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Connecting characterizations of equivalence of expressions: design research in Grade 5 by bridging graphical and symbolic representations

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Abstract
One typical challenge in algebra education is that many students justify the equivalence of expressions only by referring to transformation rules that they perceive as arbitrary without being able to justify these rules. A good algebraic understanding involves connecting the transformation rules to other characterizations of equivalence of expressions (e.g., description equivalence that both expressions describe the same situation or figure). In order to overcome this disconnection even before variables are introduced, a design research study was conducted in Grade 5 to design and investigate an early algebra learning environment to establish stronger connections between different mental models and representations of equivalence of expressions. The qualitative analysis of design experiments with 14 fifth graders revealed deep insights into complexities of connecting representations. It confirmed that many students first relate the representations in ways that are too superficial without establishing deep connections. Analyzing successful students' processes helped to identify an additional characterization that can support students in bridging the connection between other characterizations, which we call restructuring equivalence. By including learning opportunities for restructuring equivalence, students can be supported to compare expressions in graphical and symbolic representation simultaneously and dynamically. The design research study disentangles the complex requirements for realizing the design principle of connecting multiple representations, which should be of relevance beyond the specific concept of equivalence and applicable to other mathematical topics.

Keywords Early algebra · Expressions · Equivalence · Connecting multiple representations · Conceptual understanding

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Empirical studies have identified multiple challenges in understanding algebraic concepts and procedures for both beginning learners (Warren, 2006; Papadopoulos & Gunnarsson, 2020; Herscovics & Linchevski, 1994; Linchevski & Livneh, 1999, overview in Kieran, 2007) and advanced learners (Sfard & Linchevski, 1994; Stylianides et al., 2004). One typical difficulty was exemplified in a compelling vignette:

Interviewer: [*After Gil states that the expressions 3(x + 2) and 3x + 6 are equivalent*]: Why? What makes these two things equivalent?

…

Gil: Yeah ... because they're just rules ... they are there so that you can follow them, so that everybody'll do the same thing.

…

Interviewer: I prefer to do it this [$3(x + 2)$] this way [writes "$= 3x + 2$"]. Why this [$3x + 6$] and not this one [$3x + 2$]?

…

Gil: Because of the rules. But I don't know how we got the rules.

(Kieran & Sfard, 1999, p. 1)

Like Gil, many students know that they can write $3(x+2)=3x+6$ due to transformation rules, but she cannot justify these rules as she does not understand what equivalence of expressions means, in other words, that both expressions describe the same situation (ibid.). *Algebraic understanding*, though, includes this ability to connect algebraic procedures in symbolic representations to the meanings of algebraic concepts in graphical, verbal, or contextual representations (Kieran, 2007; Kilpatrick et al., 2001), because otherwise the rules cannot be explained and hence they stay arbitrary.

For developing students' algebraic understanding, learning environments have been designed based on the design principle of *connecting mental models and representations* (e.g., Friedlander & Tabach, 2011; Kaput, 2008; Mason et al., 1985), in particular for two algebraic concepts and variables and equations, but less so far for a third concept, equivalence of expressions (exceptions are, e.g., Kieran & Sfard, 1999; McNeil et al., 2019). However, Gil's challenge is not only the variable: She would have similar difficulties justifying the transformation of arithmetic expressions, for instance, from $3\times(10+2)$ to $3\times10+3\times2$, with reference to structures beyond the fact that both expressions yield the same value, that is, the same result, so these expressions without variables are the focus of this paper.

Early algebra settings were developed to provide learning opportunities for meanings of algebraic concepts and their underlying algebraic structures even before variables are introduced (Kieran, 2022). These early algebra settings provide effective approaches for preparing variables through rich generalization activities (Cai & Knuth, 2011; Lins & Kaput, 2004) and a relational use of the equal sign for preparing equations (Kieran, 1981; Rittle-Johnson & Alibali, 1999). However, they rarely focus on equivalence of expressions by studying the relations between arithmetic expressions and their underlying deeper structures. Kieran characterizes the state of research as follows:

> The dominant focus on generalizing in the development of algebraic thinking has to a large extent obscured the process of seeing structure. While generalization-oriented activity remains highly important in algebra and early algebra, and in fact includes a structural component, equal attention needs to be paid to the comple-

> mentary process of looking through mathematical objects and to decomposing and recomposing them in various structural ways. (Kieran, 2018, p. 79)

In this paper, we follow Kieran's call for developing and investigating opportunities in students' seeing and using of structures with respect to the equivalence of expressions. In Sect. 1, we will present a framework by which this gap within early algebra approaches on equivalence of expressions can be articulated and located more concisely. The design research study presented in this paper contributes to reducing this gap by pursuing the following design research question:

> *How can the design of an early algebra learning environment enhance students' connections of mental models and representations for equivalence of expressions?*

The methodological framework of the design research study is outlined in Sect. 2, and empirical insights into the design experiments are provided in Sect. 3 and discussed in Sect. 4.

1 Theoretical background

In this section, we will define more concisely what we mean by a focus on equivalence of expressions by studying the relations between arithmetic expressions and their underlying deeper structures. We start by introducing the terms and three characterizations for equivalence (Sect. 1.1), summarizing the existing approaches to enhancing students' understanding of equivalence (Sect. 1.2), and articulating the gap that needs to be bridged and the refined design research question (Sect. 1.3).

1.1 Three characterizations for equivalence of expressions

In her discussion of multiple meanings of structure, Kieran (2018) emphasizes that structure does not necessarily require pattern generalization, but is also useful for capturing students' ways of connecting representations, among other things. The articulation of our epistemological background builds upon Kieran's (1989) conceptualizations of structure. Starting from a collegiate definition of structure as aggregate of elements of an entity in the relationships to each other, Kieran (1989) defines the *surface structure* of an expression as referring "to the arrangement of the terms and operations," determining the order of operations and the *systemic structure* as "relating to the mathematical system from which the expression inherits its properties" of and between operations, such as commutativity or distributivity (p. 34). Third, Kieran also refers to structures in word problems for emphasizing the sub-entities and relationships in the situation to which expressions and equations can be related. Analogically, we will refer to structured figures as figures in which sub-entities and their relationships are considered. We denote systemic structures as *deeper (systemic) structures* when their involved properties are (explicitly or implicitly) also connected to structures in other representations, for instance, in these structured figures. Thus, our third conceptualization of structure reflects the need to also connect the systemic properties to other representations (Cooper & Warren, 2011; Mason et al., 1985) and Kieran's (2018) call to focus on seeking, using, and expressing (deeper) structures.

From the rich state of algebra education research (as summarized in several surveys, e.g., Stacey et al., 2004; Kieran, 2007), we extract three typical perspectives that students adopt on expressions. In order to harmonize the multiple terms used, we introduce the following terms:

- In a *relational perspective*, students see rich relations between the different objects involved. For expressions, that means particularly what expressions can describe (Mason et al., 1985; Kieran & Sfard, 1999; Kaput, 2008; Radford, 2011; Blanton et al., 2019). The focus is on systemic structures within the symbolic representations (relying on properties) and on specific deeper systemic structures between several representations (so that the symbolic surface structures are connected, e.g., to structured figures, see Mason et al., 1985).
- In an *operational perspective,* students view expressions mainly as requests to evaluate the result, meaning as uncompleted tasks rather than as reified object descriptions (Malle, 1993). We borrow the term "operational" from the research on the equal sign (Knuth et al., 2006; Sfard & Linchevski, 1994) and extend it to expressions. The focus is solely on surface structures to determine which operation to complete first, yet not on systemic structures.
- In a *transformational perspective*, students view expressions mainly as sets of symbols that are to be manipulated according to transformation rules, which can be arbitrary or well-justified by drawing upon robust understandings of structure (Kieran, 2004; Linchevski & Livneh, 1999; Papadopoulos & Gunnarsson, 2020). For correctly enacting transformations, students need to recognize the surface structures of the expression. The justification of the transformation rules requires references to systemic or deeper structures.

