Delta f, f\rangle_{L^2(\mathbb{R}^d)} \leq \langle Hf, f\rangle_{L^2(\mathbb{R}^d)} + \langle |V| f, f\rangle_{L^2(\mathbb{R}^d)}.$$

By Lemma 6.5 (b) and Young's inequality we have

$$\begin{aligned}\langle |V| f, f\rangle_{L^2(\mathbb{R}^d)} &\leq \frac{\varrho}{2}\|Vf\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2\varrho}\|f\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \frac{\varrho\lambda_1}{2}\|Vf\|^2_{L^2(\mathbb{R}^d)} + \Big(\frac{1}{2\varrho} + \lambda_2\Big)\|f\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

for arbitrary $\varrho > 0$. Hence,

$$\langle -\Delta f, f\rangle_{L^2(\mathbb{R}^d)} \leq \frac{1}{1 - \frac{\varrho\lambda_1}{2}}\langle Hf, f\rangle_{L^2(\mathbb{R}^d)} + \frac{\frac{1}{2\varrho} + \lambda_2}{1 - \frac{\varrho\lambda_1}{2}}\|f\|^2_{L^2(\mathbb{R}^d)}$$

and choosing $\varrho = 1/(1 + \lambda_1)$ we have thus shown

$$\langle -\Delta f, f\rangle_{L^2(\mathbb{R}^d)} \leq \Big(2\lambda + (\lambda_1 + 2\lambda_2)\Big)\|f\|^2_{L^2(\mathbb{R}^d)} \leq 2\Big(\lambda + (1 + \lambda_1 + \lambda_2)\Big)\|f\|^2_{L^2(\mathbb{R}^d)}. \quad \square$$

6.3. Carleman estimates with singular admissible potentials

Here we present two Carleman estimates valid for Schrödinger operators with singular admissible potentials. Both Carleman estimates improve or complement earlier results in the literature. The first Carleman estimate, see Theorem 6.9 below, goes back to [Ves03, EV03], where an inequality of this kind is proven for a class of second order parabolic operators. In the elliptic setting, quantitative versions are proven for the pure Laplacian in [BK05, KT16], and for second order elliptic operators in [NRT19]. The second Carleman estimate, see Theorem 6.10

below, supplements the Carleman estimate of [LR95] where second order elliptic operators are considered.

Roughly speaking, our main observation is that we can add a singular admissible potential in an existing Carleman estimate. For our purposes we implement this for the Carleman estimate given in [KT16] and a special case of the one in [LR95].

In the following, we denote by ∇_{d+1} and Δ_{d+1} the gradient and the Laplacian on $\mathbb{R}^{d+1}$, while ∇ and Δ denote the corresponding expressions on $\mathbb{R}^d$. For the partial derivative in the $(d+1)$-coordinate we write ∂_t and for $V\colon \mathbb{R}^d \to \mathbb{R}$ we use the same symbol to denote $V\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ with $V(x,t) = V(x)$ for $t \in \mathbb{R}$. This construction is necessary since in the proof in Section 6.4 below we will turn a function f in the spectral subspace of the Schrödinger operator $H = -\Delta + V$ into a function F defined on a domain in $\mathbb{R}^{d+1}$ satisfying $(-\Delta_{d+1} + V)F = 0$ almost everywhere.

Let $\rho > 0$ and define on $\mathbb{R}^{d+1}$ the weight function w by

$$w(y) = \phi(|y|/\rho) \quad \text{with} \quad \phi(r) = r\exp\Big(-\int_0^r \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t\Big).$$

For future reference we note that the such defined weight function w satisfies

$$|y|/(\rho\mathrm{e}) \le w(y) \le |y|/\rho \quad \text{and} \quad |\nabla w(y)|^2 \le w^2(y)/|y|^2 \le 1/\rho^2 \tag{6.4}$$

for all $y \in B(0,\rho) \setminus \{0\}$.

THEOREM 6.9. *Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be singular admissible. Then there is a constant $K_1 \ge 1$, depending only on the dimension d, such that for all $\alpha \ge \alpha_0 = K_1(1 + \lambda_1\rho^2 + \lambda_2^{1/3}\rho^{4/3})$ and all $F \in H^2(\mathbb{R}^{d+1})$ with support in $B(0,\rho)\setminus\{0\}$ we have*

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} \alpha\rho^2 w^{1-2\alpha}|\nabla_{d+1}F|^2 + \alpha^3 w^{-1-2\alpha}|F|^2 \\ \le K_d\rho^4 \int_{\mathbb{R}^{d+1}} w^{2-2\alpha}|(-\Delta_{d+1}+V)F|^2.\end{aligned} \tag{6.5}$$

PROOF. The case $V \equiv 0$ in the theorem agrees with [KT16, Lemma 2.1]. Let us denote the constant in this case by $\tilde\alpha_0 \ge 1$ and let K_d' be the constant on the right-hand side. It remains to show that we can insert V on the right-hand side of (6.5). To this end, we estimate $|\Delta_{d+1}F|^2 \le 2|(-\Delta_{d+1}+V)F|^2 + 2|VF|^2$ and subsume the resulting term $2K_d'\rho^4 I$ with

$$I = \int_{\mathbb{R}^{d+1}} w^{2-2\alpha}|VF|^2 = \int_{\mathbb{R}} \|(Vw^{1-\alpha}F)(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t$$

in the left-hand side of (6.5) by appropriate choices of K_d and α_0 that do not depend on F. More precisely, since w is smooth on the support of F, we have $w^{1-\alpha}(\cdot,t)F(\cdot,t) \in H^1(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. Thus,

$$I \le \int_{\mathbb{R}} \lambda_1\|\nabla(w^{1-1\alpha}(\cdot,t)F(\cdot,t))\|^2_{L^2(\mathbb{R}^d)} + \lambda_2\|w^{1-1\alpha}(\cdot,t)F(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t.$$

The product rule and (6.4) imply that the inequality

$$
\begin{aligned}
|\nabla_{d+1}(w^{1-\alpha}F)|^2 &\le 2w^{2-2\alpha}|\nabla_{d+1}F|^2 + 2(\alpha-1)^2|F|^2w^{-2\alpha}/\rho^2 \\
&\le 2w^{1-2\alpha}|\nabla_{d+1}F|^2 + 2(\alpha/\rho)^2w^{-1-2\alpha}|F|^2
\end{aligned}
\tag{6.6}
$$

holds almost everywhere, where we have taken into account that both sides vanish outside of $B(0,\rho)\setminus\{0\}$. Plugging this into the estimate for I, we see that we have proven (6.5) with $K_d = 4K_d'$, provided that

$$
\alpha\rho^2 - 4\lambda_1K_d'\rho^4 \ge \alpha\rho^2/2 \quad \text{and} \quad \alpha^3 - 2K_d'\rho^4\big(2\lambda_1(\alpha/\rho)^2 + \lambda_2\big) \ge \alpha^3/2.
$$

The latter is clearly satisfied for all $\alpha \ge \alpha_0$ with

$$
\alpha_0 = \max\{\tilde\alpha_0, 8K_d'\lambda_1\rho^2 + (4K_d'\lambda_2\rho^4)^{1/3}\}. \tag{6.7}
$$

Since the maximum is clearly bounded by $K_1(1+\lambda_1\rho^2+\lambda_2^{1/3}\rho^{4/3})$ for some constant $K_1 \ge 1$ depending only on the dimension d, this proves the claim. ☐

For the second Carleman estimate we define the weight function

$$
u\colon \mathbb{R}^{d+1} \ni (x,t) \mapsto -t + t^2/2 - |x|^2/4 \in \mathbb{R}. \tag{6.8}
$$

Furthermore, for $\rho > 0$ we let $B_\rho^+ = \{x \in \mathbb{R}^{d+1} \colon |x| < \rho,\ x_{d+1} \ge 0\}$ and we denote by $C_{c,0}^\infty(B_\rho^+)$ the set of all functions $F\colon \mathbb{R}_+^{d+1} = \mathbb{R}^d \times [0,\infty) \to \mathbb{C}$ that satisfy $F(x,0) = 0$ for all $x \in \mathbb{R}^d$ and for which there exists a smooth function $\tilde F$ on $\mathbb{R}^{d+1}$ with $\operatorname{supp}\tilde F \subset B(0,\rho) \subset \mathbb{R}^{d+1}$ satisfying $F = \tilde F$ on $\mathbb{R}_+^{d+1}$.

THEOREM 6.10. *Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be singular admissible, let $\rho \in (0, 2-\sqrt{2})$, and let u be the weight function given in* (6.8)*. Then there is a constant $K_2 \ge 1$, depending only on the dimension d, such that for all $\beta \ge \beta_0 = K_2(1+\lambda_1\rho^2+\lambda_2^{1/3})$ and all $F \in C_{c,0}^\infty(B_\rho^+)$ we have*

$$
\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \mathrm{e}^{2\beta u}\Big(\beta|\nabla_{d+1}F|^2 + \beta^3|F|^2\Big) \\
&\quad\le K_d\left(\int_{\mathbb{R}_+^{d+1}} \mathrm{e}^{2\beta u}|(-\Delta_{d+1}+V)F|^2 + \beta\int_{\mathbb{R}^d} \mathrm{e}^{2\beta u(\cdot,0)}|(\partial_t F)(\cdot,0)|^2\right).
\end{aligned}
\tag{6.9}
$$

In the particular case where $V \equiv 0$, Theorem 6.10 follows from the Carleman estimate given in [LR95, Proposition 1 in the appendix]. This Carleman estimate is formulated for arbitrary real-valued weight functions $u \in C^\infty(\mathbb{R}^{d+1})$ which satisfy

- $(\partial_t u)(x) \ne 0$ for all $x \in B_\rho^+$, and
- for all $\xi \in \mathbb{R}^{d+1}$ and $x \in B_\rho^+$ the implication

$$
\text{if} \quad \left\{\begin{array}{l} 2\langle \xi, \nabla u\rangle = 0, \\ |\xi|^2 = |\nabla u|^2 \end{array}\right\} \quad \text{then} \quad \sum_{j,k=1}^{d+1} (\partial_{jk}u)\big(\xi_j\xi_k + (\partial_j u)(\partial_k u)\big) > 0.
$$

The particular weight function (6.8) has been suggested in [JL99]. With this choice, the two above conditions are satisfied if $\rho \in (0, 2-\sqrt{2})$.

Proof of Theorem 6.10. We have already noted that the theorem holds in case that $V \equiv 0$. Let $\tilde{\beta}_0, K'_d \geq 1$ be the corresponding constants for this case. The proof of the theorem is now analogous to the one of Theorem 6.9. We only need to replace (6.6) by

$$|\nabla_{d+1}(\mathrm{e}^{\beta u} F)|^2 \leq 2\mathrm{e}^{2\beta u}\Big(|\nabla F|^2 + \beta^2\rho^2|F|^2/4\Big) \quad \text{on} \quad \operatorname{supp} F \subset B(0,\rho),$$

and choose

$$\beta_0 = \max\{\tilde{\beta}_0, 2K'_d\lambda_1\rho^2 + (4K'_d\lambda_2)^{1/3}\}. \tag{6.10}$$

Again, the maximum is bounded by $K_2(1 + \lambda_1\rho^2 + \lambda_2^{1/3})$ for some constant $K_2 \geq 1$ depending only on the dimension d which proves the claim. □

6.4. Proof of the spectral inequalities

We have collected all necessary preliminaries that enter the proofs of the spectral inequalities. Throughout this section, unless otherwise stated, let $H = -\Delta + V$ where the potential V is

(i) as in Hypothesis $(S_{\mathcal{I}})$, or (ii) singular admissible.

However, it is worth pointing out that in Subsection 6.4.1 below it suffices to focus on the situation of singular admissible potentials, since potentials as in Hypothesis $(S_{\mathcal{I}})$ locally behave like a bounded potential and we only derive local estimates therein.

6.4.1. Covering estimates for eigenfunctions. Based on the Carleman estimates we now establish local inequalities for a function F defined on some subset of $\mathbb{R}^{d+1}$ that follow as in [JL99, NTTV18, NTTV20b] from the Carleman estimates presented above. These results are used in the proof of Theorems 4.10 and 4.23 given in Subsection 6.4.3 below. In the proofs of these two theorems, the function F is constructed from an element of the spectral subspace of the Schrödinger operator $H = -\Delta + V$ by spectral calculus, see (6.18) below, and it is then always a H^2-function that satisfies $(-\Delta_{d+1} + V)F = 0$ pointwise almost everywhere and is antisymmetric in the $(d+1)$-th coordinate, cf. Subsection 6.4.2 below. In view of this, the restrictions we impose in the upcoming lemmas are always satisfied for the such defined function F. Nevertheless, we intentionally postpone the construction of F from $f \in \operatorname{Ran} P_\lambda(H)$ and first prove the local inequalities for an abstract function in order to point out that the proofs of the estimates below are detached from the ghost dimension framework.

In order to begin with the proof, we introduce for some $\theta \in (0, 1/2)$ the constants

$$
\begin{aligned}
&u_1 = -\theta^2/16, && u_2 = -\theta^2/8, && u_3 = -\theta^2/4,\\
&r_1 = \frac{1}{2} - \frac{\sqrt{16-\theta^2}}{8}, && r_2 = 1, && r_3 = 6\mathrm{e}\sqrt{d},\\
&R_1 = 1 - \frac{\sqrt{16-\theta^2}}{4}, && R_2 = 3\sqrt{d}, && R_3 = 9\mathrm{e}\sqrt{d}.
\end{aligned}
\tag{6.11}
$$

This choice is taken from [NTTV18] and accounts for the geometric conditions compiled in Lemma 6.13 below. Furthermore, we define the annuli $A_j = B(0, R_j) \setminus \overline{B(0, r_j)}$, $j \in \{1, 2, 3\}$, in $\mathbb{R}^{d+1}$ as the level sets of the weight function w from the Carleman estimate in Theorem 6.9.

From Theorem 6.9 we conclude the following three annuli inequality.

LEMMA 6.11 (Three annuli inequality). *Let $\theta \in (0, 1/2)$ and let V be singular admissible. Then for all $F \in H^2(B(0, R_3))$ with $(-\Delta_{d+1} + V)F = 0$ and all $\alpha \geq \alpha_0$ we have*

$$\|F\|^2_{H^1(A_2)} \leq K_d\Big(\frac{1}{\theta^4}\Big(\frac{\mathrm{e}R_2}{r_1}\Big)^{2\alpha-2}\|F\|^2_{H^1(A_1)} + \Big(\frac{\mathrm{e}R_2}{r_3}\Big)^{2\alpha-2}\|F\|^2_{H^1(A_3)}\Big).$$

PROOF. Let $\eta \in C_c^\infty(\mathbb{R}^{d+1})$ be a smooth cutoff function with $0 \leq \eta \leq 1$, $\eta \equiv 0$ on $B(0, r_1) \cup B(0, R_3)^{\mathsf{c}}$, and with $\eta \equiv 1$ on $B(0, r_3) \setminus \overline{B(0, R_1)}$. Then $\nabla_{d+1}\eta \not\equiv 0$ only on $A_1 \cup A_3$. Since the diameter of the annulus A_1 satisfies $R_1 - r_1 = K\theta^2$ and the diameter of the annulus A_3 satisfies $R_3 - r_3 = K_d$, we can choose η in such a way that

$$\max\{\|\Delta_{d+1}\eta\|_{L^\infty(A_1)}, \||\nabla_{d+1}\eta|\|_{L^\infty(A_1)}\} \lesssim_d \frac{1}{\theta^4}$$

and

$$\max\{\|\Delta_{d+1}\eta\|_{L^\infty(A_3)}, \||\nabla_{d+1}\eta|\|_{L^\infty(A_3)}\} \lesssim_d 1.$$

Applying Theorem 6.9 with $\rho = R_3$ to the function ηF and using the product rule we get

$$
\begin{aligned}
\int_{\mathbb{R}^{d+1}} &\alpha w^{1-2\alpha}|F\nabla_{d+1}\eta + \eta\nabla_{d+1}F|^2 + \alpha^3 w^{-1-2\alpha}|\eta F|^2\\
&\lesssim_d \int_{\mathbb{R}^{d+1}} w^{2-2\alpha}\big(4|\nabla_{d+1}\eta|^2|\nabla_{d+1}F|^2 + |F|^2|\Delta_{d+1}\eta|^2\big),
\end{aligned}
$$

since $(-\Delta_{d+1} + V)F = 0$. Using that $\eta \equiv 1$ and $\nabla_{d+1}\eta \equiv 0$ on A_2 and observing that the bounds for the weight function stated in (6.4) imply $w^{1-2\alpha} \geq w^{2-2\alpha}$ and analogously $w^{-1-2\alpha} \geq w^{2-2\alpha}$ on $B(0, R_3)$, we obtain

$$\int_{A_2} \alpha w^{1-2\alpha}|\nabla_{d+1}F|^2 + \alpha^3 w^{-1-2\alpha}|F|^2 \geq \Big(\frac{R_3}{R_2}\Big)^{2\alpha-2}\|F\|^2_{H^1(A_2)}$$

as a lower bound for the left-hand side. For the right-hand side we use that $\nabla_{d+1}\eta \not\equiv 0$ and $\Delta_{d+1}\eta \not\equiv 0$ only on $A_1 \cup A_3$. Combining this with the upper bound for the weight function, we get

$$\Big(\frac{R_3}{R_2}\Big)^{2\alpha-2}\|F\|^2_{H^1(A_2)} \le \int_{\mathbb{R}^{d+1}} w^{2-2\alpha}\big(4|\nabla_{d+1}\eta|^2|\nabla_{d+1}F|^2 + |F|^2|\Delta_{d+1}\eta|^2\big)$$
$$\lesssim_d \frac{1}{\theta^4}\Big(\frac{\mathrm{e}R_3}{r_1}\Big)^{2\alpha-2}\|F\|^2_{H^1(A_1)} + \Big(\frac{\mathrm{e}R_3}{r_3}\Big)^{2\alpha-2}\|F\|^2_{H^1(A_3)}.$$

Hence, multiplying by $(R_3/R_2)^{2\alpha-2}$ we derive the desired inequality. □

In a very analogous fashion we obtain an inequality for the levels sets

$$U_j = \{(x,t) \in \mathbb{R}^d \times [0,1] \colon u(x,t) > u_j\}, \quad j \in \{1,2,3\},$$

of the weight function of the second Carleman estimate, Theorem 6.10, where u is given in (6.8). Since $u_3 > u_2 > u_1$ we here have the inclusions $U_3 \supset U_2 \supset U_1$ and by definition the weight function satisfies $u \ge u_j$ on U_j.

LEMMA 6.12. *Let $\theta \in (0,1/2)$ and let V be singular admissible. Then for all $F \in H^2(U_3)$ with $(-\Delta_{d+1} + V)F = 0$ and $F(\cdot,0) \equiv 0$ that are antisymmetric in the $(d+1)$-th coordinate, and all $\beta \ge \beta_0$ we have*

$$\mathrm{e}^{2\beta u_1}\|F\|^2_{H^1(U_1)} \le K_d\left(\frac{\mathrm{e}^{2\beta u_2}}{\theta^8}\|F\|^2_{H^1(U_3)} + \|(\partial_t F)|_{t=0}\|^2_{L^2(B(0,\theta))}\right).$$

PROOF. Let $\eta \in C_c^\infty(\mathbb{R}^{d+1})$ be a smooth cutoff function with $0 \le \eta \le 1$. Suppose that η is antisymmetric in the $(d+1)$-th coordinate, $\eta \equiv 1$ on U_2, and $\operatorname{supp}\eta \cap \mathbb{R}^d \times [0,\infty) \subset \overline{U}_3$. With the same argument as in the proof of the previous lemma we may choose η in such a way that

$$\max\{\|\Delta_{d+1}\eta\|_{L^\infty(\mathbb{R}^{d+1})}, \||\nabla_{d+1}\eta|\|_{L^\infty(\mathbb{R}^{d+1})}\} \lesssim_d \frac{1}{\theta^4},$$

cf. also [NTTV18, Appendix B]. Since $\eta F \in H^2(\mathbb{R}^{d+1})$ is symmetric in the $(d+1)$-th coordinate by the choice of η, there is a sequence of of $(\Phi_n)_n \subset C_c^\infty(\mathbb{R}^{d+1})$ of functions that are symmetric in the $(d+1)$-th coordinate such that $\Phi_n \to \eta F$ in $H^2(\mathbb{R}^{d+1})$, see [AF03, 3.22 Theorem], and by [AF03, 5.37 Theorem] we also have $(\partial_t\Phi_n)|_{t=0} \to (\partial_t(\eta F))|_{t=0}$ in $L^2(\mathbb{R}^d)$. Therefore, Theorem 6.10 is applicable with $\rho = 1/2$ and with $F = \Phi_n$. Letting $n \to \infty$, using the same arguments as in the proof of Lemma 6.11 above, and dividing by β we thereby obtain

$$\mathrm{e}^{2\beta u_1}\|F\|^2_{H^1(U_1)} \lesssim_d \frac{\mathrm{e}^{2\beta u_2}}{\theta^8}\|F\|^2_{H^1(U_3)} + \int_{\mathbb{R}^d} \mathrm{e}^{2\beta u(\cdot,0)}|\eta(\cdot,0)(\partial_t F)|_{t=0}|^2$$

since $F(\cdot,0) \equiv 0$. For the last term we use $u(x,0) = -|x|^2/4 < 0$ and $\operatorname{supp}\eta(\cdot,0) \cap \mathbb{R}^d \subset B(0,\theta)$, which finally proves the statement. □

For some $L_k \in \mathbb{N}_0 \cup \{\infty\}$, $k \in \{1, \dots, d\}$, we set

$$\Lambda = \bigtimes_{k=1}^{d} \Big(L_k - \frac{1}{2}, L_k + \frac{1}{2} \Big) \tag{6.12}$$

and suppose that $\mathcal{K} \subset \mathbb{Z}^d$ is such that $\Lambda = \bigcup_{k\in\mathcal{K}} \Lambda_1(k)$. Moreover, let $(z_k)_{k\in\mathcal{K}}$ be a sequence with $B(z_k, \theta) \subset \Lambda_1(k)$ for all $k \in \mathcal{K}$. For $z \in \mathbb{R}^d$ and $j = 1, 2, 3$ we set $U_j(z) = U_j + (z, 0)$ and $A_j(z) = A_j + (z, 0)$ and we let

$$U_j^\bullet = \bigcup_{k\in\mathcal{K}} U_j(z_k) \quad \text{and} \quad A_j^\bullet = \bigcup_{k\in\mathcal{K}} A_j(z_k).$$

Note that $A_1^\bullet$ is a disjoint union and that the sets $U_j^\bullet$ and $A_j^\bullet$ depend on Λ (through the index set $\mathcal{K}$) although this is not indicated explicitly.

For the special choices of r_j, R_j, and u_j, $j \in \{1, 2, 3\}$, that we have made here, parts (b)–(d) of the next lemma were proven in [NTTV18, Lemma 3.3].

LEMMA 6.13. *Let*

$$\Gamma = \bigtimes_{k=1}^{d} \Big(L_k + 2R_3 - \frac{1}{2}, L_k + 2R_3 + \frac{1}{2} \Big). \tag{6.13}$$

Then, we have $U_j^\bullet \cup A_j^\bullet \subset \Gamma \times (-R_3, R_3)$ *for all* $j \in \{1, 2, 3\}$ *and*

(a) for $x \in U_j^\bullet \cup A_j^\bullet$ *we have* $|x_k| \lesssim_d L_k$*;*
(b) $A_1^\bullet \cap \mathbb{R}^{d+1}_+ \subset U_1^\bullet$*;*
(c) $\Lambda \times [-1, 1] \subset A_2^\bullet$ *if all* $L_k \geq 5$*;*
(d) $\sum_{k\in\mathcal{K}} \mathbf{1}_{A_j(z_k)} \leq K_d$.

PROOF. Using the definition of U_j it is easy to see that for $(x, t) \in U_j$ we have $|t| \leq 1$ and $|x| \leq \theta$ so that $U_j \subset B_\theta^+$. Analogously, the definition of A_j yields that $|x| \leq R_3$ and $|t| \leq R_3$ for $(x, t) \in A_j$ so that for $(x, t) \in U_j \cup A_j$ we have $|x| \leq R_3$ as well as $|t| \leq R_3$. Thus,

$$U_j^\bullet \cup A_j^\bullet \subset \left(\bigcup_{k\in\mathcal{K}} B(z_k, R_3) \right) \cup (-R_3, R_3) \subset \Gamma \times (-R_3, R_3).$$

Since $R_3 = 9\mathrm{e}\sqrt{d} \leq K_d$, this inclusion also implies (a). For the verification of parts (b)–(d) we refer to [NTTV18, Lemma 3.3]. □

Combining the last three lemmata, we now prove the pivotal result of this subsection. Here it is imperative for the proof of Theorem 4.23 below to note that Lemma 6.13 guarantees that no information of V outside of Γ (as defined in (6.13)) enters the proof.

In what follows, we write $A \sim_d B$ if $A \lesssim_d B$ and $B \lesssim_d A$.

PROPOSITION 6.14 (Covering and interpolation). *Let* Λ *be as in in* (6.12)*, let* $\mathcal{K}$*, and* $(z_k)_{k\in\mathcal{K}}$ *be as in the preceding paragraph, let* Γ *be as in* (6.13)*, and let* V *be singular admissible. Then there is* $\kappa \in (0, 1)$ *satisfying* $\kappa \sim_d 1/|\log\theta|$ *such that for* *all* $F \in H^2(\Gamma \times (-R_3, R_3))$ *satisfying* $(-\Delta_{d+1} + V)F = 0$ *and* $|F(\cdot, t)| = |F(\cdot, -t)|$ *for* $t \in (-R_3, R_3)$ *we have*

$$\|F\|_{H^1(\Lambda\times(-1,1))}^{1/\kappa} \leq \Big(\frac{1}{\theta}\Big)^{K_d(1+\lambda_1+\lambda_2^{1/3})} \|F\|_{H^1(\Gamma\times(-R_3,R_3))}^{1/\kappa-1/2} \cdot \|(\partial_t F)|_{t=0}\|_{L^2(\bigcup_{k\in\mathcal{K}} B(z_k,\theta))}^{1/2}.$$

Proof. Applying Lemma 6.12 to the translates $U_j(z_k)$ and summing over $k \in \mathcal{K}$, the definition of the sets $U_j^\bullet$ and $u_2 - u_1 = u_1$ implies that

$$\|F\|_{H^1(U_1^\bullet)}^2 \lesssim_d \frac{\mathrm{e}^{2\beta u_1}}{\theta^8} \|F\|_{H^1(U_3^\bullet)}^2 + \mathrm{e}^{-2\beta u_1} \|(\partial_t F)|_{t=0}\|_{L^2(\bigcup_{k\in\mathcal{K}} B(z_k,\theta))}^2.$$

We now interpolate this inequality using the interpolation result formulated in Lemma A.1. In this lemma we choose $r = s = -2u_1 > 0$, $\kappa = 1/2$, $t_0 = \beta_0$,

$$P = \|F\|_{H^1(U_1^\bullet)}^2, \quad Q = \|F\|_{H^1(U_3^\bullet)}^2/\theta^8, \quad \text{and} \quad R = \|(\partial_t F)|_{t=0}\|_{L^2(\bigcup_{k\in\mathcal{K}} B(z_k,\theta))}^2.$$

Then, using $\beta_0 \lesssim_d 1 + \lambda_1 + \lambda_2^{1/3}$, the prefactor in inequality (A.2) is bounded from above by $\mathrm{e}^{K_d(1+\lambda_1+\lambda_2^{1/3})} \leq \theta^{-K_d(1+\lambda_1+\lambda_2^{1/3})}$ and therefore we derive

$$\|F\|_{H^1(U_1^\bullet)}^2 \leq \theta^{-K_d(1+\lambda_1+\lambda_2^{1/3})} \|F\|_{H^1(U_3^\bullet)} \|(\partial_t F)|_{t=0}\|_{L^2(\bigcup_{k\in\mathcal{K}} B(z_k,\theta))}. \tag{6.14}$$

We proceed analogously and apply also Lemma 6.11 to the translates $A_j(z_k)$ of the annuli. Then, summing again over $k \in \mathcal{K}$, we get

$$\|F\|_{H^1(A_2^\bullet)}^2 \lesssim_d \frac{1}{\theta^4} \Big(\frac{\mathrm{e}R_2}{r_1}\Big)^{2\alpha-2} \|F\|_{H^1(A_1^\bullet)}^2 + K_d \Big(\frac{\mathrm{e}R_2}{r_3}\Big)^{2\alpha-2} \|F\|_{H^1(A_3^\bullet)}^2,$$

where we used Lemma 6.13 (d) for the second term on the right-hand side. By part (c) of the last mentioned lemma, we may bound the left-hand side from below by $\|F\|_{H^1(\Lambda\times(-1,1))}^2$ and using $|F(\cdot,t)| = |F(\cdot,-t)|$ as well as part (b) of the lemma we have $\|F\|_{H^1(A_1^\bullet)}^2 \leq 2\|F\|_{H^1(U_1^\bullet)}^2$. Hence, taking the square root, we get

$$\|F\|_{H^1(\Lambda\times(-1,1))} \lesssim_d \frac{1}{\theta^4} \Big(\frac{\mathrm{e}R_2}{r_1}\Big)^{\alpha-1} \|F\|_{H^1(U_1^\bullet)} + \Big(\frac{\mathrm{e}R_2}{r_3}\Big)^{\alpha-1} \|F\|_{H^1(\Gamma\times(-R_3,R_3))},$$

and we are again in the position to apply Lemma A.1, at this point with the choices

$$\begin{aligned} r &= -\log(\mathrm{e}R_2/r_3), & P &= \|F\|_{H^1(\Lambda\times(-1,1))}, \\ s &= \log(\mathrm{e}R_2/r_1), & Q &= (\mathrm{e}R_2/r_3)\|F\|_{H^1(\Gamma\times(-R_3,R_3))}, \\ t_0 &= \alpha_0, & R &= (\mathrm{e}R_2/r_1)\|F\|_{H^1(U_1^\bullet)}/\theta^2. \end{aligned}$$

Then

$$\kappa = \frac{\log(r_3/(\mathrm{e}R_2))}{\log(r_3/r_1)} \sim_d \frac{1}{\log(1/\theta)} \tag{6.15}$$

while $\alpha_0 \lesssim_d 1 + \lambda_1 + \lambda_2^{1/3}$. Using this to estimate the corresponding constant from inequality (A.2), we obtain

$$\|F\|_{H^1(\Lambda\times(-1,1))} \leq \theta^{-K_d(1+\lambda_1+\lambda_2^{1/3})/\log(1/\theta)} \|F\|_{H^1(U_1^\bullet)}^{\kappa} \|F\|_{H^1(\Gamma\times(-R_3,R_3))}^{1-\kappa}. \tag{6.16}$$

The statement is now immediately clear from applying inequalities (6.16) and (6.14) successively, using $U_3^\bullet \subset \Gamma \times (-R_3, R_3)$, cf. Lemma 6.13, and raising the resulting inequality to the power $1/\kappa$. □

6.4.2. Ghost dimension. The proofs of Theorems 4.10 and 4.23 are based on the so-called *ghost dimension* framework. This was introduced in [JL99] as a method to deal with spectral projections by applying some spectral calculus that transforms elements in the spectral subspace into an eigenfunction of a similar operator in higher dimensions. For Schrödinger operators, this method was used in [NTTV18, NTTV20b] for bounded potentials and is here extended to the situation where the potential is either singular admissible or as in Hypothesis $(S_\mathcal{I})$.

