Universal Partial Hyperfields of Matroids and Their Prespaces of Orderings

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Universal Partial Hyperfields of Matroids and Their Prespaces of Orderings

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Abstract

We associate a partial hyperfield $\mathbb{U}^{(0)}(M)$ with every matroid M by defining an addition on the elements of its inner Tutte group with an additional zero element such that M is representable over $\mathbb{U}^{(0)}(M)$, and every representation of M over a partial hyperfield F factors over the representation of M over $\mathbb{U}^{(0)}(M)$.

We investigate the relationship between $\mathbb{U}^{(0)}(M)$ and $\mathbb{U}^{(0)}(N)$ for minors N of M and prove that $\mathbb{U}^{(0)}(M)$ is the coproduct of $\mathbb{U}^{(0)}(M_i)$, $i = 1, \ldots, k$, where $M_1, \ldots, M_k$ are the connected components of M.

Further, we examine the possible non-trivial decompositions of $\mathbb{U}^{(0)}(M)$ as a coproduct and present sufficient geometrical conditions under which no such decomposition exists.

We develop an Artin-Schreier-Theory for partial hyperfields and show that the orderings of a partial hyperfield form a prespace of orderings, which is in general not a space of orderings in the sense of Marshall, even for the partial hyperfield $\mathbb{U}^{(0)}(M)$ of a matroid M.

Moreover, we provide examples of matroids M for which $\mathbb{U}^{(0)}(M)$ is a hyperfield and its prespace of orderings is a space of orderings in the sense of Marshall, including affine space of dimension at least 3 and affine translation planes whose kernel contains at least four elements, for which the inner Tutte group was not known before.

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1 Introduction

Oriented matroids provide a combinatorial abstraction to point configurations over the reals, real hyperplane arrangements, convex polytopes, directed graphs, as well as ordered projective spaces and affine spaces. They were introduced and studied in full generality independently by Bland and Las Vergnas and Folkman and Lawrence in 1978. An early axiomatic study was already done by Sperner in 1949.

The study of the geometrical algebra of matroids was developed by Dress and Wenzel, who introduced the Tutte groups of matroids in 1989 and provided a characterization of classes of projectively equivalent orientations of a matroid in terms of homomorphisms of the inner Tutte group. This was further elaborated by Gelfand, Rybnikov and Stone, who proved a characterization of the inner Tutte group of a finite matroid in terms of generators and relations.

A different characterization of orientations was given by Baker and Bowler in 2019 within their theory of representations of matroids over partial hyperfields, where they characterized orientations as representations over the hyperfield of signs.

Besides the classical case of projective geometries of dimension greater or equal to 3, the structure of the set of classes of projectively equivalent orientations of the matroids has only been studied for projective planes, for which Kalhoff proved in 1989 that they form a space of orderings in the sense of Marshall.

By introducing an addition on the elements of the inner Tutte group of a matroid together with an additional zero element, we will associate a partial hyperfield with every matroid, such that the matroid is representable over this partial hyperfield and every representation of it over a partial hyperfield factors over this representation.

By generalizing Marshall's characterization of spaces of orderings as a special class of hyperfields, we are able to associate a prespace of orderings with a each oriented matroid. We will further provide necessary and sufficient conditions under which these prespaces of orderings are spaces of orderings in the sense of Marshall.

Furthermore, we will use this characterization to compute the inner Tutte group for matroids, where the inner Tutte group was not known before.

1.1 Main results

In chapter 2 we will present an axiomatic characterization of partial hyperfields that were introduced by Baker and Bowler in [BB19], and provide an algebraic framework for use in later chapters. Further, for every partial hyperfield we will explicitly construct an embedding into a hyperfield.

Chapter 3 contains the definition of universal partial hyperfields of matroids and the relations between the universal partial hyperfield with those of its minors and its dual. Further, we will prove that a matroid is representable over a partial hyperfield if and only if there exists a homomorphism from its universal partial hyperfield to this partial hyperfield.

In Chapter 4 we will present an Artin-Schreier-Theory for partial hyperfields. Unfortunately, in contrast to the situation for hyperfields there exist partial hyperfields in which -1 is not a sum of squares that possess no orderings. However, we will show that every partial hyperfield is embeddable into a hyperfield in which -1 is not a sum of squares if and only if the original partial hyperfield possesses an ordering.

We will show that the orderings of a partial hyperfield form a prespace of orderings which is not necessarily a space of orderings in the sense of Marshall, even for universal partial hyperfields of matroids. Moreover, we will prove that the category of these prespaces of orderings is equivalent to the category of a certain class of partial hyperfields, generalizing the equivalence of spaces of orderings and real reduced hyperfields by Marshall.

In chapter 5 we introduce the class of artinian matroids, i. e., matroids in which every element of its inner Tutte group is a cross-ratio. We will show that they are representable over a field if and only if its universal partial hyperfield is a subfield of this field.

Futhermore, we will examine the connected components of artinian matroids and all possible decompositions of the universal partial hyperfield of a matroid as coproduct of partial hyperfields.

Chapter 6 contains examples of artinian matroids whose universal partial hyperfield is a hyperfield. First, we generalize a construction by Kalhoff used to coordinatize matroids of rank 3 to arbitrary rank and obtain a non-desarguesian analogue of vector space matroids of rank greater or equal to 4. Similar to the classical case, restricting the matroid to points whose last coordinate is equal to 1, we obtain a non-desarguesian analogue of affine spaces of dimension at least 3.

Second, we will show that the universal partial hyperfield of an affine translation planes whose kernel contains at least 4 elements is isomorphic to the universal partial hyperfield of its projective closure.

2 Partial hyperfields

In this chapter we will present an axiomatic characterization of partial hyperfields, examine several kinds of homomorphisms of partial hyperfields, and generalize various constructions for hyperfields and partial fields to our setting.

In this generality partial hyperfields were first studied by Baker and Bowler in [BB19], who defined them by restricting the operation of a hyperring.

We further generalize the techniques used by Semple in his PhD thesis ([Sem98]) to prove that every partial hyperfield is embeddable into a hyperfield.

2.1 Axiomatic characterization

Definition. A *partial hyperoperation*[1] on a non-empty set X, is a map $+$ from the set $X \times X$ to the power set of X, denoted by $+\colon X \times X \multimap X$. If A and B are subsets of X, we define

$$A + B := \bigcup_{\substack{a \in A \\ b \in B}} a + b,$$

and for any $a \in X$ and $B \subseteq X$ we set $a + B := \{a\} + B$ and $B + a := B + \{a\}$.

Let F be a set, $+\colon F \times F \multimap F$ a partial hyperoperation, and $\cdot\colon F \times F \to F$ a binary operation. We call $(F, +, \cdot)$ a *partial hyperfield* if the following axioms are satisfied:

(PH1) $a + b = b + a$ for all $a, b \in F$,

(PH2) there is an element $0 \in F$ such that $0 + a = \{a\}$ for all $a \in F$,

(PH3) there is a map $-\colon F \to F$ such that $c \in a + b$ implies $b \in c + (-a)$ for all $a, b, c \in F$,

(PH4) $(F\backslash\{0\}, \cdot)$ is an abelian group with neutral element 1 and $0 \cdot a = 0 = a \cdot 0$,

(PH5) $a \cdot (b + c) \subseteq a \cdot b + a \cdot c$ for all $a, b, c \in F$.

[1] We explicitly use the name *partial hyperoperation* instead of the shorter *hyperoperation* to emphasize that we allow $a + b$ to be empty.

We further set $F^* := F \setminus \{0\}$ if F is a partial hyperfield. As usual, we set $a - b := a + (-b)$ for all $a, b \in F$.

2.1 Lemma. Let F be a partial hyperfield. Then we have:

(a) The element $0 \in F$ is uniquely determined.

(b) For all $a, b \in F$ we have $b = -a$ if and only if $0 \in a + b$. As a consequence the map $-\colon F \to F$ is uniquely determined and $-(-a) = a$ for all $a \in F$.

(c) For all $a \in F$ we have $-a = (-1)a$ and $(-1)^2 = 1$.

(d) For all $a, b, c \in F$ such that $a \neq 0$ we have $a(b + c) = ab + ac$.

(e) For all $a, b \in F$ we have $0(a + b) = \{0\}$ if and only if $a + b \neq \emptyset$.

(f) For all $a, b, c, d \in F$ we have

$$(a + b)(c + d) \subseteq ((ac + ad) + (bc + bd)) \cap ((ac + bc) + (ad + bd)) .$$[2]

Proof. To prove (a), let $0' \in F$ be an element satisfying $0' + a = \{a\}$ for all $a \in F$. Then $0' + 0 = \{0\}$ and by (PH2) $0 + 0' = \{0'\}$. Thus, (PH1) implies $0 = 0'$.

To show (b), let $a, b \in F$. Applying (PH3) to $a \in a + 0$ implies $0 \in a + (-a)$. Therefore, $a = -b$ yields $0 \in a + b$.

Conversely, if $0 \in a + b$, it follows that $b \in 0 + (-a) = \{-a\}$ by using (PH1) and (PH3). Further, using (PH1) we get $0 \in a + (-a) = (-a) + a$ and therefore $-(-a) = a$.

In order to prove (d), let $a, b, c \in F$. $a(b + c) \subseteq ab + ac$ follows directly from (PH5). If $a \neq 0$, (PH5) implies additionally that

$$ab + ac = aa^{-1}(ab + ac) \subseteq a(a^{-1}ab + a^{-1}ac) = a(b + c).$$

Moreover, (e) follows from $0 \cdot \emptyset = \emptyset$ and $0 \cdot A = \{0\}$ for all non-empty $A \subseteq F$.

To show (c), let $a \in F$. By using (b), we get $0 \in 1 + (-1)$. Applying (d) and (e), it follows that $0 \in a(1 + (-1)) = a + (-1)a$. Using (b) again, we conclude that $-a = (-1)a$. Furthermore, (d) yields

$$0 = (-1) \cdot 0 \in (-1) \cdot (1 + (-1)) = (-1) + (-1)^2.$$

Thus, (b) implies $(-1)^2 = -(-1) = 1$.

[2] Even in the case of a hyperfield F these sets are not necessarily equal, see [Vir10] for a counter example.

Clearly, (f) holds whenever $a+b$ or $c+d$ are the empty set. Otherwise, it follows from (PH1) and

$$\begin{aligned}(a+b)(c+d) &= \bigcup_{f\in c+d} (a+b)f \subseteq \bigcup_{f\in c+d} af+bf \\ &\subseteq a(c+d)+b(c+d) \subseteq (ac+ad)+(bc+bd),\end{aligned}$$

using (d) and (e) twice. □

Definition. We call (G,ε) a *multiplicative structure* if $(G,\cdot)$ is an abelian group and $\varepsilon \in G$ with $\varepsilon^2 = 1$.

If (G,ε) and (G',ε') are multiplicative structures, we say a group homomorphism $f\colon G \to G'$ is a *multiplicative homomorphism* if $f(\varepsilon)=\varepsilon'$.

Further, if $(F,+,\cdot)$ is a partial hyperfield, we define the *underlying multiplicative structure* as $\underline{F} := (F^*,-1)$.

2.2 Proposition. Let $(F\setminus\{0\},\varepsilon)$ be a multiplicative structure, where $0\in F$ is an element such that $0\cdot a = 0 = a\cdot 0$ for all $a\in F$.

For any family $(\Delta_a)_{a\in F\setminus\{0\}}$ of subsets of $F\setminus\{0\}$ satisfying

$$b\in\Delta_a \;\Rightarrow\; a\in\Delta_b \text{ and } a^{-1}\in\Delta_{\varepsilon a^{-1}b} \tag{2.1}$$

for all $a,b\in F\setminus\{0\}$, there exists a unique partial hyperoperation $+\colon F\times F \multimap F$ such that $(F,+,\cdot)$ is a partial hyperfield with $-1=\varepsilon$ and $(1-a)\setminus\{0\}=\Delta_a$ for all $a\in F\setminus\{0\}$.

Proof. If $(F,+,\cdot)$ is a partial hyperfield, we have $0+a=\{a\}=a+0$ for all $a\in F$. Further, Lemma 2.1 (b), (c) and (d) imply that $a+b=a(1-(-a^{-1}b))$ for all $a,b\in F^*$. Thus, the partial hyperoperation $+$ is uniquely determined by the sets of the form $(1-a)\setminus\{0\}$, $a\in F^*$.

It remains to show that for any family $(\Delta_a)_{a\in F\setminus\{0\}}$ of subsets of $F\setminus\{0\}$ satisfying (2.1), defining

$$a+b := \begin{cases} \{b\} & \text{if } a=0, \\ \{a\} & \text{if } b=0, \\ a\Delta_{\varepsilon a^{-1}b} & \text{if } a\neq\varepsilon b, \\ a\Delta_{\varepsilon a^{-1}b}\cup\{0\} & \text{if } a=\varepsilon b, \end{cases}$$

yields a hyperoperation such that $(F,+,\cdot)$ is a partial hyperfield with $\varepsilon=-1$ and $(1-a)\setminus\{0\}=\Delta_a$ for all $a\in F\setminus\{0\}$. Clearly, $(F,+,\cdot)$ satisfies (PH2) and (PH4).

In order to prove (PH5) it is sufficient to show that $a(b+c) = ab + ac$ for all $a, b, c \in F$ such that $a \neq 0$, since $0 \cdot (b+c) \subseteq \{0\} = 0 \cdot b + 0 \cdot c$. If $\{b, c\} = \{x, 0\}$ for an $x \in F$, we get[3]

$$a(b+c) = a(x+0) = \{ax\} = ax + 0 = ax + a \cdot 0 = ab + ac.$$

Moreover, if $b, c \neq 0$, we have $b = \varepsilon c$ if and only if $ab = \varepsilon ac$. Thus, $0 \in a(b+c)$ if and only if $0 \in ab + ac$. Further,

$$a(b+c) \setminus \{0\} = ab\Delta_{\varepsilon b^{-1}c} = ab\Delta_{\varepsilon (ab)^{-1}ac} = (ac + bc) \setminus \{0\}.$$

To show (PH1), let $a, b \in F$. Clearly, $a + b = b + a$ if $0 \in \{a, b\}$. Since $a = \varepsilon b$ if and only if $b = \varepsilon a$ we have $0 \in a + b$ if and only if $0 \in b + a$ for all $a, b \in F \setminus \{0\}$. Thus, using (PH5) it is sufficient to prove that

$$\Delta_c = (1 + \varepsilon c) \setminus \{0\} = (\varepsilon c + 1) \setminus \{0\} = \varepsilon c \Delta_{c^{-1}}$$

for all $c \in F \setminus \{0\}$.

For $d \in \Delta_c$ it follows from (2.1) that $c^{-1} \in \Delta_{\varepsilon c^{-1}d}$. Again applying (2.1) yields $\varepsilon c^{-1} d \in \Delta_{c^{-1}}$. Therefore, $d \in \varepsilon c \Delta_{c^{-1}}$.

Conversely, if $d \in \varepsilon c \Delta_{c^{-1}}$, we have $\varepsilon c^{-1} d \in \Delta_{c^{-1}}$. Applying (2.1) twice, we get $c \in \Delta_{\varepsilon c(\varepsilon c^{-1} d)} = \Delta_d$ and thus $d \in \Delta_c$.

Finally, to prove (PH3), let $a, b, c \in F$ such that $c \in a + b$. We will show that $c \in b + (-a)$ for the map $-\colon F \to F$, $a \mapsto \varepsilon a$.

If $a = 0$ or $b = 0$, say $a = 0$, it follows that $c \in 0 + b = \{b\}$. Hence, $b = c$ and $b \in c + 0 = c + (\varepsilon \cdot 0)$.

If $c = 0$ we have $a = \varepsilon b$ and thus $b \in 0 + b = 0 + (\varepsilon a)$.

Otherwise, $a, b, c \neq 0$ and therefore $c \in a\Delta_{\varepsilon a^{-1}b}$. Since this is equivalent to $a^{-1}c \in \Delta_{\varepsilon a^{-1}b}$, we get $\varepsilon a^{-1} b \in \Delta_{a^{-1}c}$ using (2.1). Hence, using (PH1) and (PH5) we obtain

$$b \in \varepsilon a(1 + \varepsilon a^{-1} c) \subseteq \varepsilon a + c = c + (\varepsilon a).$$

□

2.3 Remark and Definition. Lemma 2.1 and (PH3) imply that for any partial hyperfield $(F, +, \cdot)$ the family of sets $(\Delta_a)_{a \in F^*}$ defined by $\Delta_a \coloneqq (1 - a) \setminus \{0\}$ for $a \in F \setminus \{0\}$ satisfies the implication (2.1).

Thus, for any multiplicative structure $(F \setminus \{0\}, \cdot)$, where $0 \in F$ is an element such that $0 \cdot a = 0 = a \cdot 0$ for all $a \in F$, Proposition 2.2 defines a one-to-one mapping between the hyperoperations $+\colon F \times F \multimap F$ such that $(F, +, \cdot)$ is a

[3] By construction, we have $0 + x = x + 0$ for all $x \in F$.

partial hyperfield with $\underline{F} = (F \setminus \{0\}, \cdot)$ and the families $(\Delta_a)_{a \in F \setminus \{0\}}$ of subsets Δ_a of $F \setminus \{0\}$, $a \in F \setminus \{0\}$ satisfying the implication (2.1).

Let $(F, +, \cdot)$ be a partial hyperfield. We call an element $a \in F$ *fundemental*[4] if $1 - a \neq \emptyset$ and denote by $\mathcal{F}(F)$ the set of fundamental elements of F. Further, we call F a *hyperneofield* if $\mathcal{F}(F) = F$ (or equivalently if $a + b \neq \emptyset$ for all $a, b \in F$) and a *hyperfield* if $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$.[5]

Definition. Let F and F' be partial hyperfields. We call a map $f\colon F \to F'$ a *homomorphism of partial hyperfields* if $f(0) = 0$, $f(1) = 1$,[6] $f(a+b) \subseteq f(a)+f(b)$, and $f(ab) = f(a)f(b)$ for all $a, b \in F$.

A homomorphism $f\colon F \to F'$ of partial hyperfields is called *strong* or *strict* if $f(a + b) = f(a) + f(b)$ holds for all $a, b \in F$.

Moreover, a homomorphism $f\colon F \to F'$ of partial hyperfields is called a *monomorphism* if f is injective, an *epimorphism*[7] if for all $a'_1, a'_2 \in F'$ and $a'_3 \in a'_1 + a'_2$ there exist $a_i \in f^{-1}(a'_i)$, $i = 1, 2$, and $a_3 \in (a_1 + a_2) \cap f^{-1}(a'_3)$, and an *isomorphism* if there is a homomorphism $g\colon F' \to F$ such that $g \circ f = \mathrm{id}_F$ and $f \circ g = \mathrm{id}_{F'}$.[8] As usual the homomorphism g is uniquely determined by f and is denoted by f^{-1}.

2.4 Lemma and Definition. Let $f\colon F \to F'$ be a homomorphism of partial hyperfields.

(a) For all $a \in F$ we have $f(a) = 0$ if and only if $a = 0$. Moreover, we have $f(-a) = -f(a)$ for all $a \in F$ and the map $\underline{f}\colon \underline{F} \to \underline{F'}$, $a \mapsto f(a)$, which we call the *underlying multiplicative homomorphism* of f, is well-defined.

(b) f is a monomorphism if and only if the *multiplicative kernel* $\ker_* f := f^{-1}(1)$ of f is trivial, i. e., $\ker_* f = \{1\}$.

(c) If f is an epimorphism, then f is surjective.[9]

(d) If f is a strong homomorphism, then f is a monomorphism.

---

[4] We adopt the terminology from [PV10] for partial fields here.

[5] Since $1 \in 1 + 0 \subseteq 1 + (a - a) = (1 + a) - a$ every hyperfield is a hyperneofield.

[6] In most of the literature on hyperfields these first two conditions are not explicitly mentioned but otherwise constant maps $F \to \mathbb{K}$ would be homomorphisms.

[7] See [DS06, Definition 2.8].

[8] Clearly all these properties are preserved under composition.

[9] Not every surjective homomorphism is an epimorphism, cf. Proposition 2.14.

Proof. Since $(F, \cdot)$ and $(F', \cdot)$ are monoids with neutral element 1 and f is a monoid homomorphism it follows that $f(F^*) \subseteq F'^*$. Additionally, applying Lemma 2.1 (b) twice yields that for any $a \in F$ we have $0 \in a - a$ and therefore $0 \in f(a) + f(-a)$. Thus, $f(-a) = -f(a)$. This proves (a).

Using (a), it follows that f is injective if and only if $\underline{f}$ is injective, which shows (b).

In order to prove (c), let $a' \in F'$. Since $a' \in a' + 0$ and f is an epimorphism there exist $a \in f^{-1}(a')$, $b \in f^{-1}(0)$ and $c \in (a + b) \cap f^{-1}(a')$. Hence, f is surjective.

Finally, to show (d), let f be a strong homomorphism and $a \in \ker_* f$. Then we have $0 \in 1 - f(a) = f(1 - a)$. So there exists a $b \in 1 - a$ such that $f(b) = 0$. By (a), it follows that $b = 0$ and therefore $a = 1$ using Lemma 2.1. □

2.5 Lemma. Let F and F' be partial hyperfields and $f\colon F \to F'$ be a map such that $f(0) = 0$ and $\underline{f}\colon \underline{F} \to \underline{F'}$, $a \mapsto f(a)$ is a multiplicative homomorphism. Then f is a homomorphism of partial hyperfields if and only if

$$b \in 1 - a \;\Rightarrow\; f(b) \in 1 - f(a)$$

for all $a, b \in F^*$.

Proof. Clearly, every homomorphism f of partial hyperfields satisfies this condition.

Conversely, let f be a map satisfying the condition above and $c \in a + b$ for $a, b \in F$. We have to show that $f(c) \in f(a) + f(b)$.

If $a = 0$ or $b = 0$, we can assume $b = 0$ by (PH1). Then $c \in a + 0 = \{a\}$ and therefore $f(c) = f(a) \in f(a) + 0 = f(a) + f(b)$.

In the case $c = 0$ it follows by Lemma 2.1 (b) that $a = -b$. Hence, we obtain $f(a) = f(-b) = f((-1)b) = -f(b)$. Thus, we get $f(c) = 0 \in f(a) + f(b)$ by applying Lemma 2.1 again.

Otherwise $a, b, c \in F^*$. It follows that $ca^{-1} \in 1 - (-ba^{-1})$ and thus

$$f(c)f(a)^{-1} = f(ca^{-1}) \in 1 - f(-ba^{-1}) = 1 + f(b)f(a)^{-1}.$$

Hence $f(c) \in f(a) + f(b)$, as desired. □

2.2 Constructions of partial hyperfields

In this section we will introduce several constructions of partial hyperfields for later usage. We will use terms and definitions from category theory (cf. [AHS06]) but not use any methods from category theory in our proofs to keep this section self-contained.

2.6 Proposition and Definition. For any partial hyperfield F the following conditions are equivalent:

(a) For every partial hyperfield F' any map $f\colon F \to F'$ is a homomorphism of partial hyperfields if and only if $f(0) = 0$ and $\underline{f}\colon \underline{F} \to \underline{F'}$ is a multiplicative homomorphism.

(b) $(a + b) \setminus \{0\} = \emptyset$ for all $a, b \in F^*$,

(c) $(1 - a) \setminus \{0\} = \emptyset$ for all $a \in F^*$.

If F satisfies one (and therefore all) of the above conditions, we say that F is the *discrete partial hyperfield* on $\underline{F}$.

Proof. By Lemma 2.1, we have $a + b = a(1 - (-ba^{-1}))$ for all $a, b \in F^*$. Therefore, (b) and (c) are equivalent. Since $(1 - a) \setminus \{0\} = \emptyset$ for all $a \in F^*$ the implication (c) $\Rightarrow$ (a) follows directly from Lemma 2.5.

Conversely, if (a) holds, Proposition 2.2 implies that there exists a unique partial hyperfield F' such that $\underline{F'} = \underline{F}$ and $(1 - a) \setminus \{0\} = \emptyset$ for all $a \in F'^*$. Thus, $\underline{f}$ is a multiplicative homomorphism for the identity map $f\colon F \to F'$, $a \mapsto a$ and therefore f is a homomorphism of partial hyperfields. Hence, $F = F'$, which yields (c). $\square$

2.7 Proposition and Definition. For any partial hyperfield F the following conditions are equivalent:

(a) For every partial hyperfield F' any map $f\colon F' \to F$ is a homomorphism of partial hyperfields if and only if $f(0) = 0$ and $\underline{f}\colon \underline{F'} \to \underline{F}$ is a multiplicative homomorphism.

(b) $(a + b) \setminus \{0\} = F^*$ for all $a, b \in F^*$,

(c) $(1 - a) \setminus \{0\} = F^*$ for all $a \in F^*$.

If F satisfies one (and therefore all) of the above conditions, we call F the *indiscrete partial hyperfield* on $\underline{F}$.

Proof. This can be proven analogously to Proposition and Definition 2.6. $\square$

2.8 Proposition and Definition. Let F be a partial hyperfield, $(F_i)_{i\in I}$ a family of partial hyperfields and $(f_i\colon F \to F_i)_{i\in I}$ a family of homomorphisms of partial hyperfields. The following conditions are equivalent:

(a) Let $g\colon F' \to F$ be a map from a partial hyperfield F' such that $g(0) = 0$ and $\underline{g}\colon \underline{F'} \to \underline{F}$ is a multiplicative homomorphism. Then g is a homomorphism of partial hyperfields if and only if $f_i \circ g$ for each $i \in I$ is one.

(b) For all $a, b \in F^*$ we have
$$(a + b) \setminus \{0\} = \{c \in F^* \mid \forall i \in I : f_i(c) \in f_i(a) + f_i(b)\}.$$

(c) For all $a \in F^*$ we have
$$(1 - a) \setminus \{0\} = \{b \in F^* \mid \forall i \in I : f_i(b) \in 1 - f_i(a)\}.$$

If one (and therefore all) of the above conditions is satisfied, we call F the *initial partial hyperfield* on $\underline{F}$ with respect to $(f_i)_{i\in I}$.

Proof. Using Proposition 2.2 and Lemma 2.5, we conclude that (b) and (c) are equivalent.

We will first prove that (c) implies (a). Let F' be a partial hyperfield and $g\colon F' \to F$ be a map such that $g(0) = 0$, $\underline{g}\colon \underline{F'} \to \underline{F}$ is a multiplicative homomorphism. It suffices to prove that g is a homomorphism of partial hyperfields if $f_i \circ g$ is one for each $i \in I$.

Let $a, b \in F'^*$ such that $b \in 1-a$. Lemma 2.5 yields that $f_i(g(b)) \in 1-f_i(g(a))$ for each $i \in I$. Using (c) we obtain $g(b) \in 1 - g(a)$ and by applying Lemma 2.5 we get that g is a homomorphism of partial hyperfields.

Finally, we show that (a) yields (c). Let F' be the partial hyperfield on the same ground set and with the same multiplication as F but addition defined by
$$(1 - a) \setminus \{0\} = \{b \in F^* \mid \forall i \in I : f_i(b) \in 1 - f_i(a)\}$$
for all $a \in F^*$. It follows from Proposition 2.2 and Lemma 2.5 that F' is indeed a partial hyperfield. By construction, the identity map $g\colon F' \to F$, $a \mapsto a$ is a multiplicative homomorphism and $f_i \circ g$ is a homomorphism of partial hyperfields for each $i \in I$.

Using (a), we get that g is a bijective homomorphism of partial hyperfields. Let $a, b \in F^*$ such that $b \in 1 - a$. Since f_i is an homomorphism of partial hyperfields it follows that $f_i(b) \in 1 - f_i(a)$ for all $i \in I$. Hence, $b \in 1 - a$ in F'. Using Proposition 2.2, we get $F = F'$, as desired. □

Definition. We call a homomorphism $f\colon F \to F'$ of partial hyperfields *initial* if F is the initial partial hyperfield with respect to f and *embedding* if it is an initial monomorphism.

Let F be a partial hyperfield and $U \subseteq F^*$ be a subgroup containing -1. The initial partial hyperfield with respect to the natural inclusion $\iota\colon U \cup \{0\} \to F$ is called the *restriction* of F to U and is denoted by $F_{|U}$.

Further, a *partial field* is defined to be the restriction of a field F to a subgroup $U \subseteq F^*$ containing -1.

We call the restriction of F to the group generated by -1 and the non-zero fundamental elements the *core* of F and denote it by $F^{(0)}$. Clearly, the inclusion map $F^{(0)} \to F$ is a strong embedding.

2.9 Proposition and Definition. Let F be a partial hyperfield, $(F_i)_{i\in I}$ a family of partial hyperfields and $(f_i\colon F_i \to F)_{i\in I}$ a family of morphisms of partial hyperfields. The following conditions are equivalent:

(a) Let $g\colon F \to F'$ be a map into a partial hyperfield F' such that $g(0) = 0$ and $\underline{g}\colon \underline{F} \to \underline{F'}$ be a multiplicative homomorphism. Then g is a homomorphism of partial hyperfields if and only if $g \circ f_i$ is one for each $i \in I$.

(b) For all $a, b \in F^*$ we have

$$(a + b) \setminus \{0\} = \big\{c \in F^* \bigm| \exists i \in I : f_i^{-1}(c) \cap \big(f_i^{-1}(a) + f_i^{-1}(b)\big) \neq \emptyset\big\}.$$

(c) For all $a \in F^*$ we have

$$(1 - a) \setminus \{0\} = \big\{b \in F^* \bigm| \exists i \in I : f_i^{-1}(b) \cap \big(1 - f_i^{-1}(a)\big) \neq \emptyset\big\}.$$

If one (and therefore all) of the above conditions is satisfied, we call F the *final partial hyperfield* on $\underline{F}$ with respect to $(f_i)_{i\in I}$.

Proof. Using Proposition 2.2 and Lemma 2.5, we see that (b) and (c) are equivalent.

In order to prove that (c) implies (a), let F' be a partial hyperfield, $g\colon F \to F'$ be a map such that $g(0) = 0$, $\underline{g}\colon \underline{F'} \to \underline{F}$ is multiplicative homomorphism. It suffices to show that g is a homomorphism of partial hyperfields if $g \circ f_i$ is one for each $i \in I$.

Let $a, b \in F^*$ such that $b \in 1 - a$. Applying (c), there exist $i \in I$, $a_i \in f_i^{-1}(a)$ and $b_i \in f_i^{-1}(b)$ such that $b_i \in 1 - a_i$. Since $g \circ f_i$ is a homomorphism of partial hyperfields $g(b) = g(f_i(b_i)) \in 1 - g(f_i(a_i)) = 1 - g(a)$. Thus, Lemma 2.5 implies that g is a homomorphism of partial hyperfields.

To prove that (a) implies (c), let F' be a partial hyperfield on the same ground set as F, with the same multiplication as F but with addition defined by

$$(1-a)\setminus\{0\} = \big\{b \in F^* \;\big|\; \exists i \in I : f_i^{-1}(b) \cap \big(1 - f_i^{-1}(a)\big) \neq \emptyset\big\}$$

for all $a \in F^*$. By construction, the identity map $g\colon F \to F'$, $a \mapsto a$ is a multiplicative homomorphism and $g \circ f_i$ is a homomorphism of partial hyperfields for for each $i \in I$.

Using (a), we get that g is a bijective homomorphism of partial hyperfields. Let $a, b \in F'^*$ such that $b \in 1-a$. By the construction of the addition, there exist $i \in I$, $a_i \in f_i^{-1}(a)$, $b_i \in f_i^{-1}(b)$ such that $b_i \in 1-a_i$. Therefore, $b = f_i(b_i) \in 1 - f_i(a_i) = 1-a$ in F. Hence, g is an isomorphism and $F = F'$, as desired. □

2.10 Remark and Definition. We call a homomorphism $f\colon F \to F'$ of partial hyperfields *final* if F' is the final partial hyperfield with respect to f.

Let F be a partial hyperfield and $U \subseteq F^*$ a subgroup. The final partial hyperfield with respect to the canonical projection $\pi\colon F \to F^*/U \cup \{0\}$ is called the *quotient* of F by U and is denoted by $F/\!_*U$.[10]

By [Mar06, Example 2.6], $F/\!_*U$ is a hyperfield if F is a hyperfield.[11]

2.11 Remark and Definition. Clearly, if F is a field and we view $a+b$ as a subset of F for all $a, b \in F$, then F is a partial hyperfield.

The indiscrete partial hyperfield on the multiplicative structure $(\{1\}, 1)$ was named the *Krasner hyperfield* by Connes and Consani ([CC11]) and is denoted by $\mathbb{K}$.

Furthermore, the discrete partial hyperfield on the multiplicative structure $(\{\pm 1\}, -1)$ is called the *regular partial field* and is denoted by $\mathbb{U}_0$ (cf. [PV10]).

Additionally, the indiscrete partial hyperfield on the multiplicative structure $(\{\pm 1\}, -1)$ is denoted by $\mathbb{W}$ and the partial hyperfield $\mathbb{S} := \mathbb{Q}/\!_*\mathbb{Q}^+$ is called the *hyperfield of signs*. (see [BB19]).

2.12 Corollary. Let $(F_i)_{i\in I}$ be family of partial hyperfields. For each $i \in I$ and $a \in F_i$ we define

$$\kappa_i(a) := \begin{cases} \big(a^{\delta_{ij}}\big)_{j\in I}{}^{12} & \text{if } a \in F_i^*, \\ (0)_{j\in I} & \text{else.} \end{cases}$$

[10] The elements of $F/\!_*U$ are the cosets of U in F and multiplication is defined in the usual way. The elements of the sum of two cosets aU and bU are the cosets cU that are contained in the setwise addition of aU and bU.

[11] Marshall uses the term multifields instead of hyperfields and a different notation.

Further, let N be the subgroup of the (multiplicative written) direct sum $\bigoplus_{i\in I} F_i^*$ of the F_i^*, $i \in I$, generated by $\kappa_i(-1)\kappa_j(-1)$, $i, j \in I$.

Then the final partial hyperfield $\coprod_{i\in I} F_i$ on the multiplicative structure $\left(\left(\bigoplus_{i\in I} F_i^*\right) N, \kappa_{i_0}(-1)N\right)$, where $i_0 \in I$, with respect to the homomorphisms

$$\iota_i \colon F_i \to \coprod_{i\in I} F_i, \quad a \mapsto \kappa_i(a)N^{13}$$

is the coproduct[14] of $(F_i)_{i\in I}$, i. e., for each family $(f_i)_{i\in I}$ of homomorphisms $f_i \colon F_i \to F$, $i \in I$, of partial hyperfields there exists a unique homomorphism $f \colon \coprod_{i\in I} F_i \to F$ such that $f \circ \iota_i = f_i$ for all $i \in I$.

Proof. First, note that since $\iota_i(-1) = \kappa_i(-1)N = \kappa_j(-1)N = \iota_j(-1)$ for all $i, j \in I$, each ι_i is a multiplicative homomorphism and thus $\coprod_{i\in I} F_i$ is well-defined.

Let $f \colon \coprod_{i\in I} F_i \to F$ be a homomorphism of partial hyperfields satisfying $f \circ \iota_i = f_i$ for all $i \in I$. Then for all $(a_i)_{i\in I}N \in \coprod_{i\in I} F_i$ we have

$$f\left((a_i)_{i\in I}N\right) = f\left(\prod_{i\in I} \iota_i(a_i)\right) = \prod_{i\in I} f_i(a_i).$$

Hence, there is at most one such f. Conversely, define $f \colon \coprod_{i\in I} F_i \to F$ by

$$f\left((a_i)_{i\in I}N\right) \coloneqq \prod_{i\in I} f_i(a_i)$$

for all $(a_i)_{i\in I}N \in \coprod_{i\in I} F_i$.

The map f is well-defined, since for all $i, j \in I$ we have

$$f(\kappa_i(-1)N) = f_i(-1) = -1 = f_j(-1) = f(\kappa_j(-1)N).$$

Further, $f(\iota_i(a)) = f(\kappa_i(a)N) = f_i(a)$ for all $i \in I$ and $a \in F_i^*$. Thus, $f \circ \iota_i = f_i$ for all $i \in I$.

Moreover, to prove that f is a multiplicative homomorphism let $(a_i)_{i\in I}N$, $(b_i)_{i\in I}N \in \coprod_{i\in I} F_i$. Then

$$\begin{aligned} f((a_i)_{i\in I}N \cdot (b_i)_{i\in I}N) &= f((a_ib_i)_{i\in I}N) \\ &= \prod_{i\in I} f_i(a_ib_i) = \left(\prod_{i\in I} f_i(a_i)\right) \cdot \left(\prod_{i\in I} f_i(b_i)\right) \\ &= f((a_i)_{i\in I}N) \cdot f((b_i)_{i\in I}N). \end{aligned}$$

[12] δ_{ij} denotes the Kronecker delta, as usual.

[13] This definition implies that $-1 = \iota_i(-1) = \kappa_i(-1)N$ for all $i \in I$.

[14] In the sense of category theory, see [AHS06, p. 10.63].

Finally, let $(a_i)_{i\in I}N, (b_i)_{i\in I}N \in \left(\coprod_{i\in I} F_i\right)^*$ such that $(b_i)_{i\in I}N \in 1 - (a_i)_{i\in I}N$. Proposition and Definition 2.9 implies that there exist $i \in I$ and $a, b \in F_i$ such that $(a_i)_{i\in I}N = \iota_i(a)$ and $(b_i)_{i\in I}N = \iota_i(b)$.

Because f_i is a homomorphism of partial hyperfields, it follows that

$$\begin{aligned} f((b_i)_{i\in I}N) &= f(\iota_i(b)) = f_i(b) \in 1 - f_i(a) \\ &= 1 - f(\iota_i(a)) = 1 - f((a_i)_{i\in I}N). \end{aligned}$$

Using Lemma 2.5, we get that f is a homomorphism of partial hyperfields. □

2.13 Remark. For aesthetical reasons we often write $F_1 \oplus F_2$ for the coproduct of two partial hyperfields F_1, F_2.

2.14 Proposition. Let $f\colon F \to F'$ be a homomorphism of partial hyperfields.

(a) If f is strong, then f is an embedding.

(b) f is epimorphism if and only if f is surjective and final.

(c) The following statements are equivalent:[15]

  (i) f is an isomorphism,

  (ii) f is a monomorphism and an epimorphism,

  (iii) f is a surjective embedding,

  (iv) f is strong and surjective.

Proof. To show (a), let f be strong. If $a, b \in F$ such that $f(a) = f(b)$ it follows from Lemma 2.1 that $0 \in f(a) - f(b) = f(a - b)$. Using Lemma 2.5, we get $0 \in a - b$, which yields that $a = b$. Hence, f is a monomorphism.

Further, let $a, b \in F^*$ such that $f(b) \in 1 - f(a) = f(1-a)$. Since f is injective, this implies $b \in 1 - a$. It follows from Proposition and Definition 2.8 that f is initial and thus an embedding.

In order to prove (b), let f be an epimorphism. By Lemma and Definition 2.4, f is surjective. Let $a', b' \in F'^*$ such that $b' \in 1 - a'$. By definition, there exist $c \in \ker_* f$, $\tilde{a} \in f^{-1}(a')$ and $\tilde{b} \in f^{-1}(b') \cap (c - a)$.

We set $a := \tilde{a}c^{-1}$ and $b := \tilde{b}c^{-1}$. Then $b \in 1 - a$, $f(a) = a'$, $f(b) = b'$. Thus, $f^{-1}(b') \cap (1 - f^{-1}(a')) \neq \emptyset$ and hence applying Proposition and Definition 2.9 we get that f is final.

[15] Although some of these implication follows directly from category theory, see [AHS06, Proposition 8.14], we will prove them directly in order to have a self-contained proof.

Conversely, let f be a surjective and final homomorphism. We show that for all $a'_1, a'_2 \in F'$ and $a'_3 \in a'_1 + a'_2$ there exist $a_i \in f^{-1}(a'_i)$, $i = 1, 2, 3$, such that $a_3 \in a_1 + a_2$.

If $0 \in \{a'_1, a'_2\}$, say $a'_2 = 0$, then $a'_3 = a'_1$. Choose $a_1 \in f^{-1}(a'_1)$ and set $a_3 \coloneqq a_1$ and $a_2 \coloneqq 0$. Thus, $a_3 \in a_1 + a_2$.

Now, let $a'_1, a'_2 \in F^*$. In the case $a'_3 = 0$, Lemma 2.1 implies that $a'_2 = -a'_1$. Choose $a_1 \in f^{-1}(a'_1)$. Thus, $-a_1 \in f^{-1}(a'_2)$, $0 \in f^{-1}(a'_3)$ and $0 \in a_1 - a_1$.

Otherwise $a'_3 \in F^*$. Set $a' \coloneqq -a'_2 a'^{-1}_1$ and $b' \coloneqq a'_3 a'^{-1}_1$. Using Lemma 2.1, we get $b' \in 1 - a'$. Since f is final there exist $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$ such that $b \in 1 - a$.