In each of these perspectives on expressions, we can characterize differently what the *equivalence* of expression entails (following Malle, 1993; Zwetzschler & Prediger, 2013). The three characterizations of equivalence are exemplified for $3\times(10+5)=3\times10+3\times5$ in Fig. 1.

- In a transformational perspective, two symbolically represented expressions are characterized as *transformation equivalent* if one can be transformed into the other by a rule-based, innersymbolic treatment (Duval, 2006). The transformation equivalence is a *dynamic* characterization (Cooper & Warren, 2011). It is characterized by the active modification process, transforming the expression E_A into E_B (later abbreviated E_A E_B, in Fig. 1 marked by the double arrow). To justify the transformations within the symbolic representation, students can refer to systemic structures relating to the properties of operations. The justification of these properties themselves requires the connection to other representations, therefore to deeper structures (Kieran & Sfard, 1999).
- In an operational perspective, two expressions are characterized as *result equivalent* by evaluating their results and comparing both expressions by their results. The result equivalence can be characterized as *static* as it remains indirect and refers to a third object (the result) as a static object of comparison (Cooper & Warren, 2011). (For expressions with variables, the result equivalence involves evaluating the expression for all numbers substituting the variables. In our research, we concentrate on expressions without variables.) In the scheme in Fig. 1, we depict the three links involved: From the expressions to the result, E_A–R and E_B–R, and from there derive the equivalence E_A–E_B (marked by the grey double line).
- In a relational perspective relating the symbolic representation to a context situation or a graphical representation, two expressions are characterized as *description equivalent* when both describe the same situation or figure (Kieran & Sfard, 1999; Malle, 1993;

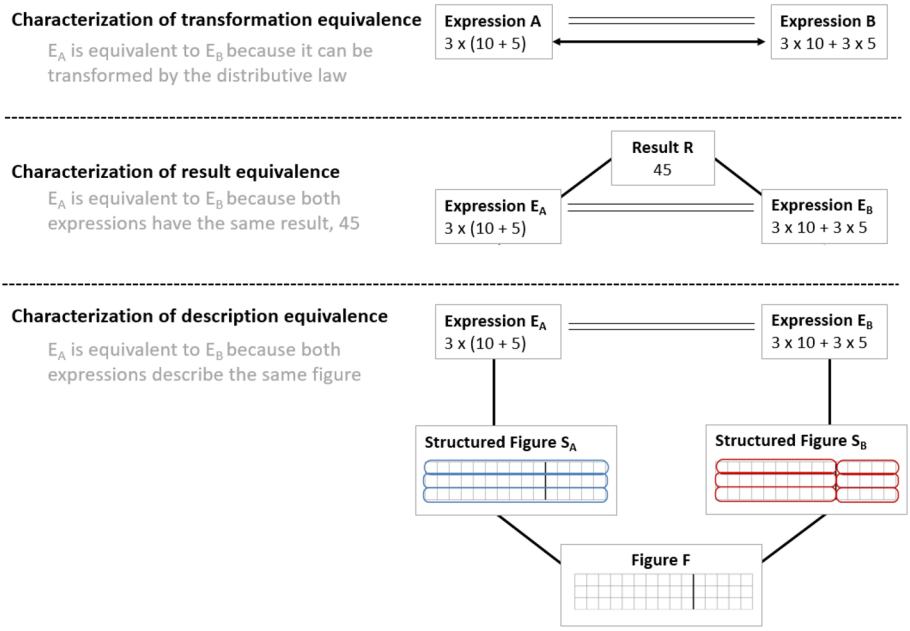


Fig. 1 Epistemological background: Three characterizations for equivalence of expressions in three perspectives

Wilkie & Clarke, 2016; Zwetzschler & Prediger, 2013). The description equivalence is also a static characterization since it is based on a third object comparison. To check if the expressions E_A and E_B describe the same Figure F, the deeper structures of E_A and E_B need to be recognized in the geometric structuring of Figure F (for the example in Fig. 1), which means that relevant subexpressions (surface structures) need to be identified and connected to related parts of the structured figures S_A and S_B (e.g., a sum involves the composition of two areas in the figure, a multiplication involves counting in groups in the figure). Thereby, understanding this characterization requires constructing five links: The link from F to E_A is constructed by the intermediate links of F–S_A (the figure is structured in parts in a particular way) and SA–E_A (identifying parts on the structured figure that are described by subexpressions of E_A) and respectively from F to E_B via F–S_B and S_B–E_B. From these two double links (F–S_A–E_A and F–S_B–E_B, marked by the four edges in the scheme), the static comparative link E_A–E_B can be derived (marked by the double line in the scheme in Fig. 1).

1.2 Existing research on and approaches for enhancing students' understanding of equivalence of expressions

Following Hiebert and Carpenter's (1992) definition of *understanding as a network of connections* (transferred to algebra by Cooper & Warren, 2011), students' understanding can be enhanced by developing mental models for all three characterizations and connecting them across graphical and symbolic representations.

Whereas the epistemological background can be clearly distinguished into three separate schemes, the mental models that students develop are not necessarily separable,

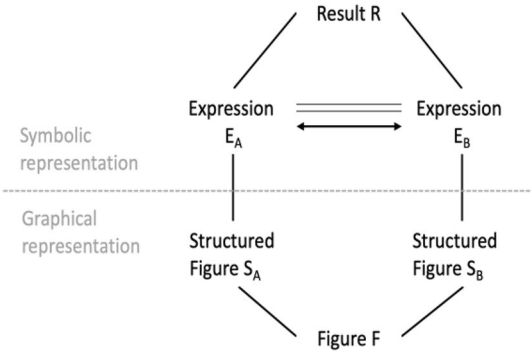


Fig. 2 Analytic framework of this paper: network of possible connections for understanding equivalence of expressions

and not all links are always established. By combining the three schemes from Fig. 1 into one scheme in Fig. 2, we provide the analytic framework by which we can concisely describe students' ideas and instructional approaches as documented in the research literature (in this subsection) and capture their learning pathways throughout the connections in our empirical data (in the next two sections).

Existing research on students' challenges Prior research has identified different typical challenges in students' mental models of equivalence of expressions:

The most often documented challenge in dealing with equalities and equations refers to students' challenges with the equal sign itself, which is often misinterpreted operationally (as calling for determining the result: $E_A = R$) rather than a more relational understanding of that symbol ($E_A = E_B$). This leads to an exclusive focus on the first number on the right side of the equal sign as the result (Herscovics & Linchevski, 1994) rather than completely capturing the surface structure of E_B. The students whose processes were investigated in this study had not internalized result equivalence but persisted in interpreting the equal sign only as a request to evaluate the result without relating the expressions to each other (E_A–R and E_B–R without E_A–E_B; Warren, 2006).