In order to introduce the ghost dimension, let $H\colon \mathcal{H} \supset \mathcal{D}(H) \to \mathcal{H}$ be any lower semibounded selfadjoint operator on some Hilbert space $\mathcal{H}$. Denote by $(\mathcal{F}_t)_{t\in\mathbb{R}}$ the family of unbounded selfadjoint operators

$$\mathcal{F}_t = \int_{-\infty}^{\infty} s_t(\mu)\, \mathrm{d}P_\mu(H), \quad s_t(\mu) = \begin{cases} \frac{\sinh(\sqrt{\mu}t)}{\sqrt{\mu}} & \mu > 0 \\ t & \mu = 0 \\ \frac{\sin(\sqrt{-\mu}t)}{\sqrt{-\mu}} & \mu < 0 \end{cases}, \tag{6.17}$$

in $\mathcal{H}$. For fixed $f \in \operatorname{Ran} P_\lambda(H)$ we define the function $\tilde{F}\colon \mathbb{R} \to \mathcal{H}$ by

$$\tilde{F}(t) := F_t := \mathcal{F}_t f \in \operatorname{Ran} P_\lambda(H) \subset \mathcal{D}(H)$$

and we let

$$F\colon \mathbb{R}^d \times \mathbb{R}, \quad (x,t) \mapsto F_t(x). \tag{6.18}$$

While this construction works for arbitrary lower semibounded, selfadjoint operators, it is particularly accessible for Schrödinger operators as in this case the function F can be interpreted as a function satisfying the eigenvalue equation of a Schrödinger operator in $(d+1)$-dimensions pointwise almost everywhere. Our goal is to apply the interpolation result from Proposition 6.14 with the extended function F. However, before we show that F indeed satisfies $(-\Delta_{d+1} + V)F = 0$ almost everywhere (recall that $V(x,t) = V(x)$ for $t \in \mathbb{R}$), we state a lemma that allows us to come back to f after using Proposition 6.14 for F.

The next lemma unifies (in the case of operators on the whole of $\mathbb{R}^d$) [NTTV20b, Proposition 2.9], [DRST, Lemma 6.1], and [DSVb, Lemma 3.1], where a similar result is proven for $H = -\Delta + V$ with potentials V that are bounded, singular admissible, or have power growth, respectively.

LEMMA 6.15. *Let $\iota > 0$ and let $f \in \operatorname{Ran} P_\lambda(H)$ for some $\lambda \geq 0$. Then with the function F given in* (6.18) *we have*

$$\frac{\iota}{2}\|f\|^2_{L^2(\mathbb{R}^d)} \leq \|F\|^2_{H^1(\mathbb{R}^d\times(-\iota,\iota))} \leq 2(1+\lambda+N_2)(1+\iota)^3 \mathrm{e}^{2\sqrt{\lambda}\iota}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

PROOF. For the proof of the lower bound we observe that by spectral calculus

$$\|F\|^2_{H^1(\mathbb{R}^d\times(-\iota,\iota))} \geq \int_{-\iota}^{\iota} \|\partial_t F_t\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t = \int_{-\infty}^{\lambda}\left[\int_{-\iota}^{\iota}(\partial_t s_t(\mu))^2\,\mathrm{d}t\right]\mathrm{d}\|P_H(\mu)f\|^2_{L^2(\mathbb{R}^d)}.$$

We distinguish between the two cases $\mu < 0$ and $\mu \geq 0$. In the latter case, we have $\partial_t s_t(\mu) \geq 1$ and, therefore, the inner integral is bounded from below by 2ι. However, in the other case, we have $\partial_t s_t(\mu) = \cos^2(\sqrt{-\mu}t)$ and we calculate

$$\int_{-\iota}^{\iota}(\partial_t s_t(\mu))^2\,\mathrm{d}t = \iota + \frac{\sin(2\sqrt{-\mu}\iota)}{2\sqrt{-\mu}}.$$

Since for small $\iota < \pi/(2\sqrt{-\mu})$ the second summand is nonnegative and for all larger ι we have $|\sin(2\sqrt{-\mu}\iota)|/(2\sqrt{-\mu}) < \iota/2$ by the mean value theorem, we conclude that in both cases

$$\|F\|^2_{H^1(\mathbb{R}^d\times(-\iota,\iota))} \geq \frac{\iota}{2}\int_{-\infty}^{\lambda} 1\,\mathrm{d}\|P_H(\mu)f\|^2_{L^2(\mathbb{R}^d)} = \frac{\iota}{2}\|f\|^2_{L^2\mathbb{R}^d)}.$$

Concerning the upper bound we first write

$$\begin{aligned}\|F\|^2_{H^1(\mathbb{R}^d\times(-\iota,\iota))} &= \int_{-\iota}^{\iota}\|F_t\|^2_{L^2(\mathbb{R}^d)} + \|\partial_t F_t\|^2_{L^2(\mathbb{R}^d)} + \|\nabla F_t\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \\ &= \int_{-\iota}^{\iota}\|F_t\|^2_{L^2(\mathbb{R}^d)} + \|\partial_t F_t\|^2_{L^2(\mathbb{R}^d)} + \langle -\Delta F_t, F_t\rangle_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \\ &\leq 2\int_{-\iota}^{\iota}(1+\lambda+N_2)\|F_t\|^2_{L^2(\mathbb{R}^d)} + \|\partial_t F_t\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t,\end{aligned}$$

where we used Lemma 6.8 for the last inequality. By spectral calculus, we therefore have

$$\|F\|^2_{H^1(\mathbb{R}^d\times(-\iota,\iota))} \leq 2(1+\lambda+N_2)\int_{-\infty}^{\lambda}\left[\int_0^{\iota} s_t(\mu)^2 + (\partial_t s_t(\mu))^2\,\mathrm{d}t\right]\mathrm{d}\|P_\mu(H)f\|^2_{L^2(\mathbb{R}^d)}$$

since $f \in \operatorname{Ran} P_\lambda(H)$. In order to estimate the inner integral, we again distinguish between $\mu < 0$ and $\mu \geq 0$.

- Let $\mu < 0$ and $t \geq 0$. Then $s_t(\mu) = \sin(\sqrt{-\mu}t)/\sqrt{-\mu} \leq t$ by the mean value theorem and $\partial_t s_t(\mu) = \cos(\sqrt{-\mu}t) \leq 1 + t$.
- Let $\mu \geq 0$ and $t \geq 0$. Then $s_t(\mu) = \sinh(\sqrt{\mu}t)/\sqrt{\mu} \leq t\cosh(\sqrt{\mu}t) \leq t\mathrm{e}^{\sqrt{\mu}t}$ by the mean value theorem and $\partial_t s_t(\mu) = \cosh(\sqrt{\mu}t) \leq \mathrm{e}^{\sqrt{\mu}t}$.

In the first case we thus have

$$\int_0^{\iota} s_t(\mu)^2 + (\partial_t s_t(\mu))^2\,\mathrm{d}t \leq \int_0^{\iota} 2(1+t)^2\,\mathrm{d}t \leq (1+\iota)^3$$

while in the second case

$$\int_0^{\iota} s_t(\mu)^2 + (\partial_t s_t(\mu))^2\,\mathrm{d}t \leq \int_0^{\iota}(1+t^2)\mathrm{e}^{2\sqrt{\mu}t}\,\mathrm{d}t \leq (\iota + \iota^3/3)\mathrm{e}^{2\sqrt{\mu}\iota} \leq (1+\iota)^3\mathrm{e}^{2\sqrt{\mu}\iota}.$$

Hence, with $\mu_+ = \max\{0,\mu\}$, we have

$$\begin{aligned}\|F\|^2_{H^1(\mathbb{R}^d\times(-\iota,\iota))} &\leq 2(1+\lambda+N_2)\int_{-\infty}^{\lambda}(1+\iota)^3\mathrm{e}^{2\sqrt{\mu_+}\iota}\,\mathrm{d}\|P_\mu(H)f\|^2_{L^2(\mathbb{R}^d)}\\ &\leq 2(1+\lambda+N_2)(1+\iota)^3\mathrm{e}^{2\sqrt{\lambda}\iota}\|f\|^2_{L^2(\mathbb{R}^d)}. \qquad \square\end{aligned}$$

The next lemma now establishes the almost everywhere identity needed for Proposition 6.14.

LEMMA 6.16. *We have $F \in H^2(\mathbb{R}^d\times(-\iota,\iota))$ for every $\iota > 0$. Moreover, F is antisymmetric in t and satisfies $(\partial_t F)(\cdot,0) = f$ as well as*

$$(-\Delta_{d+1} + V)F = 0 \tag{6.19}$$

almost everywhere on $\mathbb{R}^d$.

PROOF. The antisymmetry in the $(d+1)$-th entry follows from the antisymmetry of $t \mapsto s_t(\mu)$. It is shown in Lemma A.9 in the appendix that F is (infinitely) weakly as well as $L^2(\mathbb{R}^d)$-differentiable with respect to t and that the derivatives agree. More precisely, we have

$$\partial_t^k F(\cdot,t) = \Bigl(\int_{[\kappa,\lambda]}\partial_t^k s_t(\mu)\,\mathrm{d}P_\mu(H)\Bigr)f \in \mathcal{D}(H) \quad \text{for} \quad k\in\mathbb{N}, \tag{6.20}$$

and this formula also implies $(\partial_t F)(\cdot,0) = f$ since $\partial_t s_t(\mu)|_{t=0} = 1$. Moreover, as in [NTTV20b, Proof of Lemma 2.5], we have

$$H\mathcal{F}_t P_\lambda(H)f = \int_\kappa^\lambda \mu s_t(\mu)\,\mathrm{d}P_\mu(H)f$$

and the above formula for the derivatives of F therefore yields $H(F(\cdot,t)) = (\partial_t^2 F)(\cdot,t)$ since $\mu s_t(\mu) - \partial_t^2 s_t(\mu) = 0$. Hence, (6.19) holds almost everywhere.

It remains to show that $F \in H^2(\mathbb{R}^d\times(-\iota,\iota))$. To this end, we first note that Lemma 6.15 implies $F\in H^1(\mathbb{R}^d\times(-\iota,\iota))$. Thus, it remains to show

$$\sum_{\substack{\beta\in\mathbb{N}_0^{d+1}\\|\beta|=2}}\|\partial^\beta F\|_{L^2(\mathbb{R}^d\times(-\iota,\iota))} < \infty.$$

We write

$$\begin{aligned}\sum_{\substack{\beta\in\mathbb{N}_0^{d+1}\\|\beta|=2}}\|\partial^\beta F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} &= \|\partial_t^2 F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))}\\ &\quad+\|\nabla_d\partial_t F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} + \sum_{\substack{\beta'\in\mathbb{N}_0^{d}\\|\beta'|=2}}\|\partial^{\beta'} F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))}.\end{aligned} \tag{6.21}$$

In order to bound the first summand on the right-hand side, we use $F_t \in \operatorname{Ran} P_\lambda(H)$ as well as $HF_t = (\partial_t^2 F)(\cdot, t)$ to obtain

$$\begin{aligned}\|\partial_t^2 F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} &= \int_{-\iota}^{\iota} \|(\partial_t^2 F)(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t = \int_{-\iota}^{\iota} \|HF_t\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \\ &\leq \lambda^2 \int_{-\iota}^{\iota} \|F_t\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t = \lambda^2 \|F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} < \infty.\end{aligned}$$

Based on Lemma 6.8 we bound the second and third term on the right-hand side of (6.21): Using that $\partial_t F_t \in \operatorname{Ran} P_\lambda(H)$ by spectral calculus, we have

$$\begin{aligned}\|\nabla_d \partial_t F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} &= \int_{-\iota}^{\iota} \|\nabla_d(\partial_t F)(\cdot,t)\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t = \int_{-\iota}^{\iota} \langle -\Delta\partial_t F_t, \partial_t F_t\rangle_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \\ &\leq 2(\lambda+N_2)\int_{-\iota}^{\iota} \|F_t\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t = 2(\lambda+N_2)\|F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} < \infty,\end{aligned}$$

so that the second term is bounded. For the last term we estimate

$$\sum_{\substack{\beta'\in\mathbb{N}_0^d\\ |\beta'|=2}} \|\partial^{\beta'} F_t\|^2_{L^2(\mathbb{R}^d)} \leq 2 \sum_{\substack{\beta'\in\mathbb{N}_0^d\\ |\beta'|=2}} \frac{1}{\beta'!}\|\partial^{\beta'} F_t\|^2_{L^2(\mathbb{R}^d))} = \|\Delta F_t\|^2_{L^2(\mathbb{R}^d))},$$

where the identity follows from integration by parts, cf. [See21, Example 4.2]. Applying Lemma 6.8 and integrating over $t \in (-\iota,\iota)$ finally gives

$$\sum_{\substack{\beta'\in\mathbb{N}_0^d\\ |\beta'|=2}} \|\partial^{\beta'} F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))} \leq 4(\lambda^2+N_1)\int_{-\iota}^{\iota} \|F_t\|^2_{L^2(\mathbb{R}^d))}\,\mathrm{d}t = 4(\lambda^2+N_1)\|F\|^2_{L^2(\mathbb{R}^d\times(-\iota,\iota))},$$

which proves the lemma. □

REMARK 6.17. In an analogous way it is possible to also consider Schrödinger operators $H_\Lambda^\bullet = -\Delta_\Lambda^\bullet + V$ with Dirichlet, Neumann, or periodic boundary conditions (if applicable) $\bullet \in \{D, N, \mathrm{per}\}$ on generalized rectangles $\Lambda = \times_{j=1}^d (a_j, b_j)$ with $-\infty \leq a_j < b_j \leq \infty$, $j \in \{1,\dots,d\}$. This requires that V is *singular admissible on* Λ, that is, $\mathcal{D}(V) \supset H^1(\Lambda)$. The above proofs (in particular the one of Lemma 6.5) can easily be adapted to this situation. However, in order to apply Proposition 6.14, it is necessary to extend the potential, the operator in $(d+1)$-dimension, and the function F to a large enough region that contains the set $\Gamma \times (-R_3, R_3)$ defined in (6.13). We here avoid these merely technical details and refer the reader to [DRST, Lemma 4.5] for the extension procedure needed for the potential and to [NTTV18, NTTV20b] for the extension of the operator and the function F.

6.4.3. Proof of Theorems 4.10 and 4.23. We are now in the position to give the proofs of our main results. First, we examine the situation for singular admissible potentials.

Proof of Theorem 4.10. It suffices to prove the statement in the case where $G = 1$, since the general case follows from this by a classical scaling argument, see Subsection A.1.6 in Appendix A where it is analyzed how the parameters λ_1 and λ_2 behave under this scaling. Recall that V is singular admissible and therefore $N_2 = 1 + \lambda_1 + \lambda_2$ in Lemma 6.8. Without loss of generality let $\lambda \geq 0$. Fix $f \in \operatorname{Ran} P_\lambda(H)$ and define F as in (6.18) above. Using Lemma 6.15 twice (with $\iota = R_3 = 9\mathrm{e}\sqrt{d}$ as in (6.11) above and with $\iota = 1$) we have

$$\begin{aligned}\frac{\|F\|^2_{H^1(\mathbb{R}^d\times(-R_3,R_3))}}{\|F\|^2_{H^1(\mathbb{R}^d\times(-1,1))}} &\leq 4(1+\lambda+N_2)(1+R_3)^3\mathrm{e}^{2\sqrt{\lambda}\iota} \\ &\leq \mathrm{e}^{K_d\cdot(\log(1+\lambda+\lambda_1+\lambda_2)+\lambda^{1/2})}.\end{aligned} \tag{6.22}$$

Let $\mathcal{K} = \mathbb{Z}^d$ and let $(z_k)_{k\in\mathbb{Z}^d}$ with $z_k \in \Lambda_1(k)$ be such that $\omega \supset \bigcup_{k\in\mathcal{K}} B(z_k,\delta)$. Applying Proposition 6.14 for the function F with $\theta = \delta$ we obtain

$$\begin{aligned}\|F\|^{1/\kappa}_{H^1(\mathbb{R}^d\times(-1,1))} &\leq \theta^{-K_d(1+\lambda_1+\lambda_2^{1/3})}\|F\|^{1/\kappa-1/2}_{H^1(\mathbb{R}^d\times(-R_3,R_3))}\|(\partial_t F)|_{t=0}\|^{1/2}_{L^2(\bigcup_{k\in\mathbb{Z}^d}B(z_k,\delta))} \\ &\leq \theta^{-K_d(1+\lambda_1+\lambda_2^{1/3})}\|F\|^{1/\kappa-1/2}_{H^1(\mathbb{R}^d\times(-R_3,R_3))}\|f\|^{1/2}_{L^2(\omega)}.\end{aligned}$$

Plugging in (6.22) and dividing by $\|F\|^{1/\kappa-1/2}_{H^1(\mathbb{R}^d\times(-R_3,R_3))}$ we get

$$\|F\|^{1/2}_{H^1(\mathbb{R}^d\times(-R_3,R_3))} \leq \theta^{-K_d\cdot(1+\lambda_1+\lambda_2^{1/3})}\mathrm{e}^{K_d\cdot(\log(1+\lambda+\lambda_1+\lambda_2)+\lambda^{1/2})/\kappa}\|f\|^{1/2}_{L^2(\omega)}$$

and we further estimate the constant using

$$\log(1+\lambda+\lambda_1+\lambda_2) + \lambda^{1/2} \leq 8\cdot\bigl(1+\lambda_1+\lambda_2^{1/3}+\lambda^{1/2}\bigr),$$

which is a simple consequence of $\log(1+r) \leq 2r^{1/3}$ for $r \geq 0$. Thereby, in light of $1/\kappa \sim_d \log(1/\theta)$ (cf. equation (6.15)), the prefactors match exactly and we get

$$\|F\|_{H^1(\mathbb{R}^d\times(-R_3,R_3))} \leq \delta^{-K_d\cdot(1+\lambda_1+\lambda_2^{1/3}+\lambda^{1/2})}\|f\|_{L^2(\omega)}$$

since $\theta = \delta$. Applying Lemma 6.15 (with $\iota = R_3$) once again, we bound the left-hand side from below by the norm of f on the whole of $\mathbb{R}^d$ and finally conclude

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \delta^{-K_d\cdot(1+\lambda_1+\lambda_2^{1/3}+\lambda^{1/2})}\|f\|_{L^2(\omega)}.$$

□

The proof of the spectral inequality for Schrödinger operators with potentials as in Hypothesis ($S_\mathcal{I}$) proceeds in an analogous way, but implements a cut-off procedure which allows to only apply Proposition 6.14 on a region where V is bounded by part (i) of Hypothesis ($S_\mathcal{I}$).

Proof of Theorem 4.23. Let V be as in Hypothesis ($S_\mathcal{I}$) and let F be as in (6.18). Then $N_2 = 0$ in Lemma 6.8 and therefore, similar to (6.22) above, we have

$$\frac{\|F\|^2_{H^1(\mathbb{R}^d\times(-R_3,R_3))}}{\|F\|^2_{H^1(\mathbb{R}^d\times(-1,1))}} \leq \mathrm{e}^{K_d\cdot\lambda^{1/2}}. \tag{6.23}$$

Moreover, since $f \in \operatorname{Ran} P_\lambda(H)$ and F is defined as in (6.18), the functions F_t and $\partial_t F_t$ are also contained in $\operatorname{Ran} P_\lambda(H)$ for all $t \in \mathbb{R}$. As in the proof of Lemma 6.8 above, we assume without loss of generality that $\mathcal{I} = \{1, \dots, d_1\}$ with $d_1 \leq d$. Furthermore, we let $d_2 = d - d_1$ and write $H = H_1 \otimes I_1 + I_2 \otimes H_2$ with $H_1 = -\Delta + V$ in $L^2(\mathbb{R}^{d_1})$ and $H_2 = -\Delta$ in $L^2(\mathbb{R}^{d_2})$ according to Lemma 2.2, where V satisfies Hypothesis (S). If $d_2 = 0$ the second summand of H can be dropped here. The representation for $h \in \operatorname{Ran} P_\lambda(H)$ proven in Corollary 2.4 applied with $h = F(\cdot, t)$ and with $h = \partial_t F(\cdot, t)$ then yields that $\partial_t F(\cdot, x_2, t)$ and $\partial_{x_2}^\alpha F(\cdot, x_2, t)$ for $\alpha \in \mathbb{N}_0^{d_2}$, $|\alpha| \leq 1$, belong to $\operatorname{Ran} P_\lambda(H)$ for all $t \in \mathbb{R}$ and almost all $x_2 \in \mathbb{R}^{d_2}$. Hence, expanding the H^1-norm as

$$\|F\|^2_{H^1(\mathbb{R}^d \times (-1,1))} = \int_{-1}^{1} \int_{\mathbb{R}^{d_2}} \Big[\|F(\cdot, x_2, t)\|^2_{H^1(\mathbb{R}^{d_1})} + \|\nabla_{x_2} F(\cdot, x_2, t)\|^2_{L^2(\mathbb{R}^{d_1})} + \|\partial_t F(\cdot, x_2, t)\|^2_{L^2(\mathbb{R}^{d_1})} \Big] \,\mathrm{d}x_2 \,\mathrm{d}t$$

and estimating each of the three summands according to Theorem 5.1, we get

$$\|F\|^2_{H^1(\mathbb{R}^d \times (-1,1))} \leq 2 \cdot \|F\|^2_{H^1(B^{(d_1)}(0, C_0 \lambda^{1/\tau_1}) \times \mathbb{R}^{d_2} \times (-1,1))}. \tag{6.24}$$

In combination with (6.23) we derive at

$$\begin{aligned} \|F\|^2_{H^1(\mathbb{R}^d \times (-R_3, R_3))} &\leq 2\mathrm{e}^{K_d \cdot \lambda^{1/2}} \|F\|^2_{H^1(B^{(d_1)}(0, C_0 \lambda^{1/\tau_1}) \times \mathbb{R}^{d_2} \times (-1,1))} \\ &\leq 2\mathrm{e}^{K_d \cdot \lambda^{1/2}} \|F\|^2_{H^1(\Lambda^{(d_1)} \times \mathbb{R}^{d_2} \times (-1,1))}, \end{aligned}$$

where $\Lambda^{(d_1)} \subset \mathbb{R}^{d_1}$ denotes the smallest cube of odd integer sidelength that contains $B^{(d_1)}(0, C_0 \lambda^{1/\tau_1})$. Let $\mathcal{K} = \mathcal{K}(\lambda) = \big(\mathbb{Z}^{d_1} \cap \Lambda^{(d_1)}\big) \times \mathbb{Z}^{d_2}$ and let $(z_k)_{k \in \mathcal{K}}$ be points such that

$$\omega \cap \big(\Lambda^{(d_1)} \times \mathbb{R}^{d_2}\big) \supset \bigcup_{k \in \mathcal{K}} B\big(z_k, \delta^{1+|k|^\alpha}\big),$$

which is possible according to the assumption on ω. Clearly, by the definition $\Lambda^{(d_1)}$, for all $k \in \mathcal{K}$ we have $|k_\mathcal{I}| \leq K_d C_0 \lambda^{1/\tau_1}$. Therefore,

$$\inf_{k \in \mathcal{K}} \delta^{1+|k_\mathcal{I}|^\alpha} \geq \delta^{1+(K_d C_0 \lambda^{1/\tau_1})^\alpha} =: \theta \quad \text{and} \quad \omega \cap \big(\Lambda^{(d_1)} \times \mathbb{R}^{d_2}\big) \supset \bigcup_{k \in \mathcal{K}} B(z_k, \theta).$$

Now we apply Proposition 6.14 with this choice for θ and recall from the discussion preceding Proposition 6.14 that only information of V on Γ as defined in (6.13) enters the proof. However, since the last lemma implies $\Gamma \subset K_d \Lambda \times \mathbb{R}^{d_2}$, part (i) of Hypothesis (S) yields

$$\|V|_\Gamma\|_\infty \leq (K_d C_0 \lambda^{1/\tau_1})^{\tau_2} \leq C_1 \lambda^{\tau_2/\tau_1},$$

where $C_1 = K_d^{\tau_2} C_0^{\tau_2/\tau_1} \geq 1$. Thus, V behaves like a bounded potential and therefore we may follow the proof of Theorem 4.10 with $\lambda_1 = 0$ and $\lambda_2 = C_1^2 \lambda^{2\tau_2/\tau_1}$. This gives

$$\begin{aligned}\|F\|_{H^1(\mathbb{R}^d\times(-R_3,R_3))} &\le \theta^{-K_d\cdot(1+C_1^2\lambda^{2\tau_2/3\tau_1}+\lambda^{1/2})}\|f\|_{L^2(\bigcup_{k\in\mathcal{K}}B(z_k,\theta))}\\ &\le \theta^{-K_d\cdot(1+C_1^2\lambda^{2\tau_2/3\tau_1})}\|f\|_{L^2(\omega)}\end{aligned}$$

since $2\tau_2/3\tau_1 \ge 2/3 \ge 1/2$. Plugging in $\theta = \delta^{1+(K_dC_0\lambda^{1/\tau_1})^\alpha}$ and using Lemma 6.15 we finally conclude

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}^d)} &\le \delta^{-K_d^{1+\tau_2+\alpha}C_0^{2\tau_2/\tau_1+\alpha}\cdot\lambda^{(2\tau_2/3+\alpha)/\tau_1}}\|f\|_{L^2(\omega)}\\ &\le \delta^{-C^{1+\alpha}\lambda^{(2\tau_2/3+\alpha)/\tau_1}}\|f\|_{L^2(\omega)},\end{aligned}$$

where $C = K_d^{1+\tau_2}C_0^{1+2\tau_2/\tau_1}$. □

CHAPTER 7

Uncertainty principles based on complex analysis

In this chapter we use techniques based on complex analysis to prove the spectral inequalities for partial harmonic oscillators, in particular Theorems 4.16 and 4.19, as well as the uncertainty principle with error term formulated in Theorem 4.35. We start with a short survey of the historic development which led to the approach discussed here, see Section 7.1 below. Thereafter, Section 7.2 is devoted to the proof of the spectral inequalities for the partial harmonic oscillators. More precisely, in that section we first establish technical preliminaries that enter the proof of a generalized spectral inequality that is written down and proven in Subsection 7.2.4 below. We infer the actual spectral inequalities as simple corollaries to this generalized version in Subsection 7.2.5. Some minor extension of the aforementioned results are described in Subsection 7.2.6. The chapter is concluded with the proof of Theorem 4.35 in Section 7.3. Since this proof is very similar to the proof given in Section 7.2, several arguments require only minor adjustments and we refer in this case to the corresponding arguments in Section 7.2.

7.1. Logvinenko-Sereda inequalities

Heisenberg's famous uncertainty principle states that measuring the momentum of a particle inevitably changes its position and, vice versa, measuring its position changes its momentum. This implies that it is impossible to simultaneously determine the precise position and momentum of a particle. There are various mathematical formulations of this principle, see, e.g., the monograph [HJ94] for an overview. Since the momentum of an observable can be obtained from its position via the Fourier transform, one of the mathematical formulations is, that if a function f has a compactly supported Fourier transform $\widehat{f}$, then f itself cannot have compact support. This *qualitative* fact is well-know and a direct consequence of the Paley-Wiener and the identity theorem for holomorphic functions.

In order to discuss more quantitative versions of the uncertainty principle, we follow the diction of [HJ94] and call a set $\omega \subset \mathbb{R}^d$ *determining*, if for all functions

$$f \in \mathcal{E} = \{g \in L^2(\mathbb{R}^d)\colon \operatorname{supp} \widehat{g} \subset B(x, r) \text{ for some } r > 0,\ x \in \mathbb{R}^d\}$$

there is a constant $C > 0$ such that

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(\omega)}. \tag{7.1}$$

Clearly, if ω is determining, then each $f \in \mathcal{E}$ vanishes identically if it vanishes on ω.

Conditions on the set ω guaranteeing that it is determining were first studied in [Pan61, Pan62]. Therein, necessary conditions were established in all dimension, while the sufficiency was only shown in one dimension; however, for dimensions $d \geq 2$ an alternative sufficient condition, unrelated to the necessary one, was also given. Moreover, the proof gives no information about the constant C in inequality (7.1). This was done, independently, only by Kacnel'son [Kac73] and Logvinenko & Sereda [LS74]. Actually, they give a full characterization of determining sets ω for all dimensions and even in a general $L^p(\mathbb{R}^d)$-setting, $p \in [1, \infty)$, where determining sets are defined analogously to the L^2-situation above. We restrict ourselves here to the Hilbert space situation and formulate the result from the last mentioned articles in this setting.

THEOREM 7.1 ([Kac73, LS74]). *Let $\omega \subset \mathbb{R}^d$ be measurable. Then ω is determining if and only if there are $\gamma \in (0, 1]$ and $\rho > 0$ such that ω is (γ, ρ)-thick in the sense of Definition 4.2 above.*

If one of the two equivalent statements holds, then there are constants $C_1, C_2 > 0$ depending only on γ, ρ, and d, such that

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C_1 \mathrm{e}^{C_2 r} \|f\|_{L^2(\omega)}$$

for all $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{f} \subset B(x, r)$ for some $r > 0$ and some $x \in \mathbb{R}^d$.

The constant provided by the above theorem is not explicit in the thickness parameters nor in the dimension. This drawback has been eliminated by Kovrijkine [Kov01, Kov00] using an approach based on complex analysis. This allows to derive an explicit constant which is polynomial in $1/\gamma$, improving significantly on the constant in Theorem 7.1.

In fact, [Kov01, Kov00] uses a slightly refined definition of thickness which we recall here in order to state the geometric assumptions in full generality.

DEFINITION 7.2. A measurable set $\omega \subset \mathbb{R}^d$ is said to be (γ, a)-thick for some $\gamma \in (0, 1]$ and some $a = (a_1, \dots, a_d) \in (0, \infty)^d$ if

$$\frac{|\Lambda(x, a) \cap \omega|}{|\Lambda(x, a)|} \geq \gamma \quad \text{for all} \quad x \in \mathbb{R}^d.$$

The one dimensional version of the next theorem was established in [Kov01], while the higher dimensional analogue was proven in [Kov00]. We again formulate only the L^2-version here, although it is stated in the mentioned references for all L^p-spaces with $p \in [1, \infty]$.

THEOREM 7.3 ([Kov01, Kov00]). *Let $\gamma \in (0, 1]$, $a \in (0, \infty)^d$, and let $\omega \subset \mathbb{R}^d$ be measurable. Then, for all $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{f} \subset \Lambda(x, b)$ for some $x \in \mathbb{R}^d$ and some $b \in (0, \infty)^d$ we have*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \Big(\frac{K^d}{\gamma}\Big)^{d(Ka \cdot b+1)} \|f\|_{L^2(\omega)} \tag{7.2}$$

if and only if ω is (γ, a)-thick.