Further, we choose an $a_1 \in f^{-1}(a'_1)$, and set $a_2 \coloneqq -a_1 a$ and $a_3 \coloneqq a_1 b$. Applying Lemma 2.1, we get $f(a_i) = a'_i$, for all $i = 1, 2, 3$ and $a_3 \in a_1 + a_2$. This proves (b).

To show (i $\Rightarrow$ *iv*) from (c) let g be a homomorphism of partial hyperfields such that $g \circ f = \mathrm{id}_F$ and $f \circ g = \mathrm{id}_{F'}$. Clearly, $\underline{f}$ and $\underline{g}$ are isomophisms and Lemma and Definition 2.4 implies that f and g are bijective. For all $a, b \in F$ we have further

$$f(a) + f(b) = f(g(f(a) + f(b))) \subseteq f(g(f(a)) + g(f(b))) = f(a + b).$$

Hence, f is strong.

(iv $\Rightarrow$ iii) follows directly from (a).

In order to prove (iii $\Rightarrow$ ii) let f be a surjective embedding. By definition, f is a monomorphism. Let $a', b' \in F'^*$ such that $b' \in 1 - a'$. Since f is bijective there exist unique $a, b \in F$ such that $f(a) = a'$ and $f(b) = b'$.

Thus, $f(b) \in 1 - f(a)$ and therefore Proposition and Definition 2.8 implies that $b \in 1 - a$. Hence, using Proposition and Definition 2.9 and (b) we get that f is an epimorphism.

Finally, to prove (ii $\Rightarrow$ i) let f be a mono- and epimorphism. Then f is bijective. We will first show that f is strong. It suffices to show that $f(a) + f(b) \subseteq f(a + b)$ for all $a, b \in F$.

Since f is an injective epimorphism for each $c' \in f(a) + f(b)$ we obtain $c \coloneqq f^{-1}(c') \in a + b$. Therefore, $c' = f(c) \in f(a + b)$. This implies

$$\begin{aligned} f^{-1}(a + b) &= f^{-1}(f(f^{-1}(a)) + f(f^{-1}(b))) \\ &= f^{-1}(f(f^{-1}(a) + f^{-1}(b))) = f^{-1}(a) + f^{-1}(b) \end{aligned}$$

for all $a, b \in F$. Hence, f^{-1} is a strong homomorphism of partial hyperfields and thus f is an isomorphism. $\square$

2.15 Proposition. Let $f\colon F \to F'$ be a homomorphism of partial hyperfields and $U \subseteq \ker_* f$ a subgroup. Further, let $\pi\colon F \to F/_{\!*}U$ be the canonical projection. Then there is exactly one homomorphism $\tilde{f}\colon F/_{\!*}U \to F'$ such that $f = \tilde{f} \circ \pi$.

Moreover, we have $f(F) = \tilde{f}(F)$ and $\ker_*\tilde{f} = \ker_* f/U$.

Proof. Using Lemma and Definition 2.4 and the homomorphism theorem for groups, there exists a unique map $\tilde{f}\colon F/_{\!*}U \to F'$ such that $\tilde{f}(0) = 0$ and $\underline{f} = \underline{\tilde{f}} \circ \underline{\pi}$.

Further, it follows that $\tilde{f}(aU) = f(a)$ for all $a \in F^*$, $\tilde{f}(F) = f(F)$ and $\ker_*\tilde{f} = \ker_* f/U$. Since π is final, it follows from Proposition and Definition 2.8 that f is indeed a homomorphism of partial hyperfields. □

2.16 Corollary. Let $f\colon F \to F'$ be a homomorphism of partial hyperfields.

(a) If f is initial, then $F/_{\!*}\ker_* f \cong F'_{|f(F^*)}$.

(b) If f is final, then the embedding $F'_{|f(F^*)} \to F'$ is a strong embedding.

(c) If f is an epimorphism, then $F' \cong F/_{\!*}\ker_* f$.

Proof. Let $U \coloneqq \ker_* f$, $\tilde{F} \coloneqq F/_{\!*}U$ and $\pi\colon F \to \tilde{F}$ the canonical projection. By Proposition 2.15, there exists a unique homomorphism $\tilde{f}\colon \tilde{F} \to F'$ of partial hyperfields such that $f = \tilde{f} \circ \pi$. Moreover, $\ker_*\tilde{f} = \ker_* f/U = \{1\}$, so $\tilde{f}$ is a monomorphism.

In order to prove (a), let f be initial. To show that $\tilde{f}$ is also initial, let F'' be a partial hyperfield and $g\colon F' \to F''$ a map such that $g(0) = 0$ and $\underline{g}\colon \underline{F'} \to \underline{F''}$ is a multiplicative homomorphism and $g \circ \tilde{f}$ is a homomorphism of partial hyperfields.

Since $g \circ f = g \circ \tilde{f} \circ \pi$ is a homomorphism of partial hyperfields and f is initial it follows from Proposition and Definition 2.8 that g is a homomorphism of partial hyperfields. So $\tilde{f}$ is initial itself.

Again applying Proposition and Definition 2.8 we get that $a + b \subseteq \tilde{f}(\tilde{F})$ for all $a, b \in \tilde{f}(\tilde{F})$. Therefore, $\hat{f}\colon \tilde{F} \to F'_{|f(F^*)}$, $a \mapsto \tilde{f}(a)$ is an isomorphism of partial hyperfields such that $f = \iota \circ \tilde{f}$, where $\iota\colon F'_{|f(F^*)} \to F'$ is the canonical inclusion. Hence, $\tilde{F} \cong F'_{|f(F^*)}$.

To show (b), let f be final. Let $a'_1, a'_2 \in F'$ and $a'_3 \in a'_1 + a'_2$. Then Proposition and Definition 2.9 implies that there exist $a_i \in f^{-1}(a'_i)$, $i = 1, 2$, and $a_3 \in f^{-1}(a'_3) \cap (a_1 + a_2)$. Thus, $a'_3 = f(a_3) \in f(F)$ and therefore the inclusion map $F'_{|f(F^*)} \to F'$ is a strong embedding.

Furthermore, $a_3 = \tilde{f}(\pi(a_3))$. Hence, $\tilde{f}$ is final. Therefore, applying Proposition 2.14 proves (c). □

2.17 Corollary. Let $f\colon F \to F'$ be an isomorphism of partial hyperfields. If $U \subseteq F^*$ is a subgroup, then $F/\!\!_*U \cong F'/\!\!_*f(U)$.

Proof. Let $\pi\colon F \to F/\!\!_*U$ and $\pi'\colon F' \to F'/\!\!_*f(U)$ be the canonical projections. It follows from Proposition 2.15 that there is a unique homomorphism $g\colon F/\!\!_*U \to F'/\!\!_*f(U)$ of partial hyperfields such that $\pi' \circ f = g \circ \pi$. Further, $\ker_* g = \ker_*(\pi' \circ f)/U$ and g is surjective.

Since f is a isomorphism $\ker_*(\pi' \circ f) = U$ and therefore g is a monomorphism. Using Corollary 2.16 and the fact that $\pi \circ f$ is an epimorphism, we get that g is an epimorphism. Finally, Proposition 2.14 implies that g is an isomorphism, as desired. □

2.3 $\mathcal{A}$-regular partial fields

Let $\mathcal{A}$ be a set of sets. We define $\underline{\mathcal{A}} := \bigcup_{A \in \mathcal{A}} A$ and let $\mathbb{G}_{\mathcal{A}}$ be the free abelian group generated by ε, (a), $(a,1)$ for $a \in \underline{\mathcal{A}}$ and (a,b) for $a,b \in A \in \mathcal{A}$, $a \neq b$, where $\varepsilon, 1 \notin \underline{\mathcal{A}}$ are additional elements.

2.18 Lemma. The kernel of the group homomorphism $\iota\colon \mathbb{G}_{\mathcal{A}} \to \mathbb{Q}(\underline{\mathcal{A}})$ defined by $\iota(\varepsilon) := -1$ and

$$\iota((a)) := a,\ \iota((a,1)) := a-1,\ \iota((b,c)) := b-c$$

for all $a \in \underline{\mathcal{A}}$ and $b,c \in A \in \mathcal{A}$, $b \neq c$, where $\mathbb{Q}(\underline{\mathcal{A}})$ is a purely transcendental extension of $\mathbb{Q}$, is generated by ε^2 as well as the elements $\varepsilon(a,b)(b,a)^{-1}$ for $a,b \in A \in \mathcal{A}$, $a \neq b$.

Proof. Clearly, the kernel of ι contains ε^2 and $\varepsilon(a,b)(b,a)^{-1}$ for all $a,b \in A \in \mathcal{A}$, $a \neq b$. Conversely, if $g \in \ker \iota$, there exist suitable $k, l_a, m_a, n_{b,c} \in \mathbb{Z}$, $a \in \underline{\mathcal{A}}$, $b,c \in A \in \mathcal{A}$, $b \neq c$ such that

$$g = \varepsilon^k \prod_{a \in \underline{\mathcal{A}}} (a)^{l_a}(a,1)^{m_a} \prod_{\substack{a,b \in A \in \mathcal{A} \\ a \neq b}} (a,b)^{n_{a,b}}.$$

Thus, we have

$$(-1)^k \prod_{a \in \underline{\mathcal{A}}} a^{l_a}(a-1)^{m_a} \prod_{\substack{a,b \in A \in \mathcal{A} \\ a \neq b}} (a-b)^{n_{a,b}} = \iota(g) = 1.$$

Let $a \in \underline{\mathcal{A}}$, $F_a := \mathbb{Q}(\underline{\mathcal{A}} \setminus \{a\})$, and $x \in F_a$. Thus, a is transcendental over F_a and the localization of $F_a[a]$ at the prime ideal $(a-x)$ is defined by

$$R_{a,x} := \{p/q \in \mathbb{Q}(\underline{\mathcal{A}}) \mid p,q \in F_a[a],\ q(x) \neq 0\}.$$

Further, let $\varphi_{a,x} \colon R_{a,x} \to F_a$ be the homomorphism evaluating every function of $R_{a,x}$ at x, i. e., the unique ring homomorphism such that $\varphi_{a,x}(y) = y$ for all $y \in F_a$ and $\varphi_{a,x}(a) = x$.

If $l_a \neq 0$, say $l_a > 0$ (otherwise replace g by g^{-1}), we would get the contradiction $1 = \varphi_{a,0}(1) = \varphi_{a,0}(\iota(g)) = 0$. Therefore, $l_a = 0$.

Similarly, we get $m_a = 0$ using $\varphi_{a,1}$.

Moreover, if there would exist $a, b \in A \in \mathcal{A}$, $a \neq b$, such that $n_{a,b} + n_{b,a} \neq 0$, say $n_{a,b} + n_{b,a} > 0$, we obtain $n_{a,b} = -n_{b,a}$ using $\varphi_{a,b}$.

Finally, since -1 has order 2 in the multiplicative group of $\mathbb{Q}(\underline{\mathcal{A}})$, k is even, which yields our claim. □

The PhD thesis of Semple ([Sem98]) contains a proof of the following lemma. Nevertheless, to keep this section self-contained we will provide a proof of it.

2.19 Lemma ([Sem98, 3.1.4.1]). Let $R := \mathbb{Z}[x_1, \ldots, x_n]$ be the integral polynomial ring in n variables, $X := \{x_1, \ldots, x_n\}$ and $\hat{X} := X \cup \{0, 1\}$. Furthermore, set $\mathcal{X} := \{\{a, b\} \subseteq \hat{X} \mid a \neq b\ , \{a, b\} \neq \{0, 1\}\}$.

If $p_1, p_2, p_3 \in R$ are three coprime polynomials of the form

$$p_i = (-1)^{k_i} \prod_{\{a,b\} \in \mathcal{X}} (a - b)^{l_{i,\{a,b\}}}$$

for $k_i,\ l_{i,\{a,b\}} \in \mathbb{N}_0$, $i = 1, 2, 3$, such that $p_1 + p_2 + p_3 = 0$, then there exist pairwise different $a, b, c \in \hat{X}$ such that

$$p_1 = a - b,\ p_2 = b - c,\ p_3 = c - a,$$

or pairwise different $a, b, c, d \in \hat{X}$ such that

$$p_1 = (a - b)(c - d),\ p_2 = (a - d)(b - c),\ p_3 = (a - c)(d - b).$$

Proof. For $i = 1, 2, 3$ we set

$$F_i := \{\{a, b\} \in \mathcal{X} \mid \{a, b\} \neq \{0, 1\} \text{ and } (a - b) \mid p_i\}$$

and $\hat{F}_i := F_i \cup \{\{0, 1\}\}$. Since the polynomials p_1, p_2, p_3 are coprime the sets F_1, F_2, F_3 are pairwise disjoint. Further, let $m_i := |F_i|$, $i = 1, 2, 3$. We may assume without loss of generality that $m_1 \geq m_2, m_3$.

Moreover, we have $m_1 \geq 1$ as $F_1 = F_2 = F_3 = \emptyset$ would lead to the contradiction

$$0 = (-1)^{k_1} + (-1)^{k_2} + (-1)^{k_3} \in \{\pm 1, \pm 3\}, \quad k_1, k_2, k_3 \in \mathbb{Z}.$$

Thus, let $\{a,b\} \in F_1$ such that $a \in X$. Applying the unique non-trivial ring homomorphism $\alpha\colon R \to R$ such that $\alpha(a) = b$ and $\alpha(x) = x$ for all $x \in X \setminus \{a\}$, we get that $\alpha(p_2) = -\alpha(p_3)$.

By definition of α, every irreducible factor of $\alpha(p_i)$, $i = 1, 2, 3$, is of the form $x - y$ for $\{x,y\} \in \mathcal{X}$. Since $\alpha(p_2)$ and $\alpha(p_3)$ have exactly the same irreducible factors, but p_2 and p_3 are coprime, it follows that $x = b$ or $y = b$.

Lifting this to p_2 and p_3 implies that

$$F_i \subseteq \{\{a,z\},\{b,z\} \in \mathcal{X} \mid z \in \hat{X} \setminus \{a,b\}\}, \quad i = 2,3. \tag{2.2}$$

Moreover, we have $|m_2 - m_3| = 1$, since the only possibility that $\alpha(x-y)$ is a unit for an irreducible factor $x - y$, $\{x,y\} \in \mathcal{X}$, is that $\{x,y\} = \{a,z\}$ for a suitable $z \in \hat{X}$ such that $\{b,z\} = \{0,1\}$.

In particular, there exists an $c \in \hat{X} \setminus \{a,b\}$ such that $\{b,c\} \in \hat{F}_i$ and $\{a,c\} \in \hat{F}_j$ for $\{i,j\} = \{2,3\}$. We may assume without loss of generality that $i = 2$ and $j = 3$.

If $m_1 = 1$, it follows that $m_2, m_3 \in \{0,1\}$ and there exist k_i, ℓ_i, $i = 1,2,3$, such that

$$p_1 = (-1)^{k_1}(a-b)^{\ell_1},\ p_2 = (-1)^{k_2}(b-c)^{\ell_2},\ p_3 = (-1)^{k_3}(c-a)^{\ell_3}.$$

Suppose $\ell_i > \ell_j$ for $\{i,j\} = \{1,3\}$. Then the coefficient of the monomial a^{ℓ_i} would be $(-1)^{k_i}$ in p_i but 0 in p_2 and p_j, a contradiction. Hence $\ell_1 = \ell_3 =: \ell$.

Since b or c can be zero but not a, if $\ell > 1$, the coefficient of $a^{\ell-1}b$ in p_1 resp. the coefficient of $a^{\ell-1}c$ in p_3 would be 1 but 0 in p_2 and p_3 resp. p_1 and p_2, a contradiction. Thus, $\ell = 1$. Using similar arguments we get $\ell_2 = 1$, as desired.

Finally, we examine the case that $m_1 \geq 2$. Using the unique non-trivial ring homomorphism $\beta\colon R \to R$ such that $\beta(a) = c$ and $\beta(x) = x$ for $x \in X \setminus \{a\}$, we get

$$F_i \subseteq \{\{a,z\},\{c,z\} \in \mathcal{X} \mid z \in \hat{X} \setminus \{a,c\}\}, \quad i = 1,2. \tag{2.3}$$

Therefore, for every $\{x,y\} \in F_1 \setminus \{\{a,b\}\}$ there exist a $d \in \hat{X}$ such that $\{x,y\} = \{a,d\}$ or $\{x,y\} = \{c,d\}$.

Further, (2.2) and (2.3) imply that every $\{x,y\} \in \hat{F}_2$ is of the form $\{b,c\}$ or $\{a,z\}$ for $z \in \hat{X} \setminus \{a\}$. Using the same arguments that we used to derive (2.2) from $\{a,b\} \in F_1$, we can conclude that $\hat{F}_2 = \{\{a,d\},\{b,c\}\}$ from $\{a,d\} \in F_1$ or $\{c,d\} \in F_1$.

Since $a \in X$ we get $F_1 = \{\{a,b\},\{c,d\}\}$ and $\hat{F}_3 = \{\{a,c\},\{b,d\}\}$ from $\{a,d\} \in F_2$. Hence, there exists k_i, ℓ_i, ℓ'_i, $i = 1,2,3$, such that

$$\begin{aligned}
&p_1 = (-1)^{k_1}(a-b)^{\ell_1}(c-d)^{\ell'_1}, \quad p_2 = (-1)^{k_2}(a-d)^{\ell_2}(b-c)^{\ell'_2},\\
&p_3 = (-1)^{k_3}(a-c)^{\ell_3}(d-b)^{\ell'_3}.
\end{aligned}$$

Comparing the coefficients of $a^{\ell_2}b^{\ell'_2}$ and $a^{\ell_3}b^{\ell'_3}$ (if $b \neq 0$), $a^{\ell_1}c^{\ell'_1}$ and $a^{\ell_2}c^{\ell'_2}$ (if $c \neq 0$), $a^{\ell_1}d^{\ell'_1}$ and $a^{\ell_3}d^{\ell'_3}$ (if $d \neq 0$) and using the fact that two of b, c and d must be non-zero, we can conclude that $\ell_1 = \ell_2 = \ell_3 =: \ell$ and $\ell'_1 = \ell'_2 = \ell'_3 =: \ell'$.

Additionally, since only one of these two is equal to 1, a similar argument yields that $\ell = \ell'$. Now, we procede as in the case for $m_1 = 1$ and get that $\ell = 1$. □

2.20 Theorem and Definition (cf. [Sem98, Theorem 3.1.4]). Let $\mathcal{A}$ be a set of sets. The non-trivial fundamental elements[16] of the restriction of $\mathbb{Q}(\underline{\mathcal{A}})$ to the image of ι, which we call the *$\mathcal{A}$-regular partial field* and denote by $\mathbb{U}_{\mathcal{A}}$, are the elements of the form

$$\frac{a-b}{a-c}$$

for pairwise different $a, b, c \in A \cup \{0, 1\}$, $A \in \mathcal{A}$, and the elements of the form

$$\frac{(a-c)(b-d)}{(a-d)(b-c)}$$

for pairwise different $a, b, c, d \in A \cup \{0, 1\}$, $A \in \mathcal{A}$.

Proof. Since

$$1 - \frac{a-b}{a-c} = \frac{c-b}{c-a} \tag{2.4}$$

for all pairwise different $a, b, c \in A \cup \{0, 1\}$, $A \in \mathcal{A}$, and

$$1 - \frac{(a-c)(b-d)}{(a-d)(b-c)} = \frac{(a-b)(c-d)}{(a-d)(c-b)} \tag{2.5}$$

for all pairwise different $a, b, c, d \in A \cup \{0, 1\}$, $A \in \mathcal{A}$, these elements are indeed fundamental elements.

Conversely, if $z \in \mathbb{U}_{\mathcal{A}} \setminus \{0, 1\}$ is a fundamental element, there is a $z' \in \mathbb{U}_{\mathcal{A}}$ such that $1 - z = z'$. Then $z' \neq 0, 1$ as otherwise we would have $z \in \{0, 1\}$. We write $z = \frac{p}{q}$ and $z' = \frac{p'}{q'}$ for suitable $p, q, p', q' \in \mathbb{Z}(\underline{\mathcal{A}}) \setminus \{0\}$.

Through multiplication by the greatest common divisor g of q and q' we get an equation $p_1 + p_2 + p_3 = 0$ such that $p_1 = g$, $p_2 = -\frac{pg}{q}$ and $p_3 = -\frac{p'g}{q'}$ are all elements of an integral polynomial ring with finitely many indeterminants satisfying the precondition of Lemma 2.19.

Thus, applying this yields that there exists pairwise different $a, b, c \in A \cup \{0, 1\}$, $A \in \mathcal{A}$, such that $p_1 = a - b$, $p_2 = b - c$, $p_3 = c - a$ and therefore

$$z = \frac{b-c}{b-a},\ z' = \frac{a-c}{a-b},$$

[16]Since $1 \in 1 - 0$ and $0 \in 1 - 1$, $0, 1$ are fundamental elements of every partial hyperfield.

or there exist pairwise different $a, b, c, d \in A \cup \{0, 1\}$, $A \in \mathcal{A}$, such that

$$p_1 = (a-b)(c-d),\ p_2 = (a-d)(b-c),\ p_3 = (a-c)(d-b)$$

and therefore

$$z = \frac{(a-d)(c-b)}{(a-b)(c-d)},\ z' = \frac{(a-c)(d-b)}{(a-b)(d-c)}.$$

□

2.21 Proposition. Let $\mathcal{A}$ and $\mathcal{A}'$ be sets of sets and $\varphi\colon \underline{\mathcal{A}} \to \underline{\mathcal{A}'}$ a map such that the relation

$$R_\varphi := \{(A, A') \in \mathcal{A} \times \mathcal{A}' \mid \varphi(A) \subseteq A' \text{ and } \varphi_{|A}\colon A \to A' \text{ is a bijection}\}$$

is left and right total. Then the map $\hat{\varphi}\colon \mathbb{U}_{\mathcal{A}} \to \mathbb{U}_{\mathcal{A}'}$ defined by $\hat{\varphi}(0) := 0$, $\hat{\varphi}(-1) := -1$ and

$$\hat{\varphi}(a) := \varphi(a),\ \hat{\varphi}(a-1) := \varphi(a) - 1,\ \hat{\varphi}(b-c) := \varphi(b) - \varphi(c)$$

for all $a \in \underline{\mathcal{A}}$ and $b, c \in A \in \mathcal{A}$, $b \neq c$, is an epimorphism of partial hyperfields.

The multiplicative kernel of $\hat{\varphi}$ is generated by the elements ab^{-1}, $(a-1)(b-1)^{-1}$ for all $a, b \in \underline{\mathcal{A}}$ such that $\varphi(a) = \varphi(b)$, and the elements $(a-b)(c-d)^{-1}$ for all $a, b \in A$, $c, d \in B$, $A, B \in \mathcal{A}$, $a \neq b$, $c \neq d$ such that $\varphi(a) = \varphi(c)$ and $\varphi(b) = \varphi(d)$. In particular, $\hat{\varphi}$ is an isomorphism if and only if φ is injective.

Proof. It follows from Lemma 2.18 that $\hat{\varphi}$ is a multiplicative homomorphism. Since R_φ is left total Theorem and Definition 2.20 implies that $\hat{\varphi}$ maps fundamental elements to fundamental elements. Hence, $\hat{\varphi}$ is a homomorphism of partial hyperfields by Lemma 2.5.

The right totality of R_φ implies that every fundamental element of $\mathbb{U}_{\mathcal{A}'}$ is the image of a fundamental element of $\mathbb{U}_{\mathcal{A}}$. Thus, Proposition and Definition 2.9 yields that $\hat{\varphi}$ is a final homomorphism and is therefore an epimorphism (φ is necessarily surjective if R_φ is right total and thus also $\hat{\varphi}$).

Obviously, all of the elements ab^{-1}, $(a-1)(b-1)^{-1}$ for $a, b \in \underline{\mathcal{A}}$ with $\varphi(a) = \varphi(b)$ and $(a-b)(c-d)^{-1}$ for $a, b \in A$, $c, d \in B$, $A, B \in \mathcal{A}$ with $a \neq b$, $c \neq d$ and $\varphi(a) = \varphi(c)$, $\varphi(b) = \varphi(d)$ are elements of $\ker_* \hat{\varphi}$.

Conversely, let $z \in \ker_* \hat{\varphi}$. Then there exist suitable $k, l_a, m_a, n_{b,c} \in \mathbb{Z}$, $a \in \underline{\mathcal{A}}$, $b, c \in A \in \mathcal{A}$, $b \neq c$, such that

$$z = (-1)^k \prod_{a \in \underline{\mathcal{A}}} a^{l_a} (a-1)^{m_a} \prod_{\substack{a,b \in A \in \mathcal{A} \\ a \neq b}} (a-b)^{n_{a,b}}.$$

Thus, by applying $\hat{\varphi}$ we get

$$1 = \hat{\varphi}(z) = (-1)^k \prod_{a' \in \underline{\mathcal{A}'}} a'^{l_{a'}} (a'-1)^{m_{a'}} \prod_{\substack{a',b' \in A' \in \mathcal{A}' \\ a' \neq b'}} (a'-b')^{n_{a',b'}},$$

where

$$l_{a'} = \sum_{a \in \varphi^{-1}(a')} l_a, \quad m_{a'} = \sum_{a \in \varphi^{-1}(a')} m_a, \quad n_{b',c'} = \sum_{\substack{b \in \varphi^{-1}(b'), \\ c \in \varphi^{-1}(c') \\ b,c \in A,\ (A,A') \in R_\varphi}} n_{b,c}$$

for all $a' \in \underline{\mathcal{A}'}$, $b', c' \in A' \in \mathcal{A}'$, $b' \neq c'$. Using Lemma 2.18, we get $l_{a'} = m_{a'} = 0$ for all $a' \in \underline{\mathcal{A}'}$ and $n_{b',c'} = -n_{c',b'}$ for all $b', c' \in A' \in \mathcal{A}'$, $b' \neq c'$.

If $l_a \neq 0$ for an $a \in \underline{\mathcal{A}}$, there exists a $b \in \underline{\mathcal{A}}$ such that $l_a l_b < 0$. By definition of R_φ, there exists an $A \in \mathcal{A}$ such that $a, b \in A$. Thus, we can successively split off factors of the form ac^{-1}, $a, c \in B \in \mathcal{A}$ with $\varphi(a) = \varphi(c)$.

Similarly, if $m_a \neq 0$ for an $a \in \underline{\mathcal{A}}$, we can split off factors of the form $(a-1)(b-1)^{-1}$ for $a, b \in A \in \mathcal{A}$ with $\varphi(a) = \varphi(b)$, and if $n_{a,b} \neq -n_{b,a}$, we can split off factors of the form $(a-b)(c-d)^{-1}$ for $a, b \in A$, $c, d \in B$, $A, B \in \mathcal{A}$, $a \neq b$, $c \neq d$ such that $\varphi(a) = \varphi(c)$ and $\varphi(b) = \varphi(d)$, which completes our proof. □

The following corollary generalizes the notion of k-regular partial fields, $k \in \mathbb{N}_0$, introduced by Semple in [Sem98] to arbitrary cardinal numbers:

2.22 Corollary. Let κ be a cardinal number. The isomorphy type of the partial field $\mathbb{U}_{\{A\}}$ for a set A of cardinality κ is independent of the choice of A.

We denote this partial field by $\mathbb{U}_\kappa$ and call it the *κ-regular partial field.*

Proof. Let A and A' be sets of cardinality κ. Since we have $\underline{\{X\}} = X$ for every set X, this follows directly from Proposition 2.21 applied to any bijection $\varphi\colon A \to A'$. □

2.23 Theorem and Definition. Let F be a partial hyperfield. We set

$$\mathcal{A} := \{\{(a,0)\} \mid a \in F^*\} \cup \{\{(a,b)\} \mid a,b \in F^* \text{ such that } 1 \in a+b\}$$

and $\hat{F} := \mathbb{Q}(\underline{\mathcal{A}})/_*\mathcal{R}$, where $\mathbb{Q}(\underline{\mathcal{A}})$ is a purely transcendental extension of $\mathbb{Q}$, and $\mathcal{R}$ is the subgroup of $\mathbb{Q}(\underline{\mathcal{A}})^*$ generated by $-(-1,0)$ and the elements

$$(a,0)(b,0)(ab,0)^{-1}, \quad (c,d)(c,0)^{-1}, \quad (1-(c,d))(d,0)^{-1}$$

for all $a, b, c, d \in F^*$ such that $1 \in c+d$.

Then the map $\iota\colon F \to \hat{F}$ defined by $\iota(0) := 0$ and $\iota(a) := (a,0)\mathcal{R}$, $a \in F^*$, is an embedding, which we call the *canonical embedding of F into a hyperfield.*

Proof. First, note that $(a,0)(b,0)\mathcal{R} = (ab,0)\mathcal{R}$ for all $a,b \in F^*$. In particular, we have $(1,0) \in \mathcal{R}$ and therefore $(-1,0)\mathcal{R} = (-1)\mathcal{R}$. Hence, ι is multiplicative homomorphism.

To prove that ι is a homomorphism of partial hyperfields, let $a,b \in F^*$ such that $b \in 1-a$. Then $1 \in a+b$ and thus

$$\begin{aligned}\iota(b) &= (b,0)\mathcal{R} = (1-(a,b))\mathcal{R} \in \mathcal{R} - (a,b)\mathcal{R}^{17}\\ &= (1,0)\mathcal{R} - (a,0)\mathcal{R} = 1 - \iota(a).\end{aligned}$$

In order to show that ι is injective, let $x \in \ker_* \iota$. Then $(x,0) \in \mathcal{R}$ and there exist suitable $k, l_{a,b}, m_{c,d}, n_{c,d} \in \mathbb{Z}$ for $a,b,c,d \in F^*$ satisfying $1 \in c+d$ such that

$$\begin{aligned}(x,0) = (-(-1,0))^k &\prod_{a,b\in F^*} \left((a,0)(b,0)(ab,0)^{-1}\right)^{l_{a,b}}\\ &\cdot \prod_{\substack{c,d\in F^*\\ 1\in c+d}} \left((c,d)(c,0)^{-1}\right)^{m_{c,d}} \left((1-(c,d))(d,0)^{-1}\right)^{n_{c,d}}.\end{aligned} \tag{2.6}$$

For all $c,d \in F^*$ the right-side coefficient of (c,d) is $m_{c,d}$ and the right-side coefficient of $1-(c,d)$ is $n_{c,d}$. It follows that $m_{c,d} = n_{c,d} = 0$ for all $c,d \in F^*$ such that $1 \in c+d$. Therefore, both sides of (2.6) are contained in the subgroup G of $\mathbb{Q}(\underline{\mathcal{A}})^*$ generated by the elements -1 and $(a,0)$, $a \in F^*$.

Thus, the mapping $\kappa\colon G \to F^*$ defined by $\kappa(-1) := -1$ and $\kappa((a,0)) := a$ for $a \in F^*$ is a group homomorphism by Lemma 2.18. Applying it to both sides of (2.6), we get

$$x = \kappa((x,0)) = (-1)^{2k} \prod_{a,b\in F^*} \left(ab(ab)^{-1}\right)^{l_{a,b}} = 1.$$

Hence, ι is injective.

In order to prove that ι is initial, let $a,b \in F^*$ such that $\iota(b) \in 1 - \iota(a)$. Since $\iota(F) \subseteq \mathbb{U}_{\mathcal{A}}/_*\mathcal{R}$ and $|A| = 1$ for all $A \in \mathcal{A}$ it follows from Remark and Definition 2.10 and Theorem and Definition 2.20 that we have $\iota(a) = \frac{a'-b'}{a'-c'}\mathcal{R}$ and $\iota(b) = \frac{c'-b'}{c'-a'}\mathcal{R}$ for $\{a',b',c'\} = \{0,1,(c,0)\}$, $c \in F^*$, or $\{a',b',c'\} = \{0,1,(d,e)\}$ for $d,e \in F^*$ such that $1 \in d+e$.

Since $\iota(b) \in 1-\iota(a)$ and $\iota(a) \in 1-\iota(b)$, as well as $b \in 1-a$ and $a \in 1-b$ are equivalent using Proposition 2.2 and Remark and Definition 2.3, our claim is invariant under exchange of a' and c'. Similarly, this is true for exchange of b'

[17] Where $-$ denotes the substraction in the hyperfield $\mathbb{Q}(\underline{\mathcal{A}})/_*\mathcal{R}$.

and c', as $\iota(b) \in 1 - \iota(a)$ and $\iota(a^{-1}) \in 1 - \iota(-a^{-1}b)$, as well as $b \in 1 - a$ and $a^{-1} \in 1 - (-a^{-1}b)$ are also equivalent.

Hence, we can assume without loss of generality that $a' = 0$ and $c' = 1$. Furthermore, $b' = g$ for $g = (c, 0)$, $c \in F^*$, or $g = (d, e)$ for $d, e \in F^*$, $1 \in d + e$, and $(b, 0)\mathcal{R} = \iota(b) = (1 - g)\mathcal{R}$.

Obviously, $(1-(c, 0))(b, 0)^{-1} \notin \mathcal{R}$ for all $c \in F^*$. Thus, $g = (d, e)$ for $d, e \in F^*$ such that $1 \in d + e$. Therefore, $e = b$ and using that $\iota(a) = g\mathcal{R} = (d, b)\mathcal{R}$ we further get that $d = a$. Hence, $1 \in a + b$, which implies $b \in 1 - a$, as desired. $\square$

2.24 Remark. In [Mas85a] and [Mas85b] Massouros proved that there are hyperfields that cannot be written as $F/_{*}U$ for a field F and a subgroup $U \subseteq F^*$ (and cannot be strongly embedded into hyperfields of this kind).

But as we have shown in Theorem and Definition 2.23 every partial hyperfield and therefore every hyperfield is isomorphic to a restriction of a hyperfield of this form.

3 Universal partial hyperfields of matroids

In this chapter we will introduce the universal partial hyperfield of a matroid, a partial hyperfield whose multiplicative group is its inner Tutte group, which was introduced by Dress and Wenzel in [DW89]. We will use their results to present the relation between the universal partial field of a matroid, its minors, and its dual.

Further, we will connect representations of matroids over a partial hyperfield[1] to homomorphism from certain extensions of the universal partial hyperfield to this partial hyperfields and show that they thus factor over the identity homomorphism of the universal partial hyperfield.

Moreover, we will introduce a method inspired by the work of Semple for k-regular matroids (cf. [Sem98]) to determine the universal partial field for matroids.

For later usage, we will introduce the characterization of matroids by hyperplane and base axioms and define the basic concepts. For further reference we refer the reader to [Whi86].

Definition. Let E be a set. A set $\mathcal{H}$ of subsets of E is called the set of *hyperplanes* of a *matroid* M on E if the following axioms are satisfied:

(H0) For each $X \subseteq E$ such that $X \nsubseteq H$ for all $H \in \mathcal{H}$ there exists a finite $X' \subseteq X$ such that $X' \nsubseteq H$ for all $H \in \mathcal{H}$,[2]

(H1) $E \notin \mathcal{H}$,

(H2) $H_1 \subseteq H_2 \Rightarrow H_1 = H_2$ for all $H_1, H_2 \in \mathcal{H}$,

(H3) for all $H_1, H_2 \in \mathcal{H}$, $H_1 \neq H_2$ and $x \in E \setminus (H_1 \cup H_2)$ there exists an $H_3 \in \mathcal{H}$ such that $(H_1 \cap H_2) \cup \{x\} \subseteq H_3$.

We denote the set of hyperplanes of a given matroid M by $\mathcal{H}(M)$ (or by $\mathcal{H}$ if the referenced matroid is apparent from the context).

[1] Precisely, we mean the *weak* representations in the sense of Baker and Bowler, cf. [BB19].

[2] We allow matroids to be infinite as long as their rank is finite, which is ensured by (H0).

A subset $B \subseteq E$ is a *base* of M if it is minimal with the property $B \nsubseteq H$ for all $H \in \mathcal{H}$. We denote the set of bases of a matroid M by $\mathcal{B}(M)$ (or short by $\mathcal{B}$ if the referenced matroid is apparent from the context).

Let $A \subseteq E$. Then A is called *independent* if it is contained in a base, and a *circuit* if it is not independent, but $A \setminus \{a\}$ is independent for each $a \in A$. Its *rank* is the maximum number of elements of an independent set contained in A and is denoted by $\varrho_M(A)$ (or short by $\varrho(A)$). Further, its *closure* is the (unique) maximal superset that has the same rank as A and is denoted by $\sigma_M(A)$ (or short by $\sigma(A)$).

A is called a *flat* of M, if $\sigma(A) = A$. For any two flats K_1 and K_2 of M the flat $\sigma(K_1 \cup K_2)$ is called the *join* of K_1 and K_2 and is denoted by $K_1 \vee K_2$. If there exists a subset $\{a\}$ such that $K_2 = \sigma(\{a\})$, we often write $K_1 \vee a$ instead of $K_1 \vee \sigma(\{a\})$.

The *rank* of M is the rank of the *ground set* E and is denoted by $\varrho(M)$. Clearly, the hyperplanes of M are exactly the flats of M of rank $\varrho(M) - 1$. A flat $L \subseteq E$ of rank $\varrho(M) - 2$ is called a *hyperline*. We denote by $\mathcal{H}_L$ the sets of hyperplanes that contain a given hyperline L.

The set of hyperlines is denoted by $\mathcal{L}(M)$ (or short $\mathcal{L}$). Further, we call a flat $P \subseteq E$ of rank $\varrho(M) - 3$ a *hyperpoint* and a flat $\ell \subseteq E$ of rank 2 a *line*.

A flat $K \subseteq E$ is called *modular* if $\varrho(K \vee K') + \varrho(K \cap K') = \varrho(K) + \varrho(K')$ for any flat $K' \subseteq E$. The matroid M is said to be *modular* if every flat F of M is modular.

Furthermore, a collection $\mathcal{B}$ is the set of bases of a matroid M if and only if it satisfies the following three axioms:

(B0) every set in $\mathcal{B}$ is finite,

(B1) $B_1 \subseteq B_2 \;\Rightarrow\; B_1 = B_2$ for all $B_1, B_2 \in \mathcal{B}$,

(B2) for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ there is an $y \in B_2 \setminus B_1$ such that $(B_2 \setminus y) \cup \{x\} \in \mathcal{B}$.

All bases $B \in \mathcal{B}$ have equal cardinality $\varrho(M)$.

Definition. Let $\mathbb{F}^{\mathcal{H}}(M)$ be the free abelian group generated by ε and $X_{H,a}$ for $H \in \mathcal{H}$, $a \in E \setminus H$ and $\mathbb{K}^{\mathcal{H}}(M)$ be the subgroup of $\mathbb{F}^{\mathcal{H}}(M)$ generated by ε^2 and the elements

$$\varepsilon \cdot X_{H_1,a_2} \cdot X^{-1}_{H_1,a_3} \cdot X_{H_2,a_3} \cdot X^{-1}_{H_2,a_1} \cdot X_{H_3,a_1} \cdot X^{-1}_{H_3,a_2}$$

for $H_1, H_2, H_3 \in \mathcal{H}$ containing a common hyperline L and $a_i \in H_i \setminus L$, $i = 1, 2, 3$. The *extended Tutte group* is defined as $\mathbb{T}^{\mathcal{H}}(M) := \mathbb{F}^{\mathcal{H}}(M)/\mathbb{K}^{\mathcal{H}}(M)$. Further, we set $H(a) := X_{H,a} \cdot \mathbb{K}^{\mathcal{H}}(M)$ for all $H \in \mathcal{H}$ and $a \in E \setminus H$.

3.1 Lemma ([DW89, Lemma 1.3]). Let $H_1, H_2 \in \mathcal{H}$ such that $L \coloneqq H_1 \cap H_2$ is a hyperline, $a, b \in E \setminus (H_1 \cup H_2)$ such that $L \vee a = L \vee b$. Then

$$H_1(a) \cdot H_1(b)^{-1} \cdot H_2(b) \cdot H_2(a)^{-1} = 1.$$

Definition. Let $\mathbb{F}^{\mathcal{B}}(M)$ be the free abelian group generated by ε and $(a_1, \ldots, a_n)$ for $\{a_1, \ldots, a_n\} \in \mathcal{B}$ and $\mathbb{K}^{\mathcal{B}}(M)$ be the subgroup of $\mathbb{F}^{\mathcal{H}}(M)$ generated by ε^2, the elements

$$\varepsilon \cdot (a_1, \ldots, a_n) \cdot (a_{\pi(1)}, \ldots, a_{\pi(n)})$$

for all $\{a_1, \ldots, a_n\} \in \mathcal{B}$, and $\pi \in S_n \setminus A_n$ and the elements

$$\begin{aligned}&(a_1, \ldots, a_{n-2}, b_1, c_1) \cdot (a_1, \ldots, a_{n-2}, b_2, c_2)\\ \cdot\ &(a_1, \ldots, a_{n-2}, b_2, c_1)^{-1} \cdot (a_1, \ldots, a_{n-2}, b_1, c_2)^{-1}\end{aligned}$$

for all $a_1, \ldots, a_{n-2}, b_1, b_2, c_1, c_2 \in E$ such that $\{a_1, \ldots, a_{n-2}, b_i, c_j\} \in \mathcal{B}$ for $i, j \in \{1, 2\}$ but $\{a_1, \ldots, a_{n-2}, b_1, b_2\} \notin \mathcal{B}$.