Even if students can overcome this challenge, many students do not develop relational perspectives on equivalence but remain focused on the same result as the only interpretation for equivalence of expressions, in other words, focusing on *result equivalence* (overview in Kieran, 2007, 2011). Within the analytic scheme from Fig. 2, these findings can be characterized as deriving E_A–E_B from the links E_A–R and E_B–R in the characterization of result equivalence.

Students' tendency to adopt operational rather than relational perspectives is also expressed in their focus on procedures and surface structures rather than systemic structures while transforming expressions (overview in Kieran, 2007). Their ability to correctly identify symbolic surface structures within the expressions depends on form and order of the elements of the represented expression (Papadopoulos & Gunnarsson, 2020), which applies for expressions with and without variables (Linchevski & Livneh, 1999). The former difficulty can partially be traced back to students' difficulties in connecting symbolic substructures with parts of structured figures (E_A–S_A and E_A–S_B; Malle, 1993; Wilkie & Clarke, 2016). Consequently, students' difficulties with seeing systemic structure seem to be also connected to the missing adoption of relational perspectives (Sfard & Linchevski, 1994) and the underlying deeper structures in other representations (Kieran & Sfard, 1999). This tendency also hinders them in connecting the transformations to description equivalence.

Existing instructional approaches for overcoming the challenges The analytic framework in Fig. 2 also allows description of some of the existing *instructional approaches* designed to overcome these challenges for which we identified roughly three groups and depicted the focused connections for equivalence in Fig. 3 (without claiming completeness):

The first group of instructional approaches mainly focuses on fostering result equivalence of two expressions by foregrounding the meaning of the equal sign. Some approaches use a change of representations (e.g., McNeil et al., 2019) by utilizing figures (without using structure), while others remain in the symbolic representation (Fyfe et al., 2015; Jones et al., 2013; Rittle-Johnson & Alibali, 1999). However, they all have in common that the systemic structures of the expressions are not being focused on to identify their equivalence, neither on the structured figures nor the symbolic substructure of the expressions (this is indicated in Fig. 3a on the left by grey colors for S_A and S_B as well as E_A and E_B). Although a relational perspective on the equal sign is fostered in this group of approaches, an operational perspective on equivalence is foregrounded.

The *second group* of instructional approaches can be exemplified by two approaches: promoting the characterization of description equivalence in a learning environment in which expressions describe either functional relationships (Kieran & Sfard, 1999) or geometric figures (Mason et al., 1985; Wilkie & Clarke, 2016; Zwetzschler & Prediger, 2013). In the latter, students learn to identify different ways of structuring the figure and to recognize that the structure may be described by different expressions. Thus, this perspective promotes connections between representations and pays attention to the deeper structures. The result equivalence or alternative structurings of figures can be starting points that students mobilize when developing the new characterization of description equivalence and are best with an extended focus on deeper structures. In this approach, the transformational perspective is not included; thus the characterizations of description equivalence and transformation equivalence are not connected (Fig. 3b in the middle).

The *third group* of instructional approaches aims at establishing transformation equivalence starting from students' mental models of result equivalence. Therefore, they focus in different intensities on connecting innersymbolic structures to the manipulation of expressions in the transformational perspective. Banerjee and Subramaniam (2012) aimed at fostering students' understanding of transformations by supporting surface structures and connecting the manipulation of systematic structures in properties of operations with the idea of an unchanged result of equivalent expressions, that is to say, with the characterization of

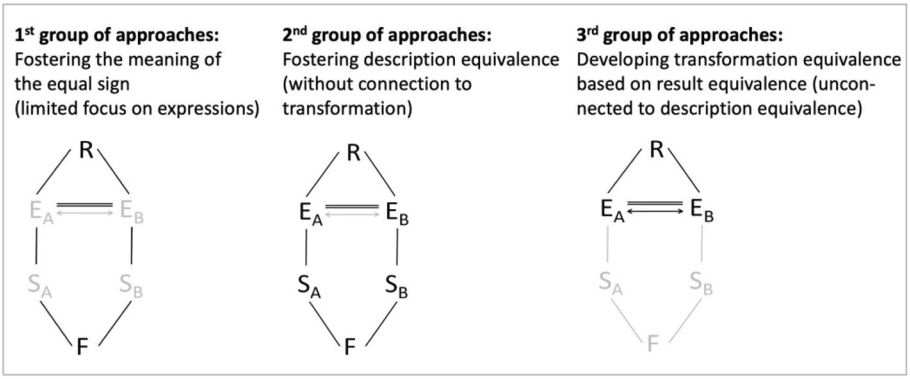


Fig. 3 Three groups of instructional approaches and their focuses in establishing connections for equivalence

result equivalence. Also Schwarzkopf, Nührenbörger, and Mayer (2018) started from the characterization of result equivalence to develop a first idea of transformation equivalence for Grade 4 students by inviting students to modify the expressions systematically considering their systemic structures (e.g., $6 \times 30 - 30 = 4 \times 30 + 30 = 150$). Blanton et al. (2019) fostered students from Grade 3 to 6 to develop their algebraic knowledge, inter alia, a relational understanding of the equal sign. The tasks combined operational aspects focusing on the same value with the idea of using systemic structure to substantiate transformation equivalence. Thus, all three perspectives aim to link operational and transformational perspectives on equivalence, yet because they remain in the symbolic representation, they do not involve deeper structures that the expressions describe (Fig. 3c on the right).

1.3 Need for bridging from description equivalence to transformation equivalence

From all three groups of instructional approaches in Sect. 1.2, we can draw important ideas for our own instructional approach:

- Developing new mental models must start from existing knowledge, such as the transformation equivalence starting from result equivalence (as in the third group).
- Dealing explicitly with systemic and surface structure seems to be promising to foster transformation of expressions (as in the third group).
- New perspectives should be explicitly introduced so that students can extend their repertoire (as in the first group of approaches for the equal sign).
- Connecting representations can be productive for developing relational understanding of equivalence by focusing deeper structures (as in the second group).

Whereas the cited existing approaches tend to prioritize the result equivalence as the natural starting point to be connected to transformation equivalence, the analysis of typical challenges suggests that the operational perspective alone is not a sufficient starting point, as it is tied to concrete numbers (in expressions without variables) and lacks the potential to generalize found relationships into properties and justify them by deeper structures. Even before introducing variables, the relational perspective bears a greater potential to generalize patterns so that the transformation rules can be justified (Kaput, 2008; Mason et al., 1985).

Both the second and third groups of instructional approaches focus on relational perspectives, but they differ in relation to the representation and the nature of the transformations undertaken. The second group of approaches focuses on the relational perspectives of expressions and supports the consideration of deeper structures across representations but within mostly static comparisons. The third group, however, focuses on dynamic transformations while relying on students having already internalized an adequate understanding of meanings about how to operate with systemic structure without a graphical representation.