The strength of the approach [Kov01, Kov00] is the fact that the assumption on the support of $\widehat{f}$ is merely used to guarantee that f is analytic and satisfies a Bernstein inequality. This allows the approach to be generalized to appropriate subspaces of analytic functions satisfying a Bernstein-type inequality. This was done, for instance, for functions having a compactly supported Hankel (or Fourier-Bessel) transform [GJ13] and for some model spaces [HJK20]. Moreover, [EV20] treated the situation of functions on a torus $\mathbb{T}_L^d = \bigtimes_{j=1}^d (0, 2\pi L_j)^d$ with sides $L = (L_1, \dots, L_d) \in (0,\infty)^d$ having a finite Fourier series. In order to formulate their result, we write $k/L := (k_1/L_1, \dots, k_d/L_d)$ for $k \in \mathbb{Z}^d$ and define the Fourier coefficients of a function $f \in L^2(\mathbb{T}_L^d)$ by

$$\widehat{f}(k/L) = \frac{1}{\prod_{j=1}^d 2\pi L_j} \int_{\mathbb{T}_L^d} f(x) \mathrm{e}^{-\mathrm{i}x\cdot(k/L)} \,\mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Hence, $\operatorname{supp} \widehat{f} \subset \mathbb{Z}^d/L \subset \mathbb{R}^d$ and f has a finite Fourier series if $\operatorname{supp} \widehat{f}$ is contained in a compact set.

THEOREM 7.4 ([EV20]). *Let $L, a \in (0,\infty)^d$ and let $\omega \subset \mathbb{R}^d$ be a (γ, a)-thick set with $0 < a_j \leq 2\pi L_j$ for all $j = 1, \dots, d$. Then, every function $f \in L^2(\mathbb{T}_L^d)$ with $\operatorname{supp} \widehat{f} \subset \Lambda(x, b)$ for some $x \in \mathbb{R}^d$ and some $b \in (0,\infty)^d$ satisfies*

$$\|f\|_{L^2(\mathbb{T}_L^d)} \leq \Big(\frac{K^d}{\gamma}\Big)^{Ka\cdot b + 3d + \frac{1}{2}} \|f\|_{L^2(\mathbb{T}_L^d \cap \omega)}. \tag{7.3}$$

This result was further extended in [Egi21] to functions on strips having compactly supported Fourier coefficients in the bounded coordinate directions and compactly supported Fourier transform in the unbounded ones.

The interest in the last two Logvinenko-Sereda inequalities from the control theory point of view stems from the observation by [EV18] that the explicit form of the constant allows to reformulate them into spectral inequalities for the Laplacian.

LEMMA 7.5. *If either*

(i) $\Omega = \mathbb{R}^d$ *and* Δ *is the Laplacian on* $\mathbb{R}^d$, *or*

(ii) $\Omega = (0, 2\pi L)^d$ *for some* $L > 0$ *and* $\Delta = \Delta_\Omega^\bullet$, $\bullet \in \{D, N, \mathrm{per}\}$, *is the Laplacian on* Ω *with periodic, Dirichlet, or Neumann boundary conditions,*

then for all $\lambda \geq 1$ *we have*

$$\operatorname{Ran} P_\lambda(H) = \big\{ f \in L^2(\Omega) \colon \operatorname{supp} \widehat{f} \subset B(0, \sqrt{\lambda}) \big\}. \tag{7.4}$$

In particular, Theorems 7.3 and 7.4 imply the spectral inequalities for the Laplacian stated in Theorem 4.3.

The restriction of compact Fourier support in Theorems 7.3 and 7.4 has subsequently been replaced by a compact Fourier support in terms of the orthonormal basis of Hermite functions [BJPS21] based on Bernstein inequalities for the Hermite functions established in the last reference, see Theorem 4.13. As already noted in the discussion before Theorem 4.13, this implies a spectral inequality for the harmonic oscillator. Furthermore, a fairly abstract framework that allows to establish a spectral inequality for a lower semibounded selfadjoint operator satisfying some abstract Bernstein-type inequality was given in [ES21]. Amongst others, this allows to treat the Laplacian (with suitably chosen boundary conditions) on some specific triangles or sectors in $\mathbb{R}^2$ as well as all the aforementioned results were ω was assumed to be thick. However, if the set ω is not assumed to be thick, it is necessary to complement the Kovrijkine approach with additional techniques. For instance, the proof of Theorem 4.15 above given in [MPS22] relies on weighted versions of the Bernstein inequalities for Hermite functions. The intention of this chapter is to show how the decay of the functions under consideration can be used to considerably weaken the geometric assumptions on the set ω.

7.2. Spectral inequality for partial harmonic oscillators

The goal of this section is to prove a generalized spectral inequality, see Theorem 7.19 below, and to show that this generalized version implies the spectral inequality for the partial harmonic oscillators we formulated in Theorem 4.19. In fact, we conclude this theorem from the generalized spectral inequality in Subsection 7.2.5 below. There we also formulate and prove the announced generalization of Theorem 4.19 to an unbounded scale which includes Theorem 4.16 as a special case. Furthermore, at the end of the present section, in Subsection 7.2.6 below, we prove a spectral inequality for the operators $H_{\mathcal{I},\mathcal{J}}$. However, there we are dealing with observation operators which are not multiplication operators by a characteristic function of a sensor set.

7.2.1. Global properties of spectral elements. We start by studying global properties of elements in the spectral subspace. To this end, recall from Lemma 2.2 that we can write the partial harmonic oscillator $H_{\mathcal{I}} = -\Delta + |x_{\mathcal{I}}|^2$ as a tensor product between the (full) harmonic oscillator $H_1 = -\Delta + |x|^2$ and the Laplacian $H_2 = -\Delta$, where, without loss of generality, $\mathcal{I} \subset \{1, \ldots, d_1\}$ for some $d_1 \leq d$. Since the spectral elements also have a tensor representation by Corollary 2.4 (a), we get global properties for $H_{\mathcal{I}}$ and for $\operatorname{Ran} P_\lambda(H_{\mathcal{I}})$ from related properties for H_1, H_2, and their respective spectral subspaces. We point out that in the present setting every $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$ can be extended to an analytic function on $\mathbb{C}^d$ which we denote again by f, cf. Corollary 2.4 (b).

The properties we are interested in here are the so-called *Bernstein inequalities*. These inequalities were first established for functions with compactly supported Fourier transform or series, see [Boa54, Theorem 11.3.3] and [MS13, Proposition 1.11], and later adapted to functions in the spectral subspace of suitable selfadjoint operators, see [ES21] and [BJPS21, Proposition 4.3]. Here we follow the concept of [ES21] and present averaged versions of the Bernstein inequalities, which allows us to give slightly more efficient estimates below.

We start with the Bernstein inequality for the Laplacian and the harmonic oscillator. The proof of both uses the technique of [ES21, Lemma 2.1].

LEMMA 7.6. *Let* $\lambda \geq 1$.

(i) Every $f_1 \in \operatorname{Ran} P_\lambda(H_1)$ *satisfies*

$$\sum_{|\alpha|=m} \frac{1}{\alpha!}\|\partial^\alpha f_1\|^2_{L^2(\mathbb{R}^{d_1})} \leq \frac{\prod_{k=0}^{m-1}(\lambda+2k)}{m!}\|f_1\|^2_{L^2(\mathbb{R}^{d_1})}, \quad m \in \mathbb{N}_0. \tag{7.5}$$

(ii) Every $f_2 \in \operatorname{Ran} P_\lambda(H_2)$ *satisfies*

$$\sum_{|\alpha|=m} \frac{1}{\alpha!}\|\partial^\alpha f_2\|^2_{L^2(\mathbb{R}^{d_2})} \leq \frac{\lambda^m}{m!}\|f_2\|^2_{L^2(\mathbb{R}^{d_2})}, \quad m \in \mathbb{N}_0. \tag{7.6}$$

PROOF. We first prove (i). To this end, recall that the Hermite functions $(\Phi_\alpha)_{\alpha\in\mathbb{N}_0^d}$ defined in (4.7) form an orthonormal basis of eigenfunctions of the harmonic oscillator corresponding to the eigenvalue $2|\alpha| + d_1$ for all $\alpha \in \mathbb{N}_0^{d_1}$. Choosing $N \in \mathbb{N}_0$ such that $2N + d_1 \leq \lambda < 2N + d_1 + 2$, we therefore have $f_1 \in \mathcal{E}_N = \{\Phi_\alpha \colon |\alpha| \leq N\}$. We show

$$\sum_{|\alpha|=m} \frac{1}{\alpha!}\|\partial^\alpha f_1\|^2_{L^2(\mathbb{R}^{d_1})} \leq \frac{\prod_{k=0}^{m-1}(2N+d_1+2k)}{m!} \cdot \|f_1\|^2_{L^2(\mathbb{R}^{d_1})}, \quad m \in \mathbb{N}_0, \tag{7.7}$$

which implies the asserted inequality by the definition of N. In order to prove (7.7), we proceed by induction and observe first that integration by parts shows

$$\sum_{j=1}^{d_1}\|\partial_{x_j} f_1\|^2_{L^2(\mathbb{R}^{d_1})} = \langle f_1, -\Delta f_1\rangle_{L^2(\mathbb{R}^{d_1})} \leq \langle f_1, H_1 f_1\rangle_{L^2(\mathbb{R}^{d_1})} \leq (2N+d)\|f_1\|^2_{L^2(\mathbb{R}^{d_1})},$$

which is the above statement for $m = 1$. Now, Lemma A.2 implies

$$\sum_{|\alpha|=m+1} \frac{1}{\alpha!}\|\partial^\alpha f_1\|^2_{L^2(\mathbb{R}^{d_1})} = \frac{1}{m+1}\sum_{|\alpha|=m}\frac{1}{\alpha!}\sum_{j=1}^{d_1}\|\partial^{\alpha+e_j} f_1\|^2_{L^2(\mathbb{R}^{d_1})}$$

and since

$$\partial^\alpha f_1 \in \mathcal{E}_{N+|\alpha|} \quad \text{for all} \quad \alpha \in \mathbb{N}_0^d, \tag{7.8}$$

we may use the induction hypothesis to further estimate the right-hand side by

$$\begin{aligned}\frac{2(N+|\alpha|)+d_1}{m+1} &\sum_{|\alpha|=m}\frac{1}{\alpha!}\|\partial^\alpha f_1\|^2_{L^2(\mathbb{R}^{d_1})} \\ &\leq \frac{2(N+|\alpha|)+d_1}{m+1} \cdot \frac{\prod_{k=0}^{m-1}(2N+d_1+2k)}{m!}\|f_1\|^2_{L^2(\mathbb{R}^{d_1})},\end{aligned}$$

which proves the asserted inequality and, therefore, the Bernstein inequality (7.5).

In case of (ii), the spectral subspace $\operatorname{Ran} P_\lambda(H_2)$ is invariant under $H_2 = -\Delta$. Since for $f_2 \in \operatorname{Ran} P_\lambda(H_2)$ we have $\langle f_2, -\Delta f_2\rangle_{L^2(\mathbb{R}^{d_2})} \leq \lambda \|f_2\|^2_{L^2(\mathbb{R}^{d_2})}$, the same induction as before proves (7.6) □

REMARK 7.7. The property (7.8) is characteristic for the finite linear combinations of Hermite functions. It can easily be proved using the so-called *raising* and *lowering operators* $\mathcal{R} = \frac{\mathrm{d}}{\mathrm{d}x} - x$ and $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x} + x$ of the harmonic oscillator, see, e.g., [BJPS21, Section 4.2].

Next, we use the previous lemma and the tensor structure to establish a Bernstein inequality for elements in the spectral subspace of the partial harmonic oscillator.

LEMMA 7.8. *Given $\lambda \geq 1$, every function $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$ satisfies*

$$\sum_{|\alpha|=m} \frac{1}{\alpha!} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{C_B(m,\lambda)}{m!} \cdot \|f\|^2_{L^2(\mathbb{R}^d)} \quad \textit{for all} \quad m \in \mathbb{N}_0,$$

where

$$C_B(m,\lambda) := 2^m \prod_{k=0}^{m-1} (\lambda + 2k). \tag{7.9}$$

PROOF. Let $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$. By Corollary 2.4, we have that $(\partial^\beta f)(x,\cdot)$ belongs to $\operatorname{Ran} P_\lambda(H_2)$ for all $x \in \mathbb{R}^{d_1}$ and all $\beta \in \mathbb{N}^d_{0,\mathcal{I}}$. In the next step we split a multi-index $\alpha \in \mathbb{N}_0^d$ as $\alpha = \beta + \nu$ with $\beta \in \mathbb{N}^d_{0,\mathcal{I}}$ and $\nu \in \mathbb{N}^d_{0,\mathcal{I}^c}$. We apply Lemma 7.6 for fixed $m \in \mathbb{N}$ and $\beta \in \mathbb{N}^d_{0,\mathcal{I}}$ to $f_1 = \partial^\beta f$ as well as Fubini's theorem to obtain

$$\sum_{|\nu|=m-|\beta|} \frac{1}{\nu!} \|\partial^\nu \partial^\beta f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\lambda^{m-|\beta|}}{(m-|\beta|)!} \|\partial^\beta f\|^2_{L^2(\mathbb{R}^d)}.$$

In the same way, $f(\cdot, y) \in \operatorname{Ran} P_\lambda(H_1)$ for all $y \in \mathbb{R}^{d_2}$, so that

$$\sum_{|\beta|=j} \frac{1}{\beta!} \|\partial^\beta f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\prod_{k=0}^{j-1}(\lambda+2k)}{j!} \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Putting the last two estimates together, we arrive at

$$\begin{aligned}\sum_{|\alpha|=m} \frac{1}{\alpha!} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^d)} &= \sum_{j=0}^m \sum_{|\beta|=j} \frac{1}{\beta!} \sum_{|\nu|=m-j} \frac{1}{\nu!} \|\partial^\nu \partial^\beta f\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \frac{1}{m!} \Bigg(\sum_{j=0}^m \binom{m}{j} \prod_{k=0}^{j-1} (\lambda+2k) \lambda^{m-j} \Bigg) \|f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

In order to complete the proof, it only remains to observe that

$$\sum_{j=0}^{m}\binom{m}{j}\prod_{k=0}^{j-1}(\lambda+2k)\cdot\lambda^{m-j}\leq\prod_{k=0}^{m-1}(\lambda+2k)\sum_{j=0}^{m}\binom{m}{j}=2^m\prod_{k=0}^{m-1}(\lambda+2k). \qquad \square$$

Since the spectral subspace of the harmonic oscillator is invariant under the Fourier transform, the Bernstein inequality allows to conclude a weighted L^2-inequality that again encodes the fast decay of elements in the spectral subspace in the growth directions of the potential in the same manner as Proposition 5.5 above. Similar bounds have already been obtained in [BJPS21, DSVc, DSVa] but the derivation we present here shows the close relationship to the Bernstein inequalities and also optimizes the constants.

LEMMA 7.9. *Let $\lambda \geq 1$ and $f \in \operatorname{Ran}_\lambda(H_{\mathcal{I}})$. Then*

$$\|\mathrm{e}^{|x_{\mathcal{I}}|^2/16}f\|^2_{L^2(\mathbb{R}^d)} \leq 4\cdot 2^\lambda\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{7.10}$$

Here the weight function is quadratic in the exponent in contrast to the linear term in the exponent of the weight function in Chapter 5, e.g., in the aforementioned Proposition 5.5. However, since the dependence of the constant on the spectral parameter λ changes similarly, this does not drastically change the radius of the ball where such functions are localized. More precisely, in view of the weighted inequality (4.6) we here have $q/p = 1/2$ with $q = 1$ and $p = 2$, whereas Proposition 5.5 (in case of $V(x) = |x|^2$) gives the same result with $q = 1/2$ and $p = 1$. We nevertheless prove Lemma 7.9 here since its weight function allows to slightly optimize the dependence of the constant on the dimension d.

PROOF OF LEMMA 7.9. Let $f_1 \in \operatorname{Ran} P_\lambda(H_1)$. By Plancherel's identity and integration by parts we have

$$\begin{aligned}\frac{1}{(2m)!}\||x|^{2m}f_1\|^2_{L^2(\mathbb{R}^{d_1})} &= \frac{1}{(2m)!}\|(-\Delta)^m\widehat{f_1}\|^2_{L^2(\mathbb{R}^{d_1})}\\ &= \frac{1}{(2m)!}\langle\widehat{f_1},(-\Delta)^{2m}\widehat{f_1}\rangle_{L^2(\mathbb{R}^{d_1})} = \sum_{|\alpha|=2m}\frac{1}{\alpha!}\|\partial^\alpha\widehat{f_1}\|^2_{L^2(\mathbb{R}^{d_1})}.\end{aligned}$$

Using that the spectral subspace $\operatorname{Ran} P_\lambda(H_1)$ is invariant under the Fourier transform, we can apply Lemma 7.6 (a) to $\widehat{f_1}$ and obtain

$$\begin{aligned}\|\mathrm{e}^{|x|^2/16}f_1\|_{L^2(\mathbb{R}^{d_1})} &\leq \sum_{m=0}^{\infty}\frac{1}{16^m m!}\||x|^{2m}f_1\|_{L^2(\mathbb{R}^{d_1})}\\ &= \sum_{m=0}^{\infty}\frac{1}{16^m m!}\Big((2m)!\cdot\sum_{|\alpha|=2m}\frac{1}{\alpha!}\|\partial^\alpha\widehat{f_1}\|^2_{L^2(\mathbb{R}^{d_1})}\Big)^{1/2}\\ &\leq \sum_{m=0}^{\infty}\frac{1}{16^m m!}\Big(\prod_{k=0}^{2m-1}(\lambda+2k)\Big)^{1/2}\|\widehat{f_1}\|_{L^2(\mathbb{R}^{d_1})}. \qquad (7.11)\end{aligned}$$

We calculate

$$\begin{aligned}\prod_{k=0}^{2m-1} (\lambda + 2k) &\leq 2^{2m} \prod_{k=0}^{2m-1} (\lceil \lambda/2 \rceil + 1 + k) = 4^m \cdot \frac{(\lceil \lambda/2 \rceil + 2m)!}{\lceil \lambda/2 \rceil !} \\ &\leq 16^m 2^{\lceil \lambda/2 \rceil} (2m)! \leq 64^m 2^\lambda (m!)^2,\end{aligned}$$

where we used $(k+j)! \leq k!j! \cdot 2^{k+j}$ for $k, j \in \mathbb{N}_0$ and $\lceil \lambda/2 \rceil \leq \lambda$ for all $\lambda \geq 1$. Plugging this into (7.11) and using Plancherel's identity once again, we get

$$\|\mathrm{e}^{|x|^2/16} f_1\|_{L^2(\mathbb{R}^{d_1})} \leq 2^{1+\lambda/2} \|f_1\|_{L^2(\mathbb{R}^{d_1})}. \tag{7.12}$$

By Corollary 2.4 we have $f(\cdot, y) \in \operatorname{Ran}_\lambda(H_1)$ for all $y \in \mathbb{R}^{d_2}$. Thus, using Fubini's theorem and applying (7.12) with $f_1 = f(\cdot, y)$ we get

$$\begin{aligned}\|\mathrm{e}^{|x_{\mathcal{I}}|^2/16} f\|_{L^2(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^{d_2}} \|\mathrm{e}^{|\cdot|^2/16} f(\cdot, y)\|^2_{L^2(\mathbb{R}^{d_1})} \, \mathrm{d}y \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^{d_2}} 2^{2+\lambda} \|f(\cdot, y)\|^2_{L^2(\mathbb{R}^{d_1})} \, \mathrm{d}y \right)^{1/2} \\ &= 2^{1+\lambda/2} \|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

□

7.2.2. Covering and localization of Bernstein inequalities. In order to treat as many sensor sets ω as possible at once, we work with an abstract covering of $\mathbb{R}^d$ and impose certain conditions on this covering in the formulation of the following lemmas. All of these conditions will be condensed in Hypothesis (H_λ) below.

Let $(Q_k)_{k \in \mathcal{K}}$ be any finite or countably infinite family of measurable, nonempty, bounded, convex subsets $Q_k \subset \mathbb{R}^d$ and let $\kappa \geq 1$ be such that

$$\Bigl| \mathbb{R}^d \setminus \bigcup_{k \in \mathcal{K}} Q_k \Bigr| = 0 \qquad \text{and} \qquad \sum_{k \in \mathcal{K}} \mathbf{1}_{Q_k}(x) \leq \kappa \quad \text{for all} \quad x \in \mathbb{R}^d. \tag{7.13}$$

We say that $(Q_k)_{k \in \mathcal{K}}$ is an *essential covering* of $\mathbb{R}^d$ of *multiplicity at most* κ. From the decay encoded in Lemma 7.9 we extract that elements of the covering that are far away from the origin carry only a neglectable amount of the mass of f. Hence, we can reduce our considerations to the covering elements close to the origin.

REMARK 7.10. We already used a similar reduction argument in inequality (6.24) in the proof of Theorem 4.23 above. However, in contrast to the situation there, we here only need to localize the function in the L^2-norm and the localization is not used to regard the potential essentially as bounded.

In what follows, $B^{(d_1)}(0, r)$ denotes the ball in $\mathbb{R}^{d_1}$.

LEMMA 7.11. *Let $\lambda \geq 1$ and $C = 6 \cdot (1 + \sqrt{\log \kappa})$. Then the subset*

$$\mathcal{K}_c := \mathcal{K}_c(\lambda) := \bigl\{ k \in \mathcal{K} \colon Q_k \cap (B^{(d_1)}(0, C\lambda^{1/2}) \times \mathbb{R}^{d_2}) \neq \emptyset \bigr\} \tag{7.14}$$

satisfies

$$\sum_{k\in\mathcal{K}_c^{\mathsf{c}}} \|f\|^2_{L^2(Q_k)} \le \frac{1}{4}\|f\|^2_{L^2(\mathbb{R}^d)} \quad \textit{for all} \quad f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}}).$$

Proof. For $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$ and $s \ge C\lambda^{1/2}$, Lemma 7.9 implies that

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^d\setminus(B^{(d_1)}(0,s)\times\mathbb{R}^{d_2}))} &= \|\mathrm{e}^{-|x_{\mathcal{I}}|^2/16}\mathrm{e}^{|x_{\mathcal{I}}|^2/16} f\|^2_{L^2(\mathbb{R}^d\setminus(B^{(d_1)}(0,s)\times\mathbb{R}^{d_2}))}\\ &\le 4\mathrm{e}^{-s^2/8}2^\lambda\|f\|^2_{L^2(\mathbb{R}^d)}\\ &\le \frac{1}{4\kappa}\|f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

By definition, $Q_k \cap (B^{(d_1)}(0, C\lambda^{1/2}) \times \mathbb{R}^{d_2}) = \emptyset$ for $k \in \mathcal{K}_c^{\mathsf{c}}$. Hence,

$$\sum_{k\in\mathcal{K}_c^{\mathsf{c}}} \|f\|^2_{L^2(Q_k)} \le \kappa\|f\|^2_{L^2(\mathbb{R}^d\setminus(B^{(d_1)}(0,CN^{1/2})\times\mathbb{R}^{d_2}))} \le \frac{1}{4}\cdot\|f\|^2_{L^2(\mathbb{R}^d)}. \qquad \square$$

We now localize the Bernstein inequality proven in Lemma 7.8 to the covering elements. This approach was established by Kovrijkine [Kov01, Kov00] and used in many works thereafter, cf. the overview presented in Section 7.1. It is clear that one can not expect that the global inequality implies a localized version on all of the covering elements. However, by choosing an additional prefactor depending on the multiplicity κ and the order of derivatives m, we can nevertheless guarantee that it holds on sufficiently many of the covering elements, the so-called *good* covering elements. Furthermore, increasing the aforementioned prefactor, it is also possible to conclude a pointwise inequality on all of those good elements.

In order to be more precise, we say that Q_k for $k \in \mathcal{K}$ is *good* with respect to a function $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$ if

$$\sum_{|\alpha|=m} \frac{1}{\alpha!}\|\partial^\alpha f\|^2_{L^2(Q_k)} \le 2^{m+1}\kappa\frac{C_B(m,\lambda)}{m!}\|f\|^2_{L^2(Q_k)} \quad \text{for all} \quad m \in \mathbb{N}, \tag{7.15}$$

and we call Q_k *bad* otherwise, i.e., if

$$\sum_{|\alpha|=m} \frac{1}{\alpha!}\|\partial^\alpha f\|^2_{L^2(Q_k)} > 2^{m+1}\kappa\frac{C_B(m,\lambda)}{m!}\|f\|^2_{L^2(Q_k)} \quad \text{for some} \quad m \in \mathbb{N}. \tag{7.16}$$

Here, the constant differs from the constant in the (global) Bernstein inequality in Lemma 7.8 only by the additional prefactor $2^{m+1}\kappa$. The latter is used in the lemma below to compensate the multiplicity of the covering and an additional summation over the order m that is needed since the upper bound for the norm $\|f\|_{L^2(Q_k)}$ given by the definition (7.16) only holds for some m.

Lemma 7.12. *Let*

$$\mathcal{K}_g := \{k \in \mathcal{K} \colon Q_k \ \textit{good}\}. \tag{7.17}$$

Then

$$\sum_{k\in\mathcal{K}_g}\|f\|^2_{L^2(Q_k)} \geq \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)} \tag{7.18}$$

and for all $k \in \mathcal{K}_g$ *there exists a point* $x_k \in Q_k$ *with*

$$\sum_{|\alpha|=m}\frac{1}{\alpha!}|\partial^\alpha f(x_k)|^2 \leq \frac{4^{m+1}\kappa C_B(m,\lambda)}{m!}\cdot\frac{\|f\|^2_{L^2(Q_k)}}{|Q_k|} \quad \textit{for all} \quad m\in\mathbb{N}_0. \tag{7.19}$$

Proof. In order to prove (7.18), we show that the bad elements Q_k, $k \in \mathcal{K}_g^{\mathsf{c}}$, contribute for at most half of the mass of f. To this end, we use the upper bound for $\|f\|_{L^2(Q_k)}$ on the bad Q_k given by (7.16), which, however, only holds for certain $m \in \mathbb{N}$. We therefore take the sum over all $m \in \mathbb{N}$ and get

$$\|f\|^2_{L^2(Q_k)} \leq \sum_{m=1}^{\infty}\frac{m!}{2^{m+1}\kappa C_B(m,\lambda)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\|\partial^\alpha f\|^2_{L^2(Q_k)}.$$

Summing also over the bad covering elements and using that the multiplicity of the covering is at most κ implies

$$\sum_{k\in\mathcal{K}_g^{\mathsf{c}}}\|f\|^2_{L^2(Q_k)} \leq \sum_{m=1}^{\infty}\frac{m!}{2^{m+1}C_B(m,\lambda)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\|\partial^\alpha f\|^2_{L^2(\mathbb{R}^d)}$$

and the Bernstein inequality from Lemma 7.8 applied to the right-hand side gives

$$\sum_{k\in\mathcal{K}_g^{\mathsf{c}}}\|f\|^2_{L^2(Q_k)} \leq \|f\|^2_{L^2(\mathbb{R}^d)}\sum_{m=1}^{\infty}\frac{1}{2^{m+1}} = \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{7.20}$$

The existence of a point $x_k \in Q_k$, $k \in \mathcal{K}_g$, such that (7.19) holds follows by a similar argument. Here, we assume for contradiction that for all $x \in Q_k$ there is $m = m(x) \in \mathbb{N}_0$ with

$$\sum_{|\alpha|=m}\frac{1}{\alpha!}|\partial^\alpha f(x)|^2 > \frac{4^{m+1}\kappa C_B(m,\lambda)}{m!|Q_k|}\|f\|^2_{L^2(Q_k)}. \tag{7.21}$$

Since this, as before, does not hold for all m, we take again the sum over $m \in \mathbb{N}$. Moreover, in order to turn the pointwise inequality (7.21) into an inequality between L^2-norms, we integrate over the Q_k so that

$$\|f\|^2_{L^2(Q_k)} < \sum_{m=0}^{\infty}\frac{m!}{4^{m+1}\kappa C_B(m,\lambda)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\|\partial^\alpha f\|^2_{L^2(Q_k)}.$$

The definition (7.15) of good elements now shows

$$\|f\|^2_{L^2(Q_k)} < \|f\|^2_{L^2(Q_k)}\sum_{m=0}^{\infty}\frac{1}{2^{m+1}} = \|f\|^2_{L^2(Q_k)},$$

contradicting (7.21). Hence, for every $k \in \mathcal{K}_g$ there is $x_k \in Q_k$ such that the pointwise bound (7.19) holds. □

REMARK 7.13. In the proof of (7.19) above we used the definition of good covering elements for all $m \in \mathbb{N}_0$, although the definition of good elements in (7.15) above is restricted to $m \in \mathbb{N}$. However, since the constant on the right-hand side of (7.15) is at least 1, it is obvious that we can also apply (7.15) in case $m = 0$.

So far, we classified the covering into elements near to or far from the origin, and into good and bad elements. In the next lemma we show that it suffices to consider good elements near the origin, i.e., covering elements Q_k with $k \in \mathcal{K}_c \cap \mathcal{K}_g$.

LEMMA 7.14. *Given $f \in \operatorname{Ran} P_\lambda(H_\mathcal{I})$, $\mathcal{K}_c$ as in (7.14), and $\mathcal{K}_g$ as in (7.17), we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq 4 \sum_{k \in \mathcal{K}_c \cap \mathcal{K}_g} \|f\|^2_{L^2(Q_k)}.$$

In particular, $\mathcal{K}_c \cap \mathcal{K}_g \neq \emptyset$.

PROOF. Subadditivity, Lemma 7.11, and (7.18) imply

$$\sum_{k \in \mathcal{K}_c^\mathsf{c} \cup \mathcal{K}_g^\mathsf{c}} \|f\|^2_{L^2(Q_k)} \leq \sum_{k \in \mathcal{K}_c^\mathsf{c}} \|f\|^2_{L^2(Q_k)} + \sum_{k \in \mathcal{K}_g^\mathsf{c}} \|f\|^2_{L^2(Q_k)} \leq \frac{3}{4} \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Passing to the complementary sum over $k \in \mathcal{K}_c \cap \mathcal{K}_g$ proves the claim. □

7.2.3. Local estimate. We next derive local estimates on the covering elements. The following lemma is implicitly contained in several recent works such as [GJ13, Theorem 4.5], [EV20, Section 5], [WWZZ19], [BJPS21, Section 3.3.3], [MPS22], [ES21, Lemma 3.5], and [Egi21]. More precisely, the formulation of the next lemma with the additional bijection Ψ was established in [ES21, Lemma 3.5], while earlier works considered only the situation where $\Psi = \mathrm{Id}$. We discuss in Subsection 7.2.6 below how the bijection can be used to optimize the constant.

LEMMA 7.15 (Local estimate). *Let $Q \subset \mathbb{R}^d$ be a nonempty, bounded, convex and open set that is contained in a hyperrectangle with sides of length $l \in (0, \infty)^d$ parallel to the coordinate axes and let $f \colon Q \to \mathbb{C}$ be a non-vanishing function that has an analytic extension $f \colon Q + D_{4l} \to \mathbb{C}$ with bounded modulus.*

Then, for every measurable set $\omega \subset \mathbb{R}^d$ and every linear bijection $\Psi \colon \mathbb{R}^d \to \mathbb{R}^d$ we have

$$\|f\|^2_{L^2(Q \cap \omega)} \geq 12 \Big(\frac{|\Psi(Q \cap \omega)|}{24 d \tau_d (\operatorname{diam} \Psi(Q))^d} \Big)^{4 \frac{\log M}{\log 2} + 1} \|f\|^2_{L^2(Q)} \tag{7.22}$$

with

$$M := \frac{\sqrt{|Q|}}{\|f\|_{L^2(Q)}} \cdot \sup_{z \in Q + D_{4l}} |f(z)|.$$

The normalized supremum in the above lemma automatically satisfies $M \geq 1$.