We define $\mathbb{T}^{\mathcal{B}}(M) \coloneqq \mathbb{F}^{\mathcal{B}}(M)/\mathbb{K}^{\mathcal{B}}(M)$. Set $[a_1, \ldots, a_n] \coloneqq (a_1, \ldots, a_n) \cdot \mathbb{K}^{\mathcal{B}}(M)$ and

$$[A \,|\, B] \coloneqq [d_1, \ldots, d_{n-1}, a] \cdot [d_1, \ldots, d_{n-1}, b]^{-1}$$

for all $A = \{d_1, \ldots, d_{n-1}, a\}$, $B = \{d_1, \ldots, d_{n-1}, b\} \in \mathcal{B}$.

Definition. We set

$$\begin{aligned}\mathcal{H}_4 \coloneqq \mathcal{H}_4(M) &\coloneqq \left\{ (H_1, H_2, H_3, H_4) \in \mathcal{H}^4 \,\middle|\, \substack{\exists L \in \mathcal{L} \text{ such that } H_i \cap H_j = L \\ \text{for all } i=1,2,\, j=3,4} \right\}\\ \mathcal{H}_4^+ \coloneqq \mathcal{H}_4^+(M) &\coloneqq \left\{ (H_1, H_2, H_3, H_4) \in \mathcal{H}_4 \,\middle|\, \substack{H_1, H_2, H_3, H_4 \\ \text{pairwise different}} \right\}.\end{aligned}$$

Following [GRS95], let $\mathbb{F}^{(0)}(M)$ be the free abelian group generated by ε and the elements $(H_1, H_2 \,|\, H_3, H_4)$ for $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4$ and $\mathbb{K}^{(0)}(M)$ the subgroup of M generated by the elements

(CR0) ε^2,

(CR1) $(H_1, H_2 \,|\, H_3, H_3)$ for $(H_1, H_2, H_3, H_3) \in \mathcal{H}_4$,

(CR2)

$$(H_1, H_2 \,|\, H_3, H_4) \cdot (H_1, H_2 \,|\, H_4, H_5) \cdot (H_1, H_2 \,|\, H_5, H_3)$$

for (H_1, H_2, H_3, H_4), $(H_1, H_2, H_3, H_5) \in \mathcal{H}_4$,

(CR3)

$$(H_1, H_2 \mid H_3, H_4) \cdot (H_3, H_4 \mid H_2, H_1)$$

for $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4$,

(CR4)

$$\varepsilon \cdot (H_1, H_2 \mid H_3, H_4) \cdot (H_1, H_3 \mid H_4, H_2) \cdot (H_1, H_4 \mid H_2, H_3)$$

for $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+$,

(CR5) ε if the Fano matroid or its dual is a minor of M,

(CR6)

$$(H_1, H_2 \mid H_6, H_9) \cdot (H_2, H_3 \mid H_4, H_7) \cdot (H_3, H_1 \mid H_5, H_8)$$

for $H_1, \dots, H_9 \in \mathcal{H}$ such that

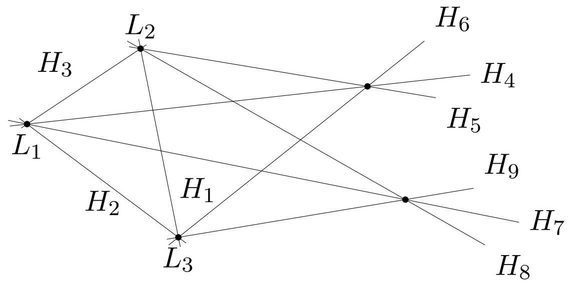
(i) $L_i \coloneqq H_j \cap H_k \in \mathcal{L}$ for $\{i, j, k\} = \{1, 2, 3\}$,

(ii) $\varrho(H_1 \cap H_2 \cap H_3) = \varrho(M) - 3$,

(iii) $L_i \subseteq H_{i+3}, H_{i+6}$ for $i = 1, 2, 3$,

(iv) $H_4 \cap H_5 \cap H_6, H_7 \cap H_8 \cap H_9 \in \mathcal{L}$,

(v) $\{H_1, H_2, H_3\} \cap \{H_4, \dots, H_9\} = \emptyset$.



Then $\mathbb{T}^{(0)}(M) \coloneqq \mathbb{F}^{(0)}(M)/\mathbb{K}^{(0)}(M)$ is called *inner Tutte group* of M. Furthermore, let

$$[H_1, H_2 \mid H_3, H_4] \coloneqq (H_1, H_2 \mid H_3, H_4)\mathbb{K}^{(0)}(M)$$

be the *cross-ratio* of $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4$.

Further, by abuse of notation, we write ε instead of $\varepsilon \cdot \mathbb{K}^{(0)}(M)$.

For finite matroids M it was already proven by Gelfand, Rybnikov and Stone that their definition of the inner Tutte group in [GRS95] is equivalent to the original definition given by Dress and Wenzel in [DW89]. The following proposition proves that this is also true if M is infinite:

3.2 Proposition. The maps $\iota_M^{\mathcal{H}} \colon \mathbb{T}^{(0)}(M) \to \mathbb{T}^{\mathcal{H}}(M)$, $\iota_M^{\mathcal{B}} \colon \mathbb{T}^{(0)}(M) \to \mathbb{T}^{\mathcal{B}}(M)$ defined by $\iota_M^{\mathcal{H}}(\varepsilon) \coloneqq \varepsilon$, $\iota_M^{\mathcal{B}}(\varepsilon) \coloneqq \varepsilon$ and

$$\iota_M^{\mathcal{H}}([H_1, H_2 \mid H_3, H_4]) \coloneqq H_1(a_3) \cdot H_1(a_4)^{-1} \cdot H_2(a_4) \cdot H_2(a_3)^{-1},$$
$$\iota_M^{\mathcal{B}}([H_1, H_2 \mid H_3, H_4]) \coloneqq [A_{13} \mid A_{14}] \cdot [A_{24} \mid A_{23}]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$, where $L \coloneqq \bigcap_{i=1}^4 H_i$, $I \subseteq L$ a maximal independent set and $a_i \in H_i \setminus L$, $A_{ij} \coloneqq I \cup \{a_i, a_j\}$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, are group monomorphisms.

Furthermore, $\mathbb{T}^{\mathcal{H}}(M)$ and $\mathbb{T}^{\mathcal{B}}(M)$ are free extensions of $\mathbb{T}^{(0)}(M)$.

Proof. First, let G is the subgroup of $\mathbb{T}^{\mathcal{H}}(M)$ generated by ε and the elements $H(a) \cdot H(b)^{-1}$ for all $a, b \in E \setminus H$, $H \in \mathcal{H}(M)$, and G' the subgroup of $\mathbb{T}^{\mathcal{B}}(M)$ generated by ε and the elements $[A \mid B]$ for all $A, B \in \mathcal{B}(M)$, $|A \triangle B| = 2$. Then [DW89, Theorem 1.1 and Theorem 1.2] imply that the map $\psi \colon G \to G'$, defined by $\psi(\varepsilon) \coloneqq \varepsilon$ and

$$\psi\left(H(a) \cdot H(b)^{-1}\right) \coloneqq [A \mid B]$$

for all $a, b \in E \setminus H$, $H \in \mathcal{H}(M)$, and $A \coloneqq I \cup \{a\}$, $B \coloneqq I \cup \{b\}$ for any maximal independent set $I \subseteq H$, is a group isomorphism. Since $\iota_M^{\mathcal{B}} = \psi \circ \iota_M^{\mathcal{H}}$ it suffices to show that $\iota_M^{\mathcal{H}}$ is a monomorphism.

It follows from [DW90, Proposition 1.1, Lemma 2.4, and Proposition 2.5] and [GRS95, Theorem 4] that $\iota_M^{\mathcal{H}}$ is a well-defined group homomorphism, which is injective if M is a finite matroid.

In order to show that this also true for infinite M, let $g \in \ker \iota_M^{\mathcal{H}}$. Then there exist $k, n \in \mathbb{N}_0$ and $\left(H_1^{(i)}, H_2^{(i)}, H_3^{(i)}, H_4^{(i)}\right) \in \mathcal{H}_4(M)$, $i = 1, \ldots, n$, such that $g = \varepsilon^k \prod_{i=1}^n \left[H_1^{(i)}, H_2^{(i)} \,\middle|\, H_3^{(i)}, H_4^{(i)}\right]$. Then $L^{(i)} \coloneqq \bigcap_{j=1}^4 H_j^{(i)}$ is a hyperline.

Choose a maximal independent set $I^{(i)} \subseteq L^{(i)}$, $a_j^{(i)} \in H_j^{(i)} \setminus L^{(i)}$, $j = 1, 2, 3, 4$, for all $i = 1, \ldots, n$, let $F \subseteq E$ be a finite set such that $I^{(i)}, a_j^{(i)} \subseteq F$ for all $i = 1, \ldots, n$, $j = 1, 2, 3, 4$, and $N \coloneqq M|F$.

Further, we define $K_j^{(i)} \coloneqq \sigma_N\left(I^{(i)} \cup \{a_j^{(i)}\}\right)$, $i = 1, \ldots, n$, $j = 1, 2, 3, 4$, and $h \coloneqq \varepsilon^k \prod_{i=1}^n \left[K_1^{(i)}, K_2^{(i)} \,\middle|\, K_3^{(i)}, K_4^{(i)}\right]$. By [DW89, Proposition 4.1] there exist group homomorphisms $f \colon \mathbb{T}^{(0)}(N) \to \mathbb{T}^{(0)}(M)$ and $f^{\mathcal{H}} \colon \mathbb{T}^{\mathcal{H}}(N) \to \mathbb{T}^{\mathcal{H}}(M)$ such that $f^{\mathcal{H}} \circ \iota_N^{\mathcal{H}} = \iota_M^{\mathcal{H}} \circ f$.

Since $M|F$ is a restriction, f is injective and therefore $h = 1$, which in turn implies $g = f(h) = 1$. Thus, $\iota_M^{\mathcal{H}}$ is injective for all matroids M.

Finally, [DW89, Theorem 1.5] yields that $\mathbb{T}^{\mathcal{H}}(M)$ and $\mathbb{T}^{\mathcal{B}}(M)$ are free extensions of $\mathbb{T}^{(0)}(M)$. □

3.3 Lemma ([DW90, Lemma 2.4, Proposition 2.5]).
Let $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4$, $L \coloneqq \bigcap_{i=1}^4 H_i$, and for every permutation $\pi \in S_4$ set

$$[H_1, H_2 \,|\, H_3, H_4]^\pi \coloneqq [H_{\pi(1)}, H_{\pi(2)} \,|\, H_{\pi(3)}, H_{\pi(4)}].$$

(a) For $\pi \in \{(12)(34), (13)(24), (14)(23)\}$ we have

$$[H_1, H_2 \,|\, H_3, H_4]^\pi = [H_1, H_2 \,|\, H_3, H_4],$$

(b) for $\pi \in \{(12), (34), (1324), (1423)\}$ we get

$$[H_1, H_2 \,|\, H_3, H_4]^\pi = [H_1, H_2 \,|\, H_3, H_4]^{-1},$$

(c) for $H_0 \in \mathcal{H}_L \setminus \{H_3, H_4\}$ and $H_5 \in \mathcal{H}_L \setminus \{H_1, H_2\}$ we have

$$\begin{aligned}[H_1, H_2 \,|\, H_3, H_4] &= [H_1, H_0 \,|\, H_3, H_4] \cdot [H_0, H_2 \,|\, H_3, H_4],\\ &= [H_1, H_2 \,|\, H_3, H_5] \cdot [H_1, H_2 \,|\, H_5, H_4],\end{aligned}$$

(d) if H_1, H_2, H_3, H_4 are pairwise distinct and $\pi \in S_4$ is an element of order 3,

$$[H_1, H_2 \,|\, H_3, H_4] \cdot [H_1, H_2 \,|\, H_3, H_4]^\pi \cdot [H_1, H_2 \,|\, H_3, H_4]^{\pi^2} = \varepsilon.$$

3.4 Proposition and Definition. The family $(\Delta_a)_{a \in \mathbb{T}^{(0)}(M)}$ defined by

$$\Delta_a \coloneqq \{[H_1, H_3 \,|\, H_2, H_4] \mid \exists (H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+ : a = [H_1, H_2 \,|\, H_3, H_4]\}$$

for all $a \in \mathbb{T}^{(0)}(M)$ satisfies (2.1).

We denote the unique partial hyperfield on $\mathbb{T}^{(0)}(M) \cup \{0\}$, such that $-1 = \varepsilon$ and $(1 - a) \setminus \{0\} = \Delta_a$ for all $a \in \mathbb{T}^{(0)}(M)$ by $\mathbb{U}^{(0)}(M)$. We further call $\mathbb{U}^{(0)}(M)$ the *universal partial hyperfield* of M.[3] Its fundamental elements are 0, 1, and the cross-ratios of M.

[3] The name is justified by Corollary 3.17.

Proof. To prove that the family $(\Delta_a)_{a\in\mathbb{T}^{(0)}(M)}$ satifies (2.1), we have to show that for all $a, b \in \mathbb{T}^{(0)}(M)$ such that $b \in \Delta_a$ it follows that $a \in \Delta_b$ and $a^{-1} \in \Delta_{\varepsilon a^{-1}b}$.

Let $a \in \mathbb{T}^{(0)}(M)$. If $b \in \Delta_a$, there exists a tuple $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+$ such that $a = [H_1, H_2 \,|\, H_3, H_4]$ and $b = [H_1, H_3 \,|\, H_2, H_4]$.

By construction, we have $a \in \Delta_b$. Using Lemma 3.3, we obtain

$$\begin{aligned}\varepsilon a^{-1} b &= \varepsilon \cdot [H_1, H_2 \,|\, H_4, H_3] \cdot [H_1, H_3 \,|\, H_2, H_4]\\ &= [H_1, H_4 \,|\, H_2, H_3].\end{aligned}$$

Hence, $a^{-1} \in \Delta_{\varepsilon a^{-1}b}$.

Therefore, Proposition 2.2 and Remark and Definition 2.3 yield that there exists a unique partial hyperfield on $\mathbb{T}^{(0)}(M) \cup \{0\}$ such that $-1 = \varepsilon$ and $(1 - a) \setminus \{0\} = \Delta_a$. Clearly, its fundamental elements are 0, 1, and the cross-ratios of M. □

3.5 Corollary. We denote the final partial hyperfield[4] on the set $\mathbb{T}^{\mathcal{H}}(M) \cup \{0\}$ resp. $\mathbb{T}^{\mathcal{B}}(M) \cup \{0\}$ with respect to the map $\iota_M^{\mathcal{H}}$ resp. $\iota_M^{\mathcal{B}}$ (which we extend by setting $\iota_M^{\mathcal{H}}(0) := 0$ and $\iota_M^{\mathcal{B}}(0) := 0$) by $\mathbb{U}^{\mathcal{H}}(M)$ resp. $\mathbb{U}^{\mathcal{B}}(M)$.

The maps $\iota_M^{\mathcal{H}}$ and $\iota_M^{\mathcal{B}}$ are strong embeddings. If we identify $\mathbb{U}^{(0)}(M)$ with its image under $\iota_M^{\mathcal{H}}$ resp. $\iota_M^{\mathcal{B}}$, then $\mathbb{U}^{\mathcal{H}}(M)$ resp. $\mathbb{U}^{\mathcal{B}}(M)$ is the unique partial hyperfield such that $-1 = \varepsilon$ and $(1-a)\setminus\{0\} = \Delta_a$, where we define $(\Delta_a)_{a\in\mathbb{T}^{\mathcal{H}}(M)}$ resp. $(\Delta_a)_{a\in\mathbb{T}^{\mathcal{B}}(M)}$ as in Proposition and Definition 3.4. The core of $\mathbb{U}^{\mathcal{H}}(M)$ and $\mathbb{U}^{\mathcal{B}}(M)$ is equal to $\mathbb{U}^{(0)}(M)$.

Proof. It suffices to prove this in the case of $\mathbb{U}^{\mathcal{H}}(M)$ and $\iota_M^{\mathcal{H}}$, as the other case follows analogously.

Since $\iota_M^{\mathcal{H}}$ is injective by Proposition 3.2, Proposition and Definition 2.8 and Proposition and Definition 2.9 imply that it is initial and final. Thus, Corollary 2.16 yields that $\iota_M^{\mathcal{H}}$ is a strong embedding. Finally, the last sentence follows directly from the construction of $\mathbb{U}^{\mathcal{H}}(M)$. □

3.1 Basic properties

3.6 Proposition. Let N be a minor of the matroid M on the set $F \subseteq E(M)$. For every subset $S \subseteq E(M) \setminus F$ such that $\varrho_M(S) = \varrho(M) - \varrho(N)$ we can write $N = (M/S)|F$. Then the map $f_S\colon \mathbb{U}^{(0)}(N) \to \mathbb{U}^{(0)}(M)$ defined by $f_S(0) := 0$, $f_S(-1) := -1$ and

$$f_S([H_1, H_2 \,|\, H_3, H_4]) := [\overline{H_1}, \overline{H_2} \,|\, \overline{H_3}, \overline{H_4}]$$

[4] See Proposition and Definition 2.9.

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(N)$, where $\overline{H_i} := \sigma_M(H_i \cup S)$, $i = 1, 2, 3, 4$, is a homomorphism of partial hyperfields.

Proof. Using [DW89, Proposition 4.1], it follows that f_S is a multiplicative homomorphism. The definition of f_S, Lemma 2.5, and Proposition and Definition 3.4 imply that f_S is a homomorphism of partial hyperfields. □

3.7 Remark. The homomorphism f_S in Proposition 3.6 depends on the choice of S, e. g. if M is the uniform matroid of rank 3 on the set $\{1, 2, 3, 4, 5, 6\}$, $S_1 = \{5\}$, $S_2 = \{6\}$, and $F = \{1, 2, 3, 4\}$, $(M/S_1)|F = (M/S_2)|F$ is the uniform matroid of rank 2 on F, but f_{S_1} and f_{S_2} are different.

Otherwise, it would follow from Lemma 4.21 that $\mathbb{U}^{(0)}(U_{3,6}) \cong \mathbb{U}^{(0)}(U_{2,6})$. Since M is representable over $\mathbb{F}_4$ but $U_{2,6}$ is not (cf. [Oxl11, Section 6.5]), this contradicts Theorem and Definition 3.16.

However, we can always replace S by a maximal independent subset. Thus, in the special cases that N is rank-preserving restriction or a contraction, f_S is independent of the choice of S.

3.8 Proposition. Let $k \in \mathbb{N}$ and M be the direct sum of the matroids M_i on the ground set E_i, $i = 1, \ldots, k$. Then $\mathbb{U}^{(0)}(M)$ is the coproduct[5] of $\mathbb{U}^{(0)}(M_i)$, $i = 1, \ldots, k$ (up to isomorphism).

Proof. Let $F := \coprod_{i=1}^{k} \mathbb{U}^{(0)}(M_i)$, $\iota_i \colon \mathbb{U}^{(0)}(M_i) \to F$ be the natural inclusion, $i = 1, \ldots, k$, and define $f \colon F \to \mathbb{U}^{(0)}(M)$ by $f(0) := 0$, $f(-1) := -1$, and

$$f(\iota_i([H_1 \cap E_i, H_2 \cap E_i \,|\, H_3 \cap E_i, H_4 \cap E_i])) := [H_1, H_2 \,|\, H_3, H_4]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+(M)$ such that $E_i \nsubseteq H_j$ and $E_l \subseteq H_j$ for all $j = 1, 2, 3, 4$, $l = 1, \ldots, k$, $l \neq i$.

[DW89, Proposition 5.1] yields that f is a multiplicative isomorphism. It follows from Lemma 2.5 and Proposition and Definition 3.4 that f is an isomorphism of partial hyperfields. □

3.9 Corollary. For any matroid M of rank $n \in \mathbb{N}_0$ the universal partial hyperfield of $M \oplus p$, i. e., the direct sum of M with the rank 1 matroid with a single point p, is isomorphic to that of M.

In particular, if for any partial hyperfield F there exists a matroid M of rank $n \in \mathbb{N}_0$ whose universal partial hyperfield is isomorphic to F, there exists a matroid M' of rank n' for any $n' \in \mathbb{N}_0$ such that $n' \geq n$ whose universal partial hyperfield is isomorphic to F.

[5] See Corollary 2.12.

Proof. The only hyperplane of the rank 1 matroid with a single point p is $\emptyset$. Further, it has neither the Fano matroid nor its dual as minor. Therefore, it follows that $\mathbb{F}^{(0)}(p) = \{1, \varepsilon\}$ and $\mathbb{K}^{(0)}(p) = \{1\}$. Thus, its universal partial hyperfield is isomorphic to $\mathbb{U}_0$ and we get $\mathbb{U}^{(0)}(M \oplus p) \cong \mathbb{U}^{(0)}(M) \oplus \mathbb{U}_0$ using Proposition 3.8.

Hence, applying Proposition and Definition 2.6 and Corollary 2.12, we obtain that $\mathbb{U}^{(0)}(M \oplus p) \cong \mathbb{U}^{(0)}(M)$. Therefore, we can iteratively construct a matroid M' of rank $n' \geq n$ such that $\mathbb{U}^{(0)}(M') \cong \mathbb{U}^{(0)}(M)$, which yields our claim. $\square$

Definition. Let M be a matroid on the ground set E. M is called a *combinatorial geometry* if for all $X \subseteq E$ such that $1 \leq |X| \leq 2$, there exists a hyperplane H of M such that $|X \cap H| = |X| - 1$ (or equivalently there exists a base B of M such that $X \subseteq B$).

For every $X \subseteq E$ let $\mathrm{s}X$ be the set of rank 1 flats contained in $\sigma(X)$. Further, if $\mathcal{X}$ is a set of subsets of E, we set $\mathrm{s}\mathcal{X} := \{\mathrm{s}X \mid X \in \mathcal{X}\}$.

The matroid $\mathrm{s}M$ on the ground set $\mathrm{s}E$ whose hyperplanes are the sets $\mathrm{s}\mathcal{H}$ (or equivalently whose bases are the sets $\mathrm{s}\mathcal{B}$) is a combinatorial geometry called the *simplification* of M. Moreover, for each $X \subseteq E$ we have $\varrho_{\mathrm{s}M}(\mathrm{s}F) = \varrho_M(F)$. In particular, $\varrho(\mathrm{s}M) = \varrho(M)$.

3.10 Proposition. For every matroid M the map $\varphi\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(\mathrm{s}M)$ defined by $\varphi(0) := 0$, $\varphi(-1) := -1$, and

$$\varphi([H_1, H_2 \,|\, H_3, H_4]) := [\mathrm{s}H_1, \mathrm{s}H_2 \,|\, \mathrm{s}H_3, \mathrm{s}H_4]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$ is an isomorphism of partial hyperfields.

Proof. By definition of the simplification for all flats F_1, F_2 of M we have $\mathrm{s}(F_1 \cap F_2) = \mathrm{s}F_1 \cap \mathrm{s}F_2$ and $\mathrm{s}F_1 = \mathrm{s}F_2$ if and only if $F_1 = F_2$. Therefore, our claim follows directly from the definition of the inner Tutte group and Proposition and Definition 3.4. $\square$

3.11 Proposition. Let M be a finite matroid on the ground set E.

(a) The map $\varphi_M\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(M^*)$ defined by $\varphi_M(0) := 0$, as well as $\varphi_M(-1) := -1$, and

$$\varphi_M([H_1, H_2 \,|\, H_3, H_4]) := [H_1^*, H_2^* \,|\, H_3^*, H_4^*]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$ and

$$H_i^* := \sigma_{M^*}(E \setminus (I \cup \{a_1, \ldots, \widehat{a_i}, \ldots, a_4\})), \quad i = 1, 2, 3, 4,$$

where $I \subseteq L := \bigcap_{i=1}^4 H_i$ is a maximal independent set and $a_i \in H_i \setminus L$, $i = 1, 2, 3, 4$, is a well-defined isomorphism of partial hyperfields.

(b) Let $E = F \cup S \cup S^*$ be any partition such that S is an independent set of M, $\varrho_M(S) = \varrho(M) - \varrho(N)$, and $N = (M/S)|F$. Then S^* is an independent set of M^*, $\varrho_{M^*}(S^*) = \varrho(M^*) - \varrho(N^*)$, and $N^* = (M^*/S^*)|F$.

Further, we have $\varphi_M \circ f_S = f_{S^*} \circ \varphi_N$, where $f_S \colon \mathbb{U}^{(0)}(N) \to \mathbb{U}^{(0)}(M)$ and $f_{S^*} \colon \mathbb{U}^{(0)}(N^*) \to \mathbb{U}^{(0)}(M^*)$ are the homomorphisms of partial hyperfields from Proposition 3.6.

Proof. Throughout this proof, we will use Proposition 3.2 and identify the elements of $\mathbb{T}^{(0)}(M)$ with its image under $\iota_M^{\mathcal{B}}$.

In order to prove (a), let $\mathbb{T}(M)$ be the subgroup of $\mathbb{T}^{\mathcal{B}}(M)$ generated by ε and the elements $[A \,|\, B]$ for all $A, B \in \mathcal{B}(M)$ such that $|A \triangle B| = 2$, and $\mathbb{T}(M^*)$ be the corresponding subgroup of $\mathbb{T}^{\mathcal{B}}(M^*)$. Applying [DW89, Proposition 1.1 and Theorem 1.1], the map $\varphi\colon \mathbb{T}(M) \to \mathbb{T}(M^*)$ defined by $\varphi(\varepsilon) := \varepsilon$ and $\varphi([A \,|\, B]) := [E \setminus A \,|\, E \setminus B]$ for all $A, B \in \mathcal{B}(M)$ such that $|A \triangle B| = 2$ is a group isomorphism.

Let $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$. Then $L := \bigcap_{i=1}^4 H_i$ is a hyperline of M. Choose a maximal independent subset $I \subseteq L$, $a_i \in H_i \setminus L$, $i = 1, 2, 3, 4$, and set $A_{ij} := I \cup \{a_i, a_j\}$, $i, j = 1, 2, 3, 4$, $i \neq j$.

Further, let $I^* := E \setminus (I \cup \{a_1, a_2, a_3, a_4\})$, $A^*_{ij} := I^* \cup \{a_i, a_j\}$, as well as $H^*_i := \sigma_{M^*}(I^* \cup \{a_i\})$, $i, j = 1, 2, 3, 4$, $i \neq j$. It follows that

$$E \setminus A_{ij} = E \setminus (I \cup \{a_i, a_j\}) = I^* \cup \{a_k, a_l\} = A^*_{kl}$$

for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Thus, A^*_{ij} is a base of M^* for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, I^* is an independent set of M^*, whose closure is a hyperline of M^*, and $(H^*_1, H^*_2, H^*_3, H^*_4) \in \mathcal{H}_4(M^*)$. Moreover, Lemma 3.3 implies that

$$\begin{aligned} \varphi([H_1, H_2 \,|\, H_3, H_4]) &= \varphi([A_{13} \,|\, A_{14}] \cdot [A_{24} \,|\, A_{23}]) \\ &= [A^*_{24} \,|\, A^*_{23}] \cdot [A^*_{13} \,|\, A^*_{14}] = [H^*_1, H^*_2 \,|\, H^*_3, H^*_4] \end{aligned}$$

Hence, $\varphi_M(g) = \varphi(g)$ for all $g \in \mathbb{U}^{(0)}(M)^*$. Therefore, φ_M is a multiplicative isomorphism, and its definition and Lemma 2.5 yield (a).

To prove (b), let $E = F \cup S \cup S^*$ be a partition and $N = (M/S)|F$ such that S is an independent set of M and $\varrho_M(S) = \varrho(M) - \varrho(N)$. Applying [Oxl11, Proposition 3.1.26], we obtain

$$N^* = ((M/S)\backslash S^*)^* = (M/S)^*/S^* = (M\backslash S)/S^* = (M/S^*)|F.$$

By [Oxl11, Proposition 2.1.9], we have $\varrho_{M^*}(A) = |A| + \varrho_M(E \setminus A) - \varrho(M)$ for all subsets $A \subseteq E$. Thus, using $\varrho_M(S \cup S^*) = |S|$, we get

$$\begin{aligned} \varrho(M^*) - \varrho(N^*) &= |E| - \varrho(M) - (|F| + \varrho_M(S \cup S^*) - \varrho(M)) \\ &= |S^*| = |S^*| + \varrho_M(F \cup S) - \varrho(M) = \varrho_{M^*}(S^*). \end{aligned}$$

In particular, S^* is an independent set of M^*.

Let $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(N)$, $L := \bigcap_{i=1}^4 H_i$, choose a maximal independent set $I \subseteq L$, and $a_i \in H_i \setminus L$, $i = 1,2,3,4$. Further, set $G_i := \sigma_M(H_i \cup S)$, $H_i^* := \sigma_{N^*}(I^* \cup \{a_i\})$, $G_i^* := \sigma_{M^*}(H_i^* \cup S^*)$, $A_{ij} := I \cup \{a_i, a_j\}$, $B_{ij} := A_{ij} \cup S$, $A_{ij}^* := I^* \cup \{a_i, a_j\}$, $B_{ij}^* := A_{ij}^* \cup S^*$ for all $i, j = 1,2,3,4$, $i \neq j$, where $I^* := F \setminus (I \cup \{a_1, a_2, a_3, a_4\})$.

Then G_i resp. G_i^* is a hyperplane of M resp. M^*, A_{ij} resp. A_{ij}^* is a base of N resp. N^*, and B_{ij} resp B_{ij}^* is a base of M resp. M^*, $i, j = 1,2,3,4$, $i \neq j$. Furthermore, $(H_1^*, H_2^*, H_3^*, H_4^*) \in \mathcal{H}_4(N^*)$, $(G_1, G_2, G_3, G_4) \in \mathcal{H}_4(M)$, $(G_1^*, G_2^*, G_3^*, G_4^*) \in \mathcal{H}_4(M^*)$ and

$$E \setminus B_{ij} = E \setminus (A_{ij} \cup S) = (A_{kl}^* \cup S \cup S^*) \cap (E \setminus S) = A_{kl}^* \cup S^* = B_{kl}^*$$

for all $\{i, j, k, l\} = \{1,2,3,4\}$. Hence,

$$\begin{aligned}
&\varphi_M(f_S([H_1, H_2 \,|\, H_3, H_4])) = \varphi_M([G_1, G_2 \,|\, G_3, G_4]) \\
&= \varphi_M([B_{13} \,|\, B_{14}] \cdot [B_{24} \,|\, B_{23}]) = [B_{23}^* \,|\, B_{24}^*] \cdot [B_{13}^* \,|\, B_{14}^*] \\
&= [G_1^*, G_2^* \,|\, G_3^*, G_4^*] = f_{S^*}([H_1^*, H_2^* \,|\, H_3^*, H_4^*]) \\
&= f_{S^*}(\varphi_N([H_1, H_2 \,|\, H_3, H_4])).
\end{aligned}$$

□

Definition. Let M and N be matroids. A bijection $\varphi\colon E(M) \to E(N)$ is an *isomorphism* from M to N if $H \in \mathcal{H}(M)$ if and only if $\varphi(H) \in \mathcal{H}(N)$ (or equivalently if $B \in \mathcal{B}(M)$ if and only if $\varphi(B) \in \mathcal{B}(N)$).

3.12 Proposition. Let M and M be matroids and φ be an isomorphism from M to N. Then the map $\hat{\varphi}\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(N)$ defined by $\hat{\varphi}(0) := 0$, $\hat{\varphi}(-1) := -1$, and

$$\hat{\varphi}([H_1, H_2 \,|\, H_3, H_4]) := [\varphi(H_1), \varphi(H_2) \,|\, \varphi(H_3), \varphi(H_4)]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$ is an isomorphism of partial hyperfields.

Proof. As we have $\varphi(X \cap Y) = \varphi(X) \cap \varphi(Y)$ and $\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y)$ for all $X, Y \subseteq E$ (since φ is a bijection), our claim follows directly from the definition of the inner Tutte group and Proposition and Definition 3.4. □

3.2 Representability of matroids

3.13 Remark. Let E be a set and F be a partial hyperfield. We denote by F^E the set of functions $f\colon E \to F$, let 0 be the constant 0 function and $-f\colon E \to F$, $e \mapsto -f(e)$, define a binary operation $\cdot\colon F \times F^E \to F^E$ by $(a \cdot f)(e) := a \cdot f(e)$ for all $a \in F$, $f \in F^E$, $e \in E$, and a partial hyperoperation $+\colon F^E \times F^E \multimap F^E$ by

$$f + g := \{h \in F^E \mid h(e) \in f(e) + g(e) \text{ for all } e \in E\}$$

for all $f, g \in F^E$.

It follows directly from the definition of a partial hyperfield that

(a) $f + g = g + f$ and $0 + f = \{f\}$ for all $f, g \in F^E$,

(b) if $f \in g + h$, then $h \in f + (-g)$ for all $f, g, h \in F^E$,

(c) $(ab)f = a(bf)$, $0 \cdot f = 0$, $1 \cdot f = f$ for all $a, b \in F$, $f \in F^E$,

(d) $(a + b)f \subseteq af + bf$ for all $a, b \in F$, $f \in F^E$,

(e) $a(f + g) \subseteq af + ag$ for all $a \in F$, $f, g \in F^E$.

For the rest of this section, let M be a matroid on the ground set E, whose set of hyperplanes we denote by $\mathcal{H}$ and whose set of bases we denote by $\mathcal{B}$.

3.14 Proposition and Definition (cf. [DW89, Theorem 3.1]). Let F be a partial hyperfield and $\iota\colon F \to F'$ an embedding of F into a hyperfield F'. For a family $(f_H)_{H\in\mathcal{H}}$ of functions $f_H\colon E \to F$ such that $f_H^{-1}(0) = H$ for all $H \in \mathcal{H}$ the following statements are equivalent:

(a) For all hyperplanes H_1, H_2, H_3 containing a common hyperline L, there exist $\alpha_1, \alpha_2, \alpha_3 \in F^*$ such that

$$0 \in \iota(\alpha_1) \cdot (\iota \circ f_{H_1}) + \iota(\alpha_2) \cdot (\iota \circ f_{H_2}) + \iota(\alpha_3) \cdot (\iota \circ f_{H_3}). \qquad (3.1)$$

(b) The map $f\colon \mathbb{U}^{\mathcal{H}}(M) \to F$ defined by $f(0) := 0$, $f(-1) := -1$ and

$$f(H(a)) := f_H(a) \text{ for } H \in \mathcal{H},\ a \in E \setminus H$$

is a homomorphism of partial hyperfields.

In particular, the condition in (a) is independent of the choice of the embedding ι.[6] If $(f_H)_{H\in\mathcal{H}}$ satisfies one (and therefore both) of the preceding conditions, we call it a *family of hyperplane functions for M and F*.

[6]Such an embedding always exists, see Theorem and Definition 2.23.

Proof. We first prove the implication (a) $\Rightarrow$ (b). Let H_1, H_2, H_3 be hyperplanes of M containing a common hyperline L and $a_i \in H_i \setminus L$, $i = 1, 2, 3$. Let $\alpha_1, \alpha_2, \alpha_3 \in F^*$ such that

$$0 \in \iota(\alpha_1) \cdot (\iota \circ f_{H_1}) + \iota(\alpha_2) \cdot (\iota \circ f_{H_2}) + \iota(\alpha_3) \cdot (\iota \circ f_{H_3}).$$

Since $a_2 \in H_2$ we have $f_{H_2}(a_2) = 0$. Thus, $0 \in \alpha_1 f_{H_1}(a_2) + \alpha_3 f_{H_3}(a_2)$. It follows that $f_{H_3}(a_2) = -\alpha_1\alpha_3^{-1} f_{H_1}(a_2)$. We get $f_{H_2}(a_3) = -\alpha_1\alpha_2^{-1} f_{H_1}(a_3)$ and $f_{H_3}(a_1) = -\alpha_2\alpha_3^{-1} f_{H_2}(a_1)$ using similar arguments. This implies

$$\begin{aligned} &f_{H_1}(a_2) \cdot f_{H_1}(a_3)^{-1} \cdot f_{H_2}(a_3) \cdot f_{H_2}(a_1)^{-1} \cdot f_{H_3}(a_1) \cdot f_{H_3}(a_2)^{-1} \\ = &\left(-\frac{\alpha_3}{\alpha_1}\right) \cdot \left(-\frac{\alpha_1}{\alpha_2}\right) \cdot \left(-\frac{\alpha_2}{\alpha_3}\right) = -1. \end{aligned}$$

Hence, f is a multiplicative homomorphism.

To prove that f is a homomorphism of partial hyperfields, let H_1, H_2, H_3, H_4 be four pairwise different hyperplanes containing a common hyperline L and choose $a_i \in H_i \setminus L$, $i = 2, 3, 4$. Then (3.1) implies

$$\iota(f_{H_3}(a_4)) \in \iota\left(-\alpha_3^{-1}(\alpha_1 f_{H_1}(a_4) + \alpha_2 f_{H_2}(a_4))\right)$$

and thus using the identification of $\mathbb{T}^{\mathcal{H}}(M)$ with a subgroup of $\mathbb{T}^{(0)}(M)$ from Proposition 3.2, $-f_{H_1}(a_2) f_{H_3}(a_2)^{-1} = \alpha_1^{-1}\alpha_3$ and $\alpha_1\alpha_2^{-1} = -f_{H_1}(a_3) f_{H_2}(a_3)^{-1}$ from above we obtain

$$\begin{aligned} &\iota(f([H_1, H_3 \,|\, H_2, H_4])) \\ &= \iota\left(f_{H_1}(a_2) f_{H_1}(a_4)^{-1} f_{H_3}(a_4) f_{H_3}(a_2)^{-1}\right) \\ &\in \iota\left(-f_{H_1}(a_2) f_{H_3}(a_2)^{-1} f_{H_1}(a_4)^{-1} \alpha_3^{-1}(\alpha_1 f_{H_1}(a_4) + \alpha_2 f_{H_2}(a_4))\right) \\ &= \iota\left(1 + \alpha_1^{-1}\alpha_2 f_{H_1}(a_4)^{-1} f_{H_2}(a_4)\right) \\ &= \iota\left(1 - f_{H_1}(a_3) f_{H_1}(a_4)^{-1} f_{H_2}(a_4) f_{H_2}(a_3)^{-1}\right) \\ &= \iota(1 - f([H_1, H_2 \,|\, H_3, H_4])). \end{aligned}$$

Since ι is an embedding, $f([H_1, H_3 \,|\, H_2, H_4])) \in 1 - f([H_1, H_2 \,|\, H_3, H_4])$. Applying Lemma 2.5 and Proposition and Definition 3.4 yields that f is a homomorphism of partial hyperfields.

Conversely, let f be a homomorphism of partial hyperfields and H_1, H_2, H_3 three pairwise different hyperplanes containing a common hyperline L.

We choose $a_i \in H_i \setminus L$, $i = 1, 2, 3$, and set

$$\alpha_1 := f_{H_2}(a_3) \cdot f_{H_1}(a_3)^{-1}, \ \alpha_2 := -1, \ \alpha_3 := f_{H_2}(a_1) \cdot f_{H_3}(a_1)^{-1}.$$

In order to prove (3.1), we show that $f_{H_2} \in \alpha_1 f_{H_1} + \alpha_3 f_{H_3}$. Since all functions f_{H_i}, $i = 1, 2, 3$, are identically zero on L, we have $f_{H_2}(a) \in \alpha_1 f_{H_1}(a) + \alpha_3 f_{H_3}(a)$ for all $a \in L$.

For $a \in H_1 \setminus L$ we have $H_2(a) \cdot H_2(a_1)^{-1} = H_3(a) \cdot H_3(a_1)^{-1}$ by Lemma 3.1 and therefore

$$\begin{aligned} f_{H_2}(a) &= f_{H_3}(a) \cdot f_{H_3}(a_1)^{-1} f_{H_2}(a_1) \\ &\in 0 + f_{H_3}(a) \cdot f_{H_3}(a_1)^{-1} f_{H_2}(a_1) \\ &= \alpha_1 f_{H_1}(a) + \alpha_3 f_{H_3}(a). \end{aligned}$$

Similarly, we get $f_{H_2}(a) \in \alpha_1 f_{H_1}(a) + \alpha_3 f_{H_3}(a)$ for all $a \in H_3 \setminus L$. As f is a multiplicative homomorphism

$$f_{H_1}(a_2) \cdot f_{H_1}(a_3)^{-1} \cdot f_{H_2}(a_3) \cdot f_{H_2}(a_1)^{-1} \cdot f_{H_3}(a_1) \cdot f_{H_3}(a_2)^{-1} = -1,$$

so for all $a \in H_2 \setminus L$ we obtain

$$\begin{aligned} f_{H_2}(a) = 0 &\in f_{H_2}(a_3) \cdot f_{H_1}(a_3)^{-1} \cdot f_{H_1}(a_2) + f_{H_2}(a_1) \cdot f_{H_3}(a_1)^{-1} \cdot f_{H_3}(a_2) \\ &= \alpha_1 f_{H_1}(a_2) + \alpha_3 f_{H_3}(a_2). \end{aligned}$$

Using Lemma 3.1, we have

$$f_{H_1}(a) f_{H_3}(a)^{-1} = f_{H_1}(a_2) f_{H_3}(a_2)^{-1} = -\alpha_1^{-1}\alpha_3,$$

and it follows that $f_{H_2}(a) \in \alpha_1 f_{H_1}(a) + \alpha_3 f_{H_3}(a)$.