1.4 Summary and research question

Figure 4 summarizes both the characterizations introduced in Sect. 1.1 and deepened by the report on existing research and the existing instructional approaches in Sects. 1.2 and 1.3. Given that some of the existing approaches work across the columns in different ways, their comparison enables us to identify that not only the representation but also the dynamics are crucial for developing students' understanding. It also locates the main gap that this paper intends to work on. Based on the theoretical background and the analytic

	Result equivalence	Description equivalence		Transformation equivalence
Genera-lizability	Concrete	Concrete and generalizable		Generalized
Represen-tation	Symbolic (numerical) representation	Graphical representation	Gap to be bridged	Symbolic (possibly algebraic) representation
Dynamics	Static comparison	Static comparison		Dynamic transformation

Fig. 4 Characterizing commonalities and distinctions between the three characterizations

framework developed throughout this section, we can now refine our research question to exactly this gap:

> *How can the design of an early algebra learning environment enhance or limit students in mentally connecting description equivalence in the graphical representation to transformation equivalence in the symbolic representation?*

As usual in semiotic processes (Duval, 2006), the students' mental processes of connecting representations in treatments and conversions involve complex mental constructions of what they focus on in each representation. In order to grasp these semiotic complexities, we will use the analytic framework in Fig. 2 and in particular the algebra-specific distinction of surface, systemic, and deeper structures.

2 Methodological framework of the design research study

2.1 Topic-specific didactical design research as the methodological framework

The research aim of developing productive learning environments is twofold, as it requests (a) designing of learning environments and (b) investigation of the learning processes initiated by these learning environments. This twofold aim can best be pursued within a design research methodology (Cobb et al., 2003). Like many approaches within the methodology of design research with a focus on learning processes (Gravemeijer & Cobb, 2006), our framework of topic-specific didactical design research (Prediger & Zwetzschler, 2013) relies on the iterative interplay between four working areas: (1) specifying and structuring learning goals and contents (here, characterizations of equivalence and their connection), (2) developing the design (here, a learning environment for fifth graders on equivalence of expressions), (3) conducting and analyzing design experiments (here, in pair settings), and (4) developing local theories on the topic-specific teaching and learning processes.

2.2 Method for data collection and sampling

Conducting design experiments is the central method for data gathering in design research methodologies (Cobb et. al., 2003), and laboratory pair settings are chosen when complex cognitive processes are in view that are hard to capture in whole-class settings.

So far, two design experiment cycles have been conducted by the first author with seven pairs of fifth graders (10–11 years old). In Cycle 1, four pairs of students were selected from a small-town comprehensive school. In Cycle 2, three more pairs of students from the same class were sampled. The design experiment series lasted two sessions of 90–105 min each. In total, 27.5 h of video were recorded and partly transcribed.

2.3 Methods of data analysis

The transcripts were analyzed qualitatively in three steps:

- In Step 1, the students' utterances and drawings were coded according to the addressed external representations and the connections made explicit between selected elements of them.
- In Step 2, the underlying mental models were inferred by disentangling students' utterances in the analytic framework of Fig. 2 with respect to the addressed elements and links in each utterance (addressed by gestures, colorings, or explicit verbal articulation). The properties that students refer to can only be inferred if explicitly articulated, but the structures they see in the expressions and figures can be inferred also from gestures and colorings of elements.
- In Step 3, graphical summaries in the analytic framework marked addressed elements and links in black and not addressed elements and links in grey. The fine-grained coding of links allowed for capturing students' learning pathways *chronologically*.

All codings and graphical summaries were conducted by the first author and checked in detail by a research assistant and the second author. Cases of missing intercoder agreement were solved by consensual discussion.

3 Insights into the design experiments and their analysis

This section documents our design research journey with the empirical insights gained for the focus tasks: We start by presenting the task designed to overcome the gap between description equivalence and transformation equivalence in Cycle 1 (Sect. 3.1) and provide empirical insights into two focus students' learning pathways with this task, which caused us to discover an additional characterization and its potential for bridging the gap (Sect. 3.2). This bridging characterization was included in the refined task design in Cycle 2 (Sect. 3.3) so that all students could engage with it. Section 3.4 then provides empirical insights into affordances and constraints for students' pathways from description equivalence towards transformation equivalence.

3.1 Task design in Cycle 1: overcoming the gap between description and transformation equivalence by several steps

The designed learning environment in Cycle 1 used the context of planning children's rooms (with desks, beds, and cupboards) to connect expressions to geometric figures (see focus task in Fig. 5). Understanding of the deeper structure of the expressions is supported by marking the respective structure within the geometric figures. Discussing different ways of determining the area in the room plan revealed the opportunity to compare different expressions with respect to their description equivalence. Prior to the focus task, students had already

- described and structured figures (areas of the room plans) with their own expressions.
- interpreted given expressions E_A/E_B with respect to the way the figures S_A/S_B were structured.
- justified why an expression matches the structured figure ($E_{A/B}$—$S_{A/B}$).

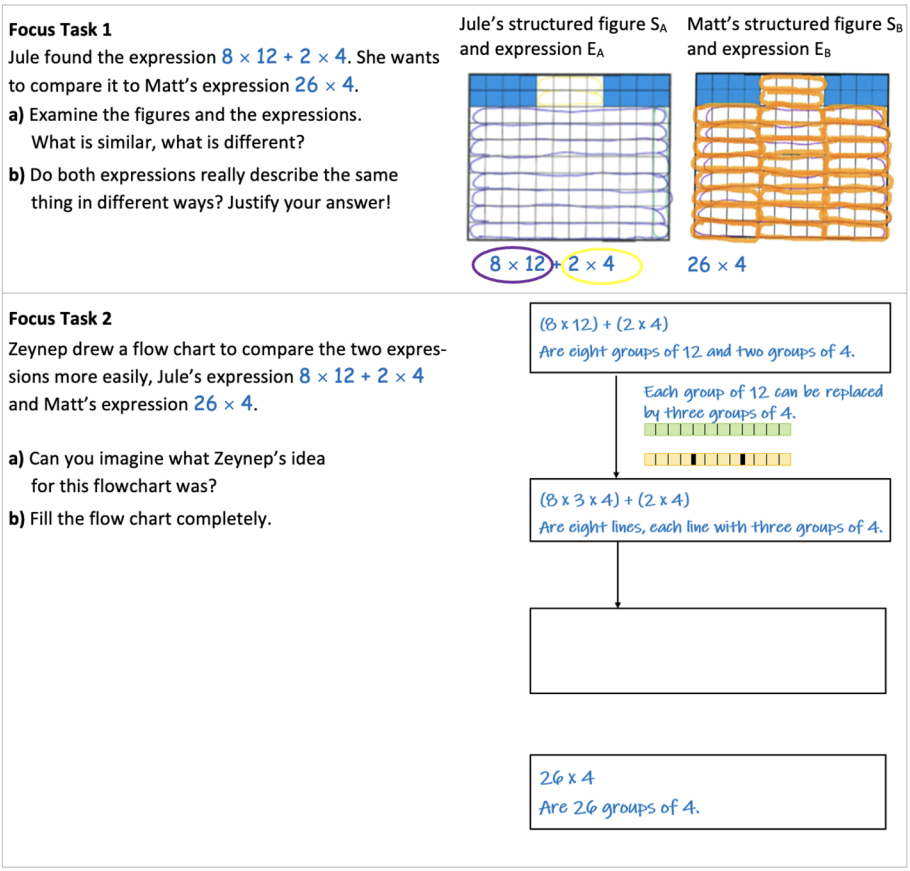


Fig. 5 Focus tasks in Cycle 1 designed to fill the gap between description and transformation equivalence (Abbreviations E_A/E_B and S_A/S_B added for the clarity of this paper)

The focus tasks (in Fig. 5) build upon these prior experiences for an already familiar room plan with two beds. Task 1 demands a comparison of structured figures with the subexpressions of the given expressions with respect to commonalities and differences in relational activities. Whereas Task 1a focuses on a comparison between E_A, E_B, S_A, and S_B, Task 1b also aims to infer the description equivalence $E_A = E_B$ by asking for a justification. The characterizations of description equivalence are deepened here by asking students to connect subexpressions of E_A/E_B to substructures in the structured figures S_A/S_B.