Since $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$ is a non-vanishing function that has an analytic extension to the whole of $\mathbb{C}^d$ by Corollary 2.4 (b), Lemma 7.15 is applicable for such f. We emphasize that only further below, when we prove an upper bound for M if $Q = Q_k$ with $k \in \mathcal{K}_g \cap \mathcal{K}_c$, we use more information about f then just the analyticity.

The proof of the above lemma we present here goes back to [Naz93, Kov00, Kov01]. It was inspired by the so-called Turan lemma, see [Naz93, Theorem I] and also [Tur84], and can be thought of as a L^2-variant of the following lemma for analytic functions.

LEMMA 7.16 (Kovrijkine's lemma [Kov01]). *Let $\varepsilon > 0$ and $\Phi\colon D(0, 4+\varepsilon) \to \mathbb{C}$ be an analytic function with $|\Phi(0)| \geq 1$. Then*

$$\sup_{t\in[0,1]} |\Phi(t)| \leq \Big(\frac{12}{|E|}\Big)^{2^{\frac{\log M_\Phi}{\log 2}}} \sup_{t\in E} |\Phi(t)| \tag{7.23}$$

for all measurable sets $E \subset [0,1]$ with $|E| > 0$, where

$$M_\Phi = \sup_{z\in D_4} |\Phi(z)|.$$

In [Kov01] the above lemma is only formulated in case $\varepsilon = 1$. Although the proof for arbitrary $\varepsilon > 0$ is essentially the same, we provide it here for the sake of completeness. Moreover, the proof below gives some details that were omitted in [Kov01].

PROOF OF LEMMA 7.16. Jensen's formula implies that the number m of zeros (a_k) of Φ inside $D(0,2)$ is at most $\log M_\Phi / \log 2$. Hence, the Blaschke condition $\sum_k (1 - a_k/2) < \infty$ is satisfied and there is a zero-free analytic function $g\colon D(0,2) \to \mathbb{C}$ with $|g(0)| \geq 1$ and $\max_{z\in D(0,2)} |g(z)| \leq M_\Phi$ such that $\Phi(z) = B(z)g(z)$ for $z \in D(0,2)$ and the Blaschke product

$$B(z) = \prod_{k=1}^{m} \frac{|a_k|}{a_k} \cdot \frac{\frac{a_k}{2} - \frac{z}{2}}{1 - \frac{\overline{a}_k}{2} \cdot \frac{z}{2}} = \prod_{k=1}^{m} \frac{|a_k|}{a_k} \cdot \frac{2(a_k - z)}{4 - \overline{a}_k z}, \quad z \in D(0,2),$$

see, e.g., [FL88, Satz 5.2]; here we used the scaling factor 1/2 since Blaschke's result is formulated for the unit disc. By construction of g, the function defined by $D(0,2) \ni z \mapsto \log(M_\Phi / |g(z)|)$ is positive and harmonic, so that Harnack's inequality is applicable and yields

$$\max_{|z|\leq 1} \log \frac{M_\Phi}{|g(z)|} \leq 3 \log \frac{M_\Phi}{|g(0)|} \leq 3 \log M_\Phi. \tag{7.24}$$

Therefore, $\min_{|z|\leq 1} |g(z)| \geq M_\Phi^{-2}$ and since $[0,1] \subset D(0,1)$ we get

$$\frac{\max_{x\in[0,1]} |g(z)|}{\min_{x\in[0,1]} |g(z)|} \leq \frac{\max_{|z|\leq 1} |g(z)|}{\min_{|z|\leq 1} |g(z)|} \leq M_\Phi^3.$$

Let us write $B = P/Q$ with polynomials $P(z) = 2^m \prod_{k=1}^m |a_k|(a_k - z)$ and $Q(z) = \prod_{k=1}^m a_k(4 - \overline{a}_k z)$ of degree m. A simple calculation using the triangle inequality shows $|a_k(4-\overline{a}_k z)| \geq 2|a_k|$ and $|a_k(4-\overline{a}_k z)| \leq 6|a_k|$ for $|z| \leq 1$, so that

$$\frac{\max_{x\in[0,1]}|Q(z)|}{\min_{x\in[0,1]}|Q(z)|} \leq \frac{\max_{|z|\leq 1}|Q(z)|}{\min_{|z|\leq 1}|Q(z)|} \leq 3^m.$$

The Remez inequality [Rem36] for a polynomial P of degree m states that

$$\max_{x\in[0,1]}|P(z)| \leq \Big(\frac{4}{|E|}\Big)^m \sup_{x\in E}|P(x)|.$$

Thus, by the identity $\Phi = B \cdot g$ on $D(0,2)$ and the fact that $B = P/Q$ we conclude

$$\begin{aligned}
\max_{x\in[0,1]}|\Phi(x)| &\leq \max_{x\in[0,1]}|g(z)| \cdot \frac{\max_{x\in[0,1]}|P(z)|}{\min_{|z|\leq 1}|Q(z)|} \\
&\leq M_\Phi^3 \cdot \min_{|z|\leq 1}|g(z)| \cdot \frac{\big(\frac{4}{|E|}\big)^m \sup_{x\in E}|P(x)|}{3^{-m}\cdot \max_{|z|\leq 1}|Q(z)|} \\
&\leq M_\Phi^3 \cdot \Big(\frac{12}{|E|}\Big)^m \sup_{x\in E}|\Phi(x)|,
\end{aligned}$$

where we used $\min_{|z|\leq 1}|Q(z)| \geq 3^{-m}\max_{|z|\leq 1}|Q(z)|$ in the second inequality. Using the bound $m \leq \log M_\Phi/\log 2$ and estimating

$$M_\Phi^3 \leq \exp(3\log M_\Phi) \leq \exp\Big(\log\Big(\frac{12}{|E|}\Big)\frac{\log M_\Phi}{\log 2}\Big) = \Big(\frac{12}{|E|}\Big)^{\frac{\log M_\Phi}{\log 2}}$$

concludes the proof. □

REMARK 7.17. The above proof can also be adapted to the case where Φ is merely analytic on a disc of radius $1+\varepsilon$ with $\varepsilon > 0$. However, in that case the upper bound for the number of zeros of Φ inside $D(0,1)$ given by Jensen's formula and the right-hand side of (7.24) would take the form

$$\frac{\log \sup_{z\in D(0,1)}|\Phi(z)|}{\log(1+\varepsilon)} \quad \text{resp.} \quad \Big(1 + \frac{2}{\varepsilon}\Big)\log \sup_{z\in D(0,1)}|\Phi(z)|.$$

The idea of the proof of Lemma 7.15 is to introduce a set W where $|f|$ is pointwise smaller than a specific constant and to use the pointwise lower bound on the complement of W to obtain (7.22). Thereby, it remains to show that W makes up only a small portion of the set $Q \cap \omega$. This is then proven using Lemma 7.16, since it allows us to connect a point where $|f|$ is large via a line segment to the set W where $|f|$ is small, leading to a bound for the measure of this set, see inequality (7.28) below.

Our proof is oriented on the one given in [ES21].

PROOF OF LEMMA 7.15. We show that the set

$$W=\Big\{x\in Q\colon |f(x)|<\Big(\frac{|\Psi(Q\cap\omega)|}{24d\tau_d(\operatorname{diam}\Psi(Q))^d}\Big)^{2\frac{\log M}{\log 2}}\cdot\frac{\|f\|_{L^2(Q)}}{|Q|^{1/2}}\Big\}$$

where $|f|$ is small compared to its L^2-norm on Q and to the measure of $Q\cap\omega$ makes up at most half of the mass of $Q\cap\omega$, i.e., $|(Q\cap\omega)\setminus W|\geq|Q\cap\omega|/2$. Then, using the pointwise lower bound for $|f|$ on the complement of W, we get

$$\begin{aligned}\|f\|^2_{L^2(Q\cap\omega)}&\geq\|f\|^2_{L^2(Q\setminus W)}\geq|(Q\cap\omega)\setminus W|\cdot\inf_{x\in Q\setminus W}|f(x)|^2\\&\geq\frac{|Q\cap\omega|}{2|Q|}\cdot\Big(\frac{|\Psi(Q\cap\omega)|}{24d\tau_d(\operatorname{diam}\Psi(Q))^d}\Big)^{4\frac{\log M}{\log 2}}\cdot\|f\|^2_{L^2(Q)}\\&=12\Big(\frac{|\Psi(Q\cap\omega)|}{24d\tau_d(\operatorname{diam}\Psi(Q))^d}\Big)^{4\frac{\log M}{\log 2}+1}\|f\|^2_{L^2(Q)},\end{aligned}$$

which proves (7.22).

It remains to show $|(Q\cap\omega)\setminus W|\geq|Q\cap\omega|/2$, which follows immediately from the inequality $|Q\cap\omega|\geq 2|W|$. In order to prove the latter, fix some point $x_0\in Q$ where $|f(x_0)|\geq\|f\|_{L^2(Q)}/|Q|^{1/2}$; such x_0 exists as otherwise integrating over Q yields a contradiction. Using polar coordinates, we now choose a direction $\widehat{y}_0\in\mathbb{S}^{d-1}$ such that the longest line segment in $\Psi(Q)$ starting at $\Psi(x_0)$ in direction $\widehat{y}_0$ sees a large part of the set $\Psi(W)$. More precisely, there is $\widehat{y}_0\in\mathbb{S}^{d-1}$ such that

$$\begin{aligned}|\Psi(W)|&=\int_0^\infty t^{d-1}\int_{\mathbb{S}^{d-1}}\mathbf{1}_{\Psi(W)}(\Psi(x_0)+t\widehat{y})\,\mathrm{d}\sigma(\widehat{y})\,\mathrm{d}t\\&\leq d\tau_d\int_0^\infty t^{d-1}\mathbf{1}_{\Psi(W)}(\Psi(x_0)+t\widehat{y}_0)\,\mathrm{d}t.\end{aligned}$$

Hence, denoting the line segment by $\Psi(I)$ where $I=\{x_0+\Psi^{-1}(\widehat{y}_0)\in Q\colon s\geq 0\}$ and observing that the length $|\Psi(I)|$ of the line segment can not exceed the diameter of $\Psi(Q)$, we further estimate

$$\frac{|\Psi(W)|}{d\tau_d(\operatorname{diam}\Psi(Q))^d}\leq\frac{|\Psi(I)|^{d-1}|\Psi(W\cap I)|}{(\operatorname{diam}\Psi(Q))^d}\leq\frac{|\Psi(W\cap I)|}{|\Psi(I)|}.\tag{7.25}$$

In particular, the intersection between $\Psi(I)$ and $\Psi(W)$ is not to small.

Let $\widehat{\xi}_0=\Psi^{-1}(\widehat{y}_0)/|\Psi^{-1}(\widehat{y}_0)|$ be the unit vector in direction $\Psi^{-1}(\widehat{y}_0)$. Then, for $z\in D(4+\varepsilon)$ with sufficiently small $\varepsilon>0$ we have $x_0+t|I|\widehat{\xi}_0\in Q+D_{4l}$ and, therefore,

$$\Phi\colon D(4+\varepsilon)\to\mathbb{C},\ z\mapsto\frac{|Q|^{1/2}}{\|f\|_{L^2(Q)}}\cdot f(x_0+t|I|\widehat{\xi}_0)$$

is analytic by assumption. Moreover, the normalized supremum satisfies and

$$M_\Phi=\sup_{z\in D(0,4)}|\Phi(z)|\leq\frac{|Q|^{1/2}}{\|f\|_{L^2(Q)}}\sup_{z\in Q+D_{4l}}|f(z)|.$$

Let $E = \{t \in [0,1] \colon x_0 + t|I|\widehat{\xi_0} \in W\}$ be the part of the line segment where $|\Phi|$ is small. By the choice of x_0 we have $|\Phi(0)| \geq 1$, while, on the other hand,

$$\sup_{t\in E}|\Phi(t)| \leq \frac{|Q|^{1/2}}{\|f\|_{L^2(Q)}} \sup_{x\in W}|f(x)|. \tag{7.26}$$

By construction of E and by Lemma 7.16 we have a lower bound for $|\Phi|$ on E. Indeed, using that a simple computation shows that the measure of E is given by $|E| = |W \cap I|/|I| = |\Psi(W \cap I)|/|\Psi(I)|$, applying Lemma 7.16 with E, and using (7.25) we obtain

$$\Big(\frac{|\Psi(W)|}{12d\tau_d(\operatorname{diam}\Psi(Q))^d}\Big)^{2\log M_\Phi/\log 2} \leq \sup_{t\in E}|\Phi(t)|. \tag{7.27}$$

Inequality (7.26) now plays the upper bound for $|f|$ on W off against the lower bound on the left-hand side of (7.27), leading to the desired inequality. In fact, by the definition of W we have

$$\sup_{x\in W}|f(x)| \leq \Big(\frac{|\Psi(Q\cap\omega)|}{24d\tau_d(\operatorname{diam}\Psi(Q))^d}\Big)^{2\log M_\Phi/\log 2}\frac{\|f\|_{L^2(Q)}}{|Q|^{1/2}},$$

but using (7.27) and (7.26) on the right-hand side we derive at

$$\sup_{x\in W}|f(x)| \leq \Big(\frac{|\Psi(Q\cap\omega)|}{2|\Psi(W)|}\Big)^{2\log M_\Phi/\log 2} \cdot \sup_{x\in W}|f(x)|. \tag{7.28}$$

Since $W \neq \emptyset$ and f is continuous, we have $\sup_{x\in W}|f(x)| > 0$, and dividing both sides of the last inequality by this factor we obtain $|\Psi(Q\cap\omega)| \geq 2|\Psi(W)|$, or, equivalently, the asserted inequality $|Q\cap\omega| \geq 2|W|$. □

In order to apply Lemma 7.15 with $Q = Q_k$, $k \in \mathcal{K}$, we need to assume that each of the covering elements Q_k is contained in a hyperrectangle, i.e., that there are $l_k = (l_k^{(1)}, \dots, l_k^{(d)}) \in (0,\infty)^d$ and $z_k \in Q_k$, $k \in \mathcal{K}$, such that

$$Q_k \subset z_k + \bigtimes_{j=1}^{d}(0, l_k^{(j)}). \tag{7.29}$$

However, as indicated by the index, the sidelengths $(l_k)_{k\in\mathcal{K}}$ *do not* need to be uniformly bounded for all $k \in \mathcal{K}$ and for now we only assume the existence of l_k such that (7.29) holds. Then, for Q_k with $k \in \mathcal{K}_g$, using Taylor expansion around the point $x_k \in Q_k$ from Lemma 7.12 where we have control over all derivatives, we now obtain an estimate for the supremum

$$M_k := \frac{\sqrt{|Q_k|}}{\|f\|_{L^2(Q_k)}} \cdot \sup_{z\in Q_k + D_{4l_k}}|f(z)| \tag{7.30}$$

of a non-zero $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$.

LEMMA 7.18. *Let $k \in \mathcal{K}_g$ and suppose that there are $l_k \in (0,\infty)^d$ and $z_k \in \mathbb{R}^d$ such that* (7.29) *holds. Then, the quantity M_k in* (7.30) *satisfies*

$$M_k \le 2\kappa^{1/2} \sum_{m\in\mathbb{N}_0} C_B(m,\lambda)^{1/2} \frac{(10|l_k|)^m}{m!}.$$

PROOF. Let $x_k \in Q_k$ be a point as in (7.19). Using Taylor expansion of f around x_k, for every $z \in x_k + D_{5l_k}$ we then have

$$\begin{aligned} |f(z)| &\le \sum_{\alpha\in\mathbb{N}_0^d} \frac{|\partial^\alpha f(x_k)|}{\alpha!} |(z-x_k)^\alpha| \le \sum_{m\in\mathbb{N}_0} \sum_{|\alpha|=m} \frac{|\partial^\alpha f(x_k)|}{\alpha!} (5l_k)^\alpha \\ &\le \sum_{m\in\mathbb{N}_0} \Bigl(\sum_{|\alpha|=m} \frac{(5l_k)^{2\alpha}}{\alpha!} \Bigr)^{1/2} \Bigl(\sum_{|\alpha|=m} \frac{|\partial^\alpha f(x_k)|^2}{\alpha!} \Bigr)^{1/2} \\ &= \sum_{m\in\mathbb{N}_0} \frac{(5|l_k|)^m}{\sqrt{m!}} \Bigl(\sum_{|\alpha|=m} \frac{|\partial^\alpha f(x_k)|^2}{\alpha!} \Bigr)^{1/2} \\ &\le 2\kappa^{1/2} \frac{\|f\|_{L^2(Q_k)}}{\sqrt{|Q_k|}} \sum_{m\in\mathbb{N}_0} C_B(m,\lambda)^{1/2} \frac{(10|l_k|)^m}{m!}, \end{aligned}$$

where for the second last step we used that $\sum_{|\nu|=m} l_k^{2\nu}/\nu! = |l_k|^{2m}/m!$. Taking into account that $Q_k + D_{4l_k} \subset x_k + D_{5l_k}$, this proves the claim. □

7.2.4. Generalized spectral inequality. We are now in the position to state and prove our generalized spectral inequality. To this end, for each fixed $\lambda \ge 1$, we formulate an abstract hypothesis on the covering elements Q_k with $k \in \mathcal{K}_c(\lambda)$.

HYPOTHESIS (H_λ). Let $\mathcal{K}$ be finite or countably infinite and let $(Q_k)_{k\in\mathcal{K}}$ be an essential covering of $\mathbb{R}^d$ with multiplicity at most κ as in (7.13). For fixed $\lambda \ge 1$ and all $k \in \mathcal{K}_c$, where $\mathcal{K}_c = \mathcal{K}_c(\lambda)$ is as in (7.14), we suppose that

- (i) Q_k is nonempty, convex, open, and contained in a hyperrectangle with sides of length $l_k = (l_k^{(1)}, \dots, l_k^{(d)}) \in (0,\infty)^d$ parallel to the coordinate axes;
- (ii) the sidelengths satisfy $|l_k| \le D\lambda^{(1-\varepsilon)/2}$ for some $\varepsilon \in (0,1]$ and $D > 0$ independent of $k \in \mathcal{K}_c$;
- (iii) there is a linear bijection $\Psi_k \colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\frac{|\Psi_k(Q_k)|}{(\operatorname{diam} \Psi_k(Q_k))^d} \ge \eta \tag{7.31}$$

for some $\eta > 0$ independent of $k \in \mathcal{K}_c$.

A covering as in Hypothesis (H_λ) satisfies all the properties of the covering used in the previous Subsections 7.2.2 and 7.2.3. In addition, property (ii) gives an upper bound for the size of the covering elements, which essentially entails the sublinear dependence on the energy parameter λ. The existence of a bijection Ψ_k as required in property (iii) is used in the proof below to give a uniform lower bound for the constant in the local estimate from Lemma 7.15. It is *always* guaranteed that such bijection exists for $\eta = \tau_d/(2d)^d$, see the discussion in Subsection 7.2.6 below. However, we state it here as a hypothesis, as it may be possible to optimize the constant η by choosing better adapted Ψ_k for certain coverings, cf. Example 7.25.

With Hypothesis (H_λ) at hand, we are now ready to prove the following result, which is not a spectral inequality in the usual sense as the energy λ is fixed and the geometric condition on ω depends on λ. For this reason, it suffices to impose conditions on the sensor set ω only with respect to the bounded region covered by Q_k with $k \in \mathcal{K}_c = \mathcal{K}_c(\lambda)$.

THEOREM 7.19 (Generalized spectral inequality). *With fixed $\lambda \geq 1$ assume Hypothesis (H_λ), and let $a \geq 0$ and $\gamma \in (0,1]$ be given. Then, if $\omega \subset \mathbb{R}^d$ is a measurable set satisfying*

$$\frac{|Q_k \cap \omega|}{|Q_k|} \geq \gamma^{\lambda^{a/2}} \quad \text{for all} \quad k \in \mathcal{K}_c, \tag{7.32}$$

we have

$$\|f\|^2_{L^2(\omega)} \geq \frac{3}{\kappa}\Big(\frac{\eta\gamma}{24d\tau_d}\Big)^{7\left(1600\mathrm{e}D(D+1)+\log(4\kappa^{1/2})\right)\lambda^{1-(\varepsilon-a)/2}} \|f\|^2_{L^2(\mathbb{R}^d)}, \tag{7.33}$$

for every $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$.

REMARK 7.20. Let us emphasize that, on one hand, ε and D in condition (ii) as well as η in condition (iii) need to be uniform in $k \in \mathcal{K}_c$. On the other, formally they are allowed to depend on λ. However, in all applications presented here this will not be the case implying that the exponent in (7.33) is proportional to $\lambda^{1-\frac{\varepsilon-a}{2}}$. In this case, the relevant power satisfies $1 - \frac{\varepsilon-a}{2} < 1$ if and only if $a < \varepsilon$.

PROOF OF THEOREM 7.19. In light of Hypothesis (H_λ) and the local estimate in Lemma 7.15 we have

$$\|f\|^2_{L^2(Q_k\cap\omega)} \geq a_k \|f\|^2_{L^2(Q_k)} \quad \text{with} \quad a_k = 12\Big(\frac{|Q_k\cap\omega|}{24d\tau_d|Q_k|}\Big)^{4^{\frac{\log M_k}{\log 2}}+1}$$

for $k \in \mathcal{K}_c = \mathcal{K}_c(\lambda)$, where M_k is as in (7.30). By Lemma 7.14 we further estimate

$$\begin{aligned}\Big(\min_{k\in\mathcal{K}_c\cap\mathcal{K}_g} a_k\Big)\|f\|^2_{L^2(\mathbb{R}^d)} &\leq 4\sum_{k\in\mathcal{K}_c\cap\mathcal{K}_g} a_k\|f\|^2_{L^2(Q_k)} \\ &\leq 4\sum_{k\in\mathcal{K}_c\cap\mathcal{K}_g}\|f\|^2_{L^2(Q_k\cap\omega)} \leq 4\kappa\|f\|^2_{L^2(\omega)}\end{aligned} \tag{7.34}$$

and it suffices to establish a suitable lower bound for the minimum on the left-hand side. Using the assumption (7.32) on the set ω, we have

$$a_k \geq 12\left(\frac{\eta\gamma^{\lambda^{a/2}}}{24d\tau_d}\right)^{4\frac{\log M_k}{\log 2}+1} \quad \text{for all} \quad k \in \mathcal{K}_c. \tag{7.35}$$

In order to proceed further, we recall the definition of $C_B(m,\lambda)$ from (7.9) and show

$$2^{-m}C_B(m,\lambda) = \prod_{k=0}^{m-1}(\lambda+2k) \leq (2\delta)^{2m}\mathrm{e}^{\mathrm{e}/\delta^2}(m!)^2\mathrm{e}^{2\sqrt{\lambda}/\delta} \quad \text{for} \quad \delta > 0. \tag{7.36}$$

To this end, we follow the arguments given in the proof of [BJPS21, Proposition 4.3] along the lines and distinguish between two cases:

- Let $2(m-1) \leq \lambda$ so that $\lambda + 2k \leq \lambda + 2(m-1) \leq 2\lambda$. Hence, the product under consideration is clearly bounded by $(2\lambda)^m$ and we get $(2\lambda)^m \leq (\sqrt{2}\delta)^{2m}(\lambda^{1/2}/\delta)^{2m} \leq (m!)^2\exp(2\sqrt{\lambda}/\delta)$.
- Now, suppose that $2(m-1) \geq \lambda$ so that $\lambda + 2k \leq 2(m-1) + 2k \leq 4(m-1)$. Hence, the product under consideration is now bounded by $(4m)^m \leq 2^{2m}m!\mathrm{e}^m = (2\delta)^{2m}m!(\mathrm{e}/\delta^2)^m$. Estimating the last factor therefore gives $(4m)^m \leq (2\delta)^{2m}(m!)^2\exp(\mathrm{e}/\delta^2)$.

Moreover, we recall that condition (ii) of Hypothesis (H_λ) gives $|l_k| \leq D\lambda^{(1-\varepsilon)/2}$ for all $k \in \mathcal{K}_c$. Therefore Lemma 7.18 and the above inequality (7.36) imply

$$\begin{aligned} M_k &\leq 2\kappa^{1/2}\sum_{m\in\mathbb{N}_0} C_B(m,\lambda)^{1/2}\frac{(10D\lambda^{(1-\varepsilon)/2})^m}{m!} \\ &= 2\kappa^{1/2}\mathrm{e}^{\mathrm{e}/(2\delta^2)}\mathrm{e}^{\sqrt{\lambda}/\delta}\sum_{m\in\mathbb{N}_0}(20\sqrt{2}\delta D\lambda^{(1-\varepsilon)/2})^m \quad \text{for all} \quad k \in \mathcal{K}_c \cap \mathcal{K}_g, \end{aligned}$$

where $\delta > 0$ is arbitrary. However, for the particular choice

$$\delta = \left(40\sqrt{2}D\lambda^{(1-\varepsilon)/2}\right)^{-1},$$

the series converges and we obtain

$$\begin{aligned} M_k &\leq 4\kappa^{1/2}\exp(1600\mathrm{e}D^2\lambda^{1-\varepsilon} + 40\sqrt{2}D\lambda^{(1-\varepsilon)/2}\sqrt{\lambda}) \\ &\leq 4\kappa^{1/2}\exp(1600\mathrm{e}D(D+1)\lambda^{1-\varepsilon/2}). \end{aligned}$$

Thus,

$$\begin{aligned} \log M_k &\leq \log(4\kappa^{1/2}) + 1600\mathrm{e}D(D+1)\lambda^{1-\varepsilon/2} \\ &\leq \left(1600\mathrm{e}D(D+1) + \log(4\kappa^{1/2})\right)\lambda^{1-\varepsilon/2} \end{aligned}$$

for all $k \in \mathcal{K}_c \cap \mathcal{K}_g$. Combining the latter with (7.35), we arrive at

$$a_k \geq 12\Big(\frac{\eta\gamma}{24d\tau_d}\Big)^{7\left(1600\mathrm{e}D(D+1)+\log(4\kappa^{1/2})\right)\lambda^{1-(\varepsilon-a)/2}} \quad \text{for all} \quad k \in \mathcal{K}_c \cap \mathcal{K}_g,$$

where we used that $1 + 4/\log 2 \leq 7$. In view of (7.34), this proves the claim. □

7.2.5. Proof of the spectral inequalities. It remains to conclude the spectral inequalities for the partial harmonic oscillators from the generalized spectral inequality. To this end, we discuss examples of sets $\omega \subset \mathbb{R}^d$ where Theorem 7.19 can be applied with D and ε not depending on λ. In the situation of Theorem 4.19, these sets are characterized in terms of an explicit covering, but for Corollary 7.21 below the covering is implicitly constructed using Besicovitch's covering theorem. Both results should be regarded as corollaries to Theorem 7.19.

PROOF OF THEOREM 4.19. Let $Q_k = \Lambda_\rho(k)$ for $k \in \mathcal{K} = (\rho\mathbb{Z})^d$. We then have $\kappa = 1$ and, thus, $C = 6$ in Lemma 7.11. Moreover, with Ψ_k in condition (ii) being the identity, we may choose $\eta = 1/d^{d/2}$. Taking into account that Stirling's formula implies the asymptotic formula $\tau_d \sim (2\pi\mathrm{e}/d)^{d/2}/\sqrt{d\pi}$, we infer that $24d\tau_d/\eta \leq K^d$. Furthermore, it is easy to see that $l_k = (\rho, \dots, \rho)$ satisfies $|l_k| = d^{1/2}\rho = D\lambda^0$ with $D := d^{1/2}\rho$. Hence, $(Q_k)_{k\in\mathcal{K}} = (\Lambda_\rho(k))_{k\in(\rho\mathbb{Z})^d}$ satisfies Hypothesis (H_λ) for every $\lambda \geq 1$. Both constants D and η are independent of λ.

Now we show

$$\frac{|k_\mathcal{I}|}{2} \leq \inf_{x\in\Lambda_\rho(k)} |x_\mathcal{I}| \leq C\lambda^{1/2} \quad \text{for all} \quad k \in \mathcal{K}_c \subset (\rho\mathbb{Z})^d.$$

The upper bound follows instantly from the definition of $\mathcal{K}_c$ and we only need to prove the lower bound. To this end, let $x \in \Lambda_\rho(k)$ and $j \in \{1, \dots, d\}$. Suppose that $k_j \neq 0$. Then we have $|k_j| \geq \rho$ and $|x_j - k_j| \leq \rho/2$. Therefore, $|x_j| \geq |k_j| - |x_j - k_j| \geq |k_j| - \rho/2 \geq |k_j|/2$. Since the same inequality is trivially satisfied if $k_j = 0$, summing over $j \in \mathcal{I}$ gives $|x| \geq |k_\mathcal{I}|/2$ for all $x \in \Lambda_\rho(k)$. This proves the lower bound.

Finally, using this estimate for the infimum, we get that for $k \in \mathcal{K}_c$ we have

$$\gamma^{1+|k_\mathcal{I}|^\alpha} \geq \big(\gamma^{2^\alpha}\big)^{1+(|k_\mathcal{I}|/2)^\alpha} \geq \big(\gamma^{2^\alpha}\big)^{1+C^\alpha\lambda^{\alpha/2}} \geq \big(\gamma^{2(2C)^\alpha}\big)^{\lambda^{\alpha/2}}.$$

Using that ω satisfies

$$\frac{|\Lambda_\rho(k) \cap \omega|}{|\Lambda_\rho(k)|} \geq \gamma^{1+|k_\mathcal{I}|^\alpha} \quad \text{for all} \quad k \in (\rho\mathbb{Z})^d, \tag{4.15 revisited}$$

this shows

$$\frac{|\Lambda_\rho(k) \cap \omega|}{|\Lambda_\rho(k)|} \geq \big(\gamma^{2(2C)^\alpha}\big)^{\lambda^{\alpha/2}} \quad \text{for all} \quad k \in \mathcal{K}_c \subset (\rho\mathbb{Z})^d.$$

The claim in Theorem 4.19 now follows from Theorem 7.19 with $\varepsilon = 1$, $a = \alpha$, and γ replaced by $\gamma^{2(2C)^\alpha}$. It only remains to observe the specific constant from the simple estimate

$$2 \cdot (2C)^\alpha \cdot 7\big(1600\mathrm{e}D(D+1) + \log(4)\big) \leq K^{1+\alpha} \cdot d \cdot (1+\rho)^2. \qquad \square$$

Motivated by Theorem 4.15, we also consider sets ω that are thick with respect to a scale that is allowed to vary in the coordinate directions corresponding to $\mathcal{I}$. To this end, let $\rho_1 \colon \mathbb{R}^{d_1} \to (0,\infty)$ be any function that satisfies

$$\rho_1(x) \leq R(1+|x|^2)^{\frac{1-\varepsilon}{2}} \quad \text{for all} \quad x \in \mathbb{R}^{d_1}$$

with $R > 0$ and $\varepsilon \in (0,1]$, and let $\rho_2 > 0$. Given $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$ we set

$$Q(x) := B^{(d_1)}(x^{(1)}, \rho_1(x^{(1)})) \times \Lambda_{\rho_2}(x^{(2)}) \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

The following result generalizes Theorem 4.16. Here, if $d_1 = d$, then $d_2 = 0$ and the second factors in the tensor sets $Q(x)$ are empty. In this case the proof of the next corollary can be carried out in the same way, but is even simpler.