Finally, let $a \in E \setminus (H_1 \cup H_2 \cup H_3)$. We set $H_4 := L \vee a$ and $a_4 := a$. Applying Proposition 3.2, we get

$$\begin{aligned} \alpha_3 f_{H_3}(a) &= \big(f_{H_3}(a_4) \cdot f_{H_3}(a_1)^{-1} \cdot f_{H_2}(a_1) \cdot f_{H_2}(a_4)^{-1}\big) \cdot f_{H_2}(a_4) \\ &= f([H_3, H_2 \,|\, H_4, H_1]) \cdot f_{H_2}(a_4) \\ &\in (1 - f([H_3, H_4 \,|\, H_2, H_1]) \cdot f_{H_2}(a_4) \\ &= (1 - f([H_1, H_2 \,|\, H_4, H_3]) \cdot f_{H_2}(a_4) \\ &= \big(1 - f_{H_1}(a_4) \cdot f_{H_1}(a_3)^{-1} \cdot f_{H_2}(a_3) \cdot f_{H_2}(a_4)^{-1}\big) \cdot f_{H_2}(a_4) \\ &= f_{H_2}(a) - \alpha_1 f_{H_1}(a). \end{aligned}$$

Thus, Lemma 2.1 yields $f_{H_2}(a) \in \alpha_1 f_{H_1}(a) + \alpha_3 f_{H_3}(a)$. □

3.15 Proposition and Definition (cf. [DW89, Proposition 3.1]). Let F be a partial hyperfield, $\iota\colon F \to F'$ an embedding into a hyperfield F' and $d\colon E^n \to F$ a map such that $d(e_1, \dots, e_n) = 0$ if and only if $\{e_1, \dots, e_n\} \notin \mathcal{B}$. Then the following statements are equivalent:

(a) For all $e_1, \dots, e_n \in E$ and $\pi \in S_n$

$$d(e_{\pi(1)}, \dots, e_{\pi(n)}) = \operatorname{sign}\pi \cdot d(e_1, \dots, e_n), \tag{3.2}$$

and for all $e_0, \dots, e_n, f_2 \in E$

$$0 \in \sum_{i=0}^{2} (-1)^i \iota\left(d(e_0, \dots, \widehat{e_i}, \dots, e_n) \cdot d(e_i, f_2, e_3, \dots, e_n)\right). \tag{3.3}$$

(b) The map $\hat{d}\colon \mathbb{U}^{\mathcal{B}}(M) \to F$ defined by $\hat{d}(0) \coloneqq 0$, $\hat{d}(-1) \coloneqq -1$ and

$$\hat{d}([e_1, \dots, e_n]) \coloneqq d(e_1, \dots, e_n)$$

for all bases $B = \{e_1, \dots, e_n\}$ of M is a homomorphism of partial hyperfields.

In particular, the condition in (a) is independent of the choice of ι. If d satisfies one (and therefore both) of the preceding conditions, we call it a *Grassmann-Plücker map for M and F*.

Proof. We first prove that (a) implies (b). Let $a_1, \dots, a_{n-2}, b_1, b_2, c_1, c_2 \in E$ such that $\{a_1, \dots, a_{n-2}, b_i, c_j\} \in \mathcal{B}$ for $i, j = 1, 2$ and $\{a_1, \dots, a_{n-2}, b_1, b_2\} \notin \mathcal{B}$. Setting $f_2 \coloneqq b_1$, $e_0 \coloneqq b_2$, $e_i \coloneqq c_i$, $i = 1, 2$, and $e_{i+2} \coloneqq a_i$, $i = 1, \dots, n-2$ we obtain by using (3.2) and (3.3) that

$$\begin{aligned} 0 \in &+ \iota(d(c_1, c_2, a_1, \dots, a_{n-2}) \cdot d(b_2, b_1, a_1, \dots, a_{n-2})) \\ &- \iota(d(b_2, c_2, a_1, \dots, a_{n-2}) \cdot d(c_1, b_1, a_1, \dots, a_{n-2})) \\ &+ \iota(d(b_2, c_1, a_1, \dots, a_{n-2}) \cdot d(c_2, b_1, a_1, \dots, a_{n-2})) \\ = &+ \iota(d(a_1, \dots, a_{n-2}, b_1, c_1) \cdot d(a_1, \dots, a_{n-2}, b_2, c_2)) \\ &- \iota(d(a_1, \dots, a_{n-2}, b_2, c_1) \cdot d(a_1, \dots, a_{n-2}, b_1, c_2)). \end{aligned}$$

Using that ι is an embedding and Lemma 2.1, we get that

$$\begin{aligned} &d(a_1, \dots, a_{n-2}, b_1, c_1) \cdot d(a_1, \dots, a_{n-2}, b_2, c_2) \\ &\cdot d(a_1, \dots, a_{n-2}, b_2, c_1)^{-1} \cdot d(a_1, \dots, a_{n-2}, b_1, c_2)^{-1} = 1. \end{aligned}$$

Therefore, $\hat{d}$ is a multiplicative homomorphism.

To show that $\hat{d}$ is a homomorphism of partial hyperfields, let H_1, H_2, H_3, H_4 be four pairwise different hyperplanes containing a common hyperline L. We choose a basis $b_1, \dots, b_{n-2}$ of L and $a_i \in H_i \setminus L$, $i = 1, 2, 3, 4$. Setting $e_0 \coloneqq a_1$, $e_1 \coloneqq a_3$, $e_2 \coloneqq a_4$, $f_2 \coloneqq a_2$ and $e_i \coloneqq b_{i-2}$ for $i = 3, \dots, n$ we obtain

$$\begin{aligned} 0 \in &+ \iota(d(a_3, a_4, b_1, \dots, b_{n-2}) \cdot d(a_1, a_2, b_1, \dots, b_{n-2})) \\ &- \iota(d(a_1, a_4, b_1, \dots, b_{n-2}) \cdot d(a_3, a_2, b_1, \dots, b_{n-2})) \\ &+ \iota(d(a_1, a_3, b_1, \dots, b_{n-2}) \cdot d(a_4, a_2, b_1, \dots, b_{n-2})). \end{aligned}$$

Since H_1, H_2, H_3, H_4 are paarwise different $\{b_1, \dots, b_{n-2}, a_i, a_j\}$ is a base of M and thus $d(b_1, \dots, b_{n-2}, a_i, a_j) \neq 0$ for all $i, j = 1, 2, 3, 4$, $i \neq j$.

Therefore, using (3.3) and the fact that ι is an embedding, we get

$$\begin{aligned} \hat{d}([H_1, H_3 \mid H_2, H_4]) &= \hat{d}\left(\frac{[b_1, \dots, b_{n-2}, a_1, a_2] \cdot [b_1, \dots, b_{n-2}, a_3, a_4]}{[b_1, \dots, b_{n-2}, a_1, a_4] \cdot [b_1, \dots, b_{n-2}, a_3, a_2]}\right) \\ &\in 1 - \hat{d}\left(\frac{[b_1, \dots, b_{n-2}, a_1, a_3] \cdot [b_1, \dots, b_{n-2}, a_2, a_4]}{[b_1, \dots, b_{n-2}, a_1, a_4] \cdot [b_1, \dots, b_{n-2}, a_2, a_3]}\right) \\ &= 1 - \hat{d}([H_1, H_2 \mid H_3, H_4]). \end{aligned}$$

Hence, Lemma 2.5 and Proposition and Definition 3.4 yield that $\hat{d}$ is a homomorphism of partial hyperfields.

In order to prove that (b) implies (a), let $\hat{d}$ be a homomorphism of partial hyperfields. First, let $e_1, \dots, e_n \in E$. If $\{e_1, \dots, e_n\} \notin \mathcal{B}$, then (3.2) is trivially satisfied. Otherwise it follows from $[e_1, \dots, e_n] = \operatorname{sign} \pi \cdot [e_{\pi(1)}, \dots, e_{\pi(n)}]$ for all $\pi \in S_n$.

To show that d satisfies (3.3), let $e_0, \dots, e_n, f_2 \in E$, $L \coloneqq \sigma(\{e_3, \dots, e_n\})$, $H_1 \coloneqq L \vee e_0$, $H_2 \coloneqq L \vee f_2$ and $H_i \coloneqq L \vee e_{i-2}$ for $i = 3, 4$.

If H_2 is not a hyperplane, then $d(e_i, f_2, e_3, \dots, e_n) = 0$ for all $i = 0, 1, 2$, which implies that all the three terms of the sum in (3.3) vanish. A similiar argument shows that this is also true if one of H_1, H_3, H_4 is not a hyperplane. Therefore, (3.3) is trivially satisfied in these cases.

Thus, let H_1, H_2, H_3, H_4 be hyperplanes. If they are pairwise different, then $\{e_0, \dots, \widehat{e_i}, \dots, e_n\}$ and $\{e_i, f_2, e_3, \dots, e_n\}$ are bases of M for all $i = 0, 1, 2$ and thus

$$\begin{aligned} &\frac{d(e_3, \dots, e_n, e_0, f_2) \cdot d(e_3, \dots, e_n, e_1, e_2)}{d(e_3, \dots, e_n, e_0, e_2) \cdot d(e_3, \dots, e_n, e_1, f_2)} \\ &= \hat{d}([H_1, H_3 \mid H_2, H_4]) \in 1 - \hat{d}([H_1, H_2 \mid H_3, H_4]) \\ &= 1 - \frac{d(e_3, \dots, e_n, e_0, e_1) \cdot d(e_3, \dots, e_n, f_2, e_2)}{d(e_3, \dots, e_n, e_0, e_2) \cdot d(e_3, \dots, e_n, f_2, e_1)}. \end{aligned}$$

Hence,

$$0 \in \sum_{i=0}^{2} (-1)^i \iota(d(e_0, \ldots, \widehat{e_i}, \ldots, e_n) \cdot d(e_i, f_2, e_3, \ldots, e_n)).$$

In the case that three of the hyperplanes are equal, each of the summands in (3.3) vanishes, and therefore the inclusion is trivial.

If exactly two of the hyperplanes are equal, using Lemma 3.3, we can assume without loss of generality that $H_1, H_2 \neq H_3, H_4$ but $H_1 = H_2$ or $H_3 = H_4$, since (3.3) is invariant under cyclic permutation of e_0, e_1, e_2. Hence, $[H_1, H_2 \,|\, H_3, H_4] = 1$ and thus

$$\frac{d(e_3, \ldots, e_n, e_0, e_1) \cdot d(e_3, \ldots, e_n, f_2, e_2)}{d(e_3, \ldots, e_n, e_0, e_2) \cdot d(e_3, \ldots, e_n, f_2, e_1)} = 1.$$

Since $d(e_1, \ldots, e_n) \cdot d(e_0, f_2, e_3, \ldots, e_n) = 0$, this implies (3.3). $\square$

3.16 Theorem and Definition. Let M be a matroid and F a partial hyperfield. Then the following statements are equivalent:

(a) There exists a family of hyperplane functions for M and F,

(b) there exists a Grassmann-Plücker map for M and F,

(c) there exists a homomorphism $\mathbb{U}^{(0)}(M) \to F$ of partial hyperfields.

If M satisfies one (and therefore all) of the above conditions, we call M is *representable over F*.

Proof. First, (a) or (b) imply (c). This follows by restricting the homomorphism which we obtain from Proposition and Definition 3.14 or Proposition and Definition 3.15 to the universal partial hyperfield $\mathbb{U}^{(0)}(M)$ of M.

Conversely, Proposition 3.2 implies that we can extend every homomorphism $\mathbb{U}^{(0)}(M) \to F$ of partial hyperfields to a homomorphism $\mathbb{U}^{\mathcal{H}}(M) \to F$ and $\mathbb{U}^{\mathcal{B}}(M) \to F$ of partial hyperfields. Thus, Proposition and Definition 3.14 and Proposition and Definition 3.15 yield that (c) implies (a) and (b). $\square$

3.17 Corollary. Every matroid M is representable over its universal partial hyperfield $\mathbb{U}^{(0)}(M)$. Moreover, if M is representable over a partial hyperfield F, the resulting homomorphism of partial hyperfields $\mathbb{U}^{(0)}(M) \to F$ factors over the universal homomorphism $\mathrm{id}_{\mathbb{U}^{(0)}(M)}$ from this representation.

Proof. Follows directly from Theorem and Definition 3.16. $\square$

3.18 Remark. Theorem and Definition 3.16 generalizes the classical theory of representability of matroids over fields as well as the theory of representability of matroids over partial hyperfields by Baker and Bowler in [BB19].

In particular, we obtain a homomorphism of partial hyperfields $\mathbb{U}^{(0)}(M) \to F$ for every field F over which M is representable.

The characterization of classes of projectively equivalent representations of matroids over fields was already done by Dress and Wenzel in [DW89] and inspired our definition of the addition of the universal partial hyperfield of a matroid.

3.19 Lemma. Let F' be a partial hyperfield such that $1 - 1 = \{0\}$ in F'[7] and $f\colon F \to F'$ be a homomorphism of partial hyperfields. Then we have $\ker_* f \setminus \{1\} \subseteq F \setminus \mathcal{F}(F)$. In particular, f is injective if F is a hyperneofield.

Proof. Let $a \in \ker_* f \cap \mathcal{F}(F)$. Then there exists a $b \in F$ such that $b \in 1 - a$. It follows that $f(b) \in 1 - f(a) = 1 - 1 = \{0\}$. Thus, $f(b) = 0$ and Lemma 2.5 implies $b = 0$. Hence, Lemma 2.1 yields that $a = 1$. □

3.20 Lemma. Every partial hyperfield F such that $F = \{-1, 0, 1\}$ is isomorphic to $\mathbb{U}_0$, $\mathbb{F}_3$, $\mathbb{S}$, $\mathbb{W}$, $\mathbb{F}_2$, or $\mathbb{K}$.

Furthermore, if $F, F' \in \{\mathbb{U}_0, \mathbb{F}_3, \mathbb{S}, \mathbb{W}, \mathbb{F}_2, \mathbb{K}\}$ and $f\colon F \to F'$ is a homomorphism of partial hyperfields, then f is uniquely determined and one of the conditions $F = F'$, $F = \mathbb{U}_0$, $F' = \mathbb{K}$, or $(F, F') \in \{(\mathbb{F}_3, \mathbb{W}), (\mathbb{S}, \mathbb{W})\}$ is satisfied.

Proof. Let F be a partial hyperfield such that $F = \{-1, 0, 1\}$. By Remark and Definition 2.3, the addition of F is completely determined by the sets $(1-1) \setminus \{0\}$ and $(1+1) \setminus \{0\}$. If $1 = -1$, then $1 - 1 = 1 + 1$ and we have either $1 + 1 = \{0\}$ or $1 + 1 = \{0, 1\}$. In the former case $F \cong \mathbb{F}_2$ and in the latter case $F \cong \mathbb{K}$.

Now, let $1 \neq -1$. If $a \in 1 - 1$, we get $-a \in -1 + 1 = 1 - 1$ by Lemma 2.1. Thus, $1 - 1 = \{0\}$ or $1 - 1 = \{-1, 0, 1\}$. Further, Lemma 2.1 implies that $1 \in 1 - 1$ if and only if $1 \in 1 + 1$.

Therefore, if $1 - 1 = \{0\}$, we have the two possibilities $1 + 1 = \emptyset$, which implies $F \cong \mathbb{U}_0$, and $1 + 1 = \{-1\}$, which implies $F \cong \mathbb{F}_3$. Otherwise, either $1 + 1 = \{1\}$ and $F \cong \mathbb{S}$, or $1 + 1 = \{-1, 1\}$ and $F \cong \mathbb{W}$.

To finish the proof, let $F, F' \in \{\mathbb{U}_0, \mathbb{F}_3, \mathbb{S}, \mathbb{W}, \mathbb{F}_2, \mathbb{K}\}$ and $f\colon F \to F'$ be a homomorphism of partial hyperfields. Since, $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$, f is uniquely determined.

[7] Note that any hyperfield F' with this property is already a field, since for any $a, b \in F'^*$ such that $b \in 1 - a$ we have $1 - a = 1 + (-1 + b) = (1 - 1) + b = \{b\}$.

Clearly, if $-1 = 1$ in F, $-1 = 1$ in F' too. In the case $-1 \neq 1$ in F but $-1 = 1$ in F', $1+1 \neq \emptyset$ in F implies $1+1 = \{0,1\}$ in F'. Thus, in this case we have $F = \mathbb{U}_0$ or $F' = \mathbb{K}$.

Otherwise, $-1 \neq 1$ in F if and only if $-1 \neq 1$ in F' and f is the identity map on the set $\{-1,0,1\}$. Finally, such an f can exist if and only the set $1-a$ of F, $a \in \{-1,1\}$ is included in the corresponding set of F', which proves our claim. □

3.21 Corollary. A matroid M is binary if and only if $\mathbb{U}^{(0)}(M) \cong \mathbb{U}_0$ or $\mathbb{U}^{(0)}(M) \cong \mathbb{F}_2$, and regular if and only if $\mathbb{U}^{(0)}(M) \cong \mathbb{U}_0$.

Proof. By Tutte's representation theorem ([Tut65, Th. 5.1.1]) and Theorem and Definition 3.16, M is binary if and only if there exists a homomorphism $f\colon \mathbb{U}^{(0)}(M) \to \mathbb{F}_2$. Further, if M is binary, we have that $\mathbb{U}^{(0)}(M) = \{-1,0,1\}$ (as otherwise M would contain four pairwise different hyperplanes over a common hyperline).

Thus, Lemma 3.20 implies that M is binary if and only if $\mathbb{U}^{(0)}(M) \cong \mathbb{U}_0$ or $\mathbb{U}^{(0)}(M) \cong \mathbb{F}_2$. Since there are no homomorphisms between fields of different characteristics, we obtain, using Lemma 3.19, that M is regular if and only if the former case holds. □

3.3 Projective planes

Throughout this section let $\Pi = (\mathcal{P}, \mathcal{H})$ be a projective plane, i. e., a connected, modular combinatorial geometry of rank 3. We will coordinatise Π as in [Pic] and use the extended radical introduced in [Kal89].

For a quadrangle (o,u,v,e) of Π, i. e., $\{o,u,v,e\}$ is a circuit of Π, we set $F \coloneqq (o \vee e) \setminus (u \vee v)$. For $p \in \mathcal{P} \setminus (u \vee v)$ we set $p \eqqcolon (x,y)$ if $(p \vee v) \cap (o \vee e) = \{x\}$ and $(p \vee u) \cap (o \vee e) = \{y\}$. For $H \in \mathcal{H}$ with $v \in H$ we set $H \eqqcolon [\infty]$ if $u \in H$, and $H \eqqcolon [x]$ if $x \in H$ for $x \in F$.

For $H \in \mathcal{H}$ such that $v \notin H$ but $o \in H$ we set $H \eqqcolon [m,0]$ if $H \cap [1] = \{(1,m)\}$. Each of these lines intersects $u \vee v$ in a point different from v and we set $(u \vee v) \cap [m,0] \eqqcolon (m)$ for all $m \in F$.

Finally, for all other $H \in \mathcal{H}$ we set $H \eqqcolon [m,c]$ if $H \cap (u \vee v) = (m)$, and $H \cap [0] = (0,c)$. Furthermore, let $0 \coloneqq o$ and $1 \coloneqq e$. We define a ternary operation $T\colon F^3 \to F$ by

$$T(m,x,c) = y \;\Leftrightarrow\; (x,y) \in [m,c]$$

for all $x,y,m,c \in F$. The tuple (F,T) is called a *planar ternary ring coordinatising* Π, or a *ternary field coordinatising* Π.

We further set $a \cdot b \coloneqq T(a, b, 0)$ and $a + b \coloneqq T(1, a, b)$ for all $a, b \in F$. Then $(F, +)$ and $(F^*, \cdot)$ are loops with neutral elements 0 resp. 1. We further denote by $a - b$ the unique element such that $(a - b) + b = a$ for all $a, b \in F$.

Definition. The *radical* $R \coloneqq R(F, T)$ of (F, T) is the normal subloop of F^* generated by the elements $r \in F^*$ for which there exist $a, b, c, d, m, n, x, u \in F$ such that $a \neq b$, $m \neq n$, $x \neq u$ and $T(m, u, c) = T(n, u, d)$ that satisfy one of the following equations

$$T(m, x, a) - T(m, x, b) = r \cdot (a - b),$$
$$T(m, x, c) - T(n, x, d) = r \cdot [(n - m)(u - x)].$$

Moreover, the *extended radical* $R_a \coloneqq R_a(F, T)$ is the normal subloop of F^* generated by $R(F, T)$ and the elements $r \in F^*$ for which there exist $x, y, z \in F^*$ that satisfy one of the following equations:

$$x(yz) = r \cdot (xy)z, \quad xy = r \cdot yx.$$

3.22 Lemma. Let (F, T) be a planar ternary ring. For all $a \in F^*$ and all non-empty subsets $L, K, M \subseteq F$ satisfying $R \cdot K = K$, $R \cdot L = L$, $R \cdot M = M$ we have

(a) $L + K = K + L = K - (-L)$,

(b) $L + (K + M) = (L + K) + M$,

(c) $a(L + K) = aL + aK$,

(d) $M[(-1)(-a)] = M[-(-a)] = Ma = -M[-a]$,

(e) $Ma = aM$ if $R_a \cdot M = M$.

Proof. First note, that [Kal88, (2.8) Korollar] directly yields (b), (c), and the first equation of (a). Moreover, [Kal88, (2.2) Lemma and (2.4) Lemma] imply the last equation of (a).

Further, it follows from [Kal88, (2.2) Lemma] that $R[b - (-a)] = R[b + a]$ and $R[(-c)d] = R[c(-d)]$ for all $a, b, c, d \in F$. Substituting b for 0, 1 for c, and d for $-a$ and using that $R \cdot M = M$, we get the first two of the three equations of (d).

Since $-Re = R[-e]$ by [Kal88, (2.3) Satz] for all $e \in E$, the third equation is implied by substituting e for $-a$ and using $R \cdot M = M$.

Finally, if $x \in aM$, there exists an $m \in M$ such that $x = am$. By definition of R_a, there exists an $r \in R_a$ such that $r \cdot (ma) = am$. Therefore, we obtain $x \in r(ma) \in R_a(Ma) \subseteq Ma$. Using a similar argument, one shows that $aM \subseteq Ma$, which completes our proof. □

3.23 Proposition. Let (F,T) be a planar ternary ring. The set $\{R_a x \mid x \in F\}$ together with the partial hyperoperation $\oplus$ defined by

$$R_a x \oplus R_a y := \{R_a z \mid z \in F \text{ such that } R_a z \subseteq R_a x + R_a y\},$$

and the multiplication defined by $R_a x \cdot R_a y := R_a(xy)$ for all $x, y \in F$ is a hyperfield, denoted by $F/_{*}R_a$.

Proof. We will first prove that $F/_{*}R_a$ is a partial hyperfield. If we have $R_a z \in R_a x \oplus R_a y$ for $x, y, z \in F$, Lemma 3.22 yields that

$$R_a z \subseteq R_a x + R_a y = R_a y + R_a x.$$

Thus, $R_a x \oplus R_a y \subseteq R_a y \oplus R_a x$, which proves (PH1).

Since $R_a \cdot 0 + R_a x = \{0\} + R_a x = \{R_a x\}$, $F/_{*}R_a$ satisfies (PH2). In order to prove (PH3), let $x, y, z \in F$ such that $R_a z \in R_a x \oplus R_a y$. Using Lemma 3.22, we get

$$R_a z \subseteq R_a x + R_a y = R_a y + R_a x = R_a y - R_a(-x).$$

Therefore, there exist $r, s \in R_a$ such that $z = sy - r(-x)$. It follows that $z + r(-x) = sy$, and thus $R_a y \subseteq R_a z + R_a(-x)$. Hence, $R_a y \in R_a z \oplus R_a(-x)$.

By definition of the extended radical, $(\{R_a x \mid x \in F^*\}, \cdot)$ is an abelian group, $R_a \cdot R_a x = R_a x$, and $\{0\} \cdot R_a x = \{0\}$ for all $x \in F$, which directly yields (PH4).

To show (PH5), let $x, y, z \in F$. Then Lemma 3.22 implies that

$$R_a x(R_a y + R_a z) = R_a x \cdot R_a y + R_a x \cdot R_a z = R_a(xy) + R_a(xz).$$

Thus, $R_a x(R_a y \oplus R_a z) \subseteq R_a(xy) \oplus R_a(xz)$.

Finally, in order to show that $F/_{*}R_a$ is a hyperfield, we have to show that

$$(R_a x \oplus R_a y) \oplus R_a z \subseteq R_a x \oplus (R_a y \oplus R_a z)$$

for all $x, y, z \in F$ (then (PH1) implies that both sets are equal). Let $w \in F$ such that $R_a w \in (R_a x \oplus R_a y) \oplus R_a z$. Then there exists a $v \in F$, such that $R_a w \in R_a v \oplus R_a z$ and $R_a v \in R_a x \oplus R_a y$. Applying Lemma 3.22, we get

$$R_a w \subseteq (R_a x + R_a y) + R_a z = R_a x + (R_a y + R_a z).$$

We can conclude that there exist $r, s, t \in R_a$, such that $w = rx + (sy + tz)$. This implies that for $u := sy + tz$ we have $R_a w \subseteq R_a x + R_a u$ and $R_a u \subseteq R_a y + R_a z$. Thus, $R_a w \in R_a x \oplus (R_a y \oplus R_a z)$. □

3.24 Theorem. For every projective plane Π and any planar ternary ring (F,T) coordinatising Π the family $(f_H)_{H\in\mathcal{H}}$ of functions $f_H\colon \mathcal{P} \to F/_{\!*}R_a$ defined by $f_H(a) = 0$ for all $a \in H \in \mathcal{H}$ and

$$f_H(a) \coloneqq \begin{cases} R_a(T(m,x,c)-y) & \text{if } H = [m,c],\ a = (x,y), \\ R_a(m-n) & \text{if } H = [m,c],\ a = (n), \\ R_a(x-d) & \text{if } H = [d],\ a = (x,y), \\ R_a & \text{else}, \end{cases}$$

for all $H \in \mathcal{H}$ and $a \in E \setminus H$, is a system of hyperplane functions for Π and $F/_{\!*}R_a$ such that the map $f\colon \mathbb{U}^{(0)}(\Pi) \to F/_{\!*}R_a$ defined by $f(0) \coloneqq \{0\}$, $f(-1) \coloneqq R_a(-1)$, and

$$f([H_1,H_2 \,|\, H_3,H_4]) \coloneqq f_{H_1}(a_3)\cdot f_{H_1}(a_4)^{-1}\cdot f_{H_2}(a_3)\cdot f_{H_2}(a_4)^{-1}$$

for all $L \in \mathcal{L}$, $H_{i-2}, H_i \in \mathcal{H}_L$, $a_i \in H_i \setminus L$, $i = 3,4$, such that $H_1, H_2 \neq H_3, H_4$, is an isomorphism of partial hyperfields.

Proof. Using Proposition and Definition 3.14, it suffices to show that f is an isomorphism.

First, let Π be the projective plane of order 2. Since Π is binary and non-regular, it follows from Corollary 3.21 that $\mathbb{U}^{(0)}(\Pi) \cong \mathbb{F}_2$. Further, $F = \{0,1\}$, and $(F,+)$ is a loop. This implies $R_a = \{1\}$, and using Lemma 3.20 that $F/_{\!*}R_a \cong \mathbb{F}_2$.

For the rest of the proof, let Π a projective plane of order strictly greater than 2. By [Kal92b, Theorem 3] and since Lemma 3.22 yields $R_a[(-1)(-x)] = R_a x$ for all $x \in F$, f is a multiplicative isomorphism.

Furthermore, using our coordinatization, we obtain for any $x \in F^*$

$$f([[\infty],[0] \,|\, [1],[x]]) = f_{[0]}(x) f_{[0]}(1)^{-1} = R_a x,$$

as in [Kal92b, Theorem 3]. Therefore, using Lemma 2.5, it suffices to show that $f(1-[[\infty],[0] \,|\, [1],[x]]) = R_a \oplus R_a(-x)$ in order to prove that f is an isomorphism of partial hyperfields. It follows from Proposition and Definition 3.4) and the fact that f is surjective that

$$1 - [[\infty],[0] \,|\, [1],[x]] = \{[[\infty],[1] \,|\, [0],[y]] \mid y \in R_a x\}.$$

Further, applying Lemma 3.22 yields that for any $y \in R_a x$ we have

$$\begin{aligned} f([[\infty],[1] \,|\, [0],[y]]) &= f_{[1]}(y) f_{[1]}(0)^{-1} = R_a(1-y) \\ &\subseteq R_a - R_a y = R_a + R_a(-y) = R_a + R_a(-x). \end{aligned}$$

Converserly, if $z \in F^*$ such that $R_a z \subseteq R_a - R_a x$, we have $R_a \subseteq R_a z + R_a x$. Thus, there exist $r, s \in R_a$ such that $1 = rz + sx$. Therefore, $rz = 1 - sx$, which implies

$$R_a z \in R_a(1 - sx) = f([[\infty], [1] \mid [0], [sx]]) \in f(1 - [[\infty], [0] \mid [1], [x]]). \qquad \square$$

3.25 Proposition. The universal partial hyperfield of the projective geometry $\mathrm{PG}(d, F)$ of dimension $d \geq 2$ over a skew field F is isomorphic to $F/\!_* F^{*\prime}$, where $F^{*\prime}$ is the commutator subgroup of F^*.

Proof. First, note that by setting $T(m, x, c) := mx + c$ for all $m, x, c \in F$ we can regard F as a planar ternary ring whose extended radical R_a is equal to $F^{*\prime}$.

For each hyperplane H of $\mathrm{PG}(d, F)$ we fix a left linear form Φ_H such that $\Phi_H^{-1}(\{0\}) = H$ and set $f_H \colon E \to F/\!_* F^{*\prime}$, $e \mapsto \Phi_H(e) F^{*\prime}$. Clearly, $(f_H)_{H \in \mathcal{H}}$ is a system of hyperplane functions for $\mathrm{PG}(d, F)$ and $F/\!_* F^{*\prime}$.

It follows from the proof of Theorem and Definition 3.16 that the underlying multiplicative homomorphism of the induced homomorphism of partial hyperfields $f \colon \mathbb{U}^{(0)}(\mathrm{PG}(d, F)) \to F/\!_* F^{*\prime}$ is the same as the group homomorphism $\Phi \colon \mathbb{T}^{(0)}(\mathrm{PG}(d, F)) \to F^*/F^{*\prime}$ induced by the $(\Phi_H)_{H \in \mathcal{H}}$ via [DW90, Proposition 1.5].

Moreover, [DW90, Theorem 3.6] yields that Φ is bijective and thus also f. Finally, [DW90, Lemma 4.5] and Proposition and Definition 3.4 imply that f is an isomorphism. $\square$

3.4 Matroids of rank 2

In his PhD thesis ([Sem98]) Semple proved that for all $k \in \mathbb{N}_0$ the uniform matroid $U_{2,k+3}$ on $k + 3$ points of rank 2 is representable over the partial field $\mathbb{U}_k$ but is not representable over $\mathbb{U}_{k'}$ for all $k' \in \mathbb{N}_0$, where $k' < k$.

We will extend this result to infinite uniform matroids of rank 2. Further, we will prove that the universal partial hyperfield of a uniform matroid of rank 2 that has at least 3 points is isomorphic to $\mathbb{U}_{\kappa-3}$, where κ is the cardinality of the set of points.

Let $M = U_{2,E}$ be the uniform matroid of rank 2 on the set E, which contains at least 3 elements named $\infty, 0, 1$. Let $A := E \setminus \{\infty, 0, 1\}$ and $\mathbb{Q}(A)$ be a purely transcendental extension of $\mathbb{Q}$.

3.26 Proposition. Let $f\colon E \to \mathbb{Q}(A)^2$ be the map defined by $f(\infty) \coloneqq (1,0)^T$ and $f(e) \coloneqq (e,1)^T$ for all $e \in E \setminus \{\infty\}$. Then the map $\hat{f}\colon \mathbb{U}^{(0)}(U_{2,E}) \to \mathbb{Q}(A)$ defined by $\hat{f}(0) \coloneqq 0$, $\hat{f}(-1) \coloneqq -1$, and

$$\hat{f}\,([H_1, H_2 \mid H_3, H_4]) \coloneqq \frac{\det(f(a_1), f(a_3)) \cdot \det(f(a_2), f(a_4))}{\det(f(a_1), f(a_4)) \cdot \det(f(a_2), f(a_3))}$$

for all hyperplanes $H_i = \{a_i\}$, $i = 1, 2, 3, 4$, of $U_{2,E}$ such that $H_1, H_2 \neq H_3, H_4$ is an embedding of partial hyperfields whose image is equal to $\mathbb{U}_{\{A\}}$.

Proof. First, we define $f_{\{a\}}\colon E \to \mathbb{Q}(A)$ by $f_{\{a\}}(e) \coloneqq \det(f(a), f(e))$, $a \in E$. Using Tutte's representation theorem ([Tut65, Th. 5.1.1]), we get that $(f_{\{a\}})_{a\in E}$ is a system of hyperplane functions for $U_{2,E}$ and $\mathbb{Q}(A)$. Thus, Proposition and Definition 3.14 implies that $\hat{f}$ is a homomorphism of partial hyperfields.

Moreover, straightforward computations show that $f_{\{a\}}(a) = 0$, $f_{\{\infty\}}(b) = 1$, $f_{\{b\}}(\infty) = -1$, and $f_{\{b\}}(c) = b - c$ for all $a \in E$, $b, c \in E \setminus \{\infty\}$, $b \neq c$. It follows that

$$\hat{f}([\{\infty\}, \{a\} \mid \{b\}, \{c\}]) = \frac{a-c}{a-b} \tag{3.4}$$

for all paarwise different $a, b, c \in E \setminus \{\infty\}$, and

$$\hat{f}([\{a\}, \{b\} \mid \{c\}, \{d\}]) = \frac{(a-c)(b-d)}{(a-d)(b-c)} \tag{3.5}$$

for all paarwise different $a, b, c, d \in E \setminus \{\infty\}$. Hence, the definition of the inner Tutte group and Lemma 3.3 imply that $\hat{f}(F) \subseteq \mathbb{U}_{\{A\}}$, where $F \coloneqq \mathbb{U}^{(0)}(U_{2,E})$.

Substituting 0 for a and 1 for b in (3.4) yields $c \in \hat{f}(F)$ for all $c \in A$, and replacing b by 0 in (3.4) yields $a^{-1}(a-c) \in \hat{f}(F)$ for $a, c \in A$, $a \neq c$. Therefore, $\hat{f}(F) = \mathbb{U}_{\{A\}}$.

To prove that $\hat{f}$ is an embedding, we will show that there exists a homomorphism $g\colon \mathbb{U}_{\{A\}} \to F$ such that $g \circ \hat{f} = \mathrm{id}_F$. For $x, y \in F$ we have $|\hat{f}(x+y)| \leq |\hat{f}(x) + \hat{f}(y)| \leq 1$, since $\mathbb{U}_{\{A\}}$ is a partial field. Thus, $|x+y| \leq 1$ because f is injective.

Suppose $x + y = \emptyset$ but $|\hat{f}(x) + \hat{f}(y)| = 1$. This would yield the contradiction $|g(\hat{f}(x) + \hat{f}(y))| \leq |g(\hat{f}(x)) + g(\hat{f}(y))| = |x+y| = 0$. Hence, $\hat{f}$ is strong and thus an embedding by Proposition 2.14.

We define g by $g(0) \coloneqq 0$, $g(-1) \coloneqq -1$ and

$$\begin{aligned} g(a) &\coloneqq [\{\infty\}, \{0\} \mid \{1\}, \{a\}], \\ g(a-1) &\coloneqq -[\{\infty\}, \{1\} \mid \{0\}, \{a\}], \\ g(b-c) &\coloneqq -[\{\infty\}, \{c\} \mid \{0\}, \{b\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{c\}] \end{aligned}$$

for all $a, b, c \in A$, $b \neq c$. By using Lemma 3.3, we get

$$\begin{aligned} g(b-a) &= -[\{\infty\}, \{a\} \mid \{0\}, \{b\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{a\}] \\ &= [\{\infty\}, \{b\} \mid \{0\}, \{a\}] \cdot [\{\infty\}, \{0\} \mid \{a\}, \{b\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{a\}] \\ &= [\{\infty\}, \{b\} \mid \{0\}, \{a\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{b\}] = -g(a-b). \end{aligned}$$

Thus, Lemma 2.18 implies that g is a well-defined multiplicative homomorphism.

Moreover, for $a \in A$, it follows from Lemma 3.3 that

$$\begin{aligned} g(a-1) &= -[\{\infty\}, \{1\} \mid \{0\}, \{a\}] \\ &= -[\{\infty\}, \{1\} \mid \{0\}, \{a\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{1\}], \\ g(1-a) &= [\{\infty\}, \{1\} \mid \{0\}, \{a\}] \\ &= -[\{\infty\}, \{a\} \mid \{0\}, \{1\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{a\}]. \end{aligned}$$

Therefore, all $a, b \in E \setminus \{\infty, 0\}$, $a \neq b$, satisfy

$$g(a-b) = -[\{\infty\}, \{b\} \mid \{0\}, \{a\}] \cdot [\{\infty\}, \{0\} \mid \{1\}, \{b\}]. \tag{3.6}$$

Since g and $\hat{f}$ are both multiplicative homomorphisms, it is sufficient to show that $g(\hat{f}([H_1, H_2 \mid H_3, H_4])) = [H_1, H_2 \mid H_3, H_4]$ for all pairwise different hyperplanes $H_i = \{a_i\}$, $a_i \in E$, $i = 1, 2, 3, 4$, of $U_{2,E}$. Using Lemma 3.3, it remains to prove this for the following cases:

First, let $a_i \neq \infty, 0$ for $i = 1, 2, 3, 4$. It follows from (3.5) and (3.6) that

$$\begin{aligned} g(\hat{f}([H_1, H_2 \mid H_3, H_4])) &= g\left(\frac{(a_1-a_3)(a_2-a_4)}{(a_1-a_4)(a_2-a_3)}\right) \\ &= \frac{[\{\infty\}, H_1 \mid \{0\}, H_3] \cdot [\{\infty\}, H_2 \mid \{0\}, H_4]}{[\{\infty\}, H_1 \mid \{0\}, H_4] \cdot [\{\infty\}, H_2 \mid \{0\}, H_3]} \\ &= [H_1, H_2 \mid H_3, H_4]. \end{aligned}$$

Next, if $a_1 = \infty$ and $a_2, a_3, a_4 \neq 0$, (2.4) and (3.6) imply that

$$\begin{aligned} g(\hat{f}([H_1, H_2 \mid H_3, H_4])) &= g\left(\frac{a_2-a_4}{a_2-a_3}\right) = \frac{[\{\infty\}, H_2 \mid \{0\}, H_4]}{[\{\infty\}, H_2 \mid \{0\}, H_3]} \\ &= [H_1, H_2 \mid H_3, H_4]. \end{aligned}$$

Further, let $a_1 = 0$ and $a_2, a_3, a_4 \neq \infty$. By using (3.5) and (3.6), we get

$$\begin{aligned} g(\hat{f}([H_1, H_2 \mid H_3, H_4])) &= g\left(\frac{a_3(a_2-a_4)}{a_4(a_2-a_3)}\right) \\ &= \frac{[\{\infty\}, \{0\} \mid \{1\}, H_3] \cdot [\{\infty\}, H_2 \mid \{0\}, H_4]}{[\{\infty\}, \{0\} \mid \{1\}, H_4] \cdot [\{\infty\}, H_2 \mid \{0\}, H_3]} \\ &= [H_1, H_2 \mid H_3, H_4]. \end{aligned}$$

If $a_1 = \infty$ and $a_2 = 0$, it follows from (3.4) and (3.6) that

$$\begin{aligned} g(\hat{f}([H_1, H_2 \mid H_3, H_4])) &= g\left(\frac{a_4}{a_3}\right) = \frac{[\{\infty\}, \{0\} \mid \{1\}, H_4]}{[\{\infty\}, \{0\} \mid \{1\}, H_3]} \\ &= [H_1, H_2 \mid H_3, H_4]. \end{aligned}$$

Finally, let $a_1 = \infty$ and $a_3 = 0$. Then (3.4) and (3.6) imply that

$$\begin{aligned} g(\hat{f}([H_1, H_2 \mid H_3, H_4])) &= g\left(\frac{a_2 - a_4}{a_2}\right) = g\left(-\frac{a_4 - a_2}{a_2}\right) \\ &= \frac{[\{\infty\}, H_2 \mid \{0\}, H_4] \cdot [\{\infty\}, \{0\} \mid \{1\}, H_2]}{[\{\infty\}, \{0\} \mid \{1\}, H_2]} \\ &= [H_1, H_2 \mid H_3, H_4]. \end{aligned}$$

Therefore, $g \circ \hat{f} = \mathrm{id}_F$, which completes our proof. □

3.27 Theorem. For every cardinal number $\kappa \geq 3$ and every set E of cardinality κ the universal partial field of $U_{2,E}$ is isomorphic to $\mathbb{U}_{\kappa-3}$.