Task 1 prepares students for Task 2 by this focus on substructures and subexpressions in which the transformation steps from E_A to E_B are to be listed in a flow chart and to be justified, each by the description equivalence of the intermediate expressions and the underlying systemic structures (additional brackets were introduced to support the recognition of surface structures; see Banerjee & Subramaniam, 2012). Using this flow chart and the stepwise justification of intermediate expressions in description equivalence, the task aims at connecting symbolic transformations and graphical considerations as well as closing the gap between the static comparison in description equivalence and the dynamic transformation in the transformational equivalence.

3.2 Empirical insights into students' learning pathways in Cycle 1: discovering restructuring equivalence as a new bridging characterization

The following transcripts document excerpts from the learning process of Victoria and Mira, two 10-year-old girls selected by their comprehensive school teachers as belonging to the 30% of higher-achieving children in the class. Victoria and Mira focus on the structured figures and express them with different utterances. Each turn in the transcript in Table 1 is analyzed with respect to the substructures and subexpressions they refer to and the links drawn by the students.

Deviating from the initial analytic scheme in Fig. 2, Victoria and Mira draw a direct link between the two structured figures S_A and S_B. They compare substructures of the figures and explain how they emerge in S_B from S_A, either with clearly material articulations of the graphical representation such as "broken through" (Turn 15), "cut it here" (Turn 17), or "split by three" (in Turn 16). In this way, both girls succeed in explaining how the structured figure S_A is connected to S_B, not by indirect static comparisons, but by a direct dynamic approach of restructuring the figure.

Although Victoria and Mira stay consequently in relational activities (considering the expressions as describing the figure), they do more than comparing the commonalities and differences requested in Task 1a. Rather than drawing an indirect link via Figure F, they directly connect the structured figures by *a process of restructuring* S_B into S_A. This led us to introduce the double arrow between S_A and S_B in the analytic scheme, which had not been anticipated before.

This dynamic way in which Victoria and Mira make individual sense of the task was also identified in other cases in Cycle 1. Rather than comparing the figures with references to structural similarities and differences, several students use a dynamic language of "broken through" and "cut it here" to articulate how to restructure the one figure into the other. From these children, we learned that dynamic transformation cannot only be conducted in the symbolic representation by manipulating symbols but also as dynamic restructuring processes in the figures.

Table 1 Transcript of Victoria's and Mira's process in Task 1

Turn	Speaker	Utterance	Analysis of links	Addressed substructures
15	Victoria	He has broken this [*points at the top line of 12 in the Structured Figure S_B*] through [*points to 4 squares in the top line of 12 in S_A and then gestures through all lines from top to bottom*]		$S_A \rightarrow S_B$ lines of 12 into 3 groups of 4
16	Mira	Always split by three		$S_A \rightarrow S_B$ all lines of 12 into lines of 3 groups of 4
17	Victoria	Yes, well, these are three lines, then [*points at the 3 columns, each with 8 partial lines of 4 in S_B, then 7 s break*] Though, for example, he has cut it here. And this is together [*points at the left subfigure of S_B with eight lines of 4*] and here, he has also cut it [*points at the middle and right subfigure with 8 lines of 4*]. Otherwise, I do not know it.		$S_A \rightarrow S_B$ 3 columns, each with lines of 4

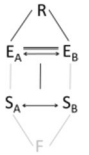
During Task 1, Victoria's and Mira's restructuring strategy is not yet connected to the symbolic expressions. When they turn to Task 2 in the transcript in Table 2, they continue pursuing their dynamic strategy and use it also to justify the symbolic manipulations.

Although Task 2 originally promoted static comparisons for description equivalence between the intermediate transformation steps, Victoria and Mira continue their dynamic approach of restructuring. They justify the transformation step $8 \times 12 = 8 \times 3 \times 4$ (in Turn 32) by explaining how S_A was restructured to reach S_B (Victoria in Turn 34) and how this is related to what happened in the expressions (Victoria in Turn 34 and Mira in Turn 35).

First, Victoria justifies the transformation in the expression of result equivalence (in Turn 32), but when asked to explain in more detail (in Turn 33), she connects the transformation of expressions to the restructuring of structured figures. For this she draws upon the ways of articulating the restructuring ("broken through" from Turn 15 in Table 1 and in Turn 34) to give meaning to the symbolic transformation. In Turn 35, Mira summarizes the result of the restructuring processes by explaining how each part of the expressions are found in the substructures of the figures. By deriving $E_A \rightarrow E_B$ from E_A–S_A and E_B–S_B and $S_A \rightarrow S_B$, the girls co-construct the justification of equivalence by an additional characterization that we later termed *restructuring equivalence*.

Results from analyzing learning pathways in Cycle 1 The analysis of students' learning pathways document the strong difference between static strategies to check equivalence (by indirectly comparing two given expressions via the result in result equivalence or by indirectly comparing them with a figure in description equivalence; see Fig. 1) and the dynamic strategy of transforming one expression into another one. However, the children

Table 2 Transcript of Victoria's and Mira's process in Task 2

Turn	Speaker	Utterance	Analysis of links	Addressed substructures/subexpressions
32	Victoria	Ok, um, Zeynep hasn't written a 12 here, only 8×3×4 [*points at 8×3×4 in expression E_B*], because 3×4 is 12, then he has expressed it slightly differently		E_B–E_A 3×4 instead of 12
33	Teacher	Mm-hmm. Explain in more detail. Try to think about it together and when you know how to say it well, then tell me [*9 s break*]. It's hard, isn't it?		
34	Victoria	Well, Zeynep has divided the 12 [*points at 3×4 in E_B*] into a multiplication, then she has, um, the 8 [*points at 8 in E_B*] – the 8 she has – um, she has left Then, she has broken through the 12 [*points at S_B while talking about 12 in S_A*], well, into a multiplication and she has simply 3×4 because 3×4 is 12 Then she has written it down		$E_A \rightarrow E_B$ 12 divided into 3×4 $E_A \rightarrow E_B$ – $S_A \rightarrow S_B$: 12 divided into 3×4 means breaking 12 into 3 groups of 4
35	Mira	This is Matt, what Matt has done [*points at 8×3×4 in E_B*]. That is to say, he has these packages of 12 [*points at a line of 3 bundles of 4 in S_B*] here, and he turned it into a multiplication [*points at E_B*]. 8, because here are 8, so [*points at the 8 lines in S_B*] And then, times 3, namely these are here, these 3 packages in each line of 12 [*points successively at the 3 bundles of 4 in the line of 12 in S_B*] and in these 12 – in these 3 packages, there are 4 inside [*points at the first bundle of 4 in the top line of 12 in S_B*]		$S_A \rightarrow E_B$: turned the packages of 12 into a multiplication E_B–S_B: 8×3×4 fit to 8 lines of 3 with groups of 4 E_A–S_A: 8×12 fit to 8 lines of 12 S_A – S_B: 12 fit to 3 groups of 4

demonstrated a non-anticipated dynamic strategy within the graphical representation that can serve as a fourth characterization for equivalence, we termed it *restructuring equivalence* (Fig. 6).