Corollary 7.21. *Let $0 \leq a < \varepsilon \leq 1$. For all measurable sets $\omega \subset \mathbb{R}^d$ satisfying*

$$\frac{|Q(x) \cap \omega|}{|Q(x)|} \geq \gamma^{1+|x_{\mathcal{I}}|^a} \quad \textit{for all} \quad x \in \mathbb{R}^d \tag{7.37}$$

and some $\gamma \in (0,1]$, we have

$$\|f\|^2_{L^2(\omega)} \geq 3\Big(\frac{\gamma}{K^d}\Big)^{K^{1+a}d^{3+a/2}(1+R+\rho_2)^2\lambda^{1-\frac{\varepsilon-a}{2}}} \|f\|^2_{L^2(\mathbb{R}^d)} \tag{7.38}$$

for all $\lambda \geq 1$ and all $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$.

In contrast to the situation in the previous proof of Theorem 4.19, the proof of Corollary 7.21 starts with the construction of the family $(Q_k)_{k\in\mathcal{K}}$, as it is this time not given explicitly in the statement of the result. For this purpose, we use the following formulation of the well-known Besicovitch covering theorem which allows to extract a countable subcovering $(Q_k)_{k\in\mathcal{K}}$ of $(Q(x))_{x\in\mathbb{R}^d}$ with bounded multiplicity.

Proposition 7.22 (Besicovitch). *If $V \subset \mathbb{R}^{d_1}$ is a bounded set and $\mathcal{B}$ is a family of closed balls such that each point in V is the center of some ball in $\mathcal{B}$, then there are at most countably many balls $(\overline{B}_k) \subset \mathcal{B}$ such that with some universal constant $C_0 \geq 1$ we have*

$$\mathbf{1}_V \leq \sum_k \mathbf{1}_{\overline{B}_k} \leq C_0^{d_1}.$$

Proof. The proof of Besicovitch's theorem in [Mat95, Theorem 2.7] shows that the statement of the proposition holds with $C_0^{d_1} = 16^{d_1} C_1$, where C_1 is chosen such that the following implication is true: *If $y_1, \ldots, y_n \in \mathbb{S}^{d_1-1}$ are points with*

$|y_r - y_s| \geq 1$ *for all* $r \neq s$, *then* $n \leq C_1$. Since for such points the spherical distance $d_{\mathbb{S}^{d_1-1}}(y_r, y_s)$ of y_r and y_s can be bounded from below by

$$d_{\mathbb{S}^{d_1-1}}(y_r, y_s) = \arccos\Big(1 - \frac{|y_r - y_s|^2}{2}\Big) \geq \arccos(1 - 1/2) = \pi/3,$$

it is easy to see that $C_1 \leq K^{d_1}$. This proves the statement with $C_0 = (16C_1)^{d_1}$. □

Proof of Corollary 7.21. We consider only the case $d_1 < d$. Let $\lambda \geq 1$ and set $V = B^{(d_1)}(0, C\lambda^{1/2}) \subset \mathbb{R}^{d_1}$, where $C = 6(1 + (\log(C_0^{d_1}))^{1/2})$. Then, the assumptions of Proposition 7.22 are fulfilled for V and the family of balls

$$\mathcal{B} = \{\overline{B^{(d_1)}(x, \rho_1(x))} \colon x \in V\}.$$

This shows that there is a subset $\mathcal{K}_* \subset \mathbb{N}$ and a collection of points $(y_j)_{j\in\mathcal{K}_*} \subset V$ such that the balls $B_j = B^{(d_1)}(y_j, \rho_1(y_j))$ satisfy $|V \setminus \bigcup_{j\in\mathcal{K}_*} B_j| = 0$. Setting $B_0 = \mathbb{R}^{d_1} \setminus \bigcup_{j\in\mathcal{K}_*} B_j$, the family $(B_j)_{j\in\mathcal{N}_0}$, $\mathcal{N}_0 = \mathcal{K}_* \cup \{0\}$, is then an essential covering of $\mathbb{R}^{d_1}$ with

$$\sum_{j\in\mathcal{N}_0} \mathbf{1}_{B_j} \leq C_0^{d_1} =: \kappa.$$

Set $\mathcal{K} := \mathcal{N}_0 \times (\rho_2\mathbb{Z})^{d_2}$ and $Q_k := B_{k^{(1)}} \times \Lambda_{\rho_2}(k^{(2)})$ for $k = (k^{(1)}, k^{(2)}) \in \mathcal{K}$. Then, $(Q_k)_{k\in\mathcal{K}}$ is an essential covering of $\mathbb{R}^d$ with with multiplicity at most κ. By construction we have

$$\mathcal{K}_c = \{k \in \mathcal{K} \colon Q_k \cap (B^{(d_1)}(0, C\lambda^{1/2}) \times \mathbb{R}^{d_2}) \neq \emptyset\} = \mathcal{K}_* \times (\rho_2\mathbb{Z})^{d_2}$$

and $Q_k = Q((y_{k^{(1)}}, k^{(2)}))$ for $k \in \mathcal{K}_c$.

We show that $(Q_k)_{k\in\mathcal{K}}$ satisfies Hypothesis (H_λ): It is easy to see that (i) is satisfied with $l_k = (2\rho_1(y_{k^{(1)}}), \ldots, 2\rho_1(y_{k^{(1)}}), \rho_2, \ldots, \rho_2)$. In order to verify condition (ii), we follow [ES21, Lemma C.1] and consider for $k \in \mathcal{K}_c$ the linear bijections $\Psi_k \colon \mathbb{R}^d \to \mathbb{R}^d$ with

$$\Psi_k(x) = (rx^{(1)}/\operatorname{diam} B_{k^{(1)}}, x^{(2)}/\operatorname{diam} \Lambda_{\rho_2}(k^{(2)})), \quad \text{where} \quad r^2 = d_1/d_2.$$

With $(\operatorname{diam} \Psi_k(Q_k))^2 = 1 + r^2$ it is then not difficult to show

$$\frac{|\Psi_k(Q_k)|}{(\operatorname{diam} \Psi_k(Q_k))^d} = \frac{\tau_{d_1} d_1^{d_1/2}}{2^{d_1} d^{d/2}} =: \eta,$$

which proves condition (ii). In particular, using the asymptotic formula for τ_d we have

$$\frac{24d\tau_d}{\eta} \sim \frac{d^{1+d/2}(2\pi\mathrm{e}/d)^{d/2} 2^{d_1}\sqrt{d_1\pi}}{\sqrt{d\pi}\, d_1^{d_1/2}(2\pi\mathrm{e}/d_1)^{d_1/2}} \leq K^d.$$

Since $y_{k^{(1)}} \in V$ for all $k \in \mathcal{K}_c$, we have $|y_{k^{(1)}}| \leq C\lambda^{1/2}$ and, consequently,

$$\rho_1(y_{k^{(1)}}) \leq 2RC\lambda^{(1-\varepsilon)/2} \quad \text{for all} \quad k \in \mathcal{K}_c.$$

Combining this with the identity for l_k stated above, we obtain

$$\|l_k\|_2 \leq \|l_k\|_1 \leq 2d_1\rho_1(y_{k^{(1)}}) + d_2\rho_2 \leq D\lambda^{(1-\varepsilon)/2} \quad \text{with} \quad D = d(4RC + \rho_2),$$

which proves condition (iii). Thus, Hypothesis (H_λ) is satisfied.

Using again $|y_{k^{(1)}}| \leq C\lambda^{1/2}$ for $k \in \mathcal{K}_c$, we see that the hypothesis on the set ω yields

$$\frac{|Q_k \cap \omega|}{|Q_k|} \geq \gamma^{1+(C\lambda^{1/2})^a} \geq \left(\gamma^{1+C^a}\right)^{\lambda^{a/2}} \quad \text{for} \quad k \in \mathcal{K}_c,$$

and this shows, in turn, that the assumptions of Theorem 7.19 are fulfilled with γ replaced by γ^{1+C^a}. Hence, the constant is given by

$$\begin{aligned}\frac{3}{\kappa}\Big(\frac{\eta\gamma^{1+C^a}}{24d\tau_d}\Big)^{7\left(1600\mathrm{e}D(D+1)+\log(4\kappa^{1/2})\right)\lambda^{1-(\varepsilon-a)/2}} \\ \geq 3\Big(\frac{\gamma}{K^d}\Big)^{(1+C^a)\cdot 7\left(1600\mathrm{e}D(D+1)+\log(4\kappa^{1/2})\right)\lambda^{1-(\varepsilon-a)/2}}.\end{aligned}$$

In order to get the constant from the statement of the corollary, we collect from the previous computations that $\kappa \leq K^d$, $1 + C^a \leq (1 + K^a)d^{a/2}$, as well as $D \leq Kd^{3/2}(R + \rho_2)$. Thereby, it is easy to see that

$$(1 + C^a) \cdot 7\big(1600\mathrm{e}D(D + 1) + \log(4\kappa^{1/2})\big) \leq K^{1+a}d^{3+a/2}(1 + R + \rho_2)^2,$$

which gives the precise constant in the statement. □

7.2.6. Discussion and extensions. Let us note here two observations that admit us to slightly generalize the above approach. The first of these allows to get rid of property (iii) in the Hypothesis (H_λ) which, however, might slightly worsen the dependence of the constants on the dimension d. The second allows, via a partial Fourier transform, to prove a spectral inequality for the operators $H_{\mathcal{I},\mathcal{J}}$ with $\mathcal{J} \subsetneq \{1, \ldots, d\}$.

John's lemma and the bijections Ψ_k**.** As already stated above, property (iii) of Hypothesis (H_λ) is always satisfied if we choose η appropriately. While in the proofs of Theorem 4.19 and Corollary 7.21 we were able to explicitly construct suitable bijections (since the geometry of the covering sets is quite simple), it might be considerably harder for general convex coverings. However, even in this situations the following corollary to John's ellipsoid theorem guarantees the existence of suitable bijections.

PROPOSITION 7.23. *Let $\emptyset \neq Q \subset \mathbb{R}^d$ be convex, open, and bounded. Then there is an linear bijection $\Psi\colon \mathbb{R}^d \to \mathbb{R}^d$ with*

$$\eta := \frac{\tau_d}{2^d d^d} \leq \frac{|\Psi(Q)|}{(\operatorname{diam}\Psi(Q))^d} \leq \frac{\tau_d}{2^{d/2}}. \tag{7.39}$$

If, in addition, Q is centrally symmetric, i.e., if there is $x_0 \in Q$ such that $x_0 + x \in Q$ implies $x_0 - x \in Q$, then η can be replaced by $\tau_d/(4d)^{d/2}$.

Proof. For the lower bound we use John's theorem [Joh48], which states that for every convex, open, bounded $\emptyset \neq Q \subset \mathbb{R}^d$ there is a linear bijection $\Upsilon\colon \mathbb{R}^d \to \mathbb{R}^d$, some $z \in \mathbb{R}^d$, and a radius $r > 0$ such that the ellipsoid $\mathcal{E} = \Upsilon(B(0,r))$ satisfies $\mathcal{E} \subset Q + z \subset d \cdot \mathcal{E}$ or, equivalently, setting $\Psi = \Upsilon^{-1}$, we have

$$B(0,r) \subset \Psi(Q) + \Psi z \subset d \cdot B(0,r).$$

This implies that $2r \leq \operatorname{diam} \Psi(Q) \leq 2rd$, as well as

$$|\Psi(Q)| \geq \tau_d r^d \geq \tau_d \Big(\frac{\operatorname{diam}(\Psi(Q))}{2d}\Big)^d = \frac{\tau_d}{(2d)^d}\big(\operatorname{diam}(\Psi(Q))\big)^d.$$

For centrally symmetric Q, John's theorem gives $\mathcal{E} \subset Q + z \subset \sqrt{d} \cdot \mathcal{E}$ leading in the same way to the stated claim in this case.

For the upper bound we recall from Jung's theorem [Jun01] that the set $\Psi(Q)$ is contained in a ball B of radius $R > 0$ satisfying

$$R \leq \operatorname{diam}(\Psi(Q))\sqrt{1/(2+1/d)} \leq \frac{\operatorname{diam}(\Psi(Q))}{\sqrt{2}}.$$

Hence,

$$|\Psi(Q)| \leq |B| = \frac{\tau_d}{2^{d/2}}\big(\operatorname{diam} \Psi(Q)\big)^d. \qquad \square$$

Using the last proposition we get the following corollary to Theorem 7.19 by simply plugging in the above choice for η.

Corollary 7.24. *With fixed $\lambda \geq 1$ assume Hypothesis* (H_λ) *except condition (iii). Then, for all ω satisfying* (7.32) *we have*

$$\|f\|^2_{L^2(\omega)} \geq \frac{3}{\kappa}\Big(\frac{\gamma}{K_d}\Big)^{7\left(1600\mathrm{e}D(D+1)+\log(4\kappa^{1/2})\right)\lambda^{1-(\varepsilon-a)/2}} \|f\|^2_{L^2(\mathbb{R}^d)}, \tag{7.40}$$

for every $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I}})$, $\lambda \geq 1$, where

$$K_d = \begin{cases} 24 \cdot 2^d d^{1+d}, & \text{if all } Q_k,\ k \in \mathcal{K}_c, \text{ are convex;}\\ 24 \cdot 2^d d^{1+d/2}, & \text{if all } Q_k,\ k \in \mathcal{K}_c, \text{ are centrally symmetric;}\\ 24 \cdot d^{1+d/2}\tau_d, & \text{if all } Q_k,\ k \in \mathcal{K}_c \text{ are cubes.}\end{cases}$$

The next example shows that the lower bound from Proposition 7.23 can be improved by choosing a customized bijection.

Example 7.25. Let $Q = B(0,1)$, let Ψ be as in Proposition 7.23 and let $\tilde{\Psi}$ be the identity. Then, since Q is centrally symmetric, we have

$$\frac{|\Psi(Q)|}{(\operatorname{diam} \Psi(Q))^d} \geq \frac{\tau_d}{(4d)^{d/2}} \quad \text{and} \quad \frac{|\tilde{\Psi}(Q)|}{(\operatorname{diam} \tilde{\Psi}(Q))^d} \geq \frac{\tau_d}{2^d}.$$

Hence, the two situations differ by the factor $d^{d/2}$.

Spectral inequalities with parts of pure multiplication. So far we have restricted our considerations to the situation where we have a full Laplacian perturbed by some potential $V(x) = |x_{\mathcal{I}}|^2$ with $\mathcal{I} \subset \{1, \dots, d\}$. We now show that using the Fourier transform we can also prove a spectral inequality for the operator $H_{\mathcal{I},\mathcal{J}}$ with $\mathcal{I} \setminus \mathcal{J} \neq \emptyset$. However, here the observation operator $\mathcal{C}$ is not the characteristic function of a measurable set.

Suppose that $\mathcal{I} \cup \mathcal{J} = \{1, \dots, d\}$ while $\mathcal{J} \neq \{1, \dots, d\}$. Then, by Lemma 2.9 the operator $H_{\mathcal{I},\mathcal{J}}$ is unitary equivalent to the operator $H_{\mathcal{I}\cap\mathcal{J}}$ via the partial Fourier transform $\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$ from (2.7). Thus, spectral inequalities of the form $\|\mathbf{1}_\omega f\|_2^2 \geq C\|f\|_2^2$ for $H_{\mathcal{I}\cap\mathcal{J}}$ translate directly to spectral inequalities for $H_{\mathcal{I},\mathcal{J}}$ of the form $\|\mathcal{C}f\|_2^2 \geq C\|f\|_2^2$ with $\mathcal{C} = \mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}^{-1}\mathbf{1}_\omega\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$ and the same constant $C > 0$. Since $H_{\mathcal{I}\cap\mathcal{J}}$ is an operator of the form discussed in the previous parts of this section, we thus get a spectral inequality for $H_{\mathcal{I},\mathcal{J}}$. This is exemplified in the following result for the situation of Theorem 4.19.

COROLLARY 7.26. *Suppose $\mathcal{I} \cup \mathcal{J} = \{1, \dots, d\}$, $\mathcal{I} \setminus \mathcal{J} \neq \emptyset$, and let ω be as in* (4.15). *Then, there is a universal constant $K \geq 1$ such that for every $\lambda \geq 1$ and all $f \in \operatorname{Ran} P_\lambda(H_{\mathcal{I},\mathcal{J}})$ we have*

$$\|\mathcal{C}f\|_{L^2(\mathbb{R}^d)}^2 \geq 3\Big(\frac{\gamma}{K^d}\Big)^{K^{1+\alpha}d\cdot(1+\rho)^2\lambda^{(1+\alpha)/2}}\|f\|_{L^2(\mathbb{R}^d)}^2,$$

where $\mathcal{C} = \mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}^{-1}\mathbf{1}_\omega\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$.

The case $\mathcal{I} \cup \mathcal{J} \neq \{1, \dots, d\}$ can be reduced to this provided the sensor sets are chosen as appropriate Cartesian products. This is the reason why the assumption $\mathcal{I} \cup \mathcal{J} = \{1, \dots, d\}$ is not a constraint here.

REMARK 7.27. If ω is Borel measurable, then $B = \mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}^{-1}\mathbf{1}_\omega\mathcal{F}_{\mathcal{I}\setminus\mathcal{J}}$ can be interpreted by functional calculus. Let $X_1, \dots, X_d$ be the strongly commuting position operators $X_j f = x_j f$. Then the multiplication operator $\mathbf{1}_\omega$ agrees with $\mathbf{1}_\omega(X_1, \dots, X_d)$ defined by joint functional calculus, cf. [Sch12, Chapter 5.5]. Since the momentum operators $P_1, \dots, P_d$ with $P_j f = -\mathrm{i}\partial_j f$ correspond to the position operators by $\mathcal{F}_{\{j\}}^{-1}X_j\mathcal{F}_{\{j\}} = P_j$, we have

$$\mathcal{C} = \mathbf{1}_\omega(R_1, \dots, R_d), \quad \text{where} \quad R_j = \begin{cases} X_j, & j \in \mathcal{J} \setminus \mathcal{I} \\ P_j, & j \in \mathcal{I} \setminus \mathcal{J} \end{cases}.$$

In particular, the observation operator $\mathcal{C}$ is not the characteristic function of a sensor set ω.

7.3. Uncertainty principles with error term

The proof of the uncertainty principle with error term for functions in some Gelfand-Shilov space stated in Theorem 4.35 likewise uses the approach presented in the previous section. However, since in the setting of Theorem 4.35 there is no Bernstein-type inequality available, [Mar22] introduced the idea to use the definition of good elements and the additional error term as an replacement for the missing Bernstein-type inequalities needed. While [Mar22] then proceeds by using an estimate for quasianalytic functions from [NSV04] and a suitable estimate for the so-called *Bang degree*, we rely on the more standard approach from the previous section and estimate Taylor expansions around suitably chosen points. We also incorporate the decay that is guaranteed by assumption (4.22). More precisely, the inequality

$$\|(1+|x|^2)^{n/2}\partial^\beta f\|_{L^2(\mathbb{R}^d)} \leq D_1 D_2^{n+|\beta|}(n!)^\nu(|\beta|!)^\mu \tag{4.22 revisited}$$

holds for all $n \in \mathbb{N}_0$ and all $\beta \in \mathbb{N}_0^d$, in order to reduce the considerations to a bounded subset of $\mathbb{R}^d$ in a similar manner as in Lemma 7.11 above. However, since there is no norm of f on the right-hand side of (4.22), we choose this ball in such a way that the contribution of f outside of it can be subsumed into the error term, see (7.42) below. This way, we obtain a more streamlined proof while getting rid of the technical assumption on the scale ρ used in [Mar22].

7.3.1. Localization and good covering elements. Suppose f satisfies (4.22) for all $n \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^d$. Let $\delta \in (0,1]$ and

$$r := \frac{D_2}{\sqrt{\delta/2}} \geq 1 \quad \text{so that} \quad \sup_{x\in\mathbb{R}^d\setminus B(0,r)} \frac{1}{1+|x|^2} \leq \frac{\delta}{2D_2^2}. \tag{7.41}$$

Then, (4.22) with $n = 1$ and $\beta = 0$ implies that

$$\|f\|^2_{L^2(\mathbb{R}^d\setminus B(0,r))} \leq \frac{\delta}{2}\cdot\frac{\|(1+|x|^2)^{1/2}f\|^2_{L^2(\mathbb{R}^d)}}{D_2^2} \leq \frac{\delta D_1^2}{2}. \tag{7.42}$$

As in the proof of Corollary 7.21 above we use inequality (7.42) and Besicovitch's covering theorem, Proposition 7.22 above, to extract a countable subcovering of

$$\bigl\{B(x,\rho(x))\colon x \in \mathbb{R}^d \text{ s.t. } B(x,\rho(x)) \cap B(0,r) \neq \emptyset\bigr\},$$

where ρ is as in Theorem 4.35. To this end, we first note that the assumption on ρ in Theorem 4.35 implies

$$\rho(x) \leq 2R|x|^{\varepsilon/2} \leq |x|/2 \quad \text{for all} \quad |x| \geq \max\{1,(4R)^{1/(1-\varepsilon)}\} =: r_0. \tag{7.43}$$

Now we observe that the assumption on the intersection requires $|x| - \rho(x) < r$ and, thus, $|x| < r_0$ or $|x|/2 \leq |x| - \rho(x) < r$, that is, $|x| < 2r$. Hence, by Besicovitch's covering theorem there is $\mathcal{K}_0 \subset \mathbb{N}$ and a collection of points $(y_k)_{k\in\mathcal{K}_0}$ with $|y_k| < \max\{r_0, 2r\}$ such that the family of balls $Q_k = B(y_k,\rho(y_k))$, $k \in \mathcal{K}_0$, gives an essential covering of $B(0,\max\{r_0,2r\})$ with overlap at most $\kappa = C_0^d$ as in (7.13) above. With $Q_0 := \mathbb{R}^d \setminus \bigcup_{k\in\mathcal{K}_0} Q_k$ and $\mathcal{K} := \mathcal{K}_0 \cup \{0\}$, the family $(Q_k)_{k\in\mathcal{K}}$ thus gives an essential covering of $\mathbb{R}^d$ with overlap at most κ, i.e.,

$$\Bigl|\mathbb{R}^d \setminus \bigcup_{k\in\mathcal{K}} Q_k\Bigr| = 0 \quad \text{and} \quad \sum_{k\in\mathcal{K}} \mathbf{1}_{Q_k}(x) \leq \kappa \quad \text{for all} \quad x \in \mathbb{R}^d.$$

We abbreviate $w(x) = w_\varepsilon(x) = (1 + |x|^2)^{\varepsilon/2}$. If $\varepsilon < 1$ we infer from [Mar22, Lemma 5.3] that (4.22) implies (by Hölder's inequality)

$$\|w^n \partial^\beta f\|_{L^2(\mathbb{R}^d)} \leq D_1 \tilde{D}_2^{n+|\beta|} (n!)^{\varepsilon\nu} (|\beta|!)^\mu \quad \text{for all} \quad n \in \mathbb{N}_0, \beta \in \mathbb{N}_0^d, \tag{7.44}$$

with $\tilde{D}_2 = 8^\nu \mathrm{e}^\nu D_2 \geq 1$. If $\varepsilon = 1$, then (7.44) agrees with (4.22) for $\tilde{D}_2 = D_2 \geq 1$. We therefore just work with (7.44) for the remaining part, since the power of $n!$ in this bound guarantees that the series in (7.50) below converges.

We define the good elements of the covering in the same way as we did in the previous Section 7.2. More precisely, we say that Q_k, $k \in \mathcal{K}_0$, is *good* with respect to f if

$$\sum_{|\beta|=m} \frac{1}{\beta!} \|w^m \partial^\beta f\|_{L^2(Q_k)}^2 \leq 2^{m+1} \kappa \cdot \frac{2d^m q_m^2}{\delta \cdot m!} \|f\|_{L^2(Q_k)}^2 \quad \text{for all} \quad m \in \mathbb{N}_0,$$

where $q_m = \tilde{D}_2^{2m} (m!)^s$ with $s = \varepsilon\nu + \mu$. Furthermore, we again call Q_k, $k \in \mathcal{K}_0$, *bad* if it is not good and set

$$\mathcal{K}_g = \{k \in \mathcal{K} \colon Q_k \text{ good}\}.$$

Although due to the missing global Bernstein inequalities we can not show that the mass of f on the good balls covers some fixed fraction of the mass of f on the whole of $\mathbb{R}^d$, inequality (7.44) nevertheless implies that the mass of f on the bad balls is bounded by $\delta D_1^2/2$. Hence, the contribution of the bad elements can likewise be subsumed into the error term. This is summarized in the following result which is proved essentially in the same way as inequality (7.20) in the proof of Lemma 7.12 above.

LEMMA 7.28. *We have*

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq \|f\|_{L^2(\bigcup_{k \in \mathcal{K}_g} Q_k)}^2 + \delta D_1^2.$$

PROOF. Since

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq \|f\|_{L^2(\bigcup_{k \in \mathcal{K}_g} Q_k)}^2 + \|f\|_{L^2(\bigcup_{k \notin \mathcal{K}_g} Q_k)}^2 + \|f\|_{L^2(Q_0)}^2,$$

it suffices to show

$$\sum_{k \notin \mathcal{K}_g} \|f\|_{L^2(Q_k)}^2 + \|f\|_{L^2(Q_0)}^2 \leq \delta D_1^2. \tag{7.45}$$

To this end, we first note that $Q_0 \subset \mathbb{R}^d \setminus B(0,r)$ and, thus, $\|f\|_{L^2(Q_0)}^2 \leq \delta D_1^2/2$ by (7.42). As in the proof of Lemma 7.12, we now use the definition of bad covering elements, sum over $m \in \mathbb{N}_0$ and use (7.44) instead of the global Bernstein type inequality to conclude

$$\sum_{k \notin \mathcal{K}_g} \|f\|_{L^2(Q_k)}^2 \leq \frac{\delta}{2} \cdot D_1^2.$$

This proves (7.45) and, thus, the lemma. ☐

In the next lemma we use the definition of good elements to extract a pointwise estimate for the derivatives of f as in the proof of inequality (7.19) of Lemma 7.12 above.

LEMMA 7.29. *Let $k \in \mathcal{K}_g$. Then there is $x_k \in Q_k$ such that for all $m \in \mathbb{N}_0$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| = m$ we have*

$$|\partial^\beta f(x_k)| \leq \left(\frac{2\kappa}{\delta}\right)^{1/2} \cdot 2^{m+1} d^{m/2} \cdot C(k,m)^{1/2} \cdot \frac{\|f\|_{L^2(Q_k)}}{\sqrt{|Q_k|}} \tag{7.46}$$

with

$$C(k,m) = q_m^2 \sup_{x \in Q_k} w(x)^{-2m}. \tag{7.47}$$

PROOF. It suffices to observe that

$$\|\partial^\beta f\|_{L^2(Q_k)}^2 = \|w^{-m} \cdot w^m \partial^\beta f\|_{L^2(Q_k)}^2 \leq \sup_{x \in Q_k} w(x)^{-2m} \cdot \|w^m \partial^\beta f\|_{L^2(Q_k)}^2.$$

Following the proof of (7.19) in Lemma 7.12 above along the lines establishes the existence of $x_k \in Q_k$ as in (7.46). □

7.3.2. Local estimate and proof of Theorem 4.35. In order to estimate f on each Q_k, $k \in \mathcal{K}_g$, we use the local estimate from Lemma 7.15. The next lemma shows that the assumptions of Lemma 7.15 are satisfied and gives a suitable upper bound for the normalized supremum

$$M_k := \frac{\sqrt{|Q_k|}}{\|f\|_{L^2(Q_k)}} \cdot \sup_{z \in Q_k + D_{8\rho(x_k)}} |F(z)| \geq 1 \tag{7.48}$$

using the pointwise estimate (7.46). Note that here we have $l = (2\rho(x_k), \dots, 2\rho(x_k))$ for the sidelengths of the rectangle containing Q_k in Lemma 7.15 and this is why the polydisc in the previous definition of M_k has radius $4 \cdot 2\rho(x_k) = 8\rho(x_k)$.

LEMMA 7.30. *Let $k \in \mathcal{K}_g$. Then, the restriction $f|_{Q_k}$ has an analytic extension $F_k \colon Q_k + D_{8\rho(x_k)} \to \mathbb{C}$, and with with $D = 40d^{3/2}\tilde{D}_2^2 R \max\{r_0, 2\}$ the normalized supremum M_k in (7.48) satisfies*

$$\log M_k \leq \log(2C_1) + \frac{1}{2}\log\left(\frac{2\kappa}{\delta}\right) + D^{1/(1-s)},$$

where $C_1 > 0$ is a constant depending only on $s = \varepsilon\nu + \mu$.