Proof. We choose three pairwise different elements $\infty, 0, 1 \in E$. Using Proposition 3.26, we obtain an embedding $\hat{f} \colon \mathbb{U}^{(0)}(U_{2,E}) \to \mathbb{Q}(E \setminus \{\infty, 0, 1\})$ whose image is equal to $\mathbb{U}_{\{E\setminus\{\infty,0,1\}\}}$. Using Corollary 2.16, we can conclude that $\mathbb{U}^{(0)}(U_{2,E}) \cong \mathbb{U}_{\{E\setminus\{\infty,0,1\}\}}$. Hence, Corollary 2.22 completes the proof. □

3.5 Matroids of rank greater or equal to 3

Using the results of the previous section, we will prove that the universal partial hyperfield of a matroid is the quotient of an $\mathcal{A}$-regular partial field for a suitable set of sets $\mathcal{A}$. We will use this method to compute the universal partial hyperfield of the ternary affine plane $\mathrm{AG}(2,3)$ and the ternary Reid geometry R_9.

Definition. We denote by $\mathcal{L}^+$ the set of those hyperlines of M that are contained in at least four distinct hyperplanes of M, and for each $L \in \mathcal{L}^+$ we denote by $\mathcal{H}_L^{(3)}$ the set of triples of pairwise different hyperplanes containing L.

Further, we call a map $\mathcal{K} \colon \mathcal{L}^+ \to \bigcup_{L\in\mathcal{L}^+} \mathcal{H}_L^{(3)}$ a *system of hypercoordinates for M* if $\mathcal{K}(L) \in \mathcal{H}_L^{(3)}$ for all $L \in \mathcal{L}^+$, write $(\infty_L, 0_L, 1_L)$ for the elements of the triple $\mathcal{K}(L)$, $L \in \mathcal{L}^+$, and set $\mathcal{A}(\mathcal{K}) \coloneqq \{A_L \mid L \in \mathcal{L}^+\}$, where

$$A_L \coloneqq \{(L, H) \mid H \in \mathcal{H}_L \setminus \{\infty_L, 0_L, 1_L\}\}.$$

3.28 Theorem. For every system of hypercoordinates $\mathcal{K}\colon \mathcal{L}^+ \to \bigcup_{L\in\mathcal{L}^+} \mathcal{H}_L^{(3)}$ for M the universal partial hyperfield of M is isomorphic to $\mathbb{U}_{\mathcal{A}(\mathcal{K})}/\!\!_*\mathcal{R}(\mathcal{K})$, where $\mathcal{R}(\mathcal{K})$ is the subgroup of $\mathbb{U}^*_{\mathcal{A}(\mathcal{K})}$ generated by -1 if M has the Fano matroid or its dual as a minor, and the elements

$$\frac{|H_1\ H_6|_{\mathcal{K}} \cdot |H_2\ H_9|_{\mathcal{K}} \cdot |H_2\ H_4|_{\mathcal{K}} \cdot |H_3\ H_7|_{\mathcal{K}} \cdot |H_3\ H_5|_{\mathcal{K}} \cdot |H_1\ H_8|_{\mathcal{K}}}{|H_1\ H_9|_{\mathcal{K}} \cdot |H_2\ H_6|_{\mathcal{K}} \cdot |H_2\ H_7|_{\mathcal{K}} \cdot |H_3\ H_4|_{\mathcal{K}} \cdot |H_3\ H_8|_{\mathcal{K}} \cdot |H_1\ H_5|_{\mathcal{K}}}$$

for hyperplanes $H_1,\dots,H_9$ satisfying the conditions (i) - (v) of (CR6), where $|H\ H'|_{\mathcal{K}} := \det(f_L(H), f_L(H'))$[8] and

$$f_L(H) := \begin{cases} (1,0)^T & \text{if } H = \infty_L,\\ (0,1)^T & \text{if } H = 0_L,\\ (1,1)^T & \text{if } H = 1_L,\\ ((L,H),1)^T & \text{else,} \end{cases}$$

for all $H, H' \in \mathcal{H}_L$, $H \neq H'$, $L \in \mathcal{L}$.

Proof. For every hyperline L and every hyperplane H of M such that $L \subseteq H$ the set $H^L := \mathrm{s}(H \setminus L)$ is a hyperplane of $\mathrm{s}(M/L)$. Thus, applying Corollary 2.22 and Proposition 3.26, the map $\hat{f}_L\colon \mathbb{U}^{(0)}(\mathrm{s}(M/L)) \to \mathbb{U}_{\mathcal{A}(\mathcal{K})}$ defined by $\hat{f}_L(0) := 0$, $\hat{f}_L(-1) := -1$, and

$$\hat{f}_L\left(\left[H_1^L, H_2^L \,\middle|\, H_3^L, H_4^L\right]\right) := \frac{|H_1\ H_3|_{\mathcal{K}} \cdot |H_2\ H_4|_{\mathcal{K}}}{|H_1\ H_4|_{\mathcal{K}} \cdot |H_2\ H_3|_{\mathcal{K}}}$$

for all $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$, $L \in \mathcal{L}(M)$, such that $H_1, H_2 \neq H_3, H_4$ is an embedding of partial hyperfields.

Let $F := \coprod_{L\in\mathcal{L}^+} \mathbb{U}^{(0)}(\mathrm{s}(M/L))$ and $\iota_L\colon \mathbb{U}^{(0)}(\mathrm{s}(M/L)) \to F$ be the canonical injection, $L \in \mathcal{L}^+$. Using Corollary 2.12, there exists a unique homomorphism of partial hyperfields $\hat{f}\colon F \to \mathbb{U}_{\mathcal{A}(\mathcal{K})}$ such that $\hat{f} \circ \iota_L = \hat{f}_L$ for each $L \in \mathcal{L}^+$. It follows from Corollary 2.12 and Proposition 3.26 that

$$\hat{f}(F) = \bigcup_{L\in\mathcal{L}^+} \hat{f}_L(\mathbb{U}^{(0)}(\mathrm{s}(M/L))) = \bigcup_{L\in\mathcal{L}^+} \mathbb{U}_{\{A_L\}} = \mathbb{U}_{\mathcal{A}(\mathcal{K})}.$$

Thus, $\hat{f}$ is surjective. Since we have $\mathbb{U}_{\{A_L\}} \cap \mathbb{U}_{\{A_{L'}\}} = \{-1, 0, 1\}$ by construction of $\mathcal{A}(\mathcal{K})$ for all hyperlines L, L' of M such that $L \neq L'$, and $-1 \neq 1$ in F, Corollary 2.12 further yields that $\hat{f}$ is an embedding. Hence, Proposition 2.14 implies that $\hat{f}$ is an isomorphism.

[8] Since we always have $L = H \cap H'$ in this setting the hyperline L on the right side can be always inferred from H and H'.

Let $\mathcal{R}(M)$ be the subgroup of F^* generated by -1 if the Fano matroid or its dual is a minor of M, and the elements of the form

$$\left[H_1^{L_3}, H_2^{L_3} \,\middle|\, H_6^{L_3}, H_9^{L_3}\right] \cdot \left[H_2^{L_1}, H_3^{L_1} \,\middle|\, H_4^{L_1}, H_7^{L_1}\right] \cdot \left[H_3^{L_2}, H_1^{L_2} \,\middle|\, H_5^{L_2}, H_8^{L_2}\right]$$

for $H_1, \ldots, H_9 \in \mathcal{H}$ and $L_1, L_2, L_3 \in \mathcal{L}$ satisfying the conditions (i) - (v) from (CR6). Since $\hat{f}$ maps $\mathcal{R}(M)$ to $\mathcal{R}(\mathcal{K})$, we get $F /\!\!_* \mathcal{R}(M) \cong \mathbb{U}_{\mathcal{A}(\mathcal{K})} /\!\!_* \mathcal{R}(\mathcal{K})$ by applying Corollary 2.17.

To complete the proof we will show that $\mathbb{U}^{(0)}(M) \cong F /\!\!_* \mathcal{R}(M)$. Combining Proposition 3.6 and Proposition 3.10 we obtain that for each $L \in \mathcal{L}^+$ the map $g_L \colon \mathbb{U}^{(0)}(\mathrm{s}(M/L)) \to \mathbb{U}^{(0)}(M)$, defined by $g_L(0) \coloneqq 0$, $g_L(-1) \coloneqq -1$, and

$$g_L\left(\left[H_1^L, H_2^L \,\middle|\, H_3^L, H_4^L\right]\right) \coloneqq [H_1, H_2 \,|\, H_3, H_4]$$

for all $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$, such that $H_1, H_2 \neq H_3, H_4$ is a well-defined homomorphism of partial hyperfields.

Using Corollary 2.12, there exists a unique homomorphism of partial hyperfields $g \colon F \to \mathbb{U}^{(0)}(M)$ such that $g \circ \iota_L = g_L$ for all $L \in \mathcal{L}^+$. It follows immediately from the definition of the inner Tutte group that its multiplicative kernel is equal to $\mathcal{R}(M)$. Hence, Proposition 2.15 implies that the map $\hat{g} \colon F /\!\!_* \mathcal{R}(M) \to \mathbb{U}^{(0)}(M)$ defined by $\hat{g}(0 \cdot \mathcal{R}(M)) \coloneqq 0$, $\hat{g}((-1) \cdot \mathcal{R}(M)) \coloneqq -1$, and

$$\hat{g}\left(\left[H_1^L, H_2^L \,\middle|\, H_3^L, H_4^L\right] \cdot \mathcal{R}(M)\right) \coloneqq [H_1, H_2 \,|\, H_3, H_4]$$

for all $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ such that $H_1, H_2 \neq H_3, H_4$, $L \in \mathcal{L}^+$, is a bijective homomorphism of partial hyperfields.

To show that $\hat{g}$ is an isomorphism it suffices to prove that $\hat{g}^{-1}$ is a homomorphism of partial hyperfields. Clearly, $\hat{g}^{-1}$ is a multiplicative homomorphism. Let $a, b \in \mathbb{T}^{(0)}(M)$ such that $b \in 1 - a$. Then Proposition and Definition 3.4 implies that there exist $L \in \mathcal{L}^+$ and pairwise different $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ such that $a = [H_1, H_2 \,|\, H_3, H_4]$ and $b = [H_1, H_3 \,|\, H_2, H_4]$. It follows that

$$\begin{aligned} \hat{g}^{-1}(b) &= \left[H_1^L, H_3^L \,\middle|\, H_2^L, H_4^L\right] \cdot \mathcal{R}(M) \\ &\in \mathcal{R}(M) - \left[H_1^L, H_2^L \,\middle|\, H_3^L, H_4^L\right] \cdot \mathcal{R}(M) = 1 - \hat{g}^{-1}(a). \end{aligned}$$

Thus, $\hat{g}^{-1}$ is a homomorphism of partial hyperfields by Lemma 2.5. □

Definition. For any set X, $n \in \mathbb{N}$, and any $Y \subseteq X^n$ such that $(x_1, \ldots, x_n) \in Y$ implies $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in Y$ for all $\pi \in S_n$, we call an equivalence relation $\approx$ on Y a *similarity relation* if for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in Y$, and $\pi \in S_n$

$$(x_1, \ldots, x_n) \approx (y_1, \ldots, y_n) \;\Rightarrow\; (x_{\pi(1)}, \ldots, x_{\pi(n)}) \approx (y_{\pi(1)}, \ldots, y_{\pi(n)}).$$

Clearly, any relation $\sim$ on Y generates a similarity relation $\approx$, which is defined by $(x_1, \dots, x_n) \approx (y_1, \dots, y_n)$ for $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in Y$ if there exists a $\pi \in S_n$ such that $(x_{\pi(1)}, \dots, x_{\pi(n)}) \simeq (y_1, \dots, y_n)$, where $\simeq$ is the equivalence relation generated by $\sim$.

We further call a similarity relation $\approx$ on $\mathcal{H}_4^+$ a *congruence relation* if for all (H_1, H_2, H_3, H_4), $(H_1', H_2', H_3', H_4') \in \mathcal{H}_4^+$, $(H_1, H_2, H_3, H_4) \approx (H_1', H_2', H_3', H_4')$ implies that $[H_1, H_2 \,|\, H_3, H_4] = [H_1', H_2' \,|\, H_3', H_4']$.

3.29 Proposition and Definition ([DW90, Theorem 2.9]).
The relation $\sim$ on $\mathcal{H}_4^+$ defined by $(H_1, H_2, H_3, H_4) \sim (H_1', H_2', H_3', H_4')$ for all (H_1, H_2, H_3, H_4), $(H_1', H_2', H_3', H_4') \in \mathcal{H}_4^+$ if there exists a $Z \in \mathcal{H}$ such that

(i) $\bigcap_{i=1}^4 H_i \neq \bigcap_{i=1}^4 H_i'$,

(ii) for all $i = 1, 2, 3, 4$ we have $H_i = H_i'$ or $H_i \cap H_i' = H_i \cap Z = H_i' \cap Z \in \mathcal{L}$,

generates a similarity relation that is a congruence relation, called *projective equivalence*, which is denoted by $\overset{pr}{\sim}$.

3.30 Corollary. Let $\mathcal{K} \colon \mathcal{L}^+ \to \bigcup_{L \in \mathcal{L}^+} \mathcal{H}_L^{(3)}$ be a system of hypercoordinates for M, $\mathcal{A}'$ be a set of sets, and $\varphi \colon \underline{\mathcal{A}(\mathcal{K})} \to \underline{\mathcal{A}'}$ be a map such that

$$R_\varphi \coloneqq \{(A, A') \in \mathcal{A}(\mathcal{K}) \times \mathcal{A}' \mid \varphi(A) \subseteq A' \text{ and } \varphi_{|A} \colon A \to A' \text{ is a bijection}\}$$

is left and right total relation, and for all $(L, H_i), (L', H_i') \in \mathcal{A}(\mathcal{K})$ such that $\varphi((L, H_i)) = \varphi((L', H_i'))$, $i = 1, 2$, we have

$$(\infty_L, 0_L, 1_L, H_1) \overset{pr}{\sim} (\infty_{L'}, 0_{L'}, 1_{L'}, H_1'),$$
$$(\infty_L, H_1, 0_L, H_2) \overset{pr}{\sim} (\infty_{L'}, H_1', 0_{L'}, H_2').$$

Then the universal partial hyperfield of M is isomorphic to $\mathbb{U}_{\mathcal{A}'} /\!\!_* \hat{\varphi}(\mathcal{R}(\mathcal{K}))$, where $\hat{\varphi} \colon \mathbb{U}_{\mathcal{A}(\mathcal{K})} \to \mathbb{U}_{\mathcal{A}'}$ is the homomorphism of partial hyperfields from Proposition 2.21, defined by $\hat{\varphi}(0) \coloneqq 0$, $\hat{\varphi}(-1) \coloneqq -1$, $\hat{\varphi}(a) \coloneqq \varphi(a)$, $\hat{\varphi}(a-1) \coloneqq \varphi(a) - 1$, $\hat{\varphi}(b - c) \coloneqq \varphi(b) - \varphi(c)$ for all $a \in \underline{\mathcal{A}(\mathcal{K})}$ and $b, c \in A \in \mathcal{A}(\mathcal{K})$, $b \neq c$.

Proof. First, note that if (H_1, H_2, H_3, H_4), $(H_1', H_2', H_3', H_4') \in \mathcal{H}_4^+$ are projectively equivalent, we have

$$\frac{|H_1\ H_3|_\mathcal{K} \cdot |H_2\ H_4|_\mathcal{K}}{|H_1\ H_4|_\mathcal{K} \cdot |H_2\ H_3|_\mathcal{K}} \cdot \frac{|H_1'\ H_4'|_\mathcal{K} \cdot |H_2'\ H_3'|_\mathcal{K}}{|H_1'\ H_3'|_\mathcal{K} \cdot |H_2'\ H_4'|_\mathcal{K}} \in \mathcal{R}(\mathcal{K}). \tag{3.7}$$

This follows from Proposition and Definition 3.29, since if $Z \in \mathcal{H}$ such that $L \coloneqq \bigcap_{i=1}^4 H_i \neq \bigcap_{i=1}^4 H_i' \eqqcolon L'$, and for all $i = 1, 2, 3, 4$ we have $H_i = H_i'$ or

$H_i \cap Z = H'_i \cap Z =: L_i \in \mathcal{L}$, we can assume without loss of generality that $H_1 = H'_1$ using Lemma 3.3 and the proof of Theorem 3.28.

Set $\tilde{L}_1 := L_2$, $\tilde{L}_2 := L'$, $\tilde{L}_3 := L$, $\tilde{H}_i := \tilde{L}_j \vee \tilde{L}_k$ for $\{i,j,k\} = \{1,2,3\}$, $\tilde{H}_{i+3} := \tilde{L}_i \vee L_3$, and $\tilde{H}_{i+6} := \tilde{L}_i \vee L_4$ for $i = 1,2,3$.

Then $\tilde{H}_1, \ldots, \tilde{H}_9$ satisfy the conditions (i) – (v) of (CR6). Therefore, as $\tilde{H}_i = H_i$ for $i = 1,2$, $\tilde{H}_4 = H_3$, $\tilde{H}_7 = H_4$, $\tilde{H}_3 = H'_2$, $\tilde{H}_5 = H'_3$, $\tilde{H}_8 = H'_4$, and $\tilde{H}_6 = \tilde{H}_9$, (3.7) is implied by Theorem 3.28.

Moreover, Proposition 2.21 yields that $\hat{\varphi}$ is an epimorphism and $\ker_*\hat{\varphi}$ is generated by the elements ab^{-1} and $(a-1)(b-1)^{-1}$ for all $a, b \in \underline{\mathcal{A}(K)}$ such that $\varphi(a) = \varphi(b)$ and the elements $(a-b)(c-d)^{-1}$ for all $a, c \in A_L$, $b, d \in A_{L'}$, $L, L' \in \mathcal{L}^+$ such that $\varphi(a) = \varphi(c)$ and $\varphi(b) = \varphi(d)$.

For $a = (L, H)$, $b = (L', H') \in \underline{\mathcal{A}(\mathcal{K})}$ such that $\varphi(a) = \varphi(b)$, and $c = (L, H_1)$, $d = (L, H_2) \in A_L$, $e = (L', H'_1)$, $\overline{f = (L'}, H'_2) \in A_{L'}$ such that $\varphi(c) = \varphi(e)$ and $\varphi(d) = \varphi(f)$, where $L, L' \in \mathcal{L}$, we obtain using (3.7) and (3.4) from the proof of Proposition 3.26 that

$$\begin{aligned}
\frac{a}{b} &= \frac{|\infty_L\ 1_L|_\mathcal{K} \cdot |0_L\ H|_\mathcal{K} \cdot |\infty_{L'}\ H'|_\mathcal{K} \cdot |0_{L'}\ 1_{L'}|_\mathcal{K}}{|\infty_L\ H|_\mathcal{K} \cdot |0_L\ 1_L|_\mathcal{K} \cdot |\infty_{L'}\ 1_{L'}|_\mathcal{K} \cdot |0_{L'}\ H'|_\mathcal{K}} \in \mathcal{R}(\mathcal{K}),\\
\frac{a-1}{b-1} &= \frac{|\infty_L\ 0_L|_\mathcal{K} \cdot |1_L\ H|_\mathcal{K} \cdot |\infty_{L'}\ H'|_\mathcal{K} \cdot |1_{L'}\ 0_{L'}|_\mathcal{K}}{|\infty_L\ H|_\mathcal{K} \cdot |1_L\ 0_L|_\mathcal{K} \cdot |\infty_{L'}\ 0_{L'}|_\mathcal{K} \cdot |1_{L'}\ H'|_\mathcal{K}} \in \mathcal{R}(\mathcal{K}),\\
\frac{(c-d)e}{c(e-f)} &= \frac{|\infty_L\ 0_L|_\mathcal{K} \cdot |H_1\ H_2|_\mathcal{K} \cdot |\infty_{L'}\ H'_2|_\mathcal{K} \cdot |H'_1\ 0_{L'}|_\mathcal{K}}{|\infty_L\ H_2|_\mathcal{K} \cdot |H_1\ 0_L|_\mathcal{K} \cdot |\infty_{L'}\ 0_{L'}|_\mathcal{K} \cdot |H'_1\ H'_2|_\mathcal{K}} \in \mathcal{R}(\mathcal{K}).
\end{aligned}$$

Hence, $\ker_*\hat{\varphi} \subseteq \mathcal{R}(\mathcal{K})$. Therefore, it follows that $\ker_*(\pi \circ \hat{\varphi}) = \mathcal{R}(\mathcal{K})$, where $\pi\colon \mathbb{U}_{\mathcal{A}'} \to \mathbb{U}_{\mathcal{A}'}/_*\hat{\varphi}(\mathcal{R}(\mathcal{K}))$ is the canonical projection. Thus, our proof is completed by applying Corollary 2.16. □

3.31 Lemma. Let M be $\mathrm{PG}(2,3) \setminus U$ or $\mathrm{PG}(2,3) \setminus ((U \cup \{p\}) \setminus \{\omega\})$, where $U := (1,0,0)^T\mathbb{F}_3 + (0,1,0)^T\mathbb{F}_3$, $\omega := (1,1,0)^T\mathbb{F}_3$, $p := (-1,-1,1)^T\mathbb{F}_3$. Further, let $o := (0,0,1)^T\mathbb{F}_3$.

Then (H_1, H_2, H_3, H_4) and (H_1, H_3, H_4, H_2) are projectivly equivalent for all $\{H_1, H_2, H_3, H_4\} = \mathcal{H}_o$[9] such that $H_1 \cap \{\omega, p\} \neq \emptyset$.

Proof. Let E be the ground set of M and $e := (1,1,1)^T\mathbb{F}_3$. To reduce the notational overload we will do all of our computations in $\mathrm{PG}(2,3)$ instead of M.

Clearly, $H_1 = o + \omega$ and $p, e \in H_1$. Since our claim is invariant under cyclic permutation of the hyperplanes H_2, H_3, H_4, we can assume without loss of generality that

$$H_2 := o + (1,0,0)^T\mathbb{F}_3,\ H_3 := o + (0,1,0)^T\mathbb{F}_3,\ H_4 := o + (1,-1,0)^T\mathbb{F}_3.$$

[9] Since the hyperlines of M are of the form $\{e\}$, $e \in E$, we often write e instead of $\{e\}$.

Further, let

$$K_2 \coloneqq e + (1,0,0)^T\mathbb{F}_3,\ K_3 \coloneqq e + (0,1,0)^T\mathbb{F}_3,\ K_4 \coloneqq e + (1,-1,0)^T\mathbb{F}_3,$$
$$Z_1 \coloneqq \omega + (0,1,1)^T\mathbb{F}_3,\ Z_2 \coloneqq \omega + (1,0,1)^T\mathbb{F}_3.$$

By straightforward computation, we obtain

$$\begin{aligned}(-1,0,1)^T\mathbb{F}_3 &\in H_2, K_4, Z_1,\ (0,1,1)^T\mathbb{F}_3 &&\in H_3, K_2, Z_1,\\ (1,-1,1)^T\mathbb{F}_3 &\in H_4, K_3, Z_1,\ (1,0,1)^T\mathbb{F}_3 &&\in H_2, K_3, Z_2,\\ (0,-1,1)^T\mathbb{F}_3 &\in H_3, K_4, Z_2,\ (-1,1,1)^T\mathbb{F}_3 &&\in H_4, K_2, Z_2.\end{aligned}$$

It follows that

$$(H_1, H_2, H_3, H_4) \overset{pr}{\sim} (H_1, K_4, K_2, K_3) \overset{pr}{\sim} (H_1, H_3, H_4, H_2). \qquad \square$$

3.32 Proposition. The universal partial hyperfield of *the ternary Reid geometry* R_9, i. e., the combinatorial geometry obtained from $\mathrm{PG}(2,3)$ by removing four points, of which exactly three of them are collinear, is isomorphic to $\mathbb{F}_3$.[10]

Proof. We continue to use the notations from the previous lemma throughout this proof and set $R_9 = \mathrm{PG}(2,3) \setminus ((U \cup \{p\}) \setminus \{\omega\})$. Further, let $\infty_o \coloneqq H_1$, $0_o \coloneqq H_2$, $1_o \coloneqq H_3$, $a_o \coloneqq H_4$.

For every $f \in E \setminus \{o\}$ let Z_f be the restriction of a hyperplane of $\mathrm{PG}(2,3)$ to R_9 that contains $(o \vee f) \cap U$ but not p. Then three of the hyperplanes $\{H_1, H_2, H_3, H_4\}$ intersect Z_f and therefore $(\infty_o, 0_o, 1_o, a_o) \overset{pr}{\sim} (\infty_f, 0_f, 1_f, a_f)$ for suitably chosen $\{\infty_f, 0_f, 1_f, a_f\} = \mathcal{H}_f$. Setting $\mathcal{K}(f) \coloneqq (\infty_f, 0_f, 1_f)$ for all $f \in E$ defines a system of hypercoordinates $\mathcal{K}\colon \mathcal{L}^+ \to \bigcup_{L \in \mathcal{L}^+} \mathcal{H}_L^{(3)}$ for R_9.

Further, let $\varphi\colon \underline{\mathcal{A}(\mathcal{K})} \to \underline{\mathcal{A}'}$ be the map defined by $\varphi(a_f) = a$ for all $f \in E$, where $\mathcal{A}' \coloneqq \overline{\{\{a\}\}}$. By construction, φ satisfies the precondition of Corollary 3.30. Therefore, $\mathbb{U}^{(0)}(R_9) \cong \mathbb{U}_{\mathcal{A}'} /\!_* \hat{\varphi}(\mathcal{R}(\mathcal{K}))$, where $\hat{\varphi}\colon \mathbb{U}_{\mathcal{A}(\mathcal{K})} \to \mathbb{U}_{\mathcal{A}'}$ is the epimorphism of partial hyperfields defined by $\hat{\varphi}(0) \coloneqq 0$, $\hat{\varphi}(-1) \coloneqq -1$, as well as $\hat{\varphi}(a_f) \coloneqq a$ and $\hat{\varphi}(a_f - 1) \coloneqq a - 1$ for all $f \in E$.

Moreover, it follows from Lemma 3.3, Proposition and Definition 3.4 and Lemma 3.31 that for $\bar{a} \coloneqq a\hat{\varphi}(\mathcal{R}(\mathcal{K}))$ we have $\bar{a}^3 = -1$ and $1 - \bar{a} = \{\bar{a}^{-1}\}$. Hence, $\mathbb{U}^{(0)}(R_9)^*$ is a cyclic group.

[10] Together with Lemma 3.19 we obtain an alternate proof for the Lemma of Reid, i. e., the ternary Reid geometry is representable over a field F if and only if F has characteristic 3, cf. [Kun90, Lemma (2.2.1)]

Now, let $e_1 := (1,0,1)^T\mathbb{F}_3$ and (in addition to the hyperplanes H_1, H_2, H_3, H_4, Z_1, Z_2 from the proof of Lemma 3.31)

$$G_1 := e_1 + (1,1,0)^T\mathbb{F}_3,\ G_2 := e_1 + (0,1,0)^T\mathbb{F}_3,\ Z_3 := e + (1,-1,0)\mathbb{F}_3.$$

Straightforward computation yields

$$\omega \in H_1, Z_1, Z_2,\ (1,0,1)^T\mathbb{F}_3 \in H_2, G_2, Z_2,\ (0,-1,1)^T\mathbb{F}_3 \in H_3, G_1, Z_2, Z_3,$$

as well as $e \in H_1, G_2, Z_3$ and $(-1,0,1)^T\mathbb{F}_3 \in H_2, Z_1, Z_3$. Therefore, we obtain

$$(H_1, H_2, H_3, H_4) \overset{pr}{\sim} (Z_1, G_2, G_1, H_4) \overset{pr}{\sim} (H_2, H_1, H_3, H_4).$$

Thus, it follows from Lemma 3.3 that $\bar{a}^2 = 1$ and hence $\bar{a} = \bar{a}^3 = -1$. This yields $1 - 1 = \{-1\}$ and $1 \neq -1$. Finally, the proof of Lemma 3.20 implies our claim. □

3.33 Lemma and Definition. The *sixth roots of unity partial field* is the restriction of $\mathbb{C}$ to the group of sixth roots of unity and is denoted by $\sqrt[6]{1}$.[11] If $\zeta \in \mathbb{C}$ is a primitive sixth root of unity, the addition of $\sqrt[6]{1}$ is characterized[12] by

$$1 - \zeta^i = \begin{cases} \emptyset & \text{if } i = 2,3,4, \\ \{0\} & \text{if } i = 0, \\ \{\zeta^{-i}\} & \text{if } i = 1,5 \end{cases}$$

for all $i = 0, \ldots, 5$.

Proof. Clearly, $1 - \zeta^0 = 1 - 1 = \{0\}$ and $1 - \zeta^3 = 1 + 1 = \emptyset$. It follows from $X^2 - X + 1 = (X - \zeta^1)(X - \zeta^5) \in \mathbb{C}[X]$ that

$$1 - \zeta^1 = \{-\zeta^2\} = \{\zeta^5\} \text{ and } 1 - \zeta^5 = \{-\zeta^{10}\} = \{\zeta^1\}.$$

Now, suppose $1 - \zeta^i \neq \emptyset$ for $i = 2$ or $i = 4$. Since $\sqrt[6]{1}$ is a partial field it would follow $1 - \zeta^2 = \{\zeta^4\}$ and $1 - \zeta^4 = \{\zeta^2\}$. As $(\zeta^4)^2 = \zeta^2$ we could conclude that $X^2 + X - 1 = (X - \zeta^2)(X - \zeta^4) \in \mathbb{C}[X]$.

But the discriminant of $X^2 + X - 1$ is equal to $1^2 - 4 \cdot (-1) = 5$, and therefore this polynomial has two distinct real roots, a contradiction. □

[11] Since we already use the symbol $\mathbb{S}$ for the hyperfield of signs, we denote the partial fields as the corresponding class of matroids, see [SW96].

[12] Cf. Proposition 2.2.

3.34 Lemma. Let $M = (E, \mathcal{H})$ be a matroid with the following property:
For all pairwise different hyperlines L_1, L_2, L_3, L, L' of M containing a common hyperpoint P such that $L, L' \nsubseteq H_i := L_j \vee L_k$ for all $\{i, j, k\} = \{1, 2, 3\}$ and H_1, H_2, H_3 are pairwise different hyperlines, there exists an $n \in \mathbb{N}_0$ and hyperlines $L = K_0, \ldots, K_n = L' \supseteq P$ of M such that for each $i = 1, \ldots, n$ and suitable $j \in \{1, 2, 3\}$ the flat $(K_{i-1} \vee K_i) \cap H_j$ is a hyperline of M.

Then $\mathbb{K}^{(0)}(M)$ is the subgroup of $\mathbb{F}^{(0)}(M)$ generated by the elements from (CR0) – (CR5) and the elements

$$\text{(CR6')} \quad (H_1, H_2 \,|\, H_3, H_4) \cdot (H_1', H_2' \,|\, H_4', H_3')$$

for all quadruples (H_1, H_2, H_3, H_4), $(H_1', H_2', H_3', H_4') \in \mathcal{H}_4^+$ such that there exists a $Z \in \mathcal{H}$ satisfying

(i) $\bigcap_{i=1}^4 H_i \neq \bigcap_{i=1}^4 H_i'$,

(ii) for all $i = 1, 2, 3, 4$ we have $H_i \cap H_i' = H_i \cap Z = H_i' \cap Z \in \mathcal{L}$ or $H_i = H_i'$.

In particular, any affine or projective geometry has this property.

Proof. Let U be the subgroup of $\mathbb{F}^{(0)}(M)$ generated by the elements from (CR0) – (CR5) and (CR6'). Using Proposition and Definition 3.29, we get $U \subseteq \mathbb{K}^{(0)}(M)$.

To prove the remaining inclusion, let $H_1, \ldots, H_9$ be hyperplanes of M satisfying the conditions (i) – (v) from (CR6). Further, let $L_i := H_j \cap H_k$ for $\{i, j, k\} = \{1, 2, 3\}$, $L := H_4 \cap H_5 \cap H_6$, $L' := H_7 \cap H_8 \cap H_9$, and $P := L_1 \cap L_2 \cap L_3$. Then there exist $n \in \mathbb{N}_0$ and hyperlines $L = K_0, \ldots, K_n = L' \supseteq P$ of M such that for each $i = 1, \ldots, n$ there exists a $j \in \{1, 2, 3\}$ such that $(K_{i-1} \vee K_i) \cap H_j \in \mathcal{L}$. We will prove that

$$g := (H_1, H_2 \,|\, H_6, H_9) \cdot (H_2, H_3 \,|\, H_4, H_7) \cdot (H_3, H_1 \,|\, H_5, H_8) \in U$$

by induction on n. If $n = 0$, then $H_{i+3} = H_{i+6}$ for all $i = 1, 2, 3$. Thus, $(H_j, H_k, H_{i+3}, H_{i+6}) \in U$ for all $\{i, j, k\} = \{1, 2, 3\}$.

Let $n \geq 1$ and set $H_{i+9} := L_i \vee K_{n-1}$ for $i = 1, 2, 3$. Then $H_1, \ldots, H_6$, H_{10}, H_{11}, H_{12} satisfy (i) – (v) from (CR6). Hence, applying the induction hypothesis, we get

$$(H_1, H_2 \,|\, H_6, H_{12}) \cdot (H_2, H_3 \,|\, H_4, H_{10}) \cdot (H_3, H_1 \,|\, H_5, H_{11}) \in U.$$

Therefore, using (CR2) we obtain

$$gU = (H_1, H_2 \,|\, H_{12}, H_9) \cdot (H_2, H_3 \,|\, H_{10}, H_7) \cdot (H_3, H_1 \,|\, H_{11}, H_8) \cdot U.$$

If there exists $\{i,j,k\} = \{1,2,3\}$ such that $L_k \subseteq K_{n-1} \vee L' =: Z$, then $(H_i, H_j \mid H_{k+9}, H_{k+6}) \in U$. Thus, we obtain that $H_i \cap Z = L_k = H_j \cap Z$, $H_{i+6} \cap Z = L' = H_{j+6} \cap Z$, and $H_{i+9} \cap Z = K_{n-1} = H_{j+9} \cap Z$. Using (CR6'), this implies

$$gU = (H_i, H_k \mid H_{j+9}, H_{j+6}) \cdot (H_k, H_j \mid H_{i+9}, H_{i+6}) \cdot U = U.$$

Otherwise, there exists $\{i,j,k\} = \{1,2,3\}$ such that $Z \cap H_k =: L'_k$ is a hyperline of M different from L_1, L_2 and L_3. Therefore, $H_k \cap Z = L'_k = H'_k \cap Z$ for $H'_k := L_k \vee L'_k$, and using (CR6'), it follows that

$$(H_l, H_k \mid H_{m+9}, H_{m+6}) \cdot (H_l, H'_k \mid H_{m+6}, H_{m+9}) \in U$$

for all $\{l,m\} = \{i,j\}$. Hence, (CR2) and (CR3) imply that

$$gU = (H_i, H_j \mid H_{k+9}, H_{k+6}) \cdot (H_j, H_i \mid H_{k+9}, H_{k+6}) \cdot U = U.$$

To prove the last sentence, let M be an affine or projective geometry. Let L_1, L_2, L_3, L, L' be hyperlines of M containing a common hyperpoint P such that $L, L' \nsubseteq H_i := L_j \vee L_k$ for all $\{i,j,k\} = \{1,2,3\}$ and H_1, H_2, H_3 are pairwise different hyperlines.

Since $\mathrm{s}(M/P)$ is an affine resp. projective plane it follows that there exists a $j \in \{1,2,3\}$ such that $(L \vee L') \cap H_j$ is a hyperplane of M. Thus, the desired property is satisfied by setting $n := 1$, $K_0 := L$, and $K_1 := L'$. □

3.35 Corollary. The universal partial hyperfield of the ternary affine plane $\mathrm{AG}(2,3)$ is isomorphic to the sixth roots of unity partial field $\sqrt[6]{1}$.

Proof. We choose a base $\{o, e_1, e_2\}$ of $\mathrm{AG}(2,3)$ and extend it to a circuit $\{o, e_1, e, e_2\}$ of $\mathrm{AG}(2,3)$. For every $p \in E \setminus \{o\}$ choose a line Z_p that is parallel to $o \vee p$ but does not contain p, set $\infty_o := o \vee e_2$, $0_o := o \vee e_1$, $1_o := o \vee e$, and define a_o by $\mathcal{H}_o = \{\infty_o, 0_o, 1_o, a_o\}$.

Then three of the hyperplanes $\{H_1, H_2, H_3, H_4\} = \mathcal{H}_p$ intersect Z_p and therefore $(\infty_o, 0_o, 1_o, a_o) \overset{pr}{\sim} (\infty_p, 0_p, 1_p, a_p)$ for suitably chosen $\{\infty_p, 0_p, 1_p, a_p\} = \mathcal{H}_p$. Setting $\mathcal{K}(p) := (\infty_p, 0_p, 1_p)$ for all $p \in E$ thus defines a system of hypercoordinates $\mathcal{K}\colon \mathcal{L}^+ \to \bigcup_{L \in \mathcal{L}^+} \mathcal{H}_L^{(3)}$. Further, let $\varphi\colon \underline{\mathcal{A}(\mathcal{K})} \to \underline{\mathcal{A}'}$ be the map defined by $\varphi(a_p) = a$ for all $p \in E$, where $\mathcal{A}' := \{\{a\}\}$.

Hence, it follows from Corollary 3.30 that $\mathbb{U}^{(0)}(\mathrm{AG}(2,3)) \cong \mathbb{U}_{\mathcal{A}'} /\!\!_{*} \hat{\varphi}(\mathcal{R}(\mathcal{K}))$, where $\hat{\varphi}\colon \mathbb{U}_{\mathcal{A}(\mathcal{K})} \to \mathbb{U}_{\mathcal{A}'}$ is the epimorphism of partial hyperfields defined by $\hat{\varphi}(0) := 0$, $\hat{\varphi}(-1) := -1$, as well as $\hat{\varphi}(a_p) := a$ and $\hat{\varphi}(a_p - 1) := a - 1$ for $p \in E$.

Moreover, as in the proof of Proposition 3.32 it follows that $\bar{a}^3 = -1$ for $\bar{a} := a\hat{\varphi}(\mathcal{R}(\mathcal{K}))$ and $1 - \bar{a} = \{\bar{a}^{-1}\}$. Thus, $\mathbb{U}^{(0)}(\mathrm{AG}(2,3))^*$ is a cyclic group.

Let $p \in E \setminus \{o\}$. For $\{H_1, H_2, H_3, H_4\} = \mathcal{H}_o$, $\{H'_1, H'_2, H'_3, H'_4\} = \mathcal{H}_p$ and $Z \in \mathcal{H}$ such that $H_1 = H'_1$ and $H_i \cap Z = H'_i \cap Z \in \mathcal{L}$ we get that Z is parallel to $o \vee p$, since each line of $\mathrm{AG}(2,3)$ contains exactly three points.

Each equivalence class of parallel lines of $\mathrm{AG}(2,3)$ consists of three lines. Thus, there are only two possibilities for the choice of Z. Since one of them, Z_p, was already used in the construction of $\mathcal{K}$ and the other one in the proof of Lemma 3.31, Lemma 3.31 and Lemma 3.34 yield that $\hat{\varphi}(\mathcal{R}(\mathcal{K}))$ is generated by $a(1-a)$.

Therefore, $1 - \bar{a}^i = \emptyset$ for $i = 2, 3, 4$, $1 - \bar{a}^0 = \{0\}$ and $1 - \overline{a}^i = \{\overline{a}^{-i}\}$ for $i = 1, 5$. Hence, our claim follows from Lemma and Definition 3.33. $\square$

4 Universal partial hyperfields of orientable matroids

In this chapter we will present an Artin-Schreier theory of partial hyperfields. Although the orderings of a partial hyperfield do not form a space of orderings in the sense of Marshall, we will construct a real reduced hyperneofield that corresponds to a prespace of orderings. Further, we will show that the orderings of the universal partial hyperfield of a matroid correspond to its orientations up to projective equivalence.