We, as design researchers, learned from the children that working in the graphical representation does not necessarily require static comparison when the structured figures are directly compared and restructured into each other, as Mira shows when she justifies that both expressions are equivalent by drawing upon restructuring equivalence.

Victoria (in Turn 34) goes even a step further and explains how the transformation $E_A \rightarrow E_B$ is connected to the transformation $S_A \rightarrow S_B$, which leads to adding another vertical line between the horizontal links $E_A \rightarrow E_B$ and $S_A \rightarrow S_B$, symbolizing $E_A \rightarrow E_B$ – $S_A \rightarrow S_B$. This link is the remarkable connection between the transformation $E_A \rightarrow E_B$ and the restructuring $S_A \rightarrow S_B$, which not only assures the equivalence but justifies the adequacy of the transformation rules by also linking the step to restructuring equivalence in generalizable terms.

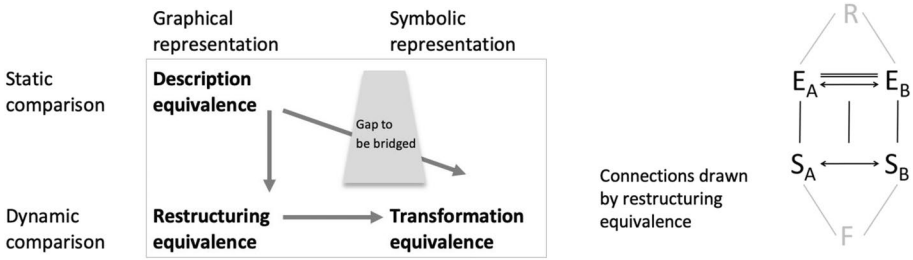


Fig. 6 Restructuring equivalence as characterization for bridging the gap between description equivalence and transformation equivalence

From these rich learning pathways of the analyzed children, we drew the hypothesis that introducing *restructuring equivalence* might also be promising for other students as the characterization that can potentially bridge the gap as indicated graphically in Fig. 6. Restructuring equivalence might be a characterization with the potential to enhance the transition from static to dynamic comparisons while persisting in the graphical representation.

3.3 Refined task design in Cycle 2: including restructuring equivalence for all students

Figure 7 documents the redesign of Tasks 1 and 2 in Cycle 2 in which we intended to explore the hypothesis by engaging all students in working with restructuring equivalence in a dynamic strategy. Whereas Task 1 was only slightly adjusted to invite activities of description equivalence (with names exchanged for technical reasons), Task 2 was redesigned so that the restructuring has to be conducted stepwise from expression E_A to E_B. This was scaffolded by offering dynamic phrases in the flow chart.

At the same time, we reduced the number of prescribed transformation steps by eliminating the intermediate $(8\times3\times4)+(2\times4)$. By this, we intended to create productive challenges and to encourage students' active reasoning but would go back to a more refined version next time.

The intention of Task 2 is that students

- recognize how the expression was transformed symbolically in each step.
- draw the structured figures for each intermediate expression.
- draw and explain how the restructuring of the figure was conducted in each step.
- by this, prepare the later connection and justification of the symbolic transformation by means of restructuring equivalence in the graphical representation.

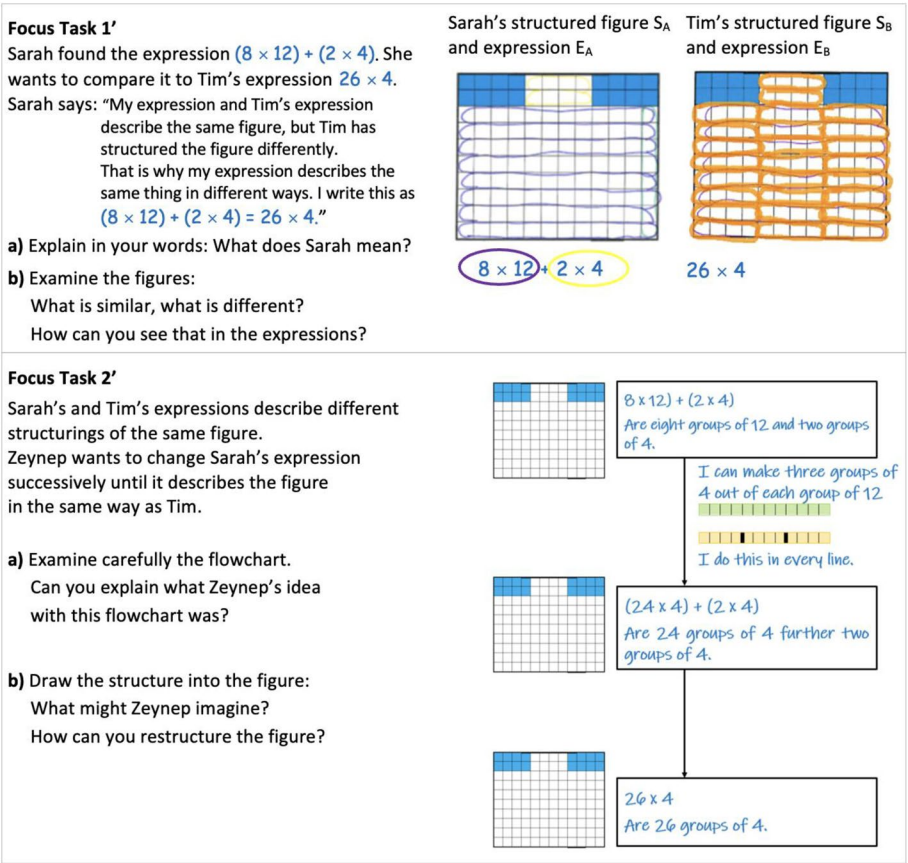


Fig. 7 Redesign of focus tasks in Cycle 2 to fill the gap between description and transformation equivalence with *restructuring equivalence* (Abbreviations E_A/E_B and S_A/S_B added here)

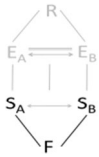
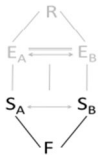
3.4 Empirical insights into students' learning pathways in Cycle 2: affordances and constraints for connecting description and transformation equivalence

For Cycle 2, we selected medium achievers (according to their teacher's comprehensive assessment) so that we could investigate whether the redesigned tasks can provide access to restructuring equivalence for more students. The need for Task 2 is illustrated by the transcript of Jannika's and Dilay's learning pathway in Table 3, which shows that in Task 1, not all students start restructuring the figures without being prompted to.

Dilay and Jannika compare the structured figures S_A and S_B ("actually, this is the same"; Turn 32) and conceive them as equivalent within the characterization of description equivalence. They also identify identical substructures (in Turn 38) but without connecting them to the subexpression 2×4. They adopt a static comparison in relational activities but without connecting the symbolic and graphical representation explicitly and without their own initiatives for dynamic restructurings.