PROOF. Let $x_k \in Q_k$ be a point as in Lemma 7.29. For every $z \in x_k + D_{10\rho(x_k)}$ we then have

$$\begin{aligned}
&\sum_{\beta\in\mathbb{N}_0^d} \frac{|\partial^\beta f(x_k)|}{\beta!}|(z-x_k)^\beta| \\
&\qquad \le \sum_{m\in\mathbb{N}_0}\sum_{|\beta|=m} \frac{1}{\beta!}\Big(\frac{2\kappa}{\delta}\Big)^{1/2} 2^{m+1} d^{m/2} C(k,m)^{1/2} (10\rho(x_k))^{|\beta|} \frac{\|f\|_{L^2(Q_k)}}{\sqrt{|Q_k|}} \\
&\qquad = 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \frac{\|f\|_{L^2(Q_k)}}{\sqrt{|Q_k|}} \sum_{m\in\mathbb{N}_0} C(k,m)^{1/2} \frac{(20d^{3/2}\rho(x_k))^m}{m!}.
\end{aligned}$$

Taking into account that $Q_k + D_{8\rho(x_k)} \subset x_k + D_{10\rho(x_k)}$ and that f is analytic by Lemma A.8, this shows that the Taylor expansion of f around x_k defines an analytic extension $F_k\colon Q_k + D_{8\rho(x_k)} \to \mathbb{C}$ of f with bounded modulus and that

$$M_k \le 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} C(k,m)^{1/2} \frac{(20d^{3/2}\rho(x_k))^m}{m!}. \tag{7.49}$$

In order to estimate the right-hand side further, suppose first that $|x_k| \le r_0$ with $r_0 \ge 1$ as in (7.43). Then, the upper bound for ρ gives

$$\rho(x_k) \le R(1+r_0^2)^{\varepsilon/2} \le R(1+r_0^2)^{1/2} \le 2Rr_0.$$

Using (7.49), (7.47), and the definition of q_m, it follows that

$$\begin{aligned}
M_k &\le 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} q_m \cdot \sup_{x\in Q_k} \frac{1}{w(x)^m} \cdot \frac{(40d^{3/2}Rr_0)^m}{m!} \\
&\le 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} \frac{(40d^{3/2}\tilde{D}_2^2 Rr_0)^m}{(m!)^{1-s}},
\end{aligned}$$

where we have taken into account that $w(x) \ge 1$ for all $x \in \mathbb{R}^d$. On the other hand, if $|x_k| \ge r_0$, then for all $x \in Q_k$ we have the lower bound $|x| \ge |x_k| - \rho(x_k) \ge |x_k|/2 > 0$ since $\rho(x) \le |x|/2$ for all $|x| \ge r_0$ by (7.43) and, thus,

$$\frac{\rho(x_k)}{w(x)} \le \frac{Rw(x_k)}{w(x)} \le 2R\Big(\frac{|x_k|}{|x|}\Big)^{\varepsilon} \le \frac{2R}{(1/2)^{\varepsilon}} \le 4R.$$

Using again (7.49) and (7.47) then gives

$$\begin{aligned}
M_k &\le 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} q_m \cdot \sup_{x\in Q_k} \frac{\rho(x_k)^m}{w(x)^m} \cdot \frac{(20d^{3/2})^m}{m!} \\
&\le 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} \frac{(80d^{3/2}\tilde{D}_2^2 R)^m}{(m!)^{1-s}}.
\end{aligned}$$

We conclude that for both cases $|x_k| \leq r_0$ and $|x_k| \geq r_0$ we have

$$M_k \leq 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} \frac{(40 d^{3/2} \tilde{D}_2^2 R \max\{r_0, 2\})^m}{(m!)^{1-s}} = 2\Big(\frac{2\kappa}{\delta}\Big)^{1/2} \sum_{m=0}^{\infty} \frac{D^m}{(m!)^{1-s}}.$$

We estimate the series using the asymptotics from [Olv97] formulated in Lemma A.3. This implies that there is a constant $C_1 \geq 1$ depending only on s such that

$$\sum_{m=0}^{\infty} \frac{D^m}{(m!)^{1-s}} \leq C_1 \mathrm{e}^{D^{1/(1-s)}}. \tag{7.50}$$

Hence,

$$M_k \leq 2C_1 \Big(\frac{2\kappa}{\delta}\Big)^{1/2} \mathrm{e}^{D^{1/(1-s)}}$$

and taking the logarithm we conclude

$$\log M_k \leq \log(2C_1) + \frac{1}{2}\log\Big(\frac{2\kappa}{\delta}\Big) + D^{1/(1-s)}. \qquad \square$$

With this preparatory steps, we are now in the position to conclude the proof of the uncertainty principle with error term.

PROOF OF THEOREM 4.35. Recall from (4.24) and the definition of Q_k above that $|Q_k \cap \omega|/|Q_k| \geq \gamma$ for all $k \in \mathcal{K}$ and, therefore,

$$\frac{|Q_k \cap \omega|}{24 d \tau_d (\operatorname{diam} Q_k)^d} = \frac{1}{24d \cdot 2^d} \cdot \frac{|Q_k \cap \omega|}{|Q_k|} \geq \frac{\gamma}{24d \cdot 2^d}.$$

Hence, applying Lemma 7.15 with $\Psi = \mathrm{Id}$ and using the estimate for $\log M_k$ with $k \in \mathcal{K}_g$ derived in Lemma 7.30, we obtain for all Q_k with $k \in \mathcal{K}_g$ that

$$\|f\|^2_{L^2(Q_k \cap \omega)} \geq \Big(\frac{\gamma}{24d \cdot 2^d}\Big)^{5 + \frac{\log C_1}{\log 2} + \frac{2}{\log 2}\log\frac{2\kappa}{\delta} + \frac{4}{\log 2} D^{1/(1-s)}} \|f\|^2_{L^2(Q_k)}.$$

Using the definition of D we further estimate

$$5 + \frac{\log C_1}{\log 2} + \frac{2}{\log 2}\log\frac{2\kappa}{\delta} + \frac{4}{\log 2} D^{1/(1-s)} \leq C_2 \cdot \Big(1 + \log\frac{1}{\delta} + D_2^{2/(1-s)}\Big)$$

for some constant $C_2 > 0$ depending on s, R, r_0, s, and d. Therefore,

$$\Big(\frac{K}{\gamma}\Big)^{C_2 \cdot \big(1 + \log\frac{1}{\delta} + D_2^{2/(1-s)}\big)} \|f\|^2_{L^2(Q_k \cap \omega)} \geq \|f\|^2_{L^2(Q_k)}, \tag{7.51}$$

and summing over all good Q_k gives

$$\begin{aligned}\sum_{k \in \mathcal{K}_g} \|f\|^2_{L^2(Q_k)} &\leq \Big(\frac{K}{\gamma}\Big)^{C_2 \cdot \big(1 + \log\frac{1}{\delta} + D_2^{2/(1-s)}\big)} \sum_{k \in \mathcal{K}_g} \|f\|^2_{L^2(Q_k \cap \omega)} \\ &\leq \kappa \Big(\frac{K}{\gamma}\Big)^{C_2 \cdot \big(1 + \log\frac{1}{\delta} + D_2^{2/(1-s)}\big)} \|f\|^2_{L^2(\omega)}.\end{aligned}$$

Together with Lemma 7.28 this proves

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{C\cdot\left(1+\log\frac{1}{\delta}+D_2^{2/(1-s)}\right)}\|f\|^2_{L^2(\omega)} + \delta D_1^2,$$

where $C = \log\kappa + C_2\big(\log K + \log\frac{1}{\gamma}\big)$. $\square$

CHAPTER 8

Dissipation estimate

Throughout this chapter, let A be a quadratic differential operator that is the generator of a semigroup $(\mathcal{T}(t))_{t\geq 0}$. The goal is to give the proof of Theorem 4.29 which is based on anisotropic smoothing effects of the semigroup. These effects describe the phenomenon that for certain generators the function $\mathcal{T}(t)g \in L^2(\mathbb{R}^d)$ with arbitrary $g \in L^2(\mathbb{R}^d)$ has a certain regularity for $t > 0$. Several recent works, see, e.g., [HPSV17, HPSV18, PSRW18, PS18, Alp21, AB], show that the smoothing effects of the semigroup are intimately related to the structure of the singular space of the generator as defined in (2.12). In particular, the article [PSRW18] (see also the discussion of it in [BPS18, last paragraph in Subsection 1.3.2]) shows that the singular space encodes the propagation of so-called Gabor wave front sets of $\mathcal{T}(t)g$. These measure the global regularity in terms of the smoothness and the decay at infinity simultaneously. Put plainly, the singular space is a subspace of $\mathbb{R}^{2d}$, since the first d-coordinates measure the decay at infinity while the second d-coordinates measure the smoothness. In particular, the Gabor wave front set contains directions in the phase space in which a tempered distribution does not behave like a Schwartz function, cf. [PS18, RT21]. It has been established in [PSRW18] that only the parts of the Gabor wave front set of the initial value inside the singular space of the generator may still occur after applying the semigroup, while all other parts get regularized by the semigroup. For instance, if $S(A) = \{0\}$ then $\mathcal{T}(t)g \in \mathcal{S}(\mathbb{R}^d)$ for $t > 0$ since no part of the Gabor wave front set of any g lies inside the (trivial) singular space of the generator, see [HPS09, Proposition 3.1.1].

For operators A with more general singular spaces the preceding discussion is made precise in Theorem 8.2 below. In order to put our results in perspective, we first state the result that was the main ingredient in the proof of the dissipation estimate in [BPS18], formulated in Proposition 4.27 above.

PROPOSITION 8.1 ([HPSV18, Proposition 4.1]). *Let $S(A) = \{0\}$ and let k_0 be the rotation exponent from (2.13). Then there are $C, C', t_0 > 0$ such that*

$$\|\mathrm{e}^{Ct^{2k_0+1}(-\Delta+|x|^2)}\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq C'\|g\|_{L^2(\mathbb{R}^d)} \quad \textit{for all} \quad 0 < t < t_0.$$

The inequality in the previous proposition implies that for some $D, t_0 > 0$ and all $\alpha, \beta \in \mathbb{N}_0^d$, $t \in (0, t_0)$, we have

$$\|x^\alpha \partial_x^\beta \mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq \frac{D^{1+|\alpha|+|\beta|}(\alpha!)^{1/2}(\beta!)^{1/2}}{t^{(k_0+1/2)(|\alpha|+|\beta|+2d)}}\|g\|_{L^2(\mathbb{R}^d)}, \tag{8.1}$$

cf. [HPSV18, Inequality (4.19)]. This establishes that the semigroup is even smoothing in the symmetric Gelfand-Shilov space $S^{1/2}_{1/2}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ in the sense of Definition 4.34 above. An alternative proof of Proposition 8.1 starting from the estimate (8.1) has been suggested in [MPS22].

In a similar way, as observed in [AB, Remark 2.9], the technique of [Alp21, Section 4.2] can be adapted to prove

$$\|\mathrm{e}^{Ct^{2k_0+1}(-\Delta)}\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq C'\|g\|_{L^2(\mathbb{R}^d)} \quad \text{for all} \quad 0 < t < t_0$$

and some constants $C, C', t_0 > 0$ using (8.1) only for $\alpha = 0$. This leads to a proof of the above Proposition 4.28. We follow the same path and establish a new version of Proposition 8.1 for the operators $H_{\mathcal{I},\mathcal{J}}$ starting with a corresponding version of (8.1). To this end, we need the following corollary to [AB, Theorem 2.6].

THEOREM 8.2. *Let $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d_{\mathcal{J}}$ for some sets $\mathcal{I}, \mathcal{J} \subset \{1, \dots, d\}$ and let k_0 be the rotation exponent from* (2.13). *Then there are constants $C_1 > 0$ and $t_0 \in (0, 1)$ such that for all $\alpha \in \mathbb{N}^d_{0,\mathcal{I}}$, $\beta \in \mathbb{N}^d_{0,\mathcal{J}}$, and all $t \in (0, t_0)$ we have*

$$\|x^\alpha \partial_x^\beta \mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq \frac{C_1^{|\alpha|+|\beta|}}{t^{(|\alpha|+|\beta|)(k_0+1/2)}}(\alpha!)^{1/2}(\beta!)^{1/2}\|g\|_{L^2(\mathbb{R}^d)}. \tag{8.2}$$

PROOF. We set $Y_k = (e_k, 0) \in \mathbb{R}^{2d}$, $k \in \mathcal{I}$, and $Y'_j = (0, e_j) \in \mathbb{R}^{2d}$, $j \in \mathcal{J}$, where $e_l \in \mathbb{R}^d$ is the l-th unit vector in $\mathbb{R}^d$. Denote by D_{Y_k} and $D_{Y'_j}$ the Weyl quantizations of the symbols $q(x, \xi) = e_k \cdot x$ and $q(x, \xi) = e_j \cdot \xi$, respectively. Then $D_{Y_k} = x_k$ and $D_{Y'_j} = -\mathrm{i}\partial_{x_j}$, cf. Lemma 2.12. Under the imposed assumptions, [AB, Theorem 2.6] implies that there are constants $C > 0$ and $t_0 \in (0, 1)$ such that for all $m \in \mathbb{N}$ and $t \in (0, t_0)$ we have

$$\|D_{Y^1} \dots D_{Y^m}\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq \frac{C^m}{t^{mk_0+m/2}}(m!)^{1/2}\|g\|_{L^2(\mathbb{R}^d)}. \tag{8.3}$$

Here each of the $Y^1, \dots, Y^m$ can be any of the vectors $\{Y_k \colon k \in \mathcal{I}\} \cup \{Y'_j \colon j \in \mathcal{J}\}$ forming a basis of $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d_{\mathcal{J}}$.

Let now $\alpha \in \mathbb{N}^d_{0,\mathcal{I}}$ and $\beta \in \mathbb{N}^d_{0,\mathcal{J}}$. For each $k \in \mathcal{I}$, we take α_k-times the vector Y_k, and, similarly, β_j-times the vector Y'_j for each $j \in \mathcal{J}$. In total, these are $m = |\alpha| + |\beta|$ many vectors. Hence, (8.3) implies

$$\|x^\alpha \partial_x^\beta \mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} \leq \frac{C^{|\alpha|+|\beta|}}{t^{(|\alpha|+|\beta|)(k_0+1/2)}}((|\alpha| + |\beta|)!)^{1/2}\|g\|_{L^2(\mathbb{R}^d)}$$

and using $(|\alpha|+|\beta|)! \leq 2^{|\alpha|+|\beta|}|\alpha|!|\beta|!$ as well as $|\alpha|! \leq d^{|\alpha|}\alpha!$ this proves the theorem with $C_1 = 2\sqrt{d}C$. □

Inequality (8.2) makes precise the above discussion that the singular space encodes the directions were smoothness and decay at infinity of $\mathcal{T}(t)g$ for fixed $t > 0$ and $g \in L^2(\mathbb{R}^d)$ is guaranteed. In particular, for the semigroup generated by the negative partial harmonic oscillator this inequality with $\beta = 0$ shows that we have decay of elements in the range of the semigroup in those directions where the potential grows unboundedly.

We now show that Theorem 8.2 implies a version of Proposition 8.1 for partial harmonic oscillators. To this end, in view of inequality (8.2), we single out the class

$$\mathcal{G}_{\mathcal{I},\mathcal{J}} := \{f \in L^2(\mathbb{R}^d)\colon x^\alpha\partial^\beta f \in L^2(\mathbb{R}^d)\ \forall\alpha\in\mathbb{N}_{0,\mathcal{I}}^d,\ \beta\in\mathbb{N}_{0,\mathcal{J}}^d\} \tag{8.4}$$

of *partially Schwartz functions* and denote $l := |\mathcal{I}\cap\mathcal{J}| \le d$. According to Theorem 8.2 the assumptions of the following lemma are natural since they are satisfied with $D_1 = \|g\|_{L^2(\mathbb{R}^d)}$ and with $D_2 = C_1 t^{-(k_0+1/2)}$ if $f = \mathcal{T}(t)g$ for an initial datum $g \in L^2(\mathbb{R}^d)$, a time $t\in(0,t_0)$, and if the singular space of the generator satisfies $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}}\times\mathbb{R}^d_{\mathcal{J}}$.

LEMMA 8.3. *Let $D_1, D_2 > 0$ be constants, and suppose that $f\in\mathcal{G}_{\mathcal{I},\mathcal{J}}$ satisfies*

$$\|x^\alpha\partial^\beta f\|_{L^2(\mathbb{R}^d)} \le D_1 D_2^{|\alpha|+|\beta|}(\alpha!)^{1/2}(\beta!)^{1/2} \quad \textit{for all} \quad \alpha\in\mathbb{N}^d_{0,\mathcal{I}}, \beta\in\mathbb{N}^d_{0,\mathcal{J}}. \tag{8.5}$$

Then, for $s\le 1/(40\mathrm{e}\cdot 2^d d D_2^2)$ we have $f\in\mathcal{D}(\mathrm{e}^{sH_{\mathcal{I},\mathcal{J}}})$ and

$$\|\mathrm{e}^{sH_{\mathcal{I},\mathcal{J}}} f\|_{L^2(\mathbb{R}^d)} \le 2\Big(\frac{2}{3}\Big)^l D_1 \le 2D_1.$$

PROOF. Define the differential expressions H_j, $j\in\{1,2,3\}$, with

$$H_1 g = (-\Delta_{\mathcal{I}\cap\mathcal{J}} + |x_{\mathcal{I}\cap\mathcal{J}}|^2)g, \quad H_2 g = -\Delta_{\mathcal{J}\setminus\mathcal{I}} g, \quad H_3 g = |x_{\mathcal{I}\setminus\mathcal{J}}|^2 g$$

for $g\in\mathcal{G}_{\mathcal{I},\mathcal{J}}$. By Lemma 2.8, we have $(H_{\mathcal{I},\mathcal{J}}+l)f = (H_1+l)f + H_2 f + H_3 f$. Since the H_j, $j\in\{1,2,3\}$, leave $\mathcal{G}_{\mathcal{I},\mathcal{J}}$ invariant and commute pairwise, this gives

$$(H_{\mathcal{I},\mathcal{J}}+l)^n f = \sum_{\substack{|\nu|=n\\ \nu\in\mathbb{N}_0^3}} \binom{n}{\nu}(H_1+l)^{\nu_1} H_2^{\nu_2} H_3^{\nu_3} f$$

for $n\in\mathbb{N}_0$. It is easy to see that

$$H_2^{\nu_2}H_3^{\nu_3} f = (-1)^{\nu_2} \sum_{\substack{|\beta|=\nu_2\\ \beta\in\mathbb{N}^d_{0,\mathcal{J}\setminus\mathcal{I}}}} \sum_{\substack{|\alpha|=\nu_3\\ \alpha\in\mathbb{N}^d_{0,\mathcal{I}\setminus\mathcal{J}}}} \binom{\nu_2}{\beta}\binom{\nu_3}{\alpha} x^{2\alpha}\partial_x^{2\beta} f \in \mathcal{G}_{\mathcal{I},\mathcal{J}}(\mathbb{R}^d).$$

Moreover, Lemma A.5 (with d replaced by l) in Appendix A shows

$$\|(H_1+l)^{\nu_1} g\|_{L^2(\mathbb{R}^d)} \le 3^{2\nu_1-d} l^{\nu_1} \sum_{\substack{|\gamma+\delta|\le 2\nu_1\\ \gamma,\delta\in\mathbb{N}^d_{0,\mathcal{I}\cap\mathcal{J}}}} (2\nu_1)^{\nu_1-|\gamma+\delta|/2}\|x^\gamma\partial_x^\delta g\|_{L^2(\mathbb{R}^d)}. \tag{8.6}$$

Hence, inserting $g = H_2^{\nu_2}H_3^{\nu_3} f$ in formula (8.6) and using the triangle inequality for operator norms, we are left with estimating

$$\sum_{\substack{|\gamma+\delta|\le 2\nu_1\\ \gamma,\delta\in\mathbb{N}^d_{0,\mathcal{I}\cap\mathcal{J}}}} \sum_{\substack{|\beta|=\nu_2\\ \beta\in\mathbb{N}^d_{0,\mathcal{J}\setminus\mathcal{I}}}} \sum_{\substack{|\alpha|=\nu_3\\ \alpha\in\mathbb{N}^d_{0,\mathcal{I}\setminus\mathcal{J}}}} 3^{2\nu_1-d} d^{\nu_1}(2\nu_1)^{\nu_1-|\gamma+\delta|/2}\binom{\nu_2}{\beta}\binom{\nu_3}{\alpha}\|x^{\gamma+2\alpha}\partial_x^{\delta+2\beta} f\|_{L^2(\mathbb{R}^d)}.$$

We may now apply the hypothesis (8.5) for each summand separately. This is possible since $\gamma + 2\alpha \in \mathbb{N}^d_{0,\mathcal{I}}$ and $\delta + 2\beta \in \mathbb{N}^d_{0,\mathcal{J}}$. Hence,

$$\begin{aligned} &3^{2\nu_1 - d} d^{\nu_1} (2\nu_1)^{\nu_1 - |\gamma+\delta|/2} \cdot \|x^{\gamma+2\alpha}\partial_x^{\delta+2\beta} f\|_{L^2(\mathbb{R}^d)} \\ &\quad \leq 3^{2\nu_1 - l} l^{\nu_1} (2\nu_1)^{(2\nu_1 - |\gamma+\delta|)/2} D_1 D_2^{|\gamma+2\alpha|+|\delta+2\beta|} ((\gamma+2\alpha)!)^{1/2} ((\delta+2\beta)!)^{1/2}. \end{aligned} \tag{8.7}$$

In order to further estimate this term, we use the simple inequality $\zeta! \leq |\zeta|^{|\zeta|}$ for multi-indices ζ and the upper bound $|\gamma + 2\alpha| + |2\beta + \delta| \leq 2n$ resulting from the conditions of the summations. Thereby,

$$D_2^{|\gamma+2\alpha|+|\delta+2\beta|} ((\gamma+2\alpha)!)^{1/2} ((\delta+2\beta)!)^{1/2} \leq D_2^{2n} (2n)^{(|\gamma+2\alpha|+|\delta+2\beta|)/2}.$$

Moreover, since $2\nu_1 - |\gamma + \delta| + |\gamma + 2\alpha| + |\delta + 2\beta| = 2n$, we get using $\nu_1 \leq n$ that

$$(2\nu_1)^{\nu_1 - |\gamma+\delta|/2} (2n)^{(|\gamma+2\alpha|+|\delta+2\beta|)/2} \leq (2n)^{(2\nu_1 - |\gamma+\delta| + |\gamma+2\alpha| + |\delta+2\beta|)/2} = (2n)^n.$$

Combining this with the elementary estimate $(2n)^n \leq (2\mathrm{e})^n n!$ and (8.7) yields

$$3^{2\nu_1 - d} d^{\nu_1} (2\nu_1)^{\nu_1 - |\gamma+\delta|/2} \cdot \|x^{\gamma+2\alpha}\partial_x^{\delta+2\beta} f\|_{L^2(\mathbb{R}^d)} \leq 3^{2\nu_1 - l} d^{\nu_1} D_1 (2\mathrm{e} D_2^2)^n n!.$$

Noting also that applying the multinomial formula twice we have

$$\begin{aligned} \sum_{\substack{|\gamma+\delta| \leq 2\nu_1 \\ \gamma,\delta \in \mathbb{N}^d_{0,\mathcal{I}\cap\mathcal{J}}}} \sum_{\substack{|\beta| = \nu_2 \\ \beta \in \mathbb{N}^d_{0,\mathcal{J}\setminus\mathcal{I}}}} \sum_{\substack{|\alpha| = \nu_3 \\ \alpha \in \mathbb{N}^d_{0,\mathcal{I}\setminus\mathcal{J}}}} \binom{\nu_2}{\beta}\binom{\nu_3}{\alpha} &\leq \#\{\gamma, \delta \in \mathbb{N}^d_{0,\mathcal{I}\cap\mathcal{J}} \colon |\gamma+\delta| \leq 2\nu_1\} \cdot d^{\nu_2+\nu_3} \\ &\leq (2\nu_1 + 1)^l d^{\nu_2+\nu_3} \leq 2^l 2^{\nu_1 d} d^{\nu_2+\nu_3}, \end{aligned}$$

we finally derive

$$\|(H_1 + l)^{\nu_1} H_2^{\nu_2} H_3^{\nu_3} f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{2}{3}\right)^l D_1 (9 \cdot 2^d)^{\nu_1} (2\mathrm{e} \cdot d D_2^2)^n n!.$$

By the multinomial formula, we have thus shown

$$\begin{aligned} \|(H_{\mathcal{I},\mathcal{J}} + l)^n f\|_{L^2(\mathbb{R}^d)} &\leq \sum_{\substack{|\nu| = n \\ \nu \in \mathbb{N}_0^3}} \binom{n}{\nu} \|(H_1 + l)^{\nu_1} H_2^{\nu_2} H_3^{\nu_3} f\|_{L^2(\mathbb{R}^d)} \\ &\leq \left(\frac{2}{3}\right)^l D_1 (20\mathrm{e} \cdot 2^d D_2^2)^n n!. \end{aligned}$$

Finally, let $s = \big(40\mathrm{e} \cdot 2^d d D_2^2\big)^{-1}$. Then $f \in \mathcal{D}(\mathrm{e}^{s(H_{\mathcal{I},\mathcal{J}}+l)})$ and

$$\|\mathrm{e}^{s(H_{\mathcal{I},\mathcal{J}}+l)} f\|_{L^2(\mathbb{R}^d)} \leq \sum_{n=0}^\infty \frac{s^n}{n!} \|(H_{\mathcal{I},\mathcal{J}} + l)^n f\|_{L^2(\mathbb{R}^d)} \leq 2 \cdot \left(\frac{2}{3}\right)^l D_1.$$

It remains to observe that $f \in \mathcal{D}(\mathrm{e}^{sH_{\mathcal{I},\mathcal{J}}})$ with

$$\|\mathrm{e}^{sH_{\mathcal{I},\mathcal{J}}} f\|_{L^2(\mathbb{R}^d)} \leq \|\mathrm{e}^{s(H_{\mathcal{I},\mathcal{J}}+l)} f\|_{L^2(\mathbb{R}^d)} \leq 2 \cdot \left(\frac{2}{3}\right)^l D_1$$

by the spectral theorem. □

The above lemma is the central tool in the proof of the following generalization and sharpening of Proposition 8.1.

THEOREM 8.4. *Let* $S(A)^\perp = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d_{\mathcal{J}}$ *for some sets* $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$ *and let* k_0 *be the rotation exponent from* (2.13). *Then we have for all* $g \in L^2(\mathbb{R}^d)$ *that* $\mathcal{T}(t)g \in \mathcal{D}(\mathrm{e}^{C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}})$ *and*

$$\|\mathrm{e}^{C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}} \mathcal{T}(t) g\|_{L^2(\mathbb{R}^d)} \leq 2\|g\|_{L^2(\mathbb{R}^d)} \quad \textit{for all} \quad t \in (0, t_0).$$

Here $C_0 = 1/(40\mathrm{e} \cdot 2^d d C_1^2)$*, and* C_1 *and* $t_0 \in (0,1)$ *are as in Theorem 8.2.*

PROOF. Recall that (8.2) in Theorem 8.2 shows that for every $0 < t < t_0$ the function $f = \mathcal{T}(t)g$ satisfies the hypotheses of Lemma 8.3 with $D_1 = \|g\|_{L^2(\mathbb{R}^d)}$ and $D_2 = C_1 t^{-(k_0+1/2)}$. The latter lemma therefore gives $\left\|\mathrm{e}^{sH_{\mathcal{I},\mathcal{J}}} f\right\|_{L^2(\mathbb{R}^d)} \leq 2\|g\|_{L^2(\mathbb{R}^d)}$ for $s \leq 1/(40\mathrm{e} \cdot 2^d d D_2^2)$, which proves the theorem. □

We have now assembled all tools needed to prove the dissipation estimate.

PROOF OF THEOREM 4.29. We have $\mathcal{T}(t)g \in \mathcal{D}(\mathrm{e}^{C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}})$ and

$$\left\|\mathrm{e}^{C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}} \mathcal{T}(t) g\right\|_{L^2(\mathbb{R}^d)} \leq 2\|g\|_{L^2(\mathbb{R}^d)} \quad \text{for} \quad t \in (0, t_0) \tag{8.8}$$

by Theorem 8.4. For those t, we therefore have

$$\mathcal{T}(t)g = \mathrm{e}^{-C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}} \mathrm{e}^{C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}} \mathcal{T}(t) g.$$

Moreover, the projections $P_\lambda = P_\lambda(H_{\mathcal{I},\mathcal{J}})$ and the operator $\mathrm{e}^{-C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}}$ commute, so that the previous identity and the spectral theorem imply

$$\begin{aligned}
\|(1 - P_\lambda)\mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)} &= \left\|\left[\mathrm{e}^{-C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}}(1 - P_\lambda)\right] \mathrm{e}^{C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}} \mathcal{T}(t) g\right\|_{L^2(\mathbb{R}^d)} \\
&\leq \left\|\mathrm{e}^{-C_0 t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}}(1 - P_\lambda)\right\|_{\mathcal{L}(L^2)} \cdot \left\|\mathrm{e}^{c t^{2k_0+1} H_{\mathcal{I},\mathcal{J}}} \mathcal{T}(t) g\right\|_{L^2(\mathbb{R}^d)} \\
&\leq 2\mathrm{e}^{-C_0 t^{2k_0+1}\lambda} \cdot \|g\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

for $t \in (0, t_0)$, where we used inequality (8.8) in the last line. □

APPENDIX A

Supplementary results and proofs

This appendix collects some additional results which are of a more technical nature.

A.1. Technical lemmas and proofs

We first recall some basic lemmas that were used in the main part of this work. Furthermore, we point out some technical details which were skipped there.

A.1.1. Basic lemmas. We start with a simple interpolation result which played an essential role in the proof of Proposition 6.14.

LEMMA A.1 (cf. [Rob95, p. 110]). *Let $P, Q, R \geq 0$ with $P \leq Q$. Suppose that there are constants $r, s, t_0 > 0$ such that*

$$P \leq \mathrm{e}^{-rt} Q + \mathrm{e}^{st} R \quad \textit{for all} \quad t \geq t_0. \tag{A.1}$$

Then

$$P \leq \max\{2, \mathrm{e}^{rt_0}\} \cdot Q^{1-\kappa} R^{\kappa} \quad \textit{with} \quad \kappa = \frac{r}{r+s}. \tag{A.2}$$

PROOF. Let $t_1 \in \mathbb{R}$ be such that $\mathrm{e}^{t_1} = (Q/R)^{1/(r+s)}$. If $t_1 \geq t_0$, we may apply (A.1) and obtain

$$P \leq \Big(\frac{R}{Q}\Big)^{r/(r+s)} Q + \Big(\frac{Q}{R}\Big)^{s/(r+s)} R \leq 2 Q^{1-\kappa} R^{\kappa}.$$

In case that $t_1 < t_0$ we have $(Q/R)^{1/(r+s)} = \mathrm{e}^{t_1} \leq \mathrm{e}^{t_0}$ and therefore $Q^{\kappa} \leq \mathrm{e}^{rt_0} R^{\kappa}$. Hence,

$$P \leq Q = Q^{1-\kappa} Q^{\kappa} \leq \mathrm{e}^{rt_0} Q^{1-\kappa} R^{\kappa}.$$

Plugging these two estimates together we obtain (A.2). □

The next lemma was used in the proof of the Bernstein inequalities in Section 7.2.

LEMMA A.2. *Let $h \colon \mathbb{N}_0^d \to \mathbb{C}$. Then, for all $m \in \mathbb{N}_0$ we have*

$$\frac{1}{m+1} \sum_{j=1}^{d} \sum_{|\beta|=m} \frac{1}{\beta!} h(\beta + e_j) = \sum_{|\alpha|=m+1} \frac{1}{\alpha!} h(\alpha).$$

PROOF. Let $\alpha \in \mathbb{N}_0^d$. For $j \in \{1, \ldots, d\}$ and $\beta \in \mathbb{N}_0^d$ with $\alpha = \beta + e_j$ it is easy to see $\alpha! = \beta! \cdot \alpha_j$. Hence,

$$\sum_{(j,\beta)\colon \alpha=\beta+e_j} \frac{1}{\beta!} = \frac{1}{\alpha!} \sum_{(j,\beta)\colon \alpha=\beta+e_j} \alpha_j = \frac{|\alpha|}{\alpha!},$$

where for the last equality we have taken into account that $\alpha = \beta + e_j$ for some $j \in \{1, \ldots, d\}$ and $\beta \in \mathbb{N}_0^d$ if and only if $\alpha_j > 0$. Therefore,

$$\frac{1}{m+1} \sum_{j=1}^{d} \sum_{|\beta|=m} \frac{1}{\beta!} h(\beta + e_j) = \frac{1}{m+1} \sum_{|\alpha|=m+1} h(\alpha) \sum_{(j,\beta)\colon \alpha=\beta+e_j} \frac{1}{\beta!} = \sum_{|\alpha|=m+1} \frac{1}{\alpha!} h(\alpha),$$

which proves the assertion. □

We also mention the following simple consequence of the asymptotics established in [Olv97] which we used in the proof of Lemma 7.30.