The Artin-Schreier theory of hyperfields and the concept of real reduced hyperfields where already introduced by Marshall in [Mar06] . For the theory of spaces of orderings we refer the reader to [Mar96].

4.1 Real partial hyperfields

Throughout this section let F denote a partial hyperfield.

Definition. A subset P of F is said to be an *ordering* if we have $P \cup -P = F$, $P \cap -P = \{0\}$, $P + P \subseteq P$, and $P \cdot P \subseteq P$. We denote the set of orderings of F by $X(F)$ and call F an *real* partial hyperfield if $X(F) \neq \emptyset$.

Further, a subset T of F is said to be a *preordering* of F if $F^2 \subseteq T$, $T \cdot T \subseteq T$, and $T + T \subseteq T$. A preordering T of F is called *proper* if $-1 \notin T$ and *real* if there exists an ordering P of F such that $T \subseteq P$.

The unique smallest preordering of F is $\sum F^2 := \bigcup_{n \in \mathbb{N}_0} \Sigma_n$, where $\Sigma_0 := F^2$ and $\Sigma_n := \Sigma_{n-1} + \Sigma_{n-1}$ for $n \in \mathbb{N}$. We call F *reduced* if $\sum F^2 = \{0, 1\}$ and *quasi-real* if $-1 \notin \sum F^2$.[1]

4.1 Lemma. Every ordering of F is a proper and real preordering. Further, for every preordering T of F the set $T^* := T \cap F^*$ is a subgroup of F^* such that $T^* + T^* \subseteq T^*$ if and only if T is proper.

[1]A hyperfield F is real if and only if it is quasi-real, cf. [Mar06].

Proof. First, let P be an ordering of F and $a \in F$. Then $a \in P$ or $a \in -P$. In the former case $a^2 \in P \cdot P \subseteq P$. In the latter case there exists a $b \in P$ such that $a = -b$. Therefore, $a^2 = (-b)^2 = b^2 \in P$. Hence, $1 \in F^2 \subseteq P$. In particular, $-1 \in -P$, which shows that P is proper.

Furthermore, if T is a preordering of F, we have $1 = 1^2 \in T^*$ and for all $a, b \in T^*$

$$ab^{-1} = ab(b^{-1})^2 \in T \cdot T \cdot F^2 \subseteq T.$$

Therefore, $ab^{-1} \in T^*$.

If we have additionally $0 \in a + b$, then Proposition 2.2 implies that $b = -a$ and therefore $-1 = -(-ab^{-1}) \in T$. Thus, T is not proper.

Conversely, if T is not proper, $\pm 1 \in T^*$, which implies $0 \in 1 - 1 \subseteq T^* + T^*$. □

4.2 Lemma. Let $f\colon F \to F'$ be a homomorphism of partial hyperfields, T' a preordering of F' and $T \coloneqq f^{-1}(T')$. Then

(a) T is a preordering of F,

(b) T is proper if and only if T' is proper,

(c) T is real if T' is real.

Proof. In order to prove (a), let $a, b \in T$. Then $f(a^2) = f(a)^2 \in F'^2 \subseteq T'$ and $f(ab) = f(a)f(b) \in T' \cdot T' \subseteq T'$. Further, $c \in a + b$ implies that

$$f(c) \in f(a) + f(b) \subseteq T' + T' \subseteq T'.$$

Thus, T is a preordering of F.

Since $f(-1) = -1$, $-1 \in T$ implies that $-1 \in T'$. Conversely, if $-1 \in T'$, there exists an $a \in T$ such that $f(a) = -1$. This yields $f(-a) = 1$. Therefore, $-1 = -aa^{-1} \in T \cdot T \subseteq T$, which proves (b).

Finally, to prove (c) let P' be an ordering of F' such that $T' \subseteq P'$ and $P \coloneqq f^{-1}(P')$. Using (a) and the fact that $f^{-1}(P' \cap -P') = f^{-1}(P) \cap -f^{-1}(P)$ and $f^{-1}(P' \cup -P') = f^{-1}(P') \cup -f^{-1}(P')$, we get that P is an ordering. Since $T = f^{-1}(T') \subseteq f^{-1}(P') = P$ it follows that T is real. □

4.3 Proposition. For every ordering P of F

$$\sigma_P\colon F \to \mathbb{S}, \quad a \mapsto \begin{cases} 1 & \text{if } a \in P^*, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a \in -P^* \end{cases}$$

is a homomorphism of partial hyperfields, and the map $X(F) \to \operatorname{Hom}(F, \mathbb{S})$, $P \mapsto \sigma_P$, is a bijection.

Proof. Lemma 2.1 and Lemma 4.1 imply that σ_P is a multiplicative homomorphism for every ordering P of F. Let $a, b \in F^*$ such that $b \in 1 - a$. If $a \in -P$, then $a = -c$ for a $c \in P$. Thus,

$$b \in 1 - a = 1 - (-c) = 1 + c \in P + P \subseteq P,$$

and therefore $\sigma_P(b) = 1 \in 1 + 1 = 1 - \sigma_P(a)$.

Otherwise $a \in P$ and $\sigma_P(b) \in \{-1, 0, 1\} = 1 - 1 = 1 - \sigma_P(a)$. Hence, it follows from Lemma 2.5 that σ_P is a homomorphism of partial hyperfields.

To prove that the map $X(F) \to \mathrm{Hom}(F, \mathbb{S})$, $P \mapsto \sigma_P$, is bijective, let $\sigma\colon F \to \mathbb{S}$ be a homomorphism of partial hyperfields. Then it follows from the proof of Lemma 4.2 that $P \coloneqq \sigma^{-1}(\{0, 1\})$ is an ordering of F (since obviously $\{0, 1\}$ is an ordering of $\mathbb{S}$) such that $\sigma = \sigma_P$. Further, for all orderings P and Q of F we have that $\sigma_P = \sigma_Q$ implies $P = Q$, which proves our claim. □

4.4 Proposition. Let $\iota\colon F \to \hat{F}$ be the canonical embedding into a hyperfield (cf. Theorem and Definition 2.23). For every real preordering T of F there exists a real preordering $\hat{T}$ of $\hat{F}$ such that $T = \iota^{-1}(\hat{T})$.

Proof. First, recall that $\hat{F} = Q(\underline{A})/_{*}\mathcal{R}$ and $\iota\colon F \to \hat{F}$ is defined by $\iota(0) \coloneqq 0$ and $\iota(a) \coloneqq (a, 0)\mathcal{R}$ for all $a \in F^*$, where

$$\mathcal{A} \coloneqq \{\{(a, 0)\} \mid a \in F^*\} \cup \{\{(a, b)\} \mid a, b \in F^* \text{ such that } 1 \in a + b\}$$

is algebraically independent over $\mathbb{Q}$, and $\mathcal{R}$ is the subgroup of $\mathbb{Q}(\underline{\mathcal{A}})^*$ generated by $-(-1, 0)$ and the elements

$$(a, 0)(b, 0)(ab, 0)^{-1}, \quad (c, d)(c, 0)^{-1}, \quad (1 - (c, d))(d, 0)^{-1}$$

for all $a, b, c, d \in F^*$ such that $1 \in c + d$.

We define $\hat{T}$ to be the subset of $\hat{F}$ that contains 0 such that $\hat{T} \setminus \{0\}$ is the subgroup of $\hat{F}^*$ generated by the elements $(c, 0)\mathcal{R}$, $c \in T^*$, $(1-(c, 0))\mathcal{R}$, $c \in -T^*$, as well as $(c, 0)^2\mathcal{R}$, $c \in F \setminus T$, and $(1 - (c, 0))^2\mathcal{R}$, $c \in F \setminus -T$.

Clearly, $\hat{T} \cdot \hat{T} \subseteq \hat{F}$. Since $\hat{F}^*$ is generated by -1, and the elements $(c, 0)\mathcal{R}$ and $(1 - (c, 0))\mathcal{R}$, $c \in F^*$, it follows that $\hat{F}^2 \subseteq \hat{T}$.

As Lemma 4.1 yields that T^* is a subgroup of F^* and $\mathcal{R}$ does not contain any element of the form $1 - (a, 0)$, $a \in F^*$, we have $x \in T^*$ if and only if $\iota(x) \in \hat{T}^*$ for all $x \in F^*$. Thus, $-1 \notin \hat{T}$ and $T = \iota^{-1}(\hat{T})$.

In order to prove $\hat{T} + \hat{T} \subseteq \hat{T}$, using the proof of Theorem and Definition 2.23 it is sufficient to show that $x \in -\hat{T}$ implies $1 - x \in T$, where $x = (c, 0)\mathcal{R}$ for a $c \in F^*$, or $x = (d, e)\mathcal{R}$ for $d, e \in F^*$ such that $1 \in d + e$. In both cases this follows directly from the definition of $\hat{T}$.

Therefore, $\hat{T}$ is a proper preordering of $\hat{F}$. Hence, [Mar06, Lemma 3.3] implies our claim. $\square$

4.5 Theorem. A partial hyperfield F is real if and only if $-1 \notin \sum \hat{F}^2$, where $\iota\colon F \to \hat{F}$ is the canonical embedding of F into a hyperfield (cf. Theorem and Definition 2.23).

Moreover, every proper preordering of a real partial hyperfield is real and is equal to the intersection of all orderings containing it.

Proof. It was already proven by Marshall that $\hat{F}$ is real if and only if $-1 \notin \sum \hat{F}^2$ (see [Mar06, Lemma 3.3]). Thus, if $-1 \notin \sum \hat{F}^2$, there exists an ordering $\hat{P}$ of $\hat{F}$. As in the proof of Lemma 4.2, $P := \iota^{-1}(\hat{P})$ is an ordering of F. Hence, F is real in this case.

Conversely, let F be a real partial hyperfield and T a real preordering of F. Using Proposition 4.4, there exists a real preordering $\hat{T}$ of $\hat{F}$ such that $T = \iota^{-1}(\hat{T})$. In particular, $\hat{F}$ is real and therefore $-1 \notin \sum \hat{F}^2$.

Further, let S be the intersection of all orderings that contain T. If there existed an $a \in S \setminus T$, we would have $\iota(a) \in \hat{S} \setminus \hat{T}$, where $\hat{S}$ is a real preordering of $\hat{F}$ such that $S = \iota^{-1}(\hat{S})$. Since $\hat{F}$ is a hyperfield, applying [Mar06, Proposition 3.4], would yield an ordering $\hat{P} \supseteq \hat{S}$ such that $a \notin \hat{P}$. As ι is an embedding and $P := \iota^{-1}(\hat{P})$ would be an ordering of F, as above, we would obtain that $a \notin S$, a contradiction. $\square$

4.6 Remark. By Lemma 4.1, $-1 \notin \sum F^2$ is a necessary condition for F to be real. However, unlike in the special case of hyperfields, it is not sufficient. Let G be a group of exponent 2 of order at least 8, $\varepsilon \in G \setminus \{1\}$ and set $F := G \cup \{0\}$. Then the sets $(\Delta_a)_{a \in G}$ defined by

$$\Delta_a := \begin{cases} \{1\} & \text{if } a = \varepsilon, \\ G & \text{if } a = 1, \\ G \setminus \{\varepsilon, a\} & \text{else} \end{cases}$$

for all $a \in G$ satisfy (2.1). Further, let $+$ be the unique partial hyperoperation on F such that $(1 - a) \setminus \{0\} = \Delta_a$ for all $a \in G$ and $\varepsilon = -1$. It follows that $(F, +, \cdot)$ is a quasi-real reduced partial hyperfield that does not possess any orderings.

Proof. In order to show that F is a partial hyperfield, using Proposition 2.2 and Remark and Definition 2.3, as well as the fact that $a^2 = 1$ for all $a \in G$, it suffices to show that for all $a, b \in G$ we have that $b \in \Delta_a$ implies $a \in \Delta_b \cap \Delta_{\varepsilon ab}$.

First, suppose there existed $a \in G$ and $b \in \Delta_a$ such that $a \notin \Delta_b$. Then $\Delta_\varepsilon = \{1\}$ and $\Delta_1 = G$ would imply that $a \neq \varepsilon$ and $b \neq 1$. Thus, $a = b$, leading to the contradiction $b \notin \Delta_a$.

Second, suppose there existed $a \in G$ and $b \in \Delta_a$ such that $a \notin \Delta_{\varepsilon ab}$. Then we would have either $a \neq 1$ and $\varepsilon ab = \varepsilon$, or $a \in \{\varepsilon, \varepsilon ab\}$ and $\varepsilon ab \neq 1, \varepsilon$. The former would imply that $a = b \neq 1$ and thus $b = a \notin \Delta_a$, a contradiction.

In the latter case we would obtain either that $a = \varepsilon$ and $b \neq 1, \varepsilon$, or $b = \varepsilon$ and $a \neq 1, \varepsilon$, contradicting $\varepsilon \in \Delta_g$ if and only if $g = 1$ for all $g \in G$.

Further, F is quasi-real reduced, as we have $-1 \notin \{0, 1\} = \sum F^2$. If there existed an ordering P of F, there would exist an $a \in -P \setminus \{0, -1\}$. Therefore, $G \setminus \{-1, a\} = 1 - a \subseteq P$, which would imply $P^* = F^*$, since P^* is a subgroup of F^*, a contradiction. □

Definition. Let F and F' be partial hyperfields. For any subgroup U of F^* we denote the set of homomorphisms $f\colon F \to F'$ such that $U \subseteq \ker_* f$ by $\mathrm{Hom}_U(F, F')$.

4.7 Proposition and Definition. Let T be a real preordering of F. For all $a \in F$ we define $\bar{a}_T\colon \mathrm{Hom}_{T^*}(F, \mathbb{S}) \to \mathbb{S}$, $\sigma \mapsto \sigma(a)$. Setting $\bar{a}_T \cdot \bar{b}_T := \overline{ab}_T$ and

$$\bar{a}_T + \bar{b}_T := \{\bar{c}_T \mid \sigma(c) \in \sigma(a) + \sigma(b) \text{ for all } \sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})\}$$

for all $a, b \in F$ defines a real reduced hyperneofield $Q_T(F)$, called the *canonical real reduced hyperneofield of F with respect to T*, on the set $\{\bar{a}_T \mid a \in F\}$.

Moreover, if $f\colon F \to F'$ is a homomorphism of partial hyperfields and T' is a real preordering of F' such that $T \subseteq f^{-1}(T')$, then $Q_{T,T'}(f)\colon Q_T(F) \to Q_{T'}(F')$, $\bar{a}_T \mapsto \overline{f(a)}_{T'}$ is also a homomorphism of partial hyperfields.

Proof. First, we will show that $Q_T(F)$ is a hyperneofield. Clearly, $Q_T(F)$ satisfies (PH1), (PH2), (PH4), and $\bar{a}_T, \bar{b}_T \in \bar{a}_T + \bar{b}_T$ for all $a, b \in F$.

To prove (PH3), let $a, b, c \in F$ such that $\bar{c}_T \in \bar{a}_T + \bar{b}_T$. It follows that $\sigma(c) \in \sigma(a) + \sigma(b)$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$. Lemma 2.1 and Lemma 2.5 imply that $\sigma(b) \in \sigma(c) + \sigma(-a)$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$. Therefore, $\bar{b}_T \in \bar{c}_T + \overline{-a}_T$.

To show (PH5), let $a, b, c, d \in F$ such that $\bar{d}_T \in \bar{a}_T(\bar{b}_T + \bar{c}_T)$. Then there exists an $e \in F$ such that $\bar{e}_T \in \bar{b}_T + \bar{c}_T$ and $\overline{ae}_T = \bar{d}_T$. Thus, for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ we obtain

$$\sigma(d) = \sigma(a)\sigma(e) \in \sigma(ab) + \sigma(ac).$$

Hence, $\bar{d}_T \in \bar{a}_T\bar{b}_T + \bar{a}_t\bar{c}_T$.

Further, our construction of Q_T implies that for any $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ the map $\bar{\sigma}\colon Q_T(F)$, $\bar{a}_T \mapsto \sigma(a)$ is a homomorphism of partial hyperfields, and for all $a \in F$ such that $\bar{a}_T \in 1+1$ we get $\sigma(a) \in 1+1 = \{1\}$. Hence, $\bar{a}_T = 1$ and therefore $Q_T(F)$ is real reduced.

Finally, we show that $Q_{T,T'}(f)$ is a homomorphism of partial hyperfields for any homomorphism of partial hyperfields $f\colon F \to F'$ and real preorderings T of F, T' of F' such that $T \subseteq f^{-1}(T')$. Let $a, b \in F$. If $\bar{a}_T = \bar{b}_T$, then for every $\sigma' \in \mathrm{Hom}_{T'^*}(F', \mathbb{S})$ we have $\sigma' \circ f \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ (since $T \subseteq f^{-1}(T')$) and therefore $\sigma'(f(a)) = \sigma'(f(b))$. Thus, $\overline{f(a)}_{T'} = \overline{f(b)}_{T'}$ and $Q_{T,T'}(f)$ is well-defined.

Further, $\overline{f(a)}_{T'} \cdot \overline{f(b)}_{T'} = \overline{f(ab)}_{T'}$ and $\overline{-1}_{T'} = \overline{f(-1)}_{T'}$. Therefore, $Q_{T,T'}$ is a multiplicative homomorphism.

To prove that $Q_{T,T'}(f)$ is a homomorphism of partial hyperfields, let $a, b, c \in F$ such that $\bar{c}_T \in \bar{a}_T + \bar{b}_T$. We have $\sigma(c) \in \sigma(a) + \sigma(b)$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$. Thus, $\sigma'(f(c)) \in \sigma'(f(a)) + \sigma'(f(b))$ for all $\sigma' \in \mathrm{Hom}_{T'^*}(F, \mathbb{S})$. Therefore, we obtain $\overline{f(c)}_{T'} \in \overline{f(a)}_{T'} + \overline{f(b)}_{T'}$. Hence, Lemma 2.5 yields that $Q_{T,T'}(f)$ is a homomorphism of partial hyperfields. □

4.8 Remark. By convention, we write $Q(F)$ instead of $Q_{\sum F^2}(F)$ and $Q(f)$ instead of $Q_{\sum F^2, \sum F'^2}(f)$, where $f\colon F \to F'$ is a homomorphism of real partial hyperfields.

If $g\colon F' \to F''$ is another homomorphism of real partial hyperfields, we have

$$Q(g \circ f)(a) = \overline{g(f(a))} = Q(g)(\overline{f(a)}) = Q(g)(Q(f)(a)).$$

for all $a \in F$. Thus, Q defines a functor from the category of real partial hyperfields to the category of real reduced partial hyperfields.

4.9 Lemma. Let $f\colon F \to F'$ be a homomorphism of real partial hyperfields and T and T' preorderings of F resp. F' such that $T \subseteq f^{-1}(T')$.

(a) If the map $f^*\colon \mathrm{Hom}_{T'^*}(F', \mathbb{S}) \to \mathrm{Hom}_{T^*}(F, \mathbb{S})$, $\sigma' \mapsto \sigma' \circ f$ is surjective, $Q_{T,T'}(f)$ is an embedding.

(b) If $Q_{T,T'}(f)$ is surjective, f^* is injective.

(c) If f is an epimorphism, $Q_{T,T'}(f)$ is also an epimorphism.

Proof. Let $a, b, c, u \in F$ such that $\overline{f(u)}_{T'} = 1$ and $\overline{f(c)}_{T'} \in \overline{f(a)}_{T'} + \overline{f(b)}_{T'}$. Then $\sigma(f(u)) = 1$ and $\sigma(f(c)) \in \sigma(f(a)) + \sigma(f(b))$ for all $\sigma \in \mathrm{Hom}_{T'^*}(F', \mathbb{S})$. Since f^* is surjective this implies that $\sigma(u) = 1$ and $\sigma(c) \in \sigma(a) + \sigma(b)$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$. Thus, $\overline{u}_T = 1$ and $\overline{c}_T \in \overline{a}_T + \overline{b}_T$. Hence, $Q_{T,T'}(f)$ is an embedding, which shows (a).

In order to prove (b), let $\sigma_1', \sigma_2' \in \mathrm{Hom}_{T'^*}(F', \mathbb{S})$ such that $\sigma_1' \circ f = \sigma_2' \circ f$. For each $a' \in F'$ there exists an $a \in F$ such that $\overline{a'}_{T'} = \overline{f(a)}_{T'}$, since $Q_{T,T'}(f)$ is surjective. Therefore, $\sigma_1'(a') = \sigma_1'(f(a)) = \sigma_2'(f(a)) = \sigma_2'(a')$.

To show (c), let f be an epimorphism, $U := \ker_* f$, $\tilde{F} := F/_*U$ and $\pi\colon F \to \tilde{F}$ be the canonical projection. Using Proposition 2.15, there exists a homomorphism $\tilde{f}\colon \tilde{F} \to F'$ such that $\tilde{f} \circ \pi = f$. It follows from Lemma 4.2 that $\tilde{T} := \tilde{f}^{-1}(T')$ is a real preordering of $\tilde{F}$. In particular, $\tilde{F}$ is real.

Thus, $Q_{\tilde{T},T'}(\tilde{f}) \circ Q_{T,\tilde{T}}(\pi) = Q_{T,T'}(f)$. Since $\tilde{f}$ is an isomorphism by Corollary 2.16, $Q_{\tilde{T},T'}(\tilde{f})$ is a surjective embedding by part (a) of this lemma. Therefore, Proposition 2.14 yields that $Q_{\tilde{T},T'}(\tilde{f})$ is an isomorphism. Hence, it suffices to show that $Q_{T,\tilde{T}}(\pi)$ is an epimorphism.

Set $\overline{U} := \{\overline{u} \mid u \in U\}$ and let $\overline{\pi}\colon Q_T(F) \to Q_T(F)/_*\overline{U}$ be the canonical projection. Clearly, $\overline{U}$ is a subgroup of $\ker_* Q(f)$. Thus, Proposition 2.15 implies that there exists a surjective homomorphism $g\colon Q_T(F)/_*\overline{U} \to Q(\tilde{F})$ such that $g \circ \overline{\pi} = Q_{T,\tilde{T}}(\pi)$.

Let $a_1, a_2, a_3 \in F$ such that $\overline{a_3 U} \in g(\overline{a_1}\overline{U}) + g(\overline{a_2}\overline{U}) = \overline{a_1 U} + \overline{a_2 U}$. For each $\sigma \in \mathrm{Hom}(Q(F)/_*\overline{U}, \mathbb{S})$, applying Proposition 2.15, there exists a unique $\tilde{\sigma} \in \mathrm{Hom}(\tilde{F}, \mathbb{S})$ such that $\tilde{\sigma} \circ \pi = \sigma \circ \overline{\pi} \circ h$, where $h\colon F \to Q(F)$, $a \mapsto \overline{a}$, since obviously $U \subseteq \ker_*(\sigma \circ \overline{\pi} \circ h)$.

Then $\tilde{\sigma}(a_3 U) \in \tilde{\sigma}(a_1 U) + \tilde{\sigma}(a_2 U)$ and therefore $\sigma(\overline{a_3}\overline{U}) \in \sigma(\overline{a_1}\overline{U}) + \sigma(\overline{a_2}\overline{U})$. Hence, $\overline{a_3 U} = g(\overline{a_3}\overline{U}) \in g(\overline{a_1}\overline{U} + \overline{a_2}\overline{U})$. Thus, g is strong and Proposition 2.14 implies that g is an isomorphism, which in turn yields that $Q_{T,\tilde{T}}(\pi)$ is an epimorphism, as desired. □

4.10 Proposition. Let T be a real preordering of F and $q_T\colon F \to Q_T(F)$, $a \mapsto \overline{a}_T$. Then $q_T^{-1}(\{0,1\}) = T$ and we further have

(a) q_T is an epimorphism if and only if for all $a \in F \setminus -T$ the set $T + Ta$ is a preordering of F and $F = T^* - T^*$,[2]

(b) the map $q_T^*\colon \mathrm{Hom}(Q_T(F), \mathbb{S}) \to \mathrm{Hom}_{T^*}(F, \mathbb{S})$, $\sigma \mapsto \sigma \circ q_T$, is a bijection. In particular, $Q(Q_T(F)) \cong Q_T(F)$.

(c) if F is a hyperfield, $Q_T(F)$ is also a hyperfield.

Proof. First, note that (c) was already proven by Marshall in [Mar06, Corollary 4.4].

By definition of $Q_T(F)$, for all $a \in F^*$ we have $\overline{a}_T = 1$ if and only if $\sigma(a) = 1$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$. Thus, it follows from Proposition 4.3 that $\ker_* q_T$ is

[2]In fact, in this case we have $Q_T(F) \cong F/_*T$, cf. Corollary 2.16. This condition is always satisfied if F is a hyperfield, cf. [Mar06].

the intersection of all P^*, where $P \supseteq T$ is an ordering of F. Hence, Theorem 4.5 yields that $\ker_* q_T = T^*$ and $q_T^{-1}(\{0,1\}) = T$.

In order to prove (a), we will use the fact that q_T is an epimorphism if and only if it is final, which follows from Proposition 2.14 since q_T is surjective by definition of $Q_T(F)$.

Let q_T be final. Clearly, $0 \in T^* - T^*$. For $a \in F^*$ we have $\sigma(a) \in 1 - 1$ for all $\sigma \in \mathrm{Hom}(F, \mathbb{S})$. Thus, $\overline{a}_T \in 1 - 1$ and therefore there exist $s, t \in T^*$ such that $as \in 1 - t$. Hence, $a \in s^{-1} - s^{-1}t \subseteq T^* - T^*$.

Further, let $a \in F \setminus -T$ and S the intersection of all orderings of F containing $T \cup \{a\}$. Clearly, $T + Ta \subseteq S$. If $b \in S^*$, then Proposition 4.3 implies that $\sigma(b) \in 1 + \sigma(a)$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ such that $\sigma(a) = 1$. Thus, $\overline{b}_T \in 1 + \overline{a}_T$.

Therefore, there exist $s, t \in T^*$ such that $bs \in 1 + ta$. This implies that $b \in s^{-1} + s^{-1}ta \subseteq T + Ta$. Hence, $T + Ta = S$ is a preordering of F.

Conversely, let $F = T^* - T^*$ and $T + Ta$ be a preordering of F for all $a \in F \setminus -T$. Let $a, b \in F^*$ such that $\overline{b}_T \in 1 + \overline{a}_T$. We have to show that there exist $a' \in F^*$ and $b' \in 1 + a'$ such that $\overline{a'}_T = \overline{a}_T$ and $\overline{b'}_T = \overline{b}_T$.

If $a \in -T$, then $\overline{a}_T = -1$. Let $s, t \in T^*$ such that $b \in s - t$. Set $b' := bs^{-1}$ and $a' := -ts^{-1}$. Thus, $\overline{a'}_T = -1$, $\overline{b'}_T = \overline{b}_T$ and $b' \in 1 + a'$.

Similarly, if $b \in T$, we set $a' := at^{-1}$ and $b' := st^{-1}$ for $s, t \in T^*$ such that $a \in s - t$, and if $b \in Ta$, we set $a' := at$ and $b' := as$ for $s, t \in T^*$ such that $a^{-1} \in s - t$.

Otherwise, $a \in F \setminus -T$ and $b \in F \setminus (T \cup Ta)$. Thus, $T + Ta$ is a proper preordering (if $-1 = s + ta$ for $s, t \in T$, we would get $t \in T^*$, as T is proper and hence $a = -t^{-1} - t^{-1}s \in -T$, a contradiction) and therefore real by Theorem 4.5.

Since $\sigma(b) \in 1 + \sigma(a)$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$, Proposition 4.3 implies that $b \in P$ for all orderings P of F such that $T \subseteq P$ and $a \in P$. Applying Theorem 4.5, their intersection is $T + Ta$. Hence, $b \in T + Ta$ and there exist $s, t \in T$ such that $b \in s + ta$.

Moreover, $s, t \in T^*$ (since $s = 0$ resp. $t = 0$ would imply that $b \in T$ resp. $b \in Ta$). Thus, $b' \in 1 + a'$ for $a' := tas^{-1}$ and $b' := bs^{-1}$. Further, $\overline{b'}_T = \overline{b}_T$ and $\overline{a'}_T = \overline{a}_T$, which proves (a).

In order to show (b), let $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ and $\overline{\sigma} \colon Q_T(F) \to \mathbb{S}$, $\overline{a}_T \mapsto \sigma(a)$. Since $T^* \subseteq \ker_* \sigma$, we get that $\overline{\sigma}$ is a multiplicative homomorphism. If $a, b, c \in F$ such that $\overline{c}_T \in \overline{a}_T + \overline{b}_T$, it follows that

$$\overline{\sigma}(\overline{c}_T) = \sigma(c) \in \sigma(a) + \sigma(b) = \overline{\sigma}(\overline{a}_T) + \overline{\sigma}(\overline{b}_T).$$

Hence, $\overline{\sigma} \in \mathrm{Hom}(Q_T(F), \mathbb{S})$ using Lemma 2.5.

Further, $\sigma = q_T^*(\overline{\sigma})$ and thus q_T^* is surjective. As this is also true for q_T the homomorphism $Q_{T,\{0,1\}}(q_T)\colon Q_T(F) \to Q(Q_T(F))$, which maps $\bar{a}_T$ to $\overline{q_T(a)}_{\{0,1\}}$,[3] is surjective. Therefore, applying Lemma 4.9 twice, we get that q_T^* is bijective and $Q_{T,\{0,1\}}(q_T)$ is a surjective embedding. Hence, Proposition 2.14 yields that it is an isomorphism. □

For the convenience of the reader we recall the definition of spaces of orderings from [Mar96]. We will use the first two axioms to define prespaces of orderings. It follows from [Mar96, Proof of Theorem 2.2.4] that a prespace of orderings in our sense is a prespace of orderings in the usual sense (cf. [ABR96, Chapter III, Proposition and Definition 1.1]).

Definition. Let X be a non-empty set and G be a subgroup of $\{-1,1\}^X$.[4] Further, let $\chi(G)$ be the group of quadratic characters of G and $\iota_X\colon X \to \chi(G)$ the function defined by $\iota(x)(a) := a(x)$ for all $x \in X$, $a \in G$.

For $a, b \in G$ we set

$$D\langle a, b\rangle := \{c \in G \mid c(x) \in \{a(x), b(x)\} \text{ for all } x \in X\}.$$

A tuple (X, G) is called a *prespace of orderings* if it satifies the following two axioms:

(AX1) X is non-empty, G is a subgroup of $\{-1,1\}^X$, contains the constant -1 function and separates points in X (i.e. for all $x, y \in X$, $x \neq y$, there is an $a \in G$ such that $a(x) \neq a(y)$),

(AX2) if $x \in \chi(G)$ satisfies $x(-1) = -1$ and $a, b \in \ker x \Rightarrow D\langle a, b\rangle \subseteq \ker x$, then x is in the image of ι_X.

It is called further a *space of orderings* if it additionally satisfies:

(AX3) For all $a_1, a_2, a_3 \in G$ and $b \in D\langle a_1, c\rangle$ for some $c \in D\langle a_2, a_3\rangle$, we have $b \in D\langle d, a_3\rangle$ for some $d \in D\langle a_1, a_2\rangle$.

If (X, G) and (Y, H) are (pre)spaces of orderings, a *morphism* $\alpha\colon (X, G) \to (Y, H)$ of (pre)spaces of orderings is a function $\alpha\colon X \to Y$ such that

$$H \to G, \quad a \mapsto a \circ f$$

is a group homomorphism.

[3] Proposition and Definition 4.7 yields that $Q_T(F)$ is real reduced.

[4] For $a, b \in \{-1,1\}^X$ we define $a \cdot b\colon X \to \{-1,1\}$, $x \mapsto a(x)b(x)$.

4.11 Theorem. The tuple $(\mathrm{Hom}_{T^*}(F,\mathbb{S}), Q_T(F)^*)$ is a prespace of orderings for any real preordering T of F. It is a space of orderings if and only if $Q_T(F)$ is a hyperfield.

Furthermore, if $f\colon F \to F'$ is a homomorphism of partial hyperfields and T' is a real preordering of F' such that $T \subseteq f^{-1}(T')$,

$$f^*\colon \mathrm{Hom}_{T'^*}(F',\mathbb{S}) \to \mathrm{Hom}_{T^*}(F,\mathbb{S}), \quad \sigma \mapsto \sigma \circ f$$

is a morphism of prespaces of orderings.

Proof. Set $X \coloneqq \mathrm{Hom}_{T^*}(F,\mathbb{S})$ and $G \coloneqq Q_T(F)^*$. Clearly, (X,G) satisfies (AX1) by construction. We will first show that $D\langle \overline{a}_T, \overline{b}_T\rangle = (\overline{a}_T + \overline{b}_T) \setminus \{0\}$ for all $a, b \in F^*$.

Let $c \in F^*$. By definition, $\overline{c}_T \in D\langle \overline{a}_T, \overline{b}_T\rangle$ if and only if $\overline{a}_T(\sigma) = \overline{b}_T(\sigma)$ implies that $\overline{c}_T(\sigma) = \overline{a}_T(\sigma) = \overline{b}_T(\sigma)$ for all $\sigma \in X$. Since $\overline{d}_T(\sigma) = \sigma(d)$ for all $d \in F$ and $\sigma \in X$, this is equivalent to $\sigma(c) \in \sigma(a) + \sigma(b)$ for all $x \in X$, which is by definition true if and only if $\overline{c}_T \in \overline{a}_T + \overline{b}_T$.

To prove (AX2), let $x \in \chi(G)$ such that $x(-1) = -1$ and $D\langle a, b\rangle \subseteq \ker x$ for all $a, b \in \ker x$. We define $\sigma\colon F \to \mathbb{S}$ by $\sigma(0) \coloneqq 0$ and $\sigma(a) \coloneqq x(\bar{a}_T)$ for all $a \in F^*$. Obviously, σ is a multiplicative homomorphism.

Let $a, b \in F^*$ such that $b \in 1 - a$. Then we get $\overline{b}_T \in 1 - \overline{a}_T$, which yields $\overline{b}_T \in D\langle 1, \overline{-a}_T\rangle$. Thus, if -$\sigma(a) = x(\overline{-a}_T) = 1$, it follows that $\sigma(b) = x(\overline{b}_T) = 1$. This implies that $\sigma(b) \in 1 - \sigma(a)$.

Hence, by Lemma 2.5 we get that $\sigma \in \mathrm{Hom}_{T^*}(F,\mathbb{S})$ and $\iota_X(\sigma) = x$, since $\bar{a}_T(\sigma) = \sigma(a) = x(\bar{a}_T)$ for all $a \in F^*$.

In order to show that (X,G) satisfies (AX3) if and only if $Q_T(F)$ is a hyperfield, let $a_1, a_2, a_3 \in F$. Note that if $0 \in \{a_1, a_2, a_3\}$, we have

$$\overline{a_1}_T + (\overline{a_2}_T + \overline{a_3}_T) = \overline{b_1}_T + \overline{b_2}_T = (\overline{a_1}_T + \overline{a_2}_T) + \overline{a_3}_T$$

for $\{a_1, a_2, a_3\} = \{0, b_1, b_2\}$.

Further, $0 \in \overline{a_1}_T + (\overline{a_2}_T + \overline{a_3}_T)$ yields that $-\overline{a_1}_T \in \overline{a_2}_T + \overline{a_3}_T$, and therefore $-\overline{a_3}_T \in \overline{a_1}_T + \overline{a_2}_T$. Thus, $0 \in (\overline{a_1}_T + \overline{a_2}_T) + \overline{a_3}_T$. Since $\overline{b}_T, \overline{c}_T \in \overline{b}_T + \overline{c}_T$ for all $b, c \in F$, we have $\overline{b_i}_T \in \overline{b_1}_T + (\overline{b_2}_T + \overline{b_3}_T)$, $i = 1, 2, 3$ for all $b_1, b_2, b_3 \in F$. Hence, (X,G) satisfies (AX3) if and only if $Q_T(F)$ is a hyperfield. Finally, Proposition and Definition 4.7 implies the last part of our claim. □

4.12 Proposition. Let (X,G) be a prespace of orderings. For all $a \in G \cup \{0\}$ we set $0 \cdot a \coloneqq a \cdot 0 \coloneqq 0$ and $a + 0 \coloneqq 0 + a \coloneqq \{a\}$. Further, let

$$a + b \coloneqq \begin{cases} D\langle a, b\rangle \cup \{0\} & \text{if } a = -b, \\ D\langle a, b\rangle & \text{if } a \neq -b \end{cases}$$

for all $a, b \in G$. Then $Q(X,G) \coloneqq (G \cup \{0\}, \cdot, +)$ is a real reduced hyperneofield whose orderings are of the form $P_x \coloneqq \{a \in G \mid a(x) = 1\} \cup \{0\}$, $x \in X$.

Moreover, if $\alpha\colon (X,G) \to (Y,H)$ is a morphism of prespaces of orderings, $Q(\alpha)\colon Q(Y,H) \to Q(X,G)$ defined by $Q(\alpha)(0) \coloneqq 0$ and $Q(\alpha)(a) \coloneqq a \circ \alpha$ for all $a \in G$ is a homomorphism of partial hyperfields.

Proof. By construction, $(G, -1)$ is a multiplicative structure and for all $a, b \in G$ we have $\{a, b\} \subseteq D\langle a, b\rangle \subseteq a + b$. In order to show that $Q(X,G)$ is a partial hyperfield using Remark and Definition 2.3 and the fact that $a^2 = 1$ for all $a \in G$, it suffices to prove that

$$b \in D\langle 1, -a\rangle \;\Rightarrow\; a \in D\langle 1, -b\rangle \cap D\langle 1, ab\rangle$$

for all $a, b \in G$. Let $a, b \in G$ such that $b \in D\langle 1, -a\rangle$. Thus, for all $x \in X$ we have that $a(x) = -1$ implies $b(x) = 1$. It follows directly that for all $x \in X$ we have that $b(x) = -1$ or $(ab)(x) = 1$ imply $a(x) = 1$. Hence, $a \in D\langle 1, -b\rangle \cap D\langle 1, ab\rangle$.

By construction of P_x, and definition of $+$ and $D\langle a, b\rangle$ for all $a, b \in G$, the sets P_x are orderings of $Q(X,G)$ for all $x \in X$.

It follows that $Q(X,G)$ is a real hyperneofield ($X \neq \emptyset$), which is reduced since (AX1) implies that $a(x) = 1$ for all $x \in X$ if and only if $a = 1$ and thus $D\langle 1, 1\rangle = \{1\}$.

Conversely, let P be an ordering and $\sigma_P\colon G \cup \{0\} \to \mathbb{S}$ be the corresponding homomorphism of partial hyperfields (see Proposition 4.3). Since $P^* + P^* \subseteq P^*$ by Lemma 4.1, the restriction of σ to G is a character of G satisfying the precondition of (AX2). Thus, there exists an $x \in X$ such that $P = P_x$.

It remains to show that $Q(\alpha)$ is a homomorphism of partial hyperfields for any morphism of prespaces of orderings $\alpha\colon (X,G) \to (Y,H)$. Clearly, $Q(\alpha)$ is by definition a multiplicative homomorphism.

Let $a, b \in H$ such that $b \in D\langle 1, -a\rangle$. Thus, for all $y \in Y$ we have that $a(y) = -1$ implies $b(y) = 1$. For $x \in X$ such that $a(\alpha(x)) = Q(\alpha)(a)(x) = -1$ we get $Q(\alpha)(b)(x) = b(\alpha(x)) = 1$. Hence, $Q(\alpha)(b) \in D\langle 1, -Q(\alpha)(a)\rangle$. Using Lemma 2.5, we get that $Q(\alpha)$ is a homomorphism of partial hyperfields. $\square$

4.13 Corollary. We have $Q(\mathrm{Hom}_{T^*}(F, \mathbb{S}), Q_T(F)^*) = Q_T(F)$ for every preordering T of F. Further, if $f\colon F \to F'$ is a homomorphism of partial hyperfields and T' is a real preordering of F' such that $T \subseteq f^{-1}(T')$, we have $Q(f^*) = Q_{T,T'}(f)$.

In particular, the category of real reduced hyperneofields and the category of prespaces of orderings, as well as the category of spaces of orderings, and the category of real reduced hyperfields are equivalent.[5]

[5] For the case of spaces of orderings and real reduced hyperfields this was already proven by Marshall, cf. [Mar06, p. 461].