Like Dilay and Jannika, the two other pairs in Cycle 2 also do not immediately start restructuring the 8 lines of 12 into 8 lines of 3 groups of 4 by themselves. Instead, they

Table 3 Transcript of Jannika's and Dilay's process in Task 1

Turn	Speaker	Utterance	Analysis of links	Addressed substructures
30	Jannika	You can see, thus, that, um, this [*points at Sarah's structured figure S_A*] is actually as this one [*points at Tim's structured figure S_B*]. Only that she – wait – Sarah has only, um, has not circled it, but		S_A – F S_B – F (refer to drawing of groups in Fig. 7: circles vs. lines for grouping)
31	Dilay	Lined		
32	Jannika	Yes, so, lined – so and, um, Tim has circled them, but actually, this is the same		
33	Teacher	Mm-hmm		
34	Dilay	So, as for Zeynep		
…				
38	Dilay	And it is also a bit the same, that they have highlighted exactly where, example here, he has highlighted this, also [*points at 2 groups of 4 in S_A and S_B in the upper lines*		S_A – S_B 2 groups of 4 in top of each structured figure
39	Teacher	Mm-hmm		
40	Dilay	And there [*points at 8 lines of 12 in S_A*] Sarah has also highlighted it and he also, just he [*points at 24 groups of 4 in S_B*] has circled it		

mainly identify the identical substructure: 2 groups of 4. For the lower part of Figure F, they compare S_A and S_B by the graphical realization of grouping (lines versus circles) but without considering the numerical relationships.

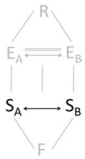
Only when explicitly prompted to think about restructuring in Task 2, they start to relate the two structured figures more intensively, in a dynamic way, as the transcript in Table 4 shows. The transcript also indicates how the students pick up the offered dynamic phrases.

Prompted by the flowchart and the offered phrases in Task 2, Dilay can explain how the figure was restructured (Turns 91 and 93), expressing the processes of restructuring by splitting into three groups. In this way, the girls can enrich their static (and rather vague) comparisons of Task 1 by a dynamic strategy for a more detailed explanation. Whereas Dilay can pick up the given phrases and appropriate them to her own words, other students stay closely with the given phrases.

Similar to Dilay and Jannika, all three pairs in Cycle 2 can overcome an initially purely static perspective and adopt (at least partially) the dynamic strategy of restructuring. This is also exemplified by the Transcript in Table 5.

Annika and Jessica also succeed in overcoming the initially static comparisons and compare the structured figures dynamically. When Jessica articulates the restructuring in Turn 82, she draws upon the verbatim articulation of the structuring from the task ("make 3 groups of 4 out of each group of 12") but changes "I can make" into "split" and includes the 8, turning into a 24. Interestingly, the 24 do not stem from a structural idea of 8 times 3, but she takes the number 24 from the next expression in the flowchart. Annica first also refers to the expression $24 \times 4 + 2 \times 4$ for justifying the 24 (in Turn 80) but then tries to validate it by counting the groups in S_B: As her empirical approach of validating by counting gets always confused, she concludes that it "does not fit" (in Turn 82). In spite of this

Table 4 Transcript of Jannika's and Dilay's process in Task 2

Turn	Speaker	Utterance	Analysis of links	Addressed substructures
88	Jannika	[*reads aloud from Task 2*] I do this in every line		Taken from task: $S_A \rightarrow S_B$ Make each group of 12 to 3 groups of 4
89	Dilay	[*reads aloud from Task 2*] I can make 3 groups of 4 out of each group of 12		
90	Teacher	Mm-hmm.		
91	Dilay	I believe, that is a 12, these are 12 [*points at the line of 12 next to the arrow in the flowchart*] and		Explained alone: $S_A \rightarrow S_B$ group of 12 is split into 3 groups of 4
92	Jannika	That is [*points at 3 groups of 4 next to the arrow in the flowchart*]		
93	Dilay	Yes, I have, in each, there he splits that through [*gestures a split with her fingers into groups of 4 each*] and these are 1, 2, 3 [*points successively at the 2 groups*]		
94	Jannika	because there are the … lines		

discovery, the two girls are not able to articulate that 24 *emerges* from splitting the entire rows into 8 rows of 3 groups. Even though they find an 8×3 structure in the given picture in a later scene, they are not able to self-explain where this structure is coming from, which also occurs in succeeding transcript turns that have not been included. Summing up, the two girls can verbalize the stepwise restructuring, but they cannot connect this

Table 5 Transcript of Annika's and Jessica's process in Task 2

Turn	Speaker	Utterance	Analysis of links	Addressed substructures
75	Teacher	Yes, exactly. But can you […] explain to me, what has, what has […] changed from here to here [*the first and second drawing in the flow chart*]		$S_A \rightarrow S_B$: 8 groups of 12 split into 24 groups of 4
78	Jessica	She has split the 8 groups of 12 into 24 groups of 4		
79	Teacher	Mm-hmm. Err, why, why are these 24 groups of 4?		
80	Annika	Because she has written it there [*points at the expression 24×4+2×4 in the flow chart*]		$E_B - S_B$ Amount 24 of groups taken from the expression without counting in the figure
81	Jessica	Because this		
82	Annika	[*starts counting in S_B* but gets confused while counting several times] That does not fit!		$E_B - S_B$: Tries to count the groups of 4 in S_B,

Table 6 Transcript of Jannika's and Dilay's process in Task 2 (continuation)

Turn	Speaker	Utterance	Analysis of links	Addressed substructures
95	Teacher	Mm-hmm.		
96	Dilay	I, this, as divided.		
97	Teacher	Mm-hmm, yes, okay. Please, explain again more exactly. What, what does she do, there now?		
98	Dilay	This green thing here [*points at the line of 12 in the flow chart*], these are 12, in total. Then, she splits it into 4 [*gestures a movement of cutting*] and this makes 3, then.		$E_A \rightarrow E_B - S_A \rightarrow S_B$: 12: 4 fits to 12 is split into 4 groups of 3

restructuring to the transformation of the expression. This is a typical state in the content trajectory for the observed students.

In contrast to Annika and Jessica, Dilay and Jannika, the girls from the transcript in Tables 3 and 4, can go beyond identifying the restructuring equivalence and use it for connecting it to the symbolic transformation (Table 6).

Dilay connects the restructuring that was already described in Turn 93 to the characterization of dividing (Turn 96) to make sense of the symbolic equality $12 = 3 \times 4$. She articulates her idea by explaining in the graphical representation with a language of symbols. She does not explicitly refer to the symbolic transformation of $E_A \rightarrow E_B$ as she talks about dividing 12 rather than $12 = 3 \times 4$. However, she succeeds in connecting the graphical with the symbolic representation by gesturing to one representation and talking about the other one.

Similar to Dilay, only two other students in Cycle 2 draw these kinds of connections between symbolic transformation and graphical restructuring by means of gesturing, while the other student stays in the stage exemplified before.

Results from analyzing learning pathways in Cycle 2 These brief insights into the learning pathways from Cycle 2 show that, indeed, medium-achieving students *can* also productively reason about restructuring structured figures and connect these graphical transformations to the symbolic transformations. Whereas some high-achieving students in Cycle 1 developed dynamic strategies on their own (and others did not), the medium-achieving students in Cycle 2 needed to be prompted to these possibilities before adopting a restructuring of the (formerly unconnected) figures. Before being prompted to restructuring, the students discovered common and different substructures in an indirect static comparison, but yet without directly connecting the structured figures. When asked to interpret the restructuring step given in the flowchart of Task 2, the students built a more direct connection of S_A and S_B, which strengthened their idea of description equivalence by backing it up by restructuring equivalence.