LEMMA A.3. *Let $p \in (0, 4]$. Then there is a constant $C_1 > 0$ such that*

$$\sum_{m=0}^{\infty} \frac{x^m}{(m!)^p} \leq C_1 \mathrm{e}^{x^{1/p}} \quad \textit{for all} \quad x \geq 1.$$

PROOF. We infer from [Olv97, Chapter 8, Eq. (8.07)] that for $p \in (0, 4]$ we have

$$F(x) := \sum_{m=0}^{\infty} \frac{x^m}{(m!)^p} = \frac{\mathrm{e}^{p x^{1/p}}}{p^{1/2} (2\pi x^{1/p})^{(p-1)/2}} \big(1 + R(p, x)\big)$$

where $R(p, x) \in \mathcal{O}(x^{-1/p})$ as $x \to \infty$. Hence, there are $C_1', x_0 \geq 1$ such that $R(p, x) \leq C_1' x^{-1/p}$ for $x \geq x_0$. If $x \geq x_0$ we directly obtain $F(x) \lesssim_p \mathrm{e}^{x^{1/p}}$. On the other hand, if $x < x_0$ we estimate $F(x) \leq F(x_0) \lesssim_{p,x_0} 1$. However, since x_0 depends only on p and since $\mathrm{e}^{x^{1/p}} \geq 1$, this also yields $F(x) \lesssim_p \mathrm{e}^{x^{1/p}}$. □

A.1.2. Proof of an observability estimate. Here we provide the simple computations that are needed to conclude the observability estimate stated in Corollary 4.36 in Section 4.3. We first recall that Theorem 4.35 implies (3.14) with

$$U_{q\tau} = \mathrm{e}^{C \cdot (1 + \log \frac{1}{\delta} + ((d+1)^{1/2} C_2 (q\tau)^{-r_2})^{2/(1-s)})} \quad \text{and} \quad E_{q\tau} = C_1^2 \delta / (q\tau)^{2 r_1} \tag{A.3}$$

for all $\delta \in (0, 1]$ and all $\tau \in (0, t_0)$, where $s = \varepsilon\nu + \mu$. The next lemma shows that these constants satisfy (3.16).

LEMMA A.4. *There is a constant $C_3 \geq 1$ depending on C, C_1, C_2, r_1, r_2, and s, and numbers $q_0 \in (1/2, 1)$ and $\tau_0 \in (0, t_1)$ such that*

$$\frac{U_{q\tau}}{(1-q)\tau} \leq \frac{1}{h(\tau)} \quad \textit{and} \quad E_{q\tau} \leq \frac{h(q\tau)}{h(\tau)} \quad \textit{for} \quad \tau \in (0, \tau_0) \tag{3.16 revisited}$$

holds with

$$h(\tau) = (1 - q_0) \exp\Big(-C_3 \cdot \tau^{-2 r_2/(1-s)}\Big).$$

Before we give the proof of the lemma we show how it implies Corollary 4.36.

PROOF OF COROLLARY 4.36. Applying Theorem 4.35 to $f = \mathcal{T}(t)g$ with D_1 and D_2 as in (4.23) shows (3.14) with the constants from (A.3). According to Lemma A.4, these constants satisfy (3.16) so that the desired observability estimate follows from Corollary 3.11. □

The following proof is essentially extracted from [Mar22, Proof of Theorem 2.11].

PROOF OF LEMMA A.4. Without loss of generality we suppose that $q \geq 1/2$ and choose $\delta = \exp\bigl(-\tau^{-2r_2/(1-s)}\bigr)$ so that

$$U_{q\tau} \leq \exp\bigl(C' \cdot \tau^{-2r_2/(1-s)}\bigr)$$

where $C' = C \cdot \bigl(1 + ((d+1)^{1/2} C_2 2^{r_2})^{2/(1-s)}\bigr)$. Moreover, the choice of δ also yields

$$E_{q\tau} \leq \frac{C_1^2}{(q\tau)^{2r_1}} \exp\bigl(-\tau^{-2r_2/(1-s)}\bigr).$$

We use twice that $\log x \leq K_\kappa x^\kappa$ for every $\kappa > 0$ and all $x \geq 1$. Then we can choose $\tau_0 \leq t_0$ sufficiently small and obtain simultaneously

$$\frac{U_{q\tau}}{\tau} = \exp\Bigl(c \cdot \tau^{-2r_2/(1-s)} + \log\frac{1}{\tau}\Bigr) \leq \exp\bigl(C_3 \cdot \tau^{-2r_2/(1-s)}\bigr)$$

with a constant $C_3 > 0$ depending on C, C_2, r_1, s and the dimension d, as well as

$$E_{q\tau} \leq \exp\Bigl(-\tau^{-4r_2/(1-s)} + \log\frac{C_1^2}{(q\tau)^{2r_1}}\Bigr) \leq \exp\Bigl(-\frac{1}{2}\tau^{-4r_2/(1-s)}\Bigr)$$

for all $\tau \in (0, \tau_0)$. Setting

$$h_q(\tau) = (1-q)\exp\bigl(-C_3 \cdot \tau^{-4r_2/(1-s)}\bigr)$$

and choosing $1/2 \leq q < 1$ in such a way that

$$\frac{h_q(q\tau)}{h_q(\tau)} = \exp\Bigl(C_3 \cdot \tau^{-4r_2/(1-s)}\bigl(1 - q^{-4r_2/(1-s)}\bigr)\Bigr) \geq \exp\Bigl(-\frac{1}{2}\tau^{-4r_2/(1-s)}\Bigr)$$

we have shown the second part of (3.16). In order to contrive the first part of the latter, we choose q such that

$$1 - q^{-4r_2/(1-s)} \geq -\frac{1}{2C_3} \quad \text{or, equivalently,} \quad q \geq \Bigl(\frac{1}{1+2C_3}\Bigr)^{(1-s)/4r_2} =: q'.$$

Then, for $q_0 = \max\{1/2, q'\}$ we get

$$\frac{U_{q_0\tau}}{(1-q_0)\tau} \leq \frac{1}{h_{q_0}(\tau)} \quad \text{and} \quad E_{q_0\tau} \leq \frac{h_{q_0}(q_0\tau)}{h_{q_0}(\tau)} \quad \text{for} \quad \tau \in (0, \tau_0),$$

which establishes (3.16) with $h(\tau) = h_{q_0}(\tau)$. □

A.1.3. Bounds for powers of the harmonic oscillator. The next lemma is used in the proof of Lemma 8.3. Here, for simplicity, we set

$$\mathcal{G} = \Big\{ f \in L^2(\mathbb{R}^d) \colon x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d) \forall \alpha, \beta \in \mathbb{N}_0^d \Big\}.$$

LEMMA A.5. *Let $f \in \mathcal{G}$ and let $H_1 = -\Delta + |x|^2$ in $L^2(\mathbb{R}^d)$. Then*

$$\|(H_1 + d)^m f\|_{L^2(\mathbb{R}^d)} \leq 3^{2m-d} d^m \sum_{\substack{\gamma,\delta \in \mathbb{N}^d \\ |\gamma+\delta| \leq 2m}} (2m)^{m-|\gamma+\delta|/2} \|x^\gamma \partial_x^\delta f\|_{L^2(\mathbb{R}^d)}. \tag{A.4}$$

This lemma is based on an one-dimensional argument we present next. The following inequality and its proof is a generalized version of the inequality stated in [CKP07, Lemma 7.5.2] and of a pointwise equality for powers of the harmonic oscillator proven explicitly in [MPS22, Eq. (4.9) and (4.11)]. It is more general then needed in the proof of Lemma A.5, however, the derivation we present shows precisely why certain factors emerge. Furthermore, this lemma might be of interest in future research when one is interested in deriving estimates of the form (A.4) for operators $H_1 = -\Delta + |x|^{2k}$ with $k > 1$.

LEMMA A.6. *Let $|\varepsilon_j| \leq 1$ and let $f \in C^\infty(\mathbb{R})$ be such that for all $j, l \in \mathbb{N}_0$ we have $x^j \partial_x^l f \in L^2(\mathbb{R})$. Then*

$$\Big\| \Big[\prod_{j=1}^{m} (\varepsilon_j \partial_x + x^k) \Big] f \Big\|_{L^2(\mathbb{R})} \leq (k+2)^m \sum_{\substack{j,l \in \mathbb{N} \\ \frac{j}{k} + l \leq m}} (km)^{\frac{km-j-kl}{k+1}} \|x^j \partial_x^l f\|_{L^2(\mathbb{R})}. \tag{A.5}$$

PROOF. The statement is obviously true for $m = 1$. We proceed by induction and suppose that the statement holds for $m \in \mathbb{N}$. Then

$$\begin{aligned}
\Big\| \Big[\prod_{j=1}^{m+1} (\varepsilon_j \partial_x + x^k) \Big] f \Big\|_{L^2(\mathbb{R})} &= \Big\| \Big[\prod_{j=1}^{m} (\varepsilon_j \partial_x + x^k) \Big] (\varepsilon_{m+1} \partial_x + x^k) f \Big\|_{L^2(\mathbb{R})} \\
&\leq (k+2)^m \sum_{\substack{j,l \in \mathbb{N} \\ \frac{j}{k} + l \leq m}} (km)^{\frac{km-j-kl}{k+1}} \|x^j \partial_x^l \big[(\varepsilon_{m+1} \partial_x + x^k) f \big] \|_{L^2(\mathbb{R})}.
\end{aligned}$$

Using the Leibniz rule, we compute

$$\|x^j \partial_x^l \big[x^k f \big] \|_{L^2(\mathbb{R})} \leq \sum_{\eta=0}^{\min\{l,k\}} (km)^\eta \|x^{j+k-\eta} \partial^{l-\eta} f\|_{L^2(\mathbb{R})}.$$

Combining this estimate with the triangle inequality we obtain

$$\|x^j \partial_x^l \big[(\varepsilon_{m+1} \partial_x + x^k) f \big] \|_{L^2(\mathbb{R})} \leq \|x^j \partial_x^{l+1} f\|_{L^2(\mathbb{R}^d)} + \sum_{\eta=0}^{\min\{l,k\}} (km)^\eta \|x^{j+k-\eta} \partial^{l-\eta} f\|_{L^2(\mathbb{R})}.$$

For each of the at most $(k+1)$-terms in the second sum we have

$$(km)^{\frac{km-j-kl}{k+1}}(km)^{\eta}\|x^{j+k-\eta}\partial^{l-\eta}f\|_{L^2(\mathbb{R})} = (km)^{\frac{k(m+1)-\tilde{j}-k\tilde{l}}{k+1}}\|x^{\tilde{j}}\partial^{\tilde{l}}f\|_{L^2(\mathbb{R})}$$

with $\tilde{j} = j+k-\eta$ and $\tilde{l} = l-\eta$. The first term can be bounded in a similar way and since we have $(k+2)$-terms in total this shows

$$(k+2)^m \sum_{\substack{j,l\in\mathbb{N}\\ \frac{j}{k}+l\le m}} (km)^{\frac{km-j-kl}{k+1}}\|x^j\partial_x^l\big[(\varepsilon_{m+1}\partial_x + x^k)f\big]\|_{L^2(\mathbb{R})}$$
$$\le (k+2)^{m+1} \sum_{\substack{\tilde{j},\tilde{l}\in\mathbb{N}\\ \frac{\tilde{j}}{k}+\tilde{l}\le m+1}} \big(k(m+1)\big)^{\frac{k(m+1)-\tilde{j}-k\tilde{l}}{k+1}}\|x^{\tilde{j}}\partial_x^{\tilde{l}}f\|_{L^2(\mathbb{R})},$$

which concludes the induction step. □

We now apply the last lemma with $k=1$.

Proof of Lemma A.5. Define the differential expressions $S_j^{\pm} = \pm\partial_{x_j} + x_j$ for $j = \{1,\dots,d\}$. Then $S_j^{\pm}$ leaves $\mathcal{G}$ invariant and are pairwise commuting. Moreover, $(\partial_{x_j}^2 + x_j^2 + 1)f = S_j^+S_j^-f$ and, therefore,

$$\big\|(H_1+d)^m f\big\|_{L^2(\mathbb{R}^d)} \le \sum_{\substack{\omega\in\mathbb{N}_0^d\\ |\omega|=m}} \binom{m}{\omega}\big\|(S_1^+S_1^-)^{\omega_1}\dots(S_d^+S_d^-)^{\omega_d}f\big\|_{L^2(\mathbb{R}^d)}$$

for $f\in\mathcal{G}$. By Lemma A.6 and Fubini's theorem we have

$$\|(S_j^+S_j^-)^{\omega_j}g\|_{L^2(\mathbb{R}^d)} \le 3^{2\omega_j-1}\sum_{\substack{\nu^{(j)}\in\mathbb{N}_0^2\\ |\nu^{(j)}|\le\omega_j}} \omega_j^{(\omega_j-|\nu^{(j)}|)/2}\|x^{\nu_1^{(j)}}\partial_x^{\nu_2^{(j)}}g\|_{L^2(\mathbb{R}^d)}$$

for $g\in\mathcal{G}$ and all $j\in\{1,\dots,d\}$. Applying this estimate repeatedly we obtain

$$\|(S_1^+S_1^-)^{\omega_1}\dots(S_d^+S_d^-)^{\omega_d}f\|_{L^2(\mathbb{R}^d)} \le 3^{2m-d}\sum_{\substack{\gamma,\delta\in\mathbb{N}_0^d\\ \gamma+\delta\le 2\omega}} (2m)^{m-|\gamma+\delta|/2}\|x^\gamma\partial_x^\delta f\|_{L^2(\mathbb{R}^d)}$$

and replacing the condition $\gamma+\delta\le 2\omega$ (where the inequality is understood entrywise) by the weaker bound $|\gamma+\delta|\le 2m$ we get rid of the dependence on ω. Hence,

$$\|(H_1+d)^m f\|_{L^2(\mathbb{R}^d)} \le 3^{2m-d}d^m\sum_{\substack{\gamma,\delta\in\mathbb{N}_0^d\\ |\gamma+\delta|\le 2m}} (2m)^{m-|\gamma+\delta|/2}\|x^\gamma\partial_x^\delta f\|_{L^2(\mathbb{R}^d)}.$$ □

REMARK A.7. A completely analogous proof shows that for the operator $H = \sum_j (-\partial_{x_j} + x_j^k)(\partial_{x_j} + x_j^k)$ we have the bound

$$\|H^n f\|_{L^2(\mathbb{R}^d)} \leq (k+2)^{2n} d^n \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^d \\ \frac{|\alpha|}{k}+|\beta|\leq 2n}} (2kn)^{\frac{2kn-|\alpha|-k|\beta|}{k+1}} \|x^\alpha \partial^\beta f\|_{L^2(\mathbb{R}^d)} \tag{A.6}$$

for all $f \in \mathcal{G}$ and all $n \in \mathbb{N}$.

A.1.4. Analyticity. We now establish that functions in the range of a semigroup that is smoothing in a Gelfand-Shilov space $\mathcal{S}_\nu^\mu(\mathbb{R}^d)$ with $0 \leq \mu < 1$ are analytic. This is required in order to follow the complex analytic approach discussed in Section 7.3.

LEMMA A.8 (see [DS22, Lemma A.1]). *Let $f \in C^\infty(\mathbb{R}^d)$ be such that*

$$\|\partial^\beta f\|_{L^2(\mathbb{R}^d)} \leq C_1 C_2^{|\beta|} \beta! \quad \textit{for all} \quad \beta \in \mathbb{N}_0^d$$

with some constants $C_1, C_2 > 0$. Then, f is analytic in $\mathbb{R}^d$.

PROOF. Choose $\sigma \in (0,1]$ with $2C_2\sigma < 1$. Let $y \in \mathbb{R}^d$, and let $B = B(y,\tau)$ with $\tau < \sigma/d$. We show that the Taylor series of f around y converges in B and agrees with f there. To this end, it suffices to establish

$$\sum_{\alpha\in\mathbb{N}_0} \frac{\|\partial^\alpha f\|_{L^\infty(B)}}{\alpha!} \tau^{|\alpha|} < \infty; \tag{A.7}$$

cf. [KP92, Theorem 2.2.5 and Proposition 2.2.10].

We proceed similarly as in the proof of [ES21, Lemma 3.2]: Since B satisfies the cone condition, by Sobolev embedding there exists a constant $C > 0$, depending only on τ and the dimension, such that $\|g\|_{L^\infty(B)} \leq C\|g\|_{W^{d,2}(B)}$ for all $g \in W^{d,2}(B)$, see, e.g., [AF03, Theorem 4.12]. Applying this to $g = \partial^\alpha f|_B$ with $|\alpha| = m \in \mathbb{N}_0$, we obtain

$$\begin{aligned}\|\partial^\alpha f\|_{L^\infty(B)}^2 &\leq C^2 \|\partial^\alpha f\|_{W^{d,2}(B)}^2 \leq C^2 \|\partial^\alpha f\|_{W^{d,2}(\mathbb{R}^d)}^2 \\ &= C^2 \sum_{|\beta|\leq d} \|\partial^{\beta+\alpha} f\|_{L^2(\mathbb{R}^d)}^2 \leq C^2 \sum_{k=m}^{m+d} \sum_{|\beta|=k} \|\partial^\beta f\|_{L^2(\mathbb{R}^d)}^2.\end{aligned}$$

Taking the square root and using the hypothesis gives

$$\|\partial^\alpha f\|_{L^\infty(B)} \leq C \sum_{k=m}^{m+d} \sum_{|\beta|=k} \|\partial^\beta f\|_{L^2(\mathbb{R}^d)} \leq CC_1 \sum_{k=m}^{m+d} C_2^k \sum_{|\beta|=k} \beta!.$$

We clearly have

$$\sum_{|\beta|=k} \beta! \leq k! \cdot \#\{\beta \in \mathbb{N}_0^d \mid |\beta| = k\} \leq k! 2^{k+d-1}.$$

In view of the choice of σ, we thus further estimate

$$\begin{aligned}\|\partial^\alpha f\|_{L^\infty(B)} &\le 2^{d-1} C C_1 \sum_{k=m}^{m+d} (2C_2)^k k! \le 2^{d-1} C C_1 \frac{(m+d)!}{\sigma^{m+d}} \sum_{k=m}^{m+d} (2C_2\sigma)^k \\ &\le \frac{2^{d-1} C C_1}{\sigma^d} \sum_{k=0}^{\infty} (2C_2\sigma)^k \cdot \frac{(m+d)!}{\sigma^m} =: C_0 \cdot \frac{(m+d)!}{\sigma^m}.\end{aligned}$$

Now,

$$\begin{aligned}\sum_{|\alpha|=m} \frac{\|\partial^\alpha f\|_{L^\infty(B)}}{\alpha!} &\le C_0 \frac{(m+d)!}{\sigma^m} \sum_{|\alpha|=m} \frac{1}{\alpha!} = C_0 \Big(\frac{d}{\sigma}\Big)^m \frac{(m+d)!}{m!} \\ &\le C_0 \Big(\frac{d}{\sigma}\Big)^m (m+d)^d,\end{aligned}$$

and since τ is chosen such that $d\tau/\sigma < 1$, this shows (A.7) and, hence, completes the proof. □

A.1.5. Remarks on ghost dimension. In the proof of Lemma 6.16 we used that the extended function F is infinitely weakly differentiable. In order to see this, we let F be as in (6.18) where the corresponding function f satisfies $f \in \operatorname{Ran} P_\lambda(H)$. The proof of the following lemma is essentially taken from [DRST, Lemma A.1].

LEMMA A.9. *F is infinitely weakly differentiable with respect to t, and the corresponding weak derivatives coincide with their $L^2(\mathbb{R}^d)$ analogues. The derivatives of F are given by the formula in equation* (6.20).

PROOF. First we show that

$$\lim_{h\to 0} \int_J \Big\| \partial_t^{k+1} F(\cdot,t) - \frac{\partial_t^k F(\cdot,t+h) - \partial_t^k F(\cdot,t)}{h} \Big\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}t = 0$$

for each bounded interval $J \subset \mathbb{R}$ and each $k \in \mathbb{N}_0$, where $\partial_t^k F(\cdot,t)$ is given by (6.20). To this end, it suffices to observe that

$$\begin{aligned}&\Big\| \partial_t^{k+1} F(\cdot,t) - \frac{\partial_t^k F(\cdot,t+h) - \partial_t^k F(\cdot,t)}{h} \Big\|_{L^2(\mathbb{R}^d)}^2 \\ &\qquad = \int_{[\kappa,E]} \Big| \partial_t^{k+1} s_t(\lambda) - \frac{\partial_t^k s_{t+h}(\lambda) - \partial_t^k s_t(\lambda)}{h} \Big|^2 \,\mathrm{d}\langle P_H(\lambda) f, f\rangle \\ &\qquad \le Ch \|f\|_{L^2(\mathbb{R}^d)}^2\end{aligned}$$

with

$$C = \sup_{(t,\lambda)\in \tilde{J}\times[\kappa,\lambda]} |\partial_t^{k+2} s_t(\lambda)|^2 < \infty, \qquad \tilde{J} = \{t \pm |h| \colon t \in J\},$$

where we have taken into account the mean value theorem of differential calculus.

Now, let $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R})$. The above then implies by Fubini's theorem and Cauchy-Schwarz inequality that

$$\int_{\mathbb{R}^d\times\mathbb{R}} (\partial_t^{k+1} F)(x,t)\varphi(x,t)\,\mathrm{d}(x,t) \\ = \lim_{h\to 0}\int_{\mathbb{R}^d\times\mathbb{R}} \frac{(\partial_t^k F)(x,t+h) - (\partial_t^k F)(x,t)}{h}\varphi(x,t)\,\mathrm{d}(x,t).$$

On the other hand, by change of variables with respect to t, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}} &\frac{(\partial_t^k F)(x,t+h) - (\partial_t^k F)(x,t)}{h}\varphi(x,t)\,\mathrm{d}(x,t) \\ &= \int_{\mathbb{R}^d\times\mathbb{R}} (\partial_t^k F)(x,t)\frac{\varphi(x,t-h)-\varphi(x,t)}{h}\,\mathrm{d}(x,t) \\ &\xrightarrow[h\to\infty]{} -\int_{\mathbb{R}^d\times\mathbb{R}} (\partial_t^k F)(x,t)(\partial_t\varphi)(x,t)\,\mathrm{d}(x,t),\end{aligned}$$

where for the latter we have taken into account Lebesgue's dominated convergence theorem. The claim then follows by induction over k. □

A.1.6. Scaling. We now give the details of the scaling procedure that were left out in the proof of Theorem 4.10 (the spectral inequality for Schrödinger operators with singular admissible potentials). To this end, recall that V is admissible and suppose that ω is (G,δ)-equidistributed, that is, each intersection $\Lambda_G(k)\cap\omega$, $k \in (G\mathbb{Z})^d$, contains a ball of radius $\delta \in (0, G/2)$. Let us denote the centers of these balls by z_k, $k \in (G\mathbb{Z})^d$, so that $\omega \supset \bigcup_k B(z_k,\delta)$. Define $S_G\colon \mathbb{R}^d \to \mathbb{R}^d$ by $S_G(x) = Gx$. Then

$$\tilde\omega := S_G^{-1}\omega \supset \bigcup_{k\in(G\mathbb{Z})^d} B(z_k/G, \delta/G) \supset \bigcup_{j\in\mathbb{Z}^d} B(y_j,\delta/G)$$

where $y_j = z_{Gj}/G \in \Lambda_1(j)$ for $j \in \mathbb{Z}^d$. Thus, $\tilde\omega$ is $(1,\delta/G)$-equidistributed.

Let $\tilde V = G^2 V\circ S_G$. Clearly $\tilde V$ is admissible and for $f \in H^1(\mathbb{R}^d)$ we have $f\circ S_G \in H^1(\mathbb{R}^d)$. Using that V is admissible we calculate

$$\begin{aligned}\|\tilde V f\|^2_{L^2(\mathbb{R}^d)} &= G^{4-d}\|V\big(f\circ S_G\big)\|^2_{L^2(\mathbb{R}^d)} \\ &\le G^{4-d}\big(\lambda_1\|\nabla(f\circ S_G)\|^2_{L^2(\mathbb{R}^d)} + \lambda_2\|f\circ S_G^{-1}\|^2_{L^2(\mathbb{R}^d)}\big) \\ &= G^2\lambda_1\|\nabla f\|^2_{L^2(\mathbb{R}^d)} + G^4\lambda_2\|f\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

and set $\tilde\lambda_1 = G^2\lambda_1$ and $\tilde\lambda_2 = G^4\lambda_2$.

If $\tilde H = -\Delta + \tilde V$ then the transformation formula for spectral measures implies that for $f \in \operatorname{Ran} P_\lambda(\tilde H)$ we have $f\circ S_G^{-1} \in \operatorname{Ran} P_{G^2\lambda}(H)$. Applying the theorem with $f\circ S_G^{-1}$ and with λ_j replaced by $\tilde\lambda_j$, $j\in\{1,2\}$, gives the asserted inequality.

A.2. Geometric properties of sensor sets

Here we give some simple additional results and calculations that relate to properties of sensor sets and that were left out in the main body of this work.

A.2.1. Different notions of thickness. We first show that it is irrelevant whether the notion of thickness is defined with respect to balls or cubes.

LEMMA A.10. *Let $\gamma \in (0,1]$ and $\rho > 0$. If a measurable set $\omega \subset \mathbb{R}^d$ is $(\gamma, \rho/2)$-thick in the sense of Definition 4.2, then it satisfies*

$$\frac{|\omega \cap \Lambda_\rho(x)|}{|\Lambda_\rho(x)|} \geq \gamma \cdot \frac{\tau_d}{2^d} \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Conversely, if ω satisfies

$$\frac{|\omega \cap \Lambda_\rho(x)|}{|\Lambda_\rho(x)|} \geq \gamma \quad \text{for all} \quad x \in \mathbb{R}^d,$$

then it is $(2^d\gamma/(d^{d/2}\tau_d), \rho\sqrt{d}/2)$-thick in the sense of Definition 4.2.

PROOF. We have $B(x, \rho/2) \subset \Lambda_\rho(x)$ and therefore

$$\frac{|\omega \cap \Lambda_\rho(x)|}{|\Lambda_\rho(x)|} \geq \frac{|\omega \cap B(x, \rho/2)|}{|B(x, \rho/2)|} \cdot \frac{|B(x, \rho/2)|}{|\Lambda_\rho(x)|} \geq \gamma \cdot \frac{\tau_d}{2^d}.$$

The second statement follows analogously using $\Lambda_\rho(x) \subset B(x, \rho\sqrt{d}/2)$. □

A.2.2. Calculations for Example 4.18. Recall $d = 1$ and $\rho(x) = (1+x^2)^{1/4}$. We set $x_n = 100n^3$ for $n \in \mathbb{N}$ and we aim to show that the set $\omega = \mathbb{R} \setminus B(x_n, n)$ satisfies

$$\frac{|B(x, \rho(x)) \cap \omega|}{|B(x, \rho(x))|} \geq \frac{1}{4}.$$

To this end, we estimate $\rho(x_n) \geq \big(100n^3\big)^{1/2} \geq 10n$ and using the binomial formula we see that $x_{n+1} \geq x_n + 4x_{n+1}^{1/2} \geq x_n + 4\rho(x_{n+1})$. Hence, for all $a \in B(x_n, \rho(x_n))$ and all $b \in B(x_{n+1}, \rho(x_{n+1}))$ we have

$$b - a \geq x_{n+1} - \rho(x_{n+1}) - (x_n + \rho(x_n)) \geq x_{n+1} - x_n - 2\rho(x_{n+1}) \geq 2\rho(x_{n+1}).$$

Therefore, $\bigcap_{n\in\mathbb{N}} B(x_n, \rho(x_n)) = \emptyset$. Now, for every $x \in \mathbb{R}$ with $|x| \geq 1$ there is at most a single $n \in \mathbb{N}$ such that $B(x, \rho(x)) \cap B(x_n, \rho(x_n)) \neq \emptyset$ and in this case we have $x + \rho(x) > x_n - n$ and $x - \rho(x) < x_n - n$ which implies $x > (x_n - n)/2$ as well as $x < 2(x_n + n)$. Using these bounds we calculate

$$\begin{aligned}\frac{|B(x, \rho(x)) \cap \omega|}{|B(x, \rho(x))|} &\geq \frac{|B(x, \rho(x))| - |B(x_n, n)|}{|B(x, \rho(x))|} \geq \frac{|x|^{1/2} - n}{|x|^{1/2}} \\ &\geq \frac{\big(\frac{x_n - n}{2}\big)^{1/2} - n}{\big(2(x_n + n)\big)^{1/2}} \geq \frac{6n^{3/2} - n}{12\sqrt{2}n^{3/2}} \geq \frac{5}{12\sqrt{2}} \geq \frac{1}{4}.\end{aligned}$$

This proves the desired bound.

A.2.3. Equidistributed and thick sets. Let us now compare the two different assumptions we imposed on our sensor sets $\omega \subset \mathbb{R}^d$. For simplicity, we consider only the situation where for some $\alpha \geq 0$, $\delta \in (0, 1/2)$, and $\gamma \in (0, 1]$ we have

(i) each intersection $\omega \cap \Lambda_1(k)$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k|^\alpha}$, or
(ii) the measure of each intersection $\omega \cap \Lambda_1(k)$, $k \in \mathbb{Z}^d$, is at least $\gamma^{1+|k|^\alpha}$.

Here, clearly, the situation in (ii) is more general. In fact, even for $\alpha = 0$ there are sets which satisfy (ii) but not (i); consider, e.g., the set $\omega = \bigcup_{k\in\mathbb{Z}}(k + M) \subset \mathbb{R}$, where $M \subset [-1/2, 1/2]$ is a measurable set with positive Lebesgue measure which has empty interior, for instance a *Smith–Volterra–Cantor set* [Smi75]. Moreover, let us also mention that α in (ii) might be chosen smaller then in (i) which might change the situation drastically since we encountered upper bounds for α in our main results. As an example, let $\beta > 1$ and suppose that ω is a set such that each intersection $\omega \cap \Lambda_1(k)$ with $k \in \mathbb{Z}^d$ contains a disjoint union of $\mathrm{e}^{1+|k|^\beta}$-many balls of radius $\mathrm{e}^{-(1+(1+|k|)^\beta)}$. Then, clearly, each of these intersections contains a ball of radius $\mathrm{e}^{-(1+(1+|k|)^\beta)}$ while the measure of the intersection satisfies

$$|\omega \cap \Lambda_1(k)| = \tau_d \mathrm{e}^{d\cdot(|k|^\beta-(1+|k|)^\beta)} \gtrsim_d \mathrm{e}^{-\beta(1+|k|^{\beta-1})}.$$

Hence, while in case (i) we need to choose $\alpha = \beta$, in case (ii) we may choose $\alpha = \beta - 1$.

On the other hand, for every set ω that satisfies (i) we have

$$|\Lambda_1(k) \cap \omega| \geq \tau_d \delta^{d\cdot(1+|k|^\alpha)} \geq \left(\frac{\delta^d}{2d^{\frac{d+1}{2}}}\right)^{1+|k|^\alpha} =: \gamma^{1+|k|^\alpha}$$

using the asymptotic formula for τ_d. Hence, ω satisfies (ii) with this choice for γ.