Proof. It follows from the proof of Theorem 4.11 and Proposition 4.12 that $Q(\mathrm{Hom}_{T^*}(F, \mathbb{S}), Q_T(F)^*) = Q_T(F)$. Further, for any $a \in F$ we have

$$Q(f^*)(\overline{a}_T)(\sigma') = \overline{a}_T(f^*(\sigma')) = \overline{a}_T(\sigma' \circ f) = \sigma'(f(a)) = \overline{f(a)}_{T'}(\sigma')$$

for all $\sigma' \in \mathrm{Hom}_{T'^*}(F', \mathbb{S})$. Thus, $Q(f^*) = Q_{T,T'}(f)$. □

4.2 Orientable Matroids

In this section, we will use the characterization of orientations of matroids by Dress and Wenzel in [DW89] to show that the classes of orientations modulo projective equivalence of a matroid M correspond to the homomorphism of partial hyperfields from the universal partial hyperfield $\mathbb{U}^{(0)}(M)$ to the sign hyperfield $\mathbb{S}$.

For the theory of oriented matroids we refer the reader to [Bjö+99].

Definition. Let $M = (E, \mathcal{B})$ be a matroid of rank $n \in \mathbb{N}_0$. A *chirotope* of M is a map $\chi\colon E^n \to \{-1, 0, 1\}$ which satisfies the following three properties:

(Ch1) The bases of M are the subsets $\{e_1, \dots, e_n\}$ of E where $\chi(e_1, \dots, e_n) \neq 0$,

(Ch2) χ is alternating, that is, for all $e_1, \dots, e_n$ and $\pi \in S_n$ we have

$$\chi(e_{\pi(1)}, \dots, e_{\pi(n)}) = \operatorname{sign} \sigma \cdot \chi(e_1 \dots, e_n)$$

(Ch3) for all $e_0, \dots, e_n, f_2 \in E$ either all of the terms

$$\xi_i \coloneqq (-1)^i \chi(e_0, \dots, \widehat{e_i}, \dots, e_n)\chi(e_i, f_2, e_3, \dots, e_n), \quad i = 0, 1, 2,$$

are equal to zero or $\xi_i = -\xi_j \neq 0$ for some $i, j = 0, 1, 2$.[6]

Let $\chi\colon E^n \to \{-1, 0, 1\}$ be a chirotope of M. For any $\alpha \in \{-1, 1\}$ and any map $\eta\colon E \to \{-1, 1\}$ the map

$$\chi_{\alpha,\eta}\colon E^n \to \{-1, 0, 1\}, \quad (e_1, \dots, e_n) \mapsto \alpha \left(\prod_{i=1}^{n} \eta(e_i) \right) \chi(e_1, \dots, e_n)$$

is also a chirotope of M. Two chirotopes χ, χ' of M are called *projectively equivalent* if there exist $\alpha \in \{-1, 1\}$ and $\eta\colon E \to \{-1, 1\}$ such that $\chi' = \chi_{\alpha,\eta}$.

M is called *orientable* if there exists a chirotope $\chi\colon E^n \to \{-1, 0, 1\}$ for M.

[6] See Remark 4.15.

4.14 Proposition. Let $M = (E, \mathcal{B})$ be a matroid of rank $n \in \mathbb{N}_0$.

(a) A map $\chi\colon E^n \to \{-1, 0, 1\}$ is a chirotope of M if and only if it is a Grassmann-Plücker map for M and $\mathbb{S}$.

(b) A multiplicative homomorphism $\sigma\colon \mathbb{U}^{(0)}(M) \to \mathbb{S}$ is a homomorphism of partial hyperfields if and only if for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+$ exactly one of the values

$$\sigma([H_1, H_2 \,|\, H_3, H_4]),\ \sigma([H_1, H_3 \,|\, H_4, H_2]),\ \sigma([H_1, H_4 \,|\, H_2, H_3])$$

is equal to -1.[7]

Proof. First, note that for any map $\chi\colon E^n \to \{-1, 0, 1\}$ that satisfies (Ch1), (Ch2) is equal to (3.2) from Proposition and Definition 3.15.

Moveover, since $0 \in a_0 + a_1 + a_2$ for all $a_i \in \mathbb{S}$, $i = 0, 1, 2$, if and only if $a_i = -a_j$ for some $i, j = 0, 1, 2$, (Ch3) is equivalent to (3.3) from Proposition and Definition 3.15. Thus, (a) follows directly from Proposition and Definition 3.15.

In order to prove (b), let $\sigma\colon \mathbb{U}^{(0)}(M) \to \mathbb{S}$ be a multiplicative homomorphism and $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+$. Thus, either all or exactly one of the values

$$\sigma([H_1, H_2 \,|\, H_3, H_4]),\ \sigma([H_1, H_3 \,|\, H_4, H_2]),\ \sigma([H_1, H_4 \,|\, H_2, H_3])$$

are equal to -1. Hence, we have $\sigma([H_1, H_3 \,|\, H_2, H_4]) \in 1 - \sigma([H_1, H_2 \,|\, H_3, H_4])$ if and only if $\sigma([H_1, H_2 \,|\, H_3, H_4]) = -1$ implies that $\sigma([H_1, H_3 \,|\, H_4, H_2]) = 1$.

Since Lemma 2.5 yields that σ is a homomorphism of partial hyperfields if and only if we have $\sigma(b) \in 1 - \sigma(a)$ for all $a, b \in \mathbb{U}^{(0)}(M)^*$ such that $b \in 1 - a$, applying Proposition and Definition 3.4 completes our proof. $\square$

4.15 Remark. Let $M = (E, \mathcal{B})$ be a matroid of rank $n \in \mathbb{N}_0$.

(a) Instead of (Ch3), frequently the following equivalent axiom is used to define orientations of matroids: for $e_0, \ldots, e_n, f_2, \ldots, f_n \in E$ either all of the terms

$$\xi_i \coloneqq (-1)^i \chi(e_0, \ldots, \widehat{e_i}, \ldots, e_0)\chi(e_i, f_2, \ldots, f_n), \quad i = 0, \ldots, n,$$

are equal to zero or $\xi_i = -\xi_j \neq 0$ for some $i, j = 0, \ldots, n$.

(b) It follows from part (b) of Proposition 4.14, [DW90, Proposition 2.19] and the last sentence of Lemma 3.34 that an ordering of the universal partial hyperfield of a projective geometry is an ordering of the projective geometry in the classical sense,[8] and vice versa.

[7] A similar characterization was already given by Dress and Wenzel, cf. [DW89, Theorem 6.1].
[8] See [KK88, p. 125].

4.3 Uniform matroids

We will use the methods developed in the two previous sections to characterize the preorderings and orderings of uniform matroids of rank 2.

We will employ this characterization to investigate the canonical real reduced hyperneofield of a uniform matroid and prove that it is not a hyperfield unless the uniform matroid is regular or isomorphic to $U_{2,4}$. This proves that there exist matroids whose canonical real reduced hyperneorfield is not a hyperfield and thus the corresponding prespace of orderings is not a space of orderings in the sense of Marshall.

4.16 Lemma. Let E be a set and $\infty, 0, 1 \in E$ be pairwise different.

(a) For any partial order $\leq$ on E satisfying $0 \leq 1$ and $a \leq \infty$ for all $a \in E$, the (multiplicative) submonoid $T_{\leq}$ of $\mathbb{U}_{\{E\}}$ generated by $\mathbb{U}^2_{\{E\}}$ and the elements $a - b$ for $a, b \in E \setminus \{\infty\}$ such that $b < a$ is a real preordering of $\mathbb{U}_{\{E\}}$.

(b) For every proper preordering T of $\mathbb{U}_{\{E\}}$ there exists a partial order $\leq$ on E such that $T = T_{\leq}$ that satisfies $0 \leq 1$ and $a \leq \infty$ for all $a \in E$. Moreover, T is an ordering if and only if $\leq$ is total.

Proof. In order to prove (a), let $\leq$ be a partial order. Then $T^*_{\leq} := T_{\leq} \setminus \{0\}$ is a subgroup of $\mathbb{U}^*_{\{E\}}$ since for each $t \in T^*_{\leq}$ we have $t^{-1} = t^{-2}t$. Therefore, similar to the proof of Lemma 2.5, it suffices to show that $1 - t \in T_{\leq}$ for all $t \in -T^*_{\leq}$. Using Theorem and Definition 2.20, we have to consider two cases:

If $-\frac{a-b}{a-c} \in T_{\leq}$ for pairwise different $a, b, c \in E \setminus \{\infty\}$, it follows that $b < a < c$ or $c < a < b$. Thus, $\frac{c-b}{c-a} \in T_{\leq}$.

Else, if $-\frac{(a-c)(b-d)}{(a-d)(b-c)} \in T_{\leq}$ for pairwise different $a, b, c, d \in E \setminus \{\infty\}$, we obtain that one of the following eight statements

$$a < c < b < d,\ a < d < b < c,\ b < c < a < d,\ b < d < a < c,$$
$$c < b < d < a,\ d < b < c < a,\ c < b < d < a,\ d < a < c < b$$

is true. Therefore, $\frac{(a-b)(c-d)}{(a-d)(c-b)} \in T_{\leq}$. Hence, $T_{\leq}$ is a preordering.

To show (b), let T be a proper preordering of $\mathbb{U}_{\{E\}}$. We define $\leq$ by $a \leq \infty$ for all $a \in E$, and $a \leq b$ if $b - a \in T$ for all $a, b \in E \setminus \{\infty\}$. Clearly, $\leq$ is reflexive.

If there existed $a, b \in E \setminus \{\infty\}$ such that $a \leq b$, $b \leq a$ and $a \neq b$, we would get $-1 = -\frac{a-b}{b-a} \in T$, a contradiction.

Moreover, let $a, b, c \in E \setminus \{\infty\}$ such that $a \leq b$ and $b \leq c$. Then

$$c - a = (c - b) + (b - a) \in T + T \subseteq T$$

and therefore $a \leq c$. Hence, $\leq$ is a partial order. By construction, we have $T = T_{\leq}$.

Since obviously $T_{\leq} \subseteq T_{\leq'}$ if and only if $\leq'$ extends $\leq$, and every partial order extends to a linear order (cf. [Szp30]), we get that $T_{\leq}$ is an ordering if and only if $\leq$ is total. Furthermore, this yields that $T_{\leq}$ is real for every partial order $\leq$. □

4.17 Lemma. Let E be a set and $\infty, 0, 1 \in E$ be pairwise different.

(a) A map $\chi\colon E^2 \to \{-1, 0, 1\}$ is a chirotope of $U_{2,E}$ such that $\chi(a, \infty) = 1$ for all $a \in E \setminus \{\infty\}$ if and only if $\leq$ defined by $a \leq b \Leftrightarrow \chi(a, b) \in \{0, 1\}$ for all $a, b \in E$ is a total order on E. Furthermore, this defines a one-to-one mapping between the chirotopes of $U_{2,E}$ and the total orders on E such that $a \leq \infty$ for all $a \in E$.

(b) A multiplicative homomorphism $\sigma\colon \mathbb{U}^{(0)}(U_{2,E}) \to \mathbb{S}$ is a homomorphism of partial hyperfields if and only if there exists a total order $\leq$ on E such that

$$\sigma([\{\infty\}, \{a\} \mid \{b\}, \{c\}]) = -1 \;\Leftrightarrow\; b < a < c \text{ or } c < a < b \qquad (4.1)$$

for all pairwise different $a, b, c \in E \setminus \{\infty\}$. Moreover, this defines a one-to-one mapping between the orderings of $\mathbb{U}^{(0)}(U_{2,E})$ and the total orders $\leq$ on E such that $0 \leq 1$ and $a \leq \infty$ for all $a \in E$.

Proof. To prove (a), let $\chi\colon E^2 \to \{-1, 0, 1\}$ be a map. Obviously, $\leq$ defined by $a \leq b$ if and only if $\chi(a, b) \in \{0, 1\}$ for all $a, b \in E$ is a reflexive, antisymmetric, and total relation if and only if χ satisfies (Ch1) and (Ch2).

Clearly, for all $a \in E \setminus \{\infty\}$ we have that $\chi(a, \infty) = 1$ is equivalent to $a \leq \infty$. Thus, it remains to show that if χ satisfies (Ch1), (Ch2), and $\chi(a, \infty) = +1$ for all $a \in E \setminus \{\infty\}$, then it satisfies (Ch3) if and only if $\leq$ is transitive.

Let $e_0, e_1, e_2 \in E$ such that $e_0 \leq e_1$ and $e_1 \leq e_2$. If $\infty \in \{e_0, e_1, e_2\}$, we have $e_2 = \infty$, and thus $e_0 \leq e_2$. Otherwise, set $f_2 \coloneqq \infty$, and let

$$\xi_0 \coloneqq \chi(e_1, e_2)\chi(e_0, f_2),\ \xi_1 \coloneqq -\chi(e_0, e_2)\chi(e_1, f_2),\ \xi_2 \coloneqq \chi(e_0, e_1)\chi(e_2, f_2).$$

Straightforward computation yields that $\xi_0, \xi_2 \in \{0, 1\}$ and $\xi_1 = -\chi(e_0, e_2)$. It follows from (Ch3) that $\xi_1 \in \{0, -1\}$. Hence, $e_0 \leq e_2$.

Conversely, let $\leq$ be transitive and $e_0, e_1, e_2, f_2 \in E$. Let ξ_i, $i = 0, 1, 2$, be defined as above. We may assume without loss of generality that $f_2 \notin \{e_0, e_1, e_2\}$, since otherwise trivially $\xi_i = 0$ for all $i = 0, 1, 2$ by (Ch1), and that e_0, e_1, e_2 are pairwise different, as otherwise (Ch3) is implied by (Ch2) in this case.

Up to multiplication by -1, the ξ_i, $i = 0, 1, 2$, are invariant under permutations of the e_i, $i = 0, 1, 2$, and exchange of e_0 and f_2. Therefore, we may further assume without loss of generality that $\chi(e_i, f_2) = 1$ for all $i = 0, 1, 2$, $\chi(e_0, e_1) = 1$ and $\chi(e_1, e_2) = 1$. Thus, $e_0 \leq e_1$ and $e_1 \leq e_2$. It follows that $e_0 \leq e_2$ and hence $\xi_1 = -1 = \xi_0$.

The last sentence of (a) follows trivially.

To show (b), first note that using Proposition 3.2, Proposition and Definition 3.15, and Proposition 4.14 we get that a multiplicative homomorphism $\sigma\colon \mathbb{U}^{(0)}(U_{2,E}) \to \mathbb{S}$ is a homomorpism of partial hyperfields if and only if there exists a chirotope $\chi\colon E^2 \to \{-1, 0, 1\}$ of M such that

$$\sigma([\{a\}, \{b\} \,|\, \{c\}, \{d\}]) \coloneqq \chi(a, c)\chi(a, d)\chi(b, c)\chi(b, d)$$

for all pairwise different $a, b, c, d \in E$.

Since projectively equivalent chirotopes induce the same σ, we can assume without loss of generality that $\chi(0, 1) = \chi(a, \infty) = 1$ for all $a \in E \setminus \{\infty\}$ (otherwise replace χ by $\chi_{\alpha,\eta}$ for $\eta\colon E \to \{-1, 1\}$ and $\alpha \in \{-1, 1\}$ defined by $\eta(a) \coloneqq \chi(a, \infty)$ for all $a \in E \setminus \{\infty\}$, and $\eta(\infty) \coloneqq \alpha \coloneqq \eta(0)\eta(1)\chi(0, 1))$.

Since $\sigma([\{\infty\}, \{a\} \,|\, \{d\}, \{c\}]) = \chi(a, b)\chi(a, c)$ for all $a, b, c \in E \setminus \{\infty\}$, it follows directly from (a) that σ is a homomorphism of partial hyperfields if and only if there exists a total order $\leq$ on E satisfying (4.1), $0 \leq 1$, and $a \leq \infty$ for all $a \in E$.

As for all $a, b \in E \setminus \{\infty, 0\}$, $a \neq b$, we have $\sigma([\{\infty\}, \{0\} \,|\, \{1\}, \{a\}]) = \chi(0, a)$ and $\sigma([\{\infty\}, \{a\} \,|\, \{0\}, \{b\}]) = \chi(a, 0)\chi(a, b)$, the total order $\leq$ is uniquely determined by these conditions, which completes our proof. $\square$

To simplify the notation for the rest of this section we will write $[a, b \,|\, c, d]$ for the cross-ratio $[\{a\}, \{b\} \,|\, \{c\}, \{d\}]$ of $U_{2,E}$, where $a, b, c, d \in E$, E a set containing at least 2 elements and further $\overline{[a, b \,|\, c, d]}$ for $\overline{[\{a\}, \{b\} \,|\, \{c\}, \{d\}]}_T \in Q_T(\mathbb{U}^{(0)}(U_{2,E}))$, where the preordering T of $\mathbb{U}^{(0)}(U_{2,E})$ is known from the context.

4.18 Example. Let $E = \{\infty, 0, 1, a, b\}$ be a set of five elements. Then $U_{2,E}$ has exactly twelve orderings σ_i, $i = 1, \ldots, 12$, corresponding to the linear orders $\leq$ on E such that $e \leq \infty$ for all $e \in E$ and $0 < 1$.

Their values on the cross-ratios $\alpha \coloneqq [\infty, 0 \,|\, 1, a]$ and $\alpha' \coloneqq [\infty, 1 \,|\, a, 0]$, as well as $\beta \coloneqq [\infty, 0 \,|\, 1, b]$, $\beta' \coloneqq [\infty, 1 \,|\, b, 0]$, and $\gamma \coloneqq [\infty, a \,|\, 0, b]$ are

i	Linear order	$\sigma_i(\alpha)$	$\sigma_i(\alpha')$	$\sigma_i(\beta)$	$\sigma_I(\beta')$	$\sigma(\gamma)$
σ_1	$0 < 1 < a < b < \infty$	$+1$	-1	$+1$	-1	$+1$
σ_2	$0 < 1 < b < a < \infty$	$+1$	-1	$+1$	-1	-1
σ_3	$0 < a < 1 < b < \infty$	$+1$	$+1$	$+1$	-1	$+1$
σ_4	$0 < b < 1 < a < \infty$	$+1$	-1	$+1$	$+1$	-1
σ_5	$0 < a < b < 1 < \infty$	$+1$	$+1$	$+1$	$+1$	$+1$
σ_6	$0 < b < a < 1 < \infty$	$+1$	$+1$	$+1$	$+1$	-1
σ_7	$a < 0 < 1 < b < \infty$	-1	$+1$	$+1$	-1	-1
σ_8	$b < 0 < 1 < a < \infty$	$+1$	-1	-1	$+1$	-1
σ_9	$a < 0 < b < 1 < \infty$	-1	$+1$	$+1$	$+1$	-1
σ_{10}	$b < 0 < a < 1 < \infty$	$+1$	$+1$	-1	$+1$	-1
σ_{11}	$a < b < 0 < 1 < \infty$	-1	$+1$	-1	$+1$	-1
σ_{12}	$b < a < 0 < 1 < \infty$	-1	$+1$	-1	$+1$	$+1.$

4.19 Lemma. Let $E = \{\infty, 0, 1, a, b\}$ be a set of five elements. For pairwise different $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in E$ such that $\{a_1, a_2, a_3, a_4\} \neq \{b_1, b_2, b_3, b_4\}$ we obtain

$$1 + \alpha = \{1, \alpha\},\ 1 - \alpha = \{1, -\alpha\alpha', -\alpha, \alpha'\},\ 1 \pm \alpha\beta = \{1, \pm\alpha\beta\}$$

for $\alpha \coloneqq \overline{[a_1, a_2 \,|\, a_3, a_4]}$, $\alpha' \coloneqq \overline{[a_1, a_3 \,|\, a_4, a_2]}$, $\beta \coloneqq \overline{[b_1, b_2 \,|\, b_3, b_4]} \in Q(U_{2,E})$.

Proof. Let $Q \coloneqq Q(U_{2,E})$. Regarded as vector space over $\mathbb{F}_2$, Q^* and the group G of all monoid homomorphism $\sigma\colon Q \to \mathbb{S}$ together with the multiplication defined by $\sigma \cdot \sigma'\colon Q \to \mathbb{S}$, $x \mapsto \sigma(x)\sigma'(x)$ for all $\sigma, \sigma' \in G$ are dual to each other and have dimension 6.

Clearly, the set $1 + x$ is the annihilator of the subspace $\{\sigma \in G \mid \sigma(x) = 1\}$ of G and contains $\{1, x\}$ for all $x \in Q$. Moreover, Proposition and Definition 4.7 implies that $\alpha' \in 1 - \alpha$. Hence, also $-\alpha\alpha' \in 1 - \alpha$.

Since $|E| = 5$, the intersection of $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ contains exactly three points. Therefore, we can assume without loss of generality that $a_1 = b_1 = \infty$, $b_2 = a_2 = 0$, $b_3 = a_3 = 1$, $a_4 = a$, and $b_4 = b$.

Using Example 4.18, we compute that $\{\sigma \in G \mid \sigma(x) = 1\}$ has dimension 4 if $x = -\alpha$, and dimension 5 if $x \in \{\alpha, \pm\alpha\beta\}$. Thus, $|1 + \alpha| = 2 = |1 + \pm\alpha\beta|$ and $|1 - \alpha| = 4$. □

4.20 Lemma. Let $E = \{\infty, 0, 1, a, b\}$ be a set of five elements. Then $Q(U_{2,E})$ is not a hyperfield.

Proof. Let $\alpha \coloneqq \overline{[\infty, 0 \,|\, 1, a]}$, $\alpha' \coloneqq \overline{[\infty, 1 \,|\, a, 0]}$, and $\alpha'' \coloneqq \overline{[\infty, a \,|\, 0, 1]}$. Further, we set $\beta \coloneqq \overline{[\infty, 0 \,|\, 1, b]}$, $\beta' \coloneqq \overline{[\infty, 1 \,|\, b, 0]}$, and $\beta'' \coloneqq \overline{[\infty, b \,|\, 0, 1]}$. It follows from Lemma 3.3 that $\alpha\alpha'\alpha'' = -1 = \beta\beta'\beta''$. Therefore, Lemma 4.19 yields that

$$\begin{aligned}
(-\alpha' + 1) - \alpha\beta' &= \{1, \alpha, -\alpha', -\alpha\alpha'\} - \alpha\beta' \\
&= \{1, -\alpha\beta'\} \cup \alpha(1 - \beta') \cup (-\alpha')(1 - \alpha''\beta') \cup \alpha''(1 + \alpha'\beta') \\
&= \{1, \alpha, \alpha\beta, -\alpha', -\alpha\alpha', -\alpha\beta', -a\beta\beta'\}, \\
&\supsetneq \{1, \alpha, -\alpha', -\alpha\alpha', -\alpha\beta'\}, \\
&= \{1 - \alpha\} \cup -\alpha'(1 - \alpha''\beta') \\
&= -\alpha' + \{1, -\alpha\beta'\} = -\alpha' + (1 - \alpha\beta').
\end{aligned}$$

Hence, $Q(U_{2,E})$ is not a hyperfield. □

4.21 Lemma. Let E be a set of at least $n \in \mathbb{N}$, $n \geq 2$, elements. Then the map $\beta\colon \mathbb{U}^{(0)}(U_{n,E}) \to \mathbb{U}^{(0)}(U_{2,E})$ defined by $\beta(0) \coloneqq 0$, $\beta(-1) \coloneqq -1$ and

$$\beta([H_1, H_2 \,|\, H_3, H_4]) \coloneqq [a_1, a_2 \,|\, a_3, a_4]$$

for every hyperline L of $U_{n,E}$, pairwise different $a_1, a_2, a_3, a_4 \in E \setminus L$ and $H_i \coloneqq L \cup \{a_i\}$, $i = 1, 2, 3, 4$, is an epimorphism of partial hyperfields.

Proof. Let $\tilde{\beta}\colon \mathbb{F}^{(0)}(U_{n,E}) \to \mathbb{T}^{(0)}(U_{2,E})$ be the group homomorphism defined by $\tilde{\beta}(\varepsilon) \coloneqq \varepsilon$ and

$$\tilde{\beta}((H_1, H_2 \,|\, H_3, H_4)) \coloneqq [a_1, a_2 \,|\, a_3, a_4]$$

for every hyperline L of $U_{n,E}$, pairwise different $a_1, a_2, a_3, a_4 \in E \setminus L$ and $H_i \coloneqq L \cup \{a_i\}$, $i = 1, 2, 3, 4$.

Obviously, the kernel of $\tilde{\beta}$ is contained in $\mathbb{K}^{(0)}(U_{n,E})$ and contains the elements of (CR0) – (CR4). Further, neither the Fano matroid nor its dual is a minor of any uniform matroid.

Finally, let $H_1, \dots, H_9$ be hyperplanes of $U_{n,E}$ that satisfy (i) – (v) from (CR6). Then there exists a subset $P \subseteq E$ of $n-3$ elements and pairwise different $a_1, a_2, a_3, a, a' \in E \setminus P$ such that $H_i \coloneqq P \cup \{a_j, a_k\}$ for all $\{i, j, k\} = \{1, 2, 3\}$, $H_{i+3} \coloneqq P \cup \{a_i, a\}$ and $H_{i+6} \coloneqq P \cup \{a_i, a'\}$ for all $i = 1, 2, 3$. Thus, we obtain

$$\begin{aligned}
&\tilde{\beta}((H_1, H_2 \,|\, H_6, H_9) \cdot (H_2, H_3 \,|\, H_4, H_7) \cdot (H_3, H_1 \,|\, H_5, H_8)) \\
&= [a_1, a_2 \,|\, a, a'] \cdot [a_2, a_3 \,|\, a, a'] \cdot [a_3, a_1 \,|\, a, a'] = 1.
\end{aligned}$$

Therefore, $\ker \tilde{\beta} = \mathbb{K}^{(0)}(U_{n,E})$.

By construction we have $\beta([H_1, H_2 \,|\, H_3, H_3]) = \tilde{\beta}((H_1, H_2 \,|\, H_3, H_4))$ for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(U_{2,E})$. Hence, Lemma 2.5 and Proposition and Definition 3.4 complete our proof. □

4.22 Theorem. For any integer $n \geq 2$ and any set E that contains at least $\max\{5, n+2\}$ elements $Q(U_{n,E})$ is not a hyperfield.

Proof. Since we obtain $\mathbb{U}^{(0)}(U_{n,E}) \cong \mathbb{U}^{(0)}(U_{|E|-n,E})$ for finite E using Proposition 3.11, it suffices to consider the case $|E| \geq 2n$.

Further, if $n \geq 3$, then $|E| - (n-2) \geq n+2 \geq 5$ and therefore there exists a hyperline L of $U_{n,E}$ and a 5-element set $E' \subseteq E \setminus L$ such that $\infty, 0, 1 \in E'$ are pairwise different. Set $F := Q(U_{n,E})$, $F' := Q(U_{2,E'})$. Using Proposition 3.6 and Proposition and Definition 4.7, the map $\overline{\alpha}\colon F' \to F$ defined by $\overline{\alpha}(0) := 0$, $\overline{\alpha}(-1) := -1$, and

$$\overline{\alpha}(\overline{[a_1, a_2 \,|\, a_3, a_4]}) := \overline{[L \cup \{a_1\}, L \cup \{a_2\} \,|\, L \cup \{a_3\}, L \cup \{a_4\}]}$$

is a homomorphism of partial hyperfields.

It is now sufficient to construct a subgroup $U \subseteq F^*$ such that $\pi \circ \overline{\alpha}$ is an isomorphism, where $\pi\colon F \to F/_{*}U$ denotes the canonical projection, because, if F was a hyperfield, then Proposition 4.10 would imply that F' is a hyperfield too, contradicting Lemma 4.20.

Fix a total order $\leq$ on $E \setminus E'$. Applying Lemma 4.17, for each $\sigma \in \mathrm{Hom}(F', \mathbb{S})$ there exists a total order $\leq_\sigma$ on E' that satisfies (4.1), $0 \leq_\sigma 1$, and $a \leq_\sigma \infty$ for all $a \in E'$. We extend this to a total order on E by defining $a \leq_\sigma b$ for all $a \in E \setminus E'$ and $b \in E'$, and $a \leq_\sigma b$ if and only if $a \leq b$ for all $a, b \in E \setminus E'$. Again using Lemma 4.17, we associate a $\tilde{\sigma} \in \mathrm{Hom}(F, \mathbb{S})$ to σ.

Let $Y := \{\tilde{\sigma} \mid \sigma \in \mathrm{Hom}(F, \mathbb{S})\}$ and $U := \bigcap_{\sigma \in Y} \ker_{*}\sigma$. Then $U \cap \overline{\alpha}(F'^{*}) = \{1\}$ by construction. In order to show that $F^* = U \cdot \overline{\alpha}(F'^{*}) =: G$, let $x = \overline{[a, b \,|\, c, d]}$ for pairwise different $a, b, c, d \in E$.

As it follows from Lemma 3.3 that $\overline{[a, b \,|\, c, d]} = \overline{[\omega, b \,|\, c, d]} \cdot \overline{[\omega, a \,|\, c, d]}$ for all $\omega \in E \setminus \{a, b, c, d\}$, we can assume without loss of generality that $a = \infty$ and – using a similar argument – also that $c = 1$. Moreover, let $b \notin E'$ or $d \notin E'$ (otherwise trivially $x \in \overline{\alpha}(F'^{*})$). We distinguish four cases:

First, if $b = 0$ and $d \notin E'$, we have $d \leq_\sigma 0 \leq_\sigma 1$ for all $\sigma \in Y$. Hence, $x = \overline{[\infty, 0 \,|\, 1, d]} \in -U \subseteq G$, since $-1 \in \hat{\alpha}(F'^{*})$.

Second, let $b \in E' \setminus \{\infty, 0\}$ and $d \notin E'$. Therefore, $d \leq_\sigma 0, b$ for all $\sigma \in Y$. It follows that $\overline{[\infty, d \,|\, 0, b]} \in U$. Thus, Lemma 3.3 and the previous case imply that $\overline{[\infty, 0 \,|\, b, d]} = \overline{[\infty, 0 \,|\, b, 1]} \cdot \overline{[\infty, 0 \,|\, 1, d]} \in G$. Applying Lemma 3.3 again, we can conclude that $x \in G$.

In the case $b \notin E'$ and $d \notin E'$, since $\sigma(x) = -1$ if and only if $d \leq b$ for all $\sigma \in Y$ and $-1 \in G$, we obtain $x \in G$.

Finally, if $d \in E'$ we get $x \in U$ because $b \leq_\sigma 0, d$ for all $\sigma \in Y$.

Thus, F^* is the direct product of U and $\overline{\alpha}(F'^{*})$, which implies that $\pi \circ \overline{\alpha}$ is a multiplicative isomorphism. Moreover, our construction of Y shows that every ordering of $F/\!_*U$ is liftable to an ordering of F'. Therefore, $\pi \circ \overline{\alpha}$ is an isomorphism, which completes our proof. $\square$

4.23 Corollary. For all $n \in \mathbb{N}_0$ and sets E containing at least n elements, $Q(U_{n,E})$ is not a hyperfield, except in the following special cases:

(a) $n \in \{0, 1\}$ or $|E| \in \{n-1, n\}$, and thus is regular,

(b) $n = 2$ and $|E| = 4$.

Proof. It follows from Theorem 4.22 that $n \in \{0,1\}$, or $|E| \in \{n-1, n\}$, or $n = 2$ and $|E| = 4$ if $Q(U_{n,E})$ is a hyperfield.

Since in the first two cases $U_{2,E}$ is regular, it follows that $Q(U_{2,E}) \cong \mathbb{S}$, which is therefore a hyperfield.

In the remaining case we have $E = \{\infty, 0, 1, a\}$ and there exist obviously three linear orders $\leq$ on E such that $0 < 1$ and $0, 1, a < \infty$. Thus, Lemma 4.17 implies that we have three orderings in this case.

Furthermore, as in Example 4.18 we get that $G := Q(U_{2,E})^*$ is generated by -1, $\alpha := \overline{[\infty, 0 \,|\, 1, a]}$ and $\alpha' := \overline{[\infty, 0 \,|\, 1, a]}$, and has order 2^3. Thus, $G = \{-1, 1\}^X$ for $X := \mathrm{Hom}(\mathbb{U}^{(0)}(U_{2,E}), \mathbb{S})$. Hence, it follows from [Mar96, Theorem 3.3.2] that (X, G) is a space of orderings.[9] Therefore, Theorem 4.11 yields that $Q(U_{2,E})$ is a hyperfield. $\square$

[9] It is the SAP space with three orderings.

5 Algebraic decomposition of matroids

If the universal partial hyperfield of a matroid is a hyperfield, then necessarily every element of the inner Tutte group is a cross-ratio. We will prove that for matroids that are representable over a field this is also a sufficient condition for their universal partial hyperfield to be a hyperfield.

Furthermore, we will show that with at most one possible exception the inner Tutte group of all connected components of these matroids contains only $\{1, \varepsilon\}$ and examine conditions of minors such that a matroid satisfies this condition.

Additionally, we will examine under which conditions the universal partial hyperfield of a matroid is the coproduct of at least two partial hyperfields.

Definition. Let M be a matroid. We denote by $\mathcal{F}(M)$ the set of fundamental elements of $\mathbb{U}^{(0)}(M)$. We call M *semiartinian* if $\mathbb{U}^{(0)}(M) \subseteq \pm\mathcal{F}(M)$, *almost artinian* if $\mathbb{U}^{(0)}(M) \setminus \{-1\} \subseteq \mathcal{F}(M)$, and *artinian* if $\mathbb{U}^{(0)}(M) \subseteq \mathcal{F}(M)$.

Further, we call M *slender* if $\mathbb{U}^{(0)}(M) \subseteq \{-1, 0, 1\}$.

5.1 Remark. Clearly, every artinian matroid is almost artinian and every almost artinian matroid is semiartinian. The reverse implications are both false, as $\mathrm{AG}(2,3)$[1] is a semiartinian matroid that is not almost artinian and every slender matroid is almost artinian, but artinian if and only if it is not regular.

However, if $1 = -1$ in $\mathbb{U}^{(0)}(M)$, these three properties are equal to each other.

5.2 Proposition. Let M be a matroid representable over a field F.

(a) M is artinian if and only if $\mathbb{U}^{(0)}(M)$ is isomorphic to a subfield of F.

(b) If M is almost artinian and one of the following conditions is satisfied, then M is regular or artinian:[2]

  (i) F is of characteristic $\neq 2, 3$,

  (ii) F is the field of two or three elements,

[1] See Corollary 3.35.

[2] It is unknown to us if there exists an almost artinian matroid that is neither regular nor artinian.

(iii) F is of characteristic 3 and M is orientable.

Proof. Since M is representable over F, Theorem and Definition 3.16 implies the existence of a homomorphism $f\colon \mathbb{U} \to F$ of partial hyperfields, where $\mathbb{U} \coloneqq \mathbb{U}^{(0)}(M)$. Clearly, if $\mathbb{U}$ is isomorphic to a subfield of F, it is itself a field. Hence, M is artinian.

Conversely, if M is artinian, Lemma 3.19 yields that f is injective. Furthermore, as F is a field, we get $|f(a+b)| \leq |f(a)+f(b)| \leq 1$ for all $a, b \in \mathbb{U}$. Because M is artinian, this implies $|a + b| = 1$. Thus, f is strong and Proposition 2.14 yields that f is an embedding. This proves (a).

In order to prove (b), let M be almost artinian but not regular. We will show that M is artinian if one of the conditions (i) – (ii) is satisfied and there are no such matroids if (iii) is satisfied. So let one of (i) – (iii) be fulfilled.

For the particular case $F = \mathbb{F}_2$ of condition (ii), we have shown this already in the proof of Corollary 3.21. Thereby, for the rest of the proof we may assume that F contains at least three elements.

Suppose f was not injective. Applying Lemma 3.19, we would get that $\ker_* f = \{1, -1\}$. Hence, we would have $-1 = 1$ in F and therefore F would be a field of characteristic 2 which is not isomorphic to $\mathbb{F}_2$, a contradiction.

If F is not of characteristic 2 or 3, by injectivity of f and Lemma 3.20 there exists an $a \in \mathbb{U} \setminus \{-1, 0, 1\}$. Since M is almost artinian and $-a \notin \{-1, 0, 1\}$, there exist $b \in 1 - a$, $c \in 1 + a$, and $d \in b + c$ (otherwise we would have $b = c$ and therefore $f(a), f(-a) \in 1 - f(b)$ by Lemma 2.5, which would imply $a = -a$, since f is injective and F is a field). It follows by Lemma 2.5 that

$$f(d) \in f(b) + f(c) = (1 - f(a)) + (1 + f(a)) = 1 + 1.$$

Since F is not of characteristic 2 or 3, $f(d) \notin \{-1, 0, 1\}$ and we get $d \notin \{-1, 0, 1\}$ using the injectivity of f. So there exists an $e \in 1 - d$. But $f(e) = 1 - f(d) = -1$ and therefore $e = -1$. Thus, Lemma 2.1 yields that M is artinian in this case.

Now, let F be a field of characteristic 3. If $F \cong \mathbb{F}_3$, then our claim directly follows from Lemma 3.20. Otherwise, suppose M were orientable. Using Theorem and Definition 3.16 and Proposition 4.14, we would get a homomorphism of partial hyperfields $\sigma\colon \mathbb{U} \to \mathbb{S}$.

It now suffices to show that there would exist $a \in \mathbb{U} \setminus \{-1, 0, 1\}$ and $b \in 1 - a$ such that $\sigma(a) = 1$ and $\sigma(b) = -1$. Then $b = -a$ would imply that $b \in 1 + b$ and thus $f(b) = 1 + f(b)$, a contradiction.

Thus, $b^{-1}a \neq -1$ and there exists a $c \in b - a = b(1 - b^{-1}a)$ for which we would get $\sigma(c) \in \sigma(b) + \sigma(-a) = \{-1\}$. Hence, $\sigma(c) = -1$. On the contrary we would have

$$f(c) \in f(b) - f(a) = (1 - f(a)) - f(a) = 1 + f(a) = f(1 + a).$$

This would imply that $c \in 1 + a$ and therefore $\sigma(c) \in 1 + \sigma(a) = \{1\}$, a contradiction.

In order to construct such $a, b \in \mathbb{U}$, let $x \in \mathbb{U} \setminus \{-1, 0, 1\}$ (exists using Lemma 3.20 and the fact that f is injective). Since $-x \notin \{-1, 0, 1\}$ we can assume without loss of generality that $\sigma(x) = 1$.

Using the fact that M is almost artinian, there exists a $y \in 1-x$. If $\sigma(y) = -1$, we set $a := x$ and $b := y$. Otherwise, we set $a := x^{-1}$ and choose a $b \in 1 - a$. Clearly, $\sigma(a) = 1$. Further, Lemma 2.5 implies that

$$f(b) \in 1 - f(a) = -f(x)^{-1}(1 - f(x)) = f(-x^{-1}y).$$

Thus, $b = -x^{-1}y$ and $\sigma(b) = -1$. □

5.1 Decomposition of the universal partial hyperfield

In Proposition 3.8 we have shown that the universal partial hyperfield of a direct sum of matroids is the coproduct of the universal partial hyperfields of these matroids.

We now will characterize geometrically when we can write the universal partial hyperfield of a matroid M as a coproduct of two partial hyperfields that are both not contained in $\{-1, 0, 1\}$. Since we will show that this is not possible if M is semiartinian, this enables us to determine the possible universal partial hyperfields of connected components of M in this case.

Definition. We call a partial hyperfield F *decomposable* if there exist partial hyperfields F_1 and F_2 such that $F \cong F_1 \oplus F_2$ and $F_i \ncong F'$ for all $i = 1, 2$ and $F' \in \{\mathbb{U}_0, \mathbb{F}_3, \mathbb{S}, \mathbb{W}, \mathbb{F}_2, \mathbb{K}\}$,[3] and else we call F *indecomposable*. Further, we call a matroid M *algebraically decomposable* resp. *algebraically indecomposable* if $\mathbb{U}^{(0)}(M)$ is decomposable resp. indecomposable.

Clearly, every slender matroid is algebraically indecomposable.

5.3 Lemma. Let $(F_i)_{i \in I}$ be a family of partial hyperfields, $F := \coprod_{i \in I} F_i$ and $\iota_i \colon F_i \to F$ the canonical injection for $i \in I$ as in Corollary 2.12. Then

(a) $\mathcal{F}(F) = \bigcup_{i \in I} \iota_i (\mathcal{F}(F_i))$,

(b) $F = \bigcup_{i \in I} \iota_i(F_i)$ if and only if there exists an $i \in I$ such that $F_j \subseteq \{-1, 0, 1\}$ for all $j \in I$, $j \neq i$.

[3] See Remark and Definition 2.11 and Lemma 3.20.

Proof. Clearly, Lemma 2.5 implies that $\bigcup_{i\in I} \iota_i(\mathcal{F}(F_i)) \subseteq \mathcal{F}(F)$.

If $a \in \mathcal{F}(F) \setminus \{0,1\}$, then there exists a $b \in F^*$ such that $b \in 1-a$. By Proposition and Definition 2.9 and Corollary 2.12, there exist $i \in I$, $a_i \in \iota_i^{-1}(a)$ and $b_i \in (1-a_i) \cap \iota_i^{-1}(b)$. Thus, $a \in \iota_i(\mathcal{F}(F_i))$, which proves (a).