The transcripts in Tables 4, 5, and 6 provide first indications that students can start filling gaps with the restructuring equivalence. Moreover, Dilay's multimodal explanations in Table 6 indicate how graphical representation can invite the use of gestures for articulating emerging characterizations. Even if most learning pathways in Cycle 2 were still constrained in their explicitness of connecting restructuring the figures to transforming the expressions, Dilay shows that the pathway can lead to justifying the transformation rules using the restructuring processes.

4 Discussion

4.1 Empirical results and theoretical contribution to algebra education research: bridging relational and transformational characterizations of restructuring equivalence

Whereas the connection of symbolic and graphical representations has often been discussed for equations (Blanton et al., 2019; Kieran, 1989, 2007), equivalence of expressions has received much less attention in the research on students' early algebraic thinking. Some instructional approaches have proven effective for connecting result equivalence and transformational equivalence (Banerjee & Subramaniam, 2012; Schwarzkopf et al., 2018), as have some approaches for developing mental models of description equivalence (Kieran & Sfard, 1999; Wilkie & Clarke, 2016; Zwetzschler & Prediger, 2013). So far, however, instructional approaches have provided only limited opportunities for students to mentally *connect* description equivalence to transformation equivalence (Kieran & Sfard, 1999). Since a flexible connection of procedures and understanding is crucial (Kieran, 2007; Kilpatrick et al., 2001), this gap should be filled, even in early algebra settings before variables are introduced. The contribution of this approach to early algebra is not on generalizing, but on seeking, using, and expressing structures, as Kieran (2018) called for.

In the presented design research project, we tried to bridge this gap in an early algebra learning environment in which description equivalence is first explored with respect to geometric figures as graphical representations in order to overcome purely operational perspectives on expressions and develop an understanding of deeper structures (Malle, 1993; Kieran, 2018), and then the shift from static comparisons in the graphical representation to dynamic transformations in the symbolic representation is encouraged (Cooper & Warren, 2011). From high-achieving students in Cycle 1, we learned that the shift from static comparison to dynamic modification can already be done *within* the graphical representation, as these children invented what we later called restructuring equivalence. Rather than comparing two expressions E_A and E_B by finding structurings S_A and S_B for a Figure F so that we can assure that both expressions describe the same figure (Step II in the content trajectory sketched in Fig. 8), the high-achieving students in Cycle 1 directly modified the structured figure S_A into a restructured S_B (Step III in Fig. 8). When students' mental model of description equivalence builds upon relationally recognizing deeper structure by connecting representations, this seems to be highly useful in fostering students' understanding of transformation equivalence.

Our main discovery is the additional characterization of restructuring equivalence, which is grounded in this relational recognition. The additional characterization helps to transfer this relational understanding to transformations. Cycle 2 was used to explore whether medium achievers can also go to this Step III and even be introduced to justifying the symbolic transformations by restructuring figures (Step IV) before extending the repertoire also to Step V, the purely symbol, rule-guided transformation equivalence. The qualitative analysis of three pairs of students revealed that students can easily reach Step III (like Dilay in Table 4 or

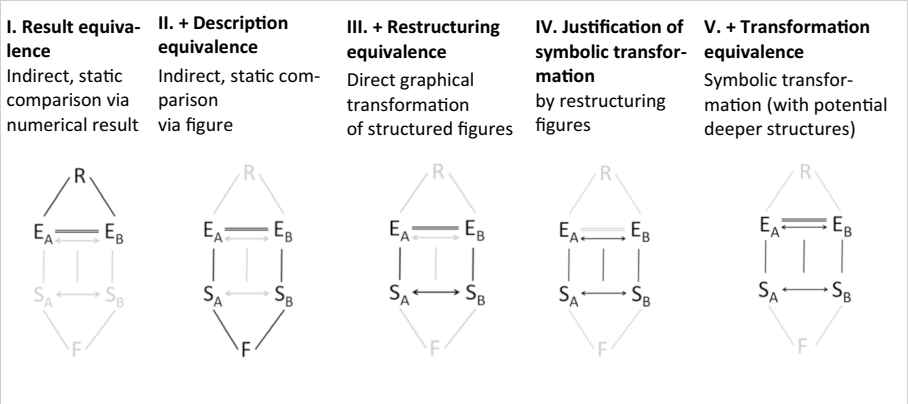


Fig. 8 Steps in a content trajectory: Successively extending the repertoire of characterizations and connecting them to each other

Jessica in Table 5), but Step IV was only accessible to some students (such as Dilay in Table 6), so it seems to require further support to be reached by all.

Although the empirical scope was still limited by a small sample size, we believe that the identified content trajectory (in Fig. 8) of successively extending and connecting the characterizations is of major relevance for algebra education research *as the discovered bridging characterization* seems to have a high epistemic potential to support students' development of conceptual understanding of transformation of expressions by perceiving deeper structures in a highly problematic field (Linchevski & Livneh, 1999; Papadopoulos & Gunnarsson, 2020).

These results were achieved based on a conceptual framework of understanding as connecting characterizations and representations (Cooper & Warren, 2011; Hiebert & Carpenter, 1992) and based on an analytic framework (see Fig. 2) that allowed the researchers to refine the general design principle for the topic in view, equivalence of expressions. We are optimistic that the analytic framework can be transferred and adapted to other parts of algebra education, so it is an important theoretical contribution in itself that helps to articulate epistemological analysis, the specification of learning goals, and content trajectories, as well as empirical analysis.

Besides the small sample size, an important limitation of the current design research project is that it has not yet included generational activities. They will be crucial in the next step of the content trajectory from arithmetic expressions to later algebraic expressions with variables (Kaput, 2008; Kieran & Sfard, 1999). As Kieran (2018) emphasized that seeing deeper structures is also a precondition for pattern generalization activities, we are optimistic that both areas of early algebra can be combined with benefit.

4.2 Zooming out: what do we learn for connecting representations for other topics?

The challenges of connecting different representations not only occur in algebra but also in many other topics wherever conceptual understanding of abstract concepts has to be developed (Kilpatrick et al., 2001; Lesh, 1979). Also in many other topics, multiple representations must not only be juxtaposed, but systematically connected in explicit construction processes (Duval, 2006). The need for active connections instead of juxtapositions has been explicitly emphasized for the transition from graphical, informal solutions to the procedural, symbolic transformations in which justifying symbolic transformation rules for multiplying fractions by connecting to part-of-part-meanings turned out to be crucial (Glade & Prediger, 2017). The current study can be used to replicate this result and extend the academic discourse on connecting representations by one more aspect: the necessary transition from static to dynamic strategies that has often been identified as difficult (Cooper & Warren, 2011), which can now be conducted in the graphical representation so that the gap between graphical and symbolic representation can be bridged. Future studies will have to explore how far this is also relevant beyond expressions and fractions. In particular, they should explore how the graphical restructurings can be expressed and explicitly related to the symbolic manipulations in order to increase students' deep understanding.

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