APPENDIX B

Unique continuation for the gradient

In this excursus we argue that the gradients of eigenfunctions of a second-order elliptic operator satisfy some quantitative unique continuation estimate. We also present an application in the theory of *random divergence-type operators*, i.e., second-order elliptic operators where the second order term is random. This depicts another scope of applications for quantitative unique continuation estimates. Since we do not go into detail here, we refer the reader to the books [Sto01, Ves08] for an overview of the theory of random divergence-type and random Schrödinger operators.

Let us now briefly motivate the findings of the authors articles [DV, Dic21]. Let $L \in \mathbb{N}$, $\Lambda_L = \Lambda_L(0)$, and let $A = (a_{j,k})_{j,k=1}^d \colon \Lambda_L \to \mathrm{Sym}(\mathbb{R}^d)$ be a matrix function that is uniformly elliptic. Consider the divergence-type operator

$$H_{\Lambda_L}^D(A) \colon L^2(\Lambda_L) \supset \mathcal{D}\big(H_{\Lambda_L}^D(A)\big) \to L^2(\Lambda_L)$$

with coefficients given by the matrix function A and with Dirichlet boundary conditions. This operator is defined as the unique selfadjoint operator associated to the lower semibounded form

$$\mathfrak{h}_{\Lambda_L}(A) \colon H_0^1(\Lambda_L) \times H_0^1(\Lambda_L) \to L^2(\mathbb{R}^d), \quad \mathfrak{h}^L[f,g] = \int_{\Lambda_L} \nabla f \cdot A \overline{\nabla g}.$$

The operator $H_{\Lambda_L}^D(A)$ has compact resolvent and therefore purely discrete spectrum. Moreover, in the sense of quadratic forms we have $H_{\Lambda_L}^D(A) = -\mathrm{div} A \nabla|_{\Lambda_L}$, where the right-hand side is understood as the restriction of the differential expression with Dirichlet boundary conditions.

The guiding question in this excursus is the following: *If $W \colon \Lambda_L \to [0,\infty)$ is a function such that $W \geq \mathbf{1}_\omega$ for some measurable set $\omega \subset \Lambda_L$, under what conditions on ω do the eigenvalues of the operator $H_{\Lambda_L}^D(A + t \cdot W)$ increase in $t > 0$?* If the eigenvalues do increase in t we say that we have an *eigenvalue lifting*. These eigenvalue liftings are of interest in the theory of random operators, since they can be used to prove so-called *initial length scale* and *Wegner estimates*. These, in turn, imply Anderson localization via the multi-scale analysis, see, e.g., the literature cited in [NTTV20b, ST20, Dic21].

In the present setting, eigenvalue liftings follow from suitable quantitative lower bounds for the derivatives of $t \mapsto E_n(H_{\Lambda_L}^D(A + t \cdot W))$, where the latter denotes the n-th eigenvalue of the operator (enumerated non-decreasingly and counting multiplicities). These derivatives can computed explicitly: With a normalized eigenfunction $f_n(t)$ corresponding to this eigenvalue we have

$$\partial_t E_n\big(H^D_{\Lambda_L}(A + t\cdot W)\big) = \|W\nabla f_n(t)\|^2_{L^2(\Lambda_L)} \geq \|\nabla f_n(t)\|^2_{L^2(\omega\cap\Lambda_L)} \tag{B.1}$$

for all except finitely many $t \in [0,1]$ if $W \geq \mathbf{1}_\omega$. In particular, a t-independent lower bound for the right-hand side implies by the fundamental theorem of calculus that the eigenvalues increase. The unique continuation estimate for the gradient we present in Corollary B.3 below provides exactly this lower bound.

An unique continuation estimate for divergence-type operators has been established in [TV20]. Here the matrix function needs to be Lipschitz-continuous, which is expected since for merely Hölder-continuous coefficients unique continuation may fail, see [Pli63, Mil73, Man98]. Furthermore, it needs to satisfy an additional technical assumption (Dir) that is required for certain extension arguments (closely related to the procedure described in Remark 6.17 above) needed in the proof. However, it is quite possible that by using a different technique the assumption (Dir) is no longer required.

In what follows, we let $\vartheta_E \geq 1$ and $\vartheta_L \geq 0$ be the ellipticity resp. Lipschitz constant of A and we let $G > 0$. We now present a result from [TV20]. Let us emphasize that the statement in [TV20] is more general than expressed in the next theorem as it also allows the operator $H^D_{\Lambda_L}(A)$ to have lower order terms.

THEOREM B.1 ([TV20, Corollary 2.8]). *Let $L \in G\mathbb{N}$ and suppose that*

$$\forall j \neq k,\ x \in \overline{\Lambda_L} \cap \overline{\Lambda_L + Le_k}\colon a_{j,k}(x) = a_{k,j}(x) = 0. \tag{Dir}$$

Then for all $\delta \in (0, G/2)$, all (G,δ)-equidistributed sets ω, and all eigenfunctions $f \in \mathcal{D}(H^D_{\Lambda_L}(A))$ corresponding to an eigenvalue $\lambda \geq 0$ we have

$$\|f\|_{L^2(\omega\cap\Lambda_L)} \geq \left(\frac{\delta}{G}\right)^{C\cdot(1+\lambda^{2/3})} \|f\|_{L^2(\Lambda_L)},$$

where $C > 0$ is a constant depending only on $\vartheta_E, G\vartheta_L$, and the dimension d.

The next lemma was proven in the authors joint work with Ivan Veselić [DV].

LEMMA B.2. *Let $0 < E_-$ and let $r > 0$. Then for all balls $B(x_0, 2r) \subset \Lambda_L$ and all eigenfunctions $f \in \mathcal{D}(H^D_{\Lambda_L}(A))$ associated to an eigenvalue $\lambda \geq E_-$ we have*

$$\|\nabla f\|^2_{L^2(B(x_0,2r))} \geq \frac{r^2 E_-^2}{16\vartheta_E} \|f\|^2_{L^2(B(x_0,r))}.$$

In the last mentioned article, the last two results were combined in order to obtain an unique continuation estimate for the gradient of an eigenfunction of the divergence-type operator $H^D_{\Lambda_L}(A)$.

COROLLARY B.3. *Let $L \in G\mathbb{N}$ and suppose* (Dir). *Then for all $\delta \in (0, G/2)$, all (G,δ)-equidistributed sets $\omega \subset \mathbb{R}^d$, and all eigenfunctions $f \in \mathcal{D}(H^D_{\Lambda_L}(A))$ associated* *to an eigenvalue* $\lambda \in [E_-, E_+] \subset (0, \infty)$ *we have*

$$\|\nabla f\|_{L^2(\omega \cap \Lambda_L)} \geq E_-^2 \cdot \left(\frac{\delta}{G}\right)^{C \cdot (1 + E_+^{2/3})} \|f\|_{L^2(\Lambda_L)}, \tag{B.2}$$

where $C > 0$ *is a constant depending only on* ϑ_E, ϑ_L, *and the dimension* d.

The constant on the right-hand side of (B.2) depends only on the energy interval $[E_-, E_+]$ and not on the particular eigenvalue. In this sense, the estimate holds uniformly over the energy interval. This guarantees that the lower bound for the right-hand side of (B.1) is independent from t.

Making the bound stated in (B.1) precise and combining it with the last corollary, one concludes the eigenvalue lifting if $W \geq 1$ at least on a (G, δ)-equidistributed set. This was proven in [DV, Theorem 4.1].

Lemma B.4. *Let* $N > 0$, $L \in G\mathbb{N}$, *and suppose* (Dir). *Then for all* $\delta \in (0, G/2)$, *all* (G, δ)*-equidistributed sets* ω, *all Lipschitz-continuous* W *with Lipschitz constant at most* N *satisfying* $\mathbf{1}_{\Lambda_L} \geq W \geq \mathbf{1}_\omega$, *all* $0 < E_- < E_+ < \infty$, *and all* $n \in \mathbb{N}$ *such that*

$$E_- \leq E_n\big(H^D_{\Lambda_L}(A)\big) \leq E_n\big(H^D_{\Lambda_L}(A + W)\big) \leq E_+,$$

we have

$$E_n\big(H^D_{\Lambda_L}(A + t \cdot W)\big) \geq E_n\big(H^D_{\Lambda_L}(A)\big) + t \cdot E_-^2 \left(\frac{\delta}{G}\right)^{C \cdot (1 + E_+^{2/3})}, \quad t \in [0, 1],$$

where $C > 0$ *is a constant that depends only on* θ_E, θ_L, *and* N.

In order to present an application of the eigenvalue lifting we recall the Wegner estimate proven by the author in [Dic21] in a simple setting. To this end, let $v(x) := (1 - |x|)_+$ on $\mathbb{R}^d$ and let $Y = (Y_j)_{j \in \mathbb{Z}^d}$ be a sequence of independent, uniformly distributed random variables taking values in the interval $[1/4, 3/4]$. Consider the random perturbation $V_Y(x) = \sum_j v((x - j)/Y_j)$ and the *random divergence-type operator*

$$H^L_Y := H^D_{\Lambda_L}(1 + V_Y) = -\mathrm{div}\big[(1 + V_Y)\nabla\big]|_{\Lambda_L}. \tag{B.3}$$

The Wegner estimate shows that the expected number of eigenvalues in an interval decreases with its length. In particular, it requires the eigenvalues to move under the influence of randomness, as otherwise the inequality in the theorem below fails. The next result is stated in [Dic21, Theorem 1.1].

Theorem B.5. *There is* $\tilde{\varepsilon} > 0$ *depending only on the dimension* d, *such that for all* $0 < E_- < E_+ < \infty$, *all* $L \in \mathbb{N}$, *all* $\varepsilon \in (0, \tilde{\varepsilon}]$, *and all* $E > 0$ *satisfying* $[E - 3\varepsilon, E + 3\varepsilon] \subset [E_-, E_+]$ *we have*

$$\mathbb{E}\big(\#\{\textit{eigenvalues of } H^L_Y \textit{ in } [E - \varepsilon, E + \varepsilon]\}\big) \lesssim_d E_+^{d/2} \varepsilon^{[K_d \cdot (1 + |\log E_-| + E_+^{2/3})]^{-1}} |\Lambda_L|^2.$$

In Theorem B.5 we not only need to remove high energies (as is the case for random Schrödinger operators), but also energies close to zero. The reason for that is, that zero is not a so-called *spectral fluctuation boundary* for the random divergence-type operators; cf. the discussion after Theorem 1.1 in [Sto98] and the dependence on E_- that was obtained in Theorem B.5 above. Theorem B.5 is proven for quite general random divergence-type operators with breather-type perturbations in [Dic21, Theorem 2.1]. The main upshot of this compared to the Wegner estimate in [DV, Theorem 4.7] is that the dependence of the random perturbation V_Y on the random variables Y might be non-linear.

Bibliography

[AB] P. Alphonse and J. Bernier. Polar decomposition of semigroups generated by non-selfadjoint quadratic differential operators and regularizing effects. To appear in Ann. Sci. ENS, Preprint: arXiv:1909.03662.

[AE13] J. Apraiz and L. Escauriaza. Null-control and measurable sets. *ESAIM, Control Optim. Calc. Var.*, 19(1):239–254, 2013.

[AEWZ14] J. Apraiz, L. Escauriaza, G. Wang, and C. Zhang. Observability inequalities and measurable sets. *J. Eur. Math. Soc.*, 16(11):2433–2475, 2014.

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[Agm82] S. Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical Notes*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[Alp] P. Alphonse. Null-controllability of evolution equations associated with fractional Shubin operators through quantitative Agmon estimates. Preprint: arXiv:2021.04374.

[Alp21] P. Alphonse. Quadratic differential equations: partial Gelfand-Shilov smoothing effect and null-controllability. *J. Inst. Math. Jussieu*, 20(6):1749–1801, 2021.

[AS82] M. Aizenman and B. Simon. Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure Appl. Math.*, 35(2):209–273, 1982.

[BEPS20] K. Beauchard, M. Egidi, and K. Pravda-Starov. Geometric conditions for the null-controllability of hypoelliptic quadratic parabolic equations with moving control supports. *C. R., Math., Acad. Sci. Paris*, 358(6):651–700, 2020.

[BJPS21] K. Beauchard, P. Jaming, and K. Pravda-Starov. Spectral estimates for finite combinations of Hermite functions and null-controllability of hypoelliptic quadratic equations. *Studia Math.*, 260(1):1–43, 2021.

[BK05] J. Bourgain and C. E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2):389–426, 2005.

[BK13] J. Bourgain and A. Klein. Bounds on the density of states for Schrödinger operators. *Invent. Math.*, 194(1):41–72, 2013.

[Boa54] R. Ph. Boas, Jr. *Entire functions*. New York: Academic Press Inc., 1954.

[BPS18] K. Beauchard and K. Pravda-Starov. Null-controllability of hypoelliptic quadratic differential equations. *J. Éc. polytech. Math.*, 5:1–43, 2018.

[BS91] F. A. Berezin and M. A. Shubin. *The Schrödinger equation*, volume 66 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the 1983 Russian edition by Yu. Rajabov, D. A. Leĭtes and N. A. Sakharova and revised by Shubin, With contributions by G. L. Litvinov and Leĭtes.

[Car45] T. Carleman. Sur la détermination d'une fonction analytique par certaines valeurs moyennes de ses dérivées. *Ark. Mat. Astron. Fys.*, 32(4):7, 1945.

[CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1987.

[CGPR19] M. Cappiello, T. Gramchev, S. Pilipovic, and L. Rodino. Anisotropic Shubin operators and eigenfunction expansions in Gelfand-Shilov spaces. *J. Anal. Math.*, 138(2):857–870, 2019.

[CHK03] J.-M. Combes, P. D. Hislop, and F. Klopp. Hölder continuity of the integrated density of states for some random operators at all energies. *Int. Math. Res. Not.*, 4:179–209, 2003.

[CKP07] R. D. Carmichael, A. Kamiński, and S. Pilipović. *Boundary Values and Convolution in Ultradistribution Spaces*, volume 1 of *Series on Analysis, Applications and Computation*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[Dav82] E. B. Davies. JWKB and related bounds on Schrödinger eigenfunctions. *Bull. Lond. Math. Soc.*, 14(4):273–284, 1982.

[Dic21] A. Dicke. Wegner estimate for random divergence-type operators monotone in the randomness. *Math. Phys. Anal. Geom.*, 24(3):22, 2021.

[Dou66] R. G. Douglas. On majorization, factorization, and range inclusion of operators on hilbert space. *Proc. Amer. Math. Soc.*, 2(17):413–415, 1966.

[DR77] S. Dolecki and D. L. Russell. A general theory of observation and control. *SIAM J. Control Optim.*, 2(15):185–220, 1977.

[DRST] A. Dicke, C. Rose, A. Seelmann, and M. Tautenhahn. Quantitative unique continuation for spectral subspaces of Schrödinger operators with singular potentials. Preprint: arXiv:2011.01801.

[DS22] A. Dicke and A. Seelmann. Uncertainty principles with error term in Gelfand–Shilov spaces. *Archiv der Mathematik*, 119(4):413–425, 2022.

[DSVa] A. Dicke, A. Seelmann, and I. Veselić. Control problem for quadratic differential operators with sensor sets of decaying density via partial harmonic oscillators. Preprint: arXiv:2201.02370.

[DSVb] A. Dicke, A. Seelmann, and I. Veselić. Spectral inequality with sensor sets of decaying density for Schrödinger operators with power growth potentials. Preprint: arXiv:2206.08682.

[DSVc] A. Dicke, A. Seelmann, and I. Veselić. Uncertainty principle for Hermite functions and null-controllability with sensor sets of decaying density. Preprint: arXiv:2201.11703.

[DV] A. Dicke and I. Veselić. Unique continuation for the gradient of eigenfunctions and Wegner estimates for random divergence-type operators. Preprint: arXiv:2003.09849.

[DZZ08] T. Duyckaerts, X. Zhang, and E. Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 25(1):1–41, 2008.

[Egi21] M. Egidi. On null-controllability of the heat equation on infinite strips and control cost estimate. *Math. Nachr.*, 294(5):843–861, 2021.

[EN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[ENS$^+$20] M. Egidi, I. Nakić, A. Seelmann, M. Täufer, M. Tautenhahn, and I. Veselić. Null-controllability and control cost estimates for the heat equation on unbounded and large bounded domains. In J. Kerner, H. Laasri, and D. Mugnolo, editors, *Control theory of infinite-dimensional systems*, volume 277 of *Oper. Theory Adv. Appl.*, pages 117–157. Springer International Publishing, Cham, 2020.

[ES21] M. Egidi and A. Seelmann. An abstract Logvinenko-Sereda type theorem for spectral subspaces. *J. Math. Anal. Appl.*, 500(1):Paper No. 125149, 32, 2021.

[ES22] M. Egidi and A. Seelmann. The reflection principle in the control problem of the heat equation. *J. Dyn. Control Syst.*, 28(3):635–655, 2022.

[EV03] L. Escauriaza and S. Vessella. Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients. In *Inverse problems: theory* *and applications (Cortona/Pisa, 2002)*, volume 333 of *Contemp. Math.*, pages 79–87. Amer. Math. Soc., Providence, RI, 2003.

[EV18] M. Egidi and I. Veselić. Sharp geometric condition for null-controllability of the heat equation on $\mathbb{R}^d$ and consistent estimates on the control cost. *Arch. Math. (Basel)*, 111(1):85–99, 2018.

[EV20] M. Egidi and I. Veselić. Scale-free unique continuation estimates and Logvinenko-Sereda theorems on the torus. *Ann. Henri Poincaré*, 21(12):3757–3790, 2020.

[FI96] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of Evolution Equations*, volume 34 of *Suhak kangŭirok*. Seoul National University, Seoul, 1996.

[FL88] W. Fischer and I. Lieb. *Ausgewählte Kapitel aus der Funktionentheorie*, volume 48 of *Vieweg Studium: Aufbaukurs Mathematik*. Friedr. Vieweg & Sohn, Braunschweig, 1988.

[Fur00] A. V. Fursikov. *Optimal control of distributed systems. Theory and applications. Transl. from the Russian by Tamara Roszkovskaya*, volume 187. Providence, RI: AMS, American Mathematical Society, 2000.

[FZ00] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: the linear case. *Adv. Differential Equ.*, 5(4–6):465–514, 2000.

[GJ13] S. Ghobber and P. Jaming. The Logvinenko-Sereda theorem for the Fourier-Bessel transform. *Integral Transforms Spec. Funct.*, 24(6):470–484, 2013.

[GK13] F. Germinet and A. Klein. A comprehensive proof of localization for continuous Anderson models with singular random potentials. *J. Eur. Math. Soc. (JEMS)*, 15(1):53–143, 2013.

[GST20] D. Gallaun, C. Seifert, and M. Tautenhahn. Sufficient criteria and sharp geometric conditions for observability in Banach spaces. *SIAM J. Control Optim.*, 58(4):2639–2657, 2020.

[Güi85] E. N. Güichal. A lower bound of the norm of the control operator for the heat equation. *J. Math. Anal. Appl.*, 110(2):519–527, 1985.

[GY12] J. Gagelman and H. Yserentant. A spectral method for Schrödinger equations with smooth confinement potentials. *Numer. Math.*, 122(2):383–398, 2012.

[HJ94] V. Havin and B. Jöricke. *The uncertainty principle in harmonic analysis*, volume 28 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1994.

[HJK20] A. Hartmann, P. Jaming, and K. Kellay. Quantitative estimates of sampling constants in model spaces. *Am. J. Math.*, 142(4):1301–1326, 2020.

[Hol01] E. Holmgren. Über Systeme von linearen partiellen Differentialgleichungen. *Stockh. Öfv.*, 58:91–103, 1901.

[Hör63] L. Hörmander. *Linear partial differential operators*. Die Grundlehren der mathematischen Wissenschaften, Band 116. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[Hör95] L. Hörmander. Symplectic classification of quadratic forms, and general Mehler formulas. *Math. Z.*, 219(3):413–449, 1995.

[Hör07] L. Hörmander. *The analysis of linear partial differential operators. III.* Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[HP57] E. Hille and R. S. Phillips. *Functional analysis and semigroups. Rev. ed*, volume 31 of *Colloq. Publ., Am. Math. Soc.* American Mathematical Society (AMS), Providence, RI, 1957.

[HPS09] M. Hitrik and K. Pravda-Starov. Spectra and semigroup smoothing for non-elliptic quadratic operators. *Math. Ann.*, 344(4):801–846, 2009.

[HPSV17] M. Hitrik, K. Pravda-Starov, and J. Viola. Short-time asymptotics of the regularizing effect for semigroups generated by quadratic operators. *Bull. Sci. Math.*, 141(7):615–675, 2017.

[HPSV18] M. Hitrik, K. Pravda-Starov, and J. Viola. From semigroups to subelliptic estimates for quadratic operators. *Trans. Am. Math. Soc.*, 370(10):7391–7415, 2018.

[Isa17] V. Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer, Cham, third edition, 2017.

[JL99] D. Jerison and G. Lebeau. Nodal sets of sums of eigenfunctions. In M. Christ, C. E. Kenig, and C. Sadosky, editors, *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 223–239. Univ. Chicago Press, Chicago, IL, Chicago, 1999.

[Joh48] F. John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.

[Jun01] H. Jung. Ueber die kleinste Kugel, die eine räumliche Figur einschliesst. *J. Reine Angew. Math.*, 123:241–257, 1901.

[Kac73] V. È. Kacnel'son. Equivalent norms in spaces of entire functions. *Mat. Sb. (N.S.)*, 92(134):34–54, 165, 1973.

[Kat72] T. Kato. Schrödinger operators with singular potentials. *Israel J. Math.*, 13(1):135–148, 1972.

[Kat80] T. Kato. *Perturbation theory for linear operators*. Springer, Berlin, 2nd edition, 1980.

[Ken87] C. E. Kenig. Carleman estimates, uniform Sobolev inequalities for second-order differential operators, and unique continuation theorems. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 948–960. Amer. Math. Soc., Providence, RI, 1987.

[Kle13] A. Klein. Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators. *Comm. Math. Phys.*, 323(3):1229–1246, 2013.

[KM82] W. Kirsch and F. Martinelli. On the density of states of Schrödinger operators with a random potential. *J. Phys. A*, 15(7):2139–2156, 1982.

[Kov00] O. Kovrijkine. *Some estimates of Fourier transforms*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–California Institute of Technology.

[Kov01] O. Kovrijkine. Some results related to the Logvinenko-Sereda theorem. *Proc. Amer. Math. Soc.*, 129(10):3037–3047, 2001.

[KP92] S. G. Krantz and H. R. Parks. *A primer of real analytic functions*, volume 4 of *Basler Lehrbücher*. Birkhäuser Verlag, Basel, 1992.

[KT16] A. Klein and C. S. Sidney Tsang. Quantitative unique continuation principle for Schrödinger operators with singular potentials. *Proc. Am. Math. Soc.*, 144(2):665–679, 2016.

[Ler19] N. Lerner. *Carleman inequalities*, volume 353 of *Grundlehren der mathematischen Wissenschaften*. Springer, Cham, 2019. An introduction and more.

[LL12] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Contr. Optim. Ca.*, 18(3):712–747, 2012.

[LM] G. Lebeau and I. Moyano. Spectral inequalities for the Schrödinger operator. Preprint: arXiv:1901.03513.

[LM16] J. Le Rousseau and I. Moyano. Null-controllability of the Kolmogorov equation in the whole phase space. *J. Differ. Equations*, 260(4):3193–3233, 2016.

[LR95] G. Lebeau and L. Robbiano. Contrôle exact de léquation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[LS74] V. N. Logvinenko and J. F. Sereda. Equivalent norms in spaces of entire functions of exponential type. *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 20:102–111, 1974.

[LT91] E. H. Lieb and W. E. Thirring. *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, pages 135–169. Springer, Berlin Heidelberg, 1991.

[LZ98] G. Lebeau and E. Zuazua. Null-controllability of a system of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 141(4):297–329, 1998.

[Man98] N. Mandache. On a counterexample concerning unique continuation for elliptic equations in divergence form. *Math. Phys. Anal. Geom.*, 1(3):273–292, 1998.

[Mar] J. Martin. Spectral inequalities for anisotropic Shubin operators. Preprint: arXiv:2205.11868.

[Mar22] J. Martin. Uncertainty principles in Gelfand-Shilov spaces and null-controllability. *J. Funct. Anal.*, 283(8):48, 2022.

[Mat95] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[Mil] L. Miller. Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones. Preprint: hal-00411840.

[Mil73] K. Miller. Non-unique continuation for certain ODE's in Hilbert space and for uniformly parabolic and elliptic equations in self-adjoint divergence form. In *Symposium on Non-Well-Posed Problems and Logarithmic Convexity (Heriot-Watt Univ., Edinburgh, 1972)*, pages 85–101. Lecture Notes in Math., Vol 316. Springer, Berlin, 1973.

[Mil04] L. Miller. Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. *J. Differ. Equations*, 204(1):202–226, 2004.

[Mil10] L. Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1465–1485, 2010.

[MPS22] J. Martin and K. Pravda-Starov. Spectral inequalities for combinations of Hermite functions and null-controllability for evolution equations enjoying Gelfand–Shilov smoothing effects. *J. Inst. Math. Jussieu*, page 1–50, 2022.

[MS13] C. Muscalu and W. Schlag. *Classical and multilinear harmonic analysis. Vol. I*, volume 137 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[Naz93] F. L. Nazarov. Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. *Algebra i Analiz*, 5(4):3–66, 1993.

[NR10] F. Nicola and L. Rodino. *Global pseudo-differential calculus on Euclidean spaces*, volume 4 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

[NRT19] I. Nakić, C. Rose, and M. Tautenhahn. A quantitative Carleman estimate for second-order elliptic operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 149(4):915–938, 2019.

[NSV04] F. Nazarov, M. Sodin, and A. Volberg. Lower bounds for quasianalytic functions. I: How to control smooth functions. *Math. Scand.*, 95(1):59–79, 2004.

[NTTV18] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Scale-free unique continuation principle for spectral projectors, eigenvalue-lifting and Wegner estimates for random Schrödinger operators. *Anal. PDE*, 11(4):1049–1081, 2018.

[NTTV20a] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Sharp estimates and homogenization of the control cost of the heat equation on large domains. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 54, 26, 2020.

[NTTV20b] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Unique continuation and lifting of spectral band edges of Schrödinger operators on unbounded domains. *J. Spectr. Theory*, 10(3):843–885, 2020.

[Olv97] F. W. J. Olver. *Asymptotics and special functions*. AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997. Reprint of the 1974 original.

[Pan61] B. P. Panejah. Some theorems of Paley-Wiener type. *Soviet Math. Dokl.*, 2:533–536, 1961.

[Pan62] B. P. Panejah. On some problems in harmonic analysis. *Dokl. Akad. Nauk SSSR*, 142:1026–1029, 1962. translated as Soviet Math. Dokl., 3:239–242, 1962.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[Pli63] A. Pliś. On non-uniqueness in Cauchy problem for an elliptic second order differential equation. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 11:95–100, 1963.

[PS18] K. Pravda-Starov. Generalized Mehler formula for time-dependent non-selfadjoint quadratic operators and propagation of singularities. *Math. Ann.*, 372(3-4):1335–1382, 2018.

[PSRW18] K. Pravda-Starov, L. Rodino, and P. Wahlberg. Propagation of Gabor singularities for Schrödinger equations with quadratic Hamiltonians. *Math. Nachr.*, 291(1):128–159, 2018.

[Rem36] E. Remes. Sur une propriété extremale des polynômes de Tchebychef. *Comm. Inst. Sci. math. mec. Univ. Kharkoff et de la Soc. Math. de Kharkoff*, 13(1):93–95, 1936.

[RMV13] C. Rojas-Molina and I. Veselić. Scale-free unique continuation estimates and applications to random Schrödinger operators. *Comm. Math. Phys.*, 320(1):245–274, 2013.

[Rob95] L. Robbiano. Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptotic Anal.*, 10(2):95–115, 1995.

[RT21] L. Rodino and S. I. Trapasso. An introduction to the Gabor wave front set. In *Anomalies in partial differential equations*, volume 43 of *Springer INdAM Ser.*, pages 369–393. Springer, Cham, 2021.

[Sch12] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2012.

[See21] A. Seelmann. The Laplacian on Cartesian products with mixed boundary conditions. *Arch. Math.*, 117(1):87–94, 2021.

[Sei84] T. I. Seidman. Two results on exact boundary control of parabolic equations. *Appl. Math. Optim.*, 11(2):145–152, 1984.

[Smi75] H. J. S. Smith. On the Integration of Discontinuous Functions. *Proc. Lond. Math. Soc.*, 6:140–153, 1874/75.

[ST20] A. Seelmann and M. Täufer. Band edge localization beyond regular Floquet eigenvalues. *Ann. Henri Poincaré*, 21(7):2151–2166, 2020.

[Sto98] P. Stollmann. Localization for random perturbations of anisotropic periodic media. *Israel J. Math.*, 107:125–139, 1998.

[Sto01] P. Stollmann. *Caught by disorder*, volume 20 of *Progress in Mathematical Physics*. Birkhäuser Boston, Inc., Boston, MA, 2001. Bound states in random media.

[Tre08] C. Tretter. *Spectral theory of block operator matrices and applications*. Imperial College Press, London, 2008.

[TT11] G. Tenenbaum and M. Tucsnak. On the null-controllability of diffusion equations. *ESAIM Control Optim. Calc. Var.*, 17(4):1088–1100, 2011.

[Tur84] P. Turán. *On a new method of analysis and its applications.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. With the assistance of G. Halász and J. Pintz, With a foreword by Vera T. Sós, A Wiley-Interscience Publication.

[TV20] M. Tautenhahn and I. Veselić. Sampling and equidistribution theorems for elliptic second order operators, lifting of eigenvalues, and applications. *J. Differ. Equations*, 268(12):7669–7714, 2020.

[TW09] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.

[Ves03] S. Vessella. Carleman estimates, optimal three cylinder inequality, and unique continuation properties for solutions to parabolic equations. *Comm. Partial Differential Equations*, 28(3–4):637–676, 2003.

[Ves08] I. Veselić. *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, volume 1917 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2008.

[Wei80] J. Weidmann. *Linear operators in Hilbert spaces*, volume 68 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1980. Translated from the German by Joseph Szücs.

[WWZZ19] G. Wang, M. Wang, C. Zhang, and Y. Zhang. Observable set, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. *J. Math. Pures Appl. (9)*, 126:144–194, 2019.

[WZ17] G. Wang and C. Zhang. Observability inequalities from measurable sets for some abstract evolution equations. *SIAM J. Control Optim.*, 55(3):1862–1886, 2017.

[Yam09] M. Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 25(12):123013, 75, 2009.

[Zab20] J. Zabczyk. *Mathematical control theory—an introduction.* Systems & Control: Foundations & Applications. Birkhäuser/Springer, Cham, 2020.

[Zha16] Y. Zhang. Unique continuation estimates for the Kolmogorov equation in the whole space. *C. R., Math., Acad. Sci. Paris*, 354(4):389–393, 2016.

[Zua07] E. Zuazua. Controllability and observability of partial differential equations: some results and open problems. In *Handbook of differential equations: evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 527–621. Elsevier/North-Holland, Amsterdam, 2007.

[Zui83] C. Zuily. *Uniqueness and nonuniqueness in the Cauchy problem*, volume 33 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1983.