Moreover, if $j \in I$ such that $F_i \subseteq \{-1,0,1\}$ for all $i \in I$, $i \neq j$, it follows that $F = \iota_j(F_j) \subseteq \bigcup_{i\in I} \iota_i(F_i)$.

Conversely, let $F \subseteq \bigcup_{i\in I} \iota_i(F_i)$. Further, for each $i \in I$ let $\kappa_i \colon F_i \to \mathbb{K} \oplus F_i$ and $\lambda_i \colon \mathbb{K} \to \mathbb{K} \oplus F_i$ be the canonical injections. Since $1+1 = \{0,1\}$ in $\mathbb{K}$, we have $\{0,1\} = \lambda_i(1+1) \subseteq 1+1$ in $\mathbb{K} \oplus F_i$. Therefore, Lemma 2.5 yields that the map $\mu_i \colon F_i \to \mathbb{K} \oplus F_i$ defined by $\mu_i(0) \coloneqq 0$ and $\mu_i(a_i) \coloneqq 1$ for all $a_i \in F_i^*$ is a homomorphism of partial hyperfields.

Moreover, Corollary 2.12 implies that for each $i \in I$ there exists a unique homomorphism $g_i \colon F \to \mathbb{K} \oplus F_i$ such that $g_i \circ \iota_i = \kappa_i$ and $g_i \circ \iota_j = \mu_j$ for all $j \in I \setminus \{i\}$.

Now, let $J \subseteq I$ be a two element set and $a_i \in F_i^*$, $i \in J$. Then there exist $k \in I$ and $a_k \in F_k^*$ such that $\iota_i(a_i)\iota_j(a_j) = \iota_k(a_k)$ for $\{i,j\} = J$. If $k \neq i,j$, applying g_k, we get $a_k \in \{-1,1\}$, as $\ker \lambda_k = \{-1,1\}$ by construction of the coproduct. Using g_i or g_j yields similarly $a_i, a_j \in \{-1,1\}$.

Otherwise, $k = i$ or $k = j$. In the former case, we get $a_i a_k^{-1}, a_j \in \{-1,1\}$ and in the latter case $a_i, a_j a_k^{-1} \in \{-1,1\}$ using g_i and g_j. Thus, it is not possible to have $F_l \subsetneq \{-1,0,1\}$ for both $l \in J$. □

5.4 Lemma. A partial hyperfield F is indecomposable if $F \subseteq \pm\mathcal{F}(F)$. In particular, a semiartinian matroid M is algebraically indecomposable.

Proof. Let $F \cong F_1 \oplus F_2$ for partial hyperfields F_1, F_2 and denote by $\iota_i \colon F_i \to F$ the canonical injection, $i = 1,2$. Then Lemma 5.3 yields that

$$\pm\mathcal{F}(F) = (\iota_1(\pm\mathcal{F}(F_1)) \cup \iota_2(\pm\mathcal{F}(F_2))) \subseteq \iota_1(F_1) \cup \iota_2(F_2).$$

Applying Lemma 5.3 again, $F \subseteq \pm\mathcal{F}(F)$ implies $F_i \subseteq \{-1,0,1\}$ for an $i \in \{1,2\}$, which proves our claim. □

5.5 Theorem. Let M be a matroid.

(a) M is algebraically indecomposable if and only if there exists a slender matroid S and a connected algebraically indecomposable matroid N such that $M \cong S \oplus N$.

(b) M is semiartinian if and only if there exists a slender matroid S and a connected semiartinian matroid N such that $M \cong S \oplus N$.

Proof. First, if M is slender, then $M \cong M \oplus N$, where N is the empty matroid and in this case (a) and (b) follow trivially, since every slender matroid is semiartinian and therefore algebraically indecomposable by Lemma 5.4.

Otherwise, Corollary 2.12 and Proposition 3.8 imply that at least one of the connected components $M_1, \dots, M_k$ of M is not slender. We may assume without loss of generality that M_1 is such a component. Set $N \coloneqq M_1$ and $S \coloneqq \bigoplus_{i=2}^k M_i$. Further, let $\mathbb{U} \coloneqq \mathbb{U}^{(0)}(N)$ and $F \coloneqq \mathbb{U}^{(0)}(S)$. Using Proposition 3.8, we have $\mathbb{U}^{(0)}(M) \cong \mathbb{U} \oplus F$. Let $\upsilon\colon \mathbb{U} \to \mathbb{U}^{(0)}(M)$ and $\iota\colon F \to \mathbb{U}^{(0)}(M)$ be the canonical injections.

Clearly, $\mathbb{U}^{(0)}(M)$ is decomposable if S is not slender and therefore M is not semiartinian by Lemma 5.3. So let S be slender. Then $\mathbb{U}^{(0)}(M) = \upsilon(\mathbb{U})$.

Now, (b) follows quickly as Lemma 5.3 yields that M is semiartinian if and only if N is semiartinian. In order to prove (a), it suffices to show that M is algebraically indecomposable if and only if N is algebraically indecomposable.

If N is algebraically decomposable, there exist partial hyperfields F_1, F_2 such that $F_i \not\subseteq \{-1, 0, 1\}$, $i = 1, 2$ and $\mathbb{U} \cong F_1 \oplus F_2$. Thus, it follows from Lemma 5.3 that $F_2 \oplus F \not\subseteq \{-1, 0, 1\}$ and $\mathbb{U}^{(0)}(M) \cong F_1 \oplus (F_2 \oplus F)$. Hence, M is algebraically decomposable.

Conversely, if M is algebraically decomposable, there exist partial hyperfields F_1 and F_2 such that $\mathbb{U}^{(0)}(M) \cong F_1 \oplus F_2$ and $F_i \not\subseteq \{-1, 0, 1\}$, $i = 1, 2$. Further, let $\iota_i\colon F_i \to \mathbb{U}^{(0)}(M)$ be the canonical injection, $i = 1, 2$. Then $\mathbb{U}^{(0)}(M) = \upsilon(\mathbb{U})$ implies $\iota_i(F_i) \subseteq \upsilon(\mathbb{U})$, $i = 1, 2$.

Further, let F_i' be the initial partial hyperfield with respect to the set inclusion $\iota_i'\colon \upsilon^{-1}(\iota_i(F_i)) \to \mathbb{U}$, $i = 1, 2$. In order to show $\mathbb{U} \cong F_1' \oplus F_2'$ and thus N is algebraically decomposable, using Corollary 2.12, it is sufficient to show that for all homomorphisms $f_i\colon F_i' \to F'$, $i = 1, 2$, into a partial hyperfield F', there exists a unique homomorphism $f\colon \mathbb{U} \to F'$ such that $f \circ \iota_i = f_i$, $i = 1, 2$.

If $f\colon \mathbb{U} \to F'$ is such an f, then using Corollary 2.12, for every $a \in \mathbb{U}$ there exist $a_i \in F_i$, $i = 1, 2$, such that $\upsilon(a) = \iota_1(a_1)\iota_2(a_2)$. Therefore, $a = a_1'a_2'$ for suitable $a_i' \in F_i'$, $i = 1, 2$, and $f(a) = f(\iota_1'(a_1')\iota_2'(a_2')) = f_1(a_1')f_2(a_2')$. Hence, there exists at most one such f.

Conversely, setting $f(a) = f_1(a_1')f_2(a_2')$ if $a = a_1'a_2'$ for $a_i' \in F_i'$, yields a well-defined multiplicative homomorphism $f\colon \mathbb{U} \to F'$, since if $a \neq 0$ and $a = a_1'a_2' = b_1'b_2'$ for $a_i', b_i' \in F_i'$, $i = 1, 2$, it follows from $F_1' \cap F_2' = \{-1, 0, 1\}$ that $a_i' = b_i'$, $i = 1, 2$, or $a_i' = -b_i'$, $i = 1, 2$. Therefore, $f_1(a_1')f_2(a_2') = f_1(b_1')f_2(b_2')$.

If $a, b \in \mathbb{U}^*$ and $b \in 1-a$, we get $\upsilon(b) \in 1-\upsilon(a)$. Applying Corollary 2.12, there exist $i \in \{1, 2\}$ and $a_i, b_i \in F_i^*$ with $\iota_i(a_i) = \upsilon(a)$, $\iota_i(b_i) = \upsilon(b)$ and $b_i \in 1 - a_i$. It follows that $a, b \in F_i'$ and therefore $f(b) = f_i(b) \in 1 - f_i(a) = 1 - f(a)$. Using Lemma 2.5, f is a homomorphism of partial hyperfields, completing our proof. □

5.6 Corollary. Let M be a matroid such that $1 \neq -1$ in $\mathbb{U}^{(0)}(M)$.

(a) M is almost artinian if and only if there exists a slender matroid S and a connected almost artinian matroid N such that $M \cong S \oplus N$.

(b) M is artinian if and only if there exists a slender matroid S and a connected almost artinian matroid N such that $M \cong S \oplus N$ and at least one of S or N is artinian.

Proof. Since every almost artinian or artinian matroid is semiartinian, it follows from Theorem 5.5 that M is neither almost artinian nor artinian, if M is not the direct sum of a slender and a connected semiartinian matroid.

Thus, it suffices to examine the case $M = S \oplus N$ where S is a slender matroid and N a connected semiartinian matroid. Set $\mathbb{U} := \mathbb{U}^{(0)}(N)$ and $F := \mathbb{U}^{(0)}(S)$, and let $\upsilon\colon \mathbb{U} \to \mathbb{U}^{(0)}(M)$ and $\iota\colon F \to \mathbb{U}^{(0)}(M)$ denote the canonical injections, as in the proof of Theorem 5.5. Further, note that $F \subseteq \{-1, 0, 1\}$ and Lemma 5.3 yields $\mathbb{U}^{(0)}(M) = \upsilon(\mathbb{U})$.

Since $-1 \neq 1$ in $\mathbb{U}^{(0)}(M)$, υ is bijective and ι injective. Therefore, Lemma 5.3 implies that $\upsilon(\mathcal{F}(\mathbb{U}) \setminus \{-1\}) = \mathcal{F}(M) \setminus \{-1\}$. Thus, M is almost artinian if and only if N is almost artinian, which proves (a).

Finally, since $-1 \in \mathcal{F}(M)$ if and only if $-1 \in \mathcal{F}(\mathbb{U})$ or $-1 \in \mathcal{F}(F)$, we obtain that M is artinian if and only if at least one of S or N is artinian. This proves (b). □

5.7 Example. The matroid $M := \mathrm{AG}(2,3) \oplus \mathrm{PG}(2,2)$ is semiartinian by Theorem 5.5. Since the Fano matroid $\mathrm{PG}(2,2)$ is a minor of M we have $-1 = 1$ in $\mathbb{U}^{(0)}(M)$. Thus, M is artinian.

Both $\mathrm{AG}(2,3)$ and $\mathrm{PG}(2,2)$ are representable over $\mathbb{F}_4$ and therefore classical matroid theory, or Proposition 3.8 and Theorem and Definition 3.16 imply that M is also representable over $\mathbb{F}_4$. Hence, applying Proposition 5.2, we get $\mathbb{U}^{(0)}(M) \cong \mathbb{F}_4$.

5.8 Remark. Unfortunately, a characterization of the universal partial hyperfields of the connected components of matroids, whose universal partial hyperfield is a hyperfield, similar to Theorem 5.5 and Corollary 5.6, does not exist.

Proof. Let F be a hyperfield. Since $\mathbb{U}_0 \oplus F \cong F$ and $\mathbb{F}_2 \oplus F \cong F/\!\!_*\langle -1 \rangle$, $F' \oplus F$ is a hyperfield if $F' \in \{\mathbb{U}_0, \mathbb{F}_2\}$.

This is not necessarily the case for $F' \in \{\mathbb{F}_3, \mathbb{S}, \mathbb{W}, \mathbb{K}\}$. If F is field of characteristic 0, then $1 + (1 + \iota(2)) = 1 + \iota(3) = \{\iota(4)\}$, where $\iota\colon F \to F' \oplus F$ is the canonical injection, but $\iota(3) \in 1 + \iota(2) \subseteq (1+1) + \iota(2)$ if $F' \in \{\mathbb{S}, \mathbb{W}, \mathbb{K}\}$, or $1 \in -1 + \iota(2) \subseteq (1+1) + \iota(2)$ if $F' \in \{\mathbb{F}_3, \mathbb{W}\}$. □

5.9 Lemma. The uniform matroid $U_{n,E}$ for any set E containing at least $n+2$ elements, $n \in \mathbb{N}$, $n \geq 2$, is not semiartinian.

In particular, a uniform matroid is semiartinian if and only if it is regular.

Proof. Let $M := U_{n,E}$, $N := U_{2,E}$ and $\infty, 0, 1 \in E$ pairwise different. Using Theorem 3.27, we obtain $\mathbb{U}^{(0)}(N) \cong \mathbb{U}_{E \setminus \{\infty,0,1\}}$. Further, Theorem and Definition 2.20 and Theorem and Definition 2.23 yield that $\pm a^2 \notin \mathcal{F}(\mathbb{U}_{E \setminus \{\infty,0,1\}})$ for any $a \in E \setminus \{\infty, 0, 1\}$ and thus N is not semiartinian.

Applying Lemma 4.21, there exists an epimorphism $f \colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(N)$. Now, if M were semiartinian, N would also be semiartinian, since for every $a' \in \mathbb{U}^{(0)}(N)$ there exists an $a \in f^{-1}(a')$. Thus, $b \in 1-a$ or $b \in 1+a$ would imply that $f(b) \in 1-a'$ or $f(b) \in 1+a'$. Hence, $a' \in \mathcal{F}(N)$.

In all other cases, i. e., $n \in \{0,1\}$ or $|E| \in \{n-1, n\}$, M is regular and therefore almost artinian. □

5.10 Proposition. Let M be a modular combinatorial geometry.

(a) If M is semiartinian, it is either regular or artinian.

(b) If M is non-slender, then M is semiartinian if and only if M is the direct sum of a slender modular combinatorial geometry and a non-slender projective geometry of dimension at least 2.

(c) M is slender if and only if M is the direct sum of matroids of the following types:

  (i) $U_{0,0}$, $U_{1,1}$, $U_{2,2}$, or $U_{2,3}$,

  (ii) $\mathrm{PG}(d, \mathbb{F}_p)$ for $d \in \mathbb{N}$, $d \geq 2$, and $p \in \{2,3\}$,

  (iii) a projective plane Π such that the extended radical of a planar ternary ring coordinatizing it is not $\{1\}$, but is either F^* or a normal subloop of F^* of index 2 not containing -1.

Proof. Since a combinatorial geometry is modular if and only if it is a direct sum of projective geometries ([Whi86, Corollary 3.6.5]) it follows from Theorem 5.5 that it is sufficient to examine the universal partial hyperfields of projective geometries.

Let Π be a projective geometry and $E := E(\Pi)$. If Π has dimension at most 1, it is uniform and therefore using Lemma 5.9 semiartinian if and only if it is equal to $U_{0,0}$, $U_{1,1}$, $U_{2,2}$, or $U_{2,3}$.

Applying Proposition 3.25, any projective geometry of dimension at least 3 is artinian. It is slender if and only if $\Pi \cong \mathrm{PG}(d, \mathbb{F}_p)$, $d \geq 3$, and $p \in \{2,3\}$.

Finally, let Π be a projective plane and (F, T) be a planar ternary ring coordinatizing it. Then Theorem 3.24 implies that Π is artinian. We have $\mathbb{U}^{(0)}(\Pi) \cong F$ for $F \in \{\mathbb{F}_2, \mathbb{F}_3\}$ if and only if $\Pi \cong \mathrm{PG}(2, \mathbb{F}_p)$ for $p \in \{2, 3\}$. Furthermore, if $\mathbb{U}^{(0)}(\Pi) \cong F$ for $F \in \{\mathbb{S}, \mathbb{W}, \mathbb{K}\}$, then Π is non-Pappian. Therefore, $R_a \neq \{1\}$ and we have either $R_a = F$ or $F = R_a \cup -R_a$ and $-1 \notin R_a$.

Conversely, if $R_a = F^*$ or $F^* = R_a \cup -R_a$ and $-1 \notin R_a$, it follows from Lemma 3.20 that Π is slender. □

5.11 Remark. Let P be an archimedean ordering of a field F and $k \in P \setminus \{0, 1\}$. Setting

$$T(m, x, c) := \begin{cases} mkx + c & \text{if } m, x \in -P, \\ mx + c & \text{else,} \end{cases}$$

we obtain a planar ternary ring (F, T) such that $\mathbb{U}^{(0)}(\Pi) \cong \mathbb{S}$ for the projective plane Π that is coordinatized by (F, T) (see [Kal92a, Proposition (4.2)]).

Let (F, T) be a planar ternary ring coordinatizing a projective plane Π. If (F, T) is finite but not a field, $\mathbb{U}^{(0)}(\Pi) \cong \mathbb{K}$ (see Corollary below). However, we have not found any planar ternary ring (F, T) such that $\mathbb{U}^{(0)}(\Pi) \cong \mathbb{W}$.

5.12 Corollary. Let M be a finite modular combinatorial geometry.

(a) M is slender if and only if M is the direct sum of matroids of the following types:
   (i) $U_{0,0}$, $U_{1,1}$, $U_{2,2}$, or $U_{2,3}$,
   (ii) $\mathrm{PG}(d, \mathbb{F}_p)$ for $d \in \mathbb{N}$, $d \geq 2$, and $p \in \{2, 3\}$,
   (iii) a non-Desarguesian finite projective plane.

(b) M is non-slender and artinian if and only if M is the direct sum of a slender modular combinatorial geometry and $\mathrm{PG}(d, F)$ for $d \in \mathbb{N}$, $d \geq 2$, and a finite field F with at least four elements.

Proof. Follows from Proposition 5.10 and [JK90, Korollar 8]. □

5.13 Proposition. For each matroid M of rank at least 3 such that $E = H \cup \ell$ for a hyperplane H and a line ℓ of M such that $\varrho(H \cap \ell) \geq 1$ we have

$$\mathbb{U}^{(0)}(M) \cong \mathbb{U}^{(0)}(M|H) \oplus \mathbb{U}^{(0)}(M|\ell).$$

Moreover, M is algebraically indecomposable if and only if $\mathrm{s}(M|\ell)$ contains at most 3 points and $M|H$ is algebraically indecomposable.

Proof. Let $s \coloneqq H \cap \ell$ and $n \coloneqq \varrho(M)$. Since E is not a hyperplane, $\varrho(s) = 1$. In particular, each line of M thus intersects H non-trivially and therefore H is modular.

We set $M_1 \coloneqq M|H$, $M_2 \coloneqq M|\ell$, $F \coloneqq \mathbb{U}^{(0)}(M)$, and $F_i \coloneqq \mathbb{U}^{(0)}(M_i)$, $i = 1, 2$. Further, we choose maximal independent sets $S_1 \subseteq \ell$ and $S_2 \subseteq H$ such that $\sigma(S_1 \cup s) = \ell$ and $\sigma(S_2 \cup s) = H$. Then $|S_1| = 1$ and $|S_2| = n - 2$. Using Proposition 3.6, it follows that $\iota_i \colon F_i \to F$ defined by $\iota_i(0) \coloneqq 0$, $\iota_i(-1) \coloneqq -1$ and

$$\iota_i([H_1, H_2 \,|\, H_3, H_4]) \coloneqq [\overline{H_1}, \overline{H_2} \,|\, \overline{H_3}, \overline{H_4}],$$

for $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M_i)$, where $\overline{H_j} \coloneqq \sigma_M(H_j \cup S_i)$, $j = 1, 2, 3, 4$, is a homomorphism of partial hyperfields for $i = 1, 2$.

Applying Corollary 2.12, $F \cong F_1 \oplus F_2$ follows if we show that for all homomorphisms $f_i \colon F_i \to F'$, $i = 1, 2$, into a partial hyperfield F' there exists a unique homomorphism $f \colon F \to F'$ such that $f \circ \iota_i = f_i$, $i = 1, 2$.

We first construct a group homomorphism $g \colon \mathbb{F}^{(0)}(M) \to F'^{*}$. Set $g(\varepsilon) \coloneqq -1$. For any $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$ we set

$$g((H_1, H_2 \,|\, H_3, H_4)) \coloneqq \begin{cases} f_1([H_1', H_2' \,|\, H_3', H_4']) & \text{if } L \nsubseteq H, \\ f_2([H_1'', H_2'' \,|\, H_3'', H_4'']) & \text{if } s \nsubseteq L \subseteq H, \\ 1 & \text{else,} \end{cases}$$

where $L \coloneqq \bigcap_{i=1}^{4} H_i$, and $H_i' \coloneqq H_i \cap H$, $H_i'' \coloneqq H_i \cap \ell$. $i = 1, 2, 3, 4$.

Then g is well-defined, as the modularity of H implies that $H_i \cap H$ is a hyperplane and $L \cap H$ is a hyperline of $M|H$, $i = 1, 2, 3, 4$, if $L \nsubseteq H$, and $H_i \cap \ell$ is a hyperplane of $M|\ell$ if $s \nsubseteq L \subseteq H$,[4] $i = 1, 2, 3, 4$.

It is now sufficient to show that all the elements of (CR0) – (CR6) are contained in $\ker g$, since this implies that $\mathbb{K}^{(0)}(M) \subseteq \ker g$ and thus there exists a unique group homomorphism $f \colon F^{*} \to F'^{*}$ such that

$$f([H_1, H_2 \,|\, H_3, H_4]) = g((H_1, H_2 \,|\, H_3, H_4)).$$

Thus, by setting $f(0) \coloneqq 0$ and using Lemma 2.5 as well as Proposition and Definition 3.4 we obtain the desired homomorphism $f \colon F \to F'$ such that $f \circ \iota_i = f_i$, $i = 1, 2$.

For all hyperlines $L \supseteq s$ of M we have $\mathcal{H}_L = \{H, L \vee \ell\}$. Therefore, by construction, all the elements of (CR0) – (CR4) are contained in $\ker g$. In order to show that $\ker g$ contains ε (this is (CR5)) if M has the Fano matroid or its dual as a minor, we prove that in this case M_1 has the Fano matroid or its dual as a minor.

[4] In this case $H_i = H$ or $H_i = L \cup p$ for a hyperplane p of $M|\ell$.

Let N be such a minor of M on the set $E' \subseteq E$. Then there exist an independent set I such that $N = (M/I)|E'$. Since $\varrho_M(\ell) = 2$, $|I \cap \ell| \leq 2$.

Suppose $I \subseteq H$. It would follow that $E' \subseteq E \setminus I$ and N would be a minor of $M' := (M/I)|(E \setminus I)$, which has the same rank as N. This would imply that E is the union of a hyperplane and a line of N, because $E \setminus I$ is the union of the hyperplane $H \setminus I$ and the line ℓ of M'.

This is a contradiction, as their union can contain at most 6 points and $|E'| = 7$ (if N is the Fano matroid both are lines, which contain 3 points, and if N is the dual of the Fano matroid, each hyperplane contains 4 and each line 2 elements). Therefore, $I \nsubseteq H$ and let $i \in I \cap \ell$.

Further, since $I \cap \ell$ contains at most 2 elements, there is at most one $j \in I \setminus H$ such that $j \neq i$. If such a j exists, we have $\ell \subseteq \sigma_M(I)$ and therefore, we can replace j by any $j' \in H \cap \ell$ such that $\varrho(\{j'\}) = 1$.

Hence, we can assume without loss of generality that $|I \setminus H| = 1$ and $E' \subseteq H$ (we have $\varrho_{M/i}(\{x, y\}) \leq 1$ for all $x, y \in \ell \setminus (H \cup \{i\})$). Thus, $N = (M/I)|E'$ is a minor of $(M/i)|H = M_1$.

Finally, we show that $\ker g$ contains all the elements of the form (CR6). Let $H_1, \dots, H_9$ be hyperplanes satisfying (i) – (v) from (CR6), i. e., $L_i := H_j \cap H_k$, $L_4 := H_4 \cap H_5 \cap H_6$, $L_5 := H_7 \cap H_8 \cap H_9$, hyperlines of M for all $\{i, j, k\} = \{1, 2, 3\}$, $P := H_1 \cap H_2 \cap H_3$ a hyperpoint of M, $L_i \subseteq H_{i+3}, H_{i+6}$ for $i = 1, 2, 3$ and $\{H_1, H_2, H_3\} \cap \{H_4, \dots, H_9\} = \emptyset$. We have to prove that

$$x := (H_1, H_2 \,|\, H_6, H_9) \cdot (H_2, H_3 \,|\, H_4, H_7) \cdot (H_3, H_2 \,|\, H_5, H_8) \in \ker g.$$

If $L_4 = L_5$, this is trivial, so assume $L_4 \neq L_5$. Then $L_1, \dots, L_5$ are pairwise different. If $P \nsubseteq H$, then the $H'_i := H_i \cap H$, $i = 1, \dots, 9$, satisfy (i) – (v) from (CR6) of M_1 and thus

$$g(x) = [H'_1, H'_2 \,|\, H'_6, H'_9] \cdot [H'_2, H'_3 \,|\, H'_4, H'_7] \cdot [H'_3, H'_1 \,|\, H'_5, H'_8] = 1.$$

Otherwise, $P \subseteq H$ implies that either three of the hyperlines L_i, $i = 1, 2, 3, 4, 5$, are contained in H or intersect $\ell \setminus H$ in a flat of rank 1 (and thus are contained in the hyperplane $P \vee \ell$) Since H_1, H_2, H_3 are pairwise different we can assume without loss of generality that these three hyperlines are L_3, L_4, L_5. This implies $H_6 = H_9$ and therefore $(H_1, H_2 \,|\, H_6, H_9) \in \ker g$.

If $L_3, L_4, L_5 \subseteq H$, we thus obtain for $H'_i := H_i \cap H$, $i = 1, 2, 3, 4, 5, 7, 8$, that $H'_1 = H'_2$, $H'_4 = H'_5$, $H'_7 = H'_8$. Hence, by definition of g, we get

$$g(x) = [H'_1, H'_3 \,|\, H'_4, H'_7] \cdot [H'_3, H'_1 \,|\, H'_4, H'_7] = 1.$$

The case $L_3, L_4, L_5 \subseteq P \vee \ell$ is proven similarly. Finally, the last sentence follows as in the proof of Theorem 5.5. □

Definition. Let M be a matroid on the ground set E. The *free extension* of M is the matroid on the set $E \cup \{\omega\}$ for any $\omega \notin E$ whose hyperplanes are the hyperplanes H of M and the sets $L \cup \{\omega\}$ for hyperlines L of M. It is denoted by $M + \omega$ and its rank is equal to that of M.

Further, let M_i be a matroid on the ground set E_i, $i = 1, 2$, such that $U := E_1 \cap E_2$ is a modular flat of M_1 and $M_1|U = M_2|U$. Then the *generalized parallel connection* of M_1 and M_2 is the matroid on the set $E_1 \cup E_2$ whose flats are the sets $K \subseteq E_1 \cup E_2$ such that $K \cap E_i$ is a flat of M_i, $i = 1, 2$. It is denoted by $P_U(M_1, M_2)$.[5] Further, the rank of a flat K of $P_U(M_1, M_2)$ can be obtained by

$$\varrho_{P_U(M_1,M_2)}(K) = \varrho_{M_1}(K \cap E_1) + \varrho_{M_2}(K \cap E_2) - \varrho(K \cap U).$$

Thus, $\varrho(P_U(M_1, M_2)) = n_1 + n_2 - k$, where $n_i = \varrho(M_i)$, $i = 1, 2$, and $k = \varrho(U)$.

5.14 Corollary. For each matroid M of rank at least 3 such that $E = H \cup \ell$ and $\varrho(H \cap \ell) = 0$ for a hyperplane H and a line ℓ of M we have

$$\mathbb{U}^{(0)}(M) \cong \mathbb{U}^{(0)}(M|H + \omega) \oplus \mathbb{U}^{(0)}(M|\ell + \omega)$$

for an $\omega \notin E$. In particular, M is algebraically indecomposable if and only if $M|H + \omega$ is algebraically indecomposable and $\mathrm{s}(M|\ell)$ contains at most 2 points.

Proof. Let $\hat{M} := P_s(M|H + \omega, M|\ell + \omega)$, where $s := \sigma_M(\emptyset) \cup \{\omega\}$, and for each flat K of M we set $\hat{K} := \sigma_{\hat{M}}(K)$. We will show that $\mathbb{U}^{(0)}(M) \cong \mathbb{U}^{(0)}(\hat{M})$. Since $\hat{M}|\hat{K} = M|K + \omega$ for $K \in \{H, \ell\}$, then Proposition 5.13 implies our claim.

Using Lemma 2.5 and Proposition and Definition 3.4, we accomplish this by proving that the group homomorphism $g\colon \mathbb{F}^{(0)}(M) \to \mathbb{T}^{(0)}(\hat{M})$, defined by $g(\varepsilon) := -1$ and

$$g((H_1, H_2 \,|\, H_3, H_4) := [\widehat{H_1}, \widehat{H_2} \,|\, \widehat{H_3}, \widehat{H_4}]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$, is an epimorphism whose kernel is equal to $\mathbb{K}^{(0)}(M)$.

For each flat K of M we have $\varrho_{\hat{M}}(\hat{K}) = \varrho_M(K)$ and thus $\hat{K} \cap E = K$. Let K_1, K_2 be flats of rank $k \in \mathbb{N}$ of M such that $\varrho_M(K_1 \cap K_2) = k - 1$. Clearly, $\widehat{K_1 \cap K_2} \subseteq \widehat{K_1} \cap \widehat{K_2}$ and $\widehat{K_1} \vee \widehat{K_2} \subseteq \widehat{K_1 \vee K_2}$. Since $\varrho_{\hat{M}}(\widehat{K_1} \cap \widehat{K_2}) = k$ would imply that $\widehat{K_1} = \widehat{K_2}$, and therefore $K_1 = \widehat{K_1} \cap E = \widehat{K_2} \cap E = K_2$, a contradiction, we get $\widehat{K_1 \cap K_2} = \widehat{K_1} \cap \widehat{K_2}$.

Thus, $\varrho_{\hat{M}}(\widehat{K_1} \vee \widehat{K_2}) = k + 1$, which also yields $\widehat{K_1} \vee \widehat{K_2} = \widehat{K_1 \vee K_2}$. In particular, $(\widehat{H_1}, \widehat{H_2}, \widehat{H_3}, \widehat{H_4}) \in \mathcal{H}_4(\hat{M})$ for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$. Hence,

[5] Cf. [Bry75, Theorem 5.3 and Proposition 5.5]. If $U = \emptyset$, then $P_U(M_1, M_2) \cong M_1 \oplus M_2$; if $\varrho(U) = 1$ and $U = \{p\}$, it is the classical parallel connection.

g is well-defined and its kernel contains all the elements from (CR0) – (CR4) and (CR6).

Further, we have $\mathbb{K}^{(0)}(M) \subseteq \ker g$. To prove that $\ker g$ contains the element from (CR5), it suffices to show that if $\hat{M}$ has the Fano matroid or its dual as a minor, then this is already a minor of M.

Let N be the Fano matroid or its dual and let it be a minor of $\hat{M}$ on the set E'. Using the proof of Proposition 5.13, we obtain that N is a minor of $M|H+\omega$. Thus, there exists an independent set $I \subseteq E \setminus E'$ such that $N = ((M|H+\omega)/I)|E'$.

We will show that $\omega \in E'$. Then we have $N = (M/I')|E'$ for $I' = I \cup \{p\}$ if $\omega \notin I$, and $I' = (I \setminus \{\omega\}) \cup \{p,q\}$ else, where $\{p,q\}$ is a maximal independent set of ℓ.

Suppose $\omega \in E'$. Every hyperplane of $M|H+\omega$ which contains ω is of the form $L \cup \{\omega\}$ for a hyperline L of $M|H$, contradicting the fact that in the case that N is the Fano matroid, every hyperplane contains 3 points and every hyperline 1 point, and if N is its dual, every hyperplane contains 4 points and every hyperline 2 points.

Finally, to show that g is surjective, let $(H_1', H_2', H_3', H_4') \in \mathcal{H}_4(\hat{M})$ and $L' = \bigcap_{i=1}^{4} H_i'$. If $\omega \in L'$, then the proof of Proposition 5.13 yields that $\mathcal{H}_{L'} = \{H \cup \{\omega\}, L \vee \ell\}$ (since $\sigma_{\hat{M}}(\omega) = s$). Thus, $H_1' = H_2'$ and $H_3' = H_4'$, and we get $[H_1', H_2' \,|\, H_3', H_4'] = 1$.

Otherwise, $L \coloneqq L' \cap E$ is a hyperline of M such that $\hat{L} = L'$ and $H_i \coloneqq H_i' \cap E$ is a hyperplane of M such that $\widehat{H_i} = H_i'$, $i = 1,2,3,4$. Hence,

$$g((H_1, H_2 \,|\, H_3, H_4)) = [H_1', H_2' \,|\, H_3', H_4'].$$

Therefore, g is surjective, which completes our claim. □

It follows from Proposition 3.11, Lemma 5.4 and Lemma 5.9 that a matroid of rank less or equal to 2 is algebraically indecomposable. Thus, it remains to study matroids of rank greater or equal to 3 whose ground set is not the union of a hyperplane and a line.

For the rest of this section we will prove that matroids of rank greater or equal to 3 whose ground set is not the union of two hyperplanes are algebraically indecomposable if certain additional assumptions are satisfied.

5.15 Lemma. Let F_i be a partial hyperfield, $i = 1,2$, $\iota_i \colon F_i \to F_1 \oplus F_2$ the canonical injection, $i = 1,2$, and E be a set which contains at least 4 elements. For any homomorphism of partial hyperfields $f \colon \mathbb{U}^{(0)}(U_{2,E}) \to F_1 \oplus F_2$ there exists a $k \in \{1,2\}$ such that $f(\mathbb{U}^{(0)}(U_{2,E})) \subseteq \iota_k(F_k)$.

Proof. There is nothing to prove if $f(\mathbb{U}^{(0)}(U_{2,E})) \subseteq \{-1,0,1\}$. So let there exist pairwise different $\infty, 0, 1, a \in E$ such that $f([\{\infty\},\{0\} \mid \{1\},\{a\}]) \neq -1, 1$.

Using Proposition 3.26, it suffices to prove that for any homomorphism of partial hyperfields $g\colon F \to F_1 \oplus F_2$, where $F \coloneqq \mathbb{U}_{\{E \setminus \{\infty,0,1\}\}}$, with $g(a) \neq -1, 1$ there exists a $k \in \{1,2\}$ such that $g(F) \subseteq \iota_k(F_k)$.

Lemma 5.3 yields that there exists a unique $k \in \{1,2\}$ such that $g(a) \in \iota_k(F_k)$. We will first show $g(b) \in \iota_k(F_k)$ for all $b \in E \setminus \{\infty, 0, 1\}$.

Let $i, j \in \{1,2\}$ such that $a^{-1}b^{-1} \in \iota_i(F_i)$ and $b \in \iota_j(F_j)$. Since i, j, k cannot be pairwise different, $\iota_l(F_l^*)$ is a subgroup of $(F_1 \oplus F_2)^*$ for all $l = 1, 2$, and as $a \cdot a^{-1}b^{-1} \cdot b = 1$ we obtain that $i = j = k$.

Further, Corollary 2.12 implies that $g(1-b) \in \iota_k(F_k)$. Thus,

$$g(b-c) \in g(b-1) + g(1-c) \in \iota_k(F_k)$$

for all $b, c \in E \setminus \{\infty, 0, 1\}$ such that $b \neq c$. Hence, $g(F) \subseteq \iota_k(F_k)$. □

Definition. Let $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ be a homomorphism of partial hyperfields, where M is a matroid, F_1, F_2 are partial hyperfields and $\iota_i\colon F_i \to F_1 \oplus F_2$ is the canonical injection, $i = 1, 2$.

For each hyperline L of M let $k_f(L)$ be the set of all $j \in \{1,2\}$ such that for all pairwise different $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ we have $f([H_1, H_2 \mid H_3, H_4]) \in \iota_j(F_j)$. It follows from Lemma 5.15 that $k_f(L) \neq \emptyset$.

Further, we say that M is *f-indecomposable* if we have $k_f(L) \subseteq k_f(L')$ or $k_f(L') \subseteq k_f(L)$ for all hyperlines L, L' of M. Clearly, M is indecomposable if M is f-indecomposable for any isomorphism $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ such that $F_i \nsubseteq \{-1,0,1\}$, $i = 1, 2$.

5.16 Lemma. Let L be a hyperline of a matroid M and $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ pairwise different such that $f([H_1, H_2 \mid H_3, H_4]) \neq -1, 1$ for a homomorphism $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ to the coproduct of two partial hyperfields F_i, $i = 1, 2$.

(a) If H is a hyperplane such that $P \coloneqq H \cap L$ is a hyperpoint, as well as $r \in \{1,2,3,4\}$ such that $H \cap H_i$ is a hyperline for $i = 1,2,3,4$, $i \neq r$, then $k_f(L') = k_f(L)$ for all hyperlines $L' \supseteq P$ such that $L' \nsubseteq H, H_i$ for $i = 1,2,3,4$, $i \neq r$.

In particular, if $H \cap H_i$ is a hyperline for all $i = 1,2,3,4$, we have $k_f(L') = k_f(L)$ for all hyperlines $L' \supseteq P$ such that $L' \nsubseteq H$.

(b) If M is the uniform matroid of rank 3 on $m \in \{5,6\}$ points, and if $m = 6$ we have additionally $-1, 1 \notin 1 - 1$ in $F_1 \oplus F_2$, then $k_f(L) \subseteq k_f(L')$ for all hyperlines L' of M.

Moreover, we have $k_f(L') = k_f(L)$ for at least two hyperlines $L' \neq L$.

Proof. First, to prove (a), let H be a hyperplane such that $P \coloneqq H \cap L$ is a hyperpoint, $r \in \{1,2,3,4\}$ such that $L_i \coloneqq H \cap H_i \in \mathcal{L}$ for all $i = 1,2,3,4$, $i \neq r$, and $L' \supseteq P$ be a hyperline such that $L' \nsubseteq H, H_i$ for all $i = 1,2,3,4$, $i \neq r$. If we set $H'_r \coloneqq H_r$ and $H'_i \coloneqq L' \vee L_i$ for all $i = 1,2,3,4$, $i \neq r$, then (H_1,H_2,H_3,H_4) and (H'_1,H'_2,H'_3,H'_4) are projectively equivalent, and Proposition and Definition 3.29 yields that

$$\alpha \coloneqq [H_1,H_2 \,|\, H_3,H_4] = [H'_1,H'_2 \,|\, H'_3,H'_4].$$

Since $f(\alpha) \neq -1,1$ there exists a $j \in \{1,2\}$ such that $f(\alpha) = \iota_j(F_j)$, where $\iota_i \colon F_i \to F_1 \oplus F_2$ is the canonical injection, $i = 1,2$. Thus, Lemma 5.15 yields that $k_f(L') = \{j\} = k_f(L)$.

To show (b), let M be the uniform matroid of rank 3 on the points $\{1,\dots,m\}$, where $m \in \{5,6\}$ and set $\overline{ij} \coloneqq \{i,j\}$ for all $i,j = 1,\dots,m$, $i \neq j$. Let L' be a hyperline such that $k_f(L') \neq k_f(L)$. It follows from Lemma 3.3

$$f([H_1,H_2 \,|\, H_3,H_4]) = f([H_1,H_2 \,|\, H_3,H_5]) \cdot f([H_1,H_2 \,|\, H_5,H_4])$$

for any $H_5 \in \mathcal{H}_L \setminus \{H_1,H_2,H_3,H_4\}$. Hence, using the fact that every permutation of $\{1,\dots,m\}$ is an automorphism of M and Proposition 3.12, we may assume without loss of generality that $L = \{1\}$, $L' = \{2\}$ and $\overline{1i} = H_{i-1}$ for all $i = 2,3,4,5$. Applying (CR6) yields that

$$f([\overline{12},\overline{13} \,|\, \overline{14},\overline{15}]) \cdot f([\overline{31},\overline{32} \,|\, \overline{34},\overline{35}]) \cdot f([\overline{23},\overline{21} \,|\, \overline{24},\overline{25}]) = 1.$$

Thus, we have

$$\{f([\overline{31},\overline{32} \,|\, \overline{34},\overline{35}]), f([\overline{23},\overline{21} \,|\, \overline{24},\overline{25}])\} \nsubseteq \{-1,1\}.$$

It follows from the proof of Lemma 5.15 that $k_f(\{1\}) = \{j\} = k_f(\{i\})$ for an $i \in \{2,3\}$. Hence, $k_f(\{1\}) = k_f(\{3\})$.

Similarly, we get $k_f(\{1\}) = k_f(\{i\})$ for an $i \in \{4,5\}$, and Lemma 3.3 yields $f([G_1,G_2 \,|\, G_3,G_4]) \in \{-1,1\}$ for all $\{G_1,G_2,G_3,G_4\} = \{\overline{21},\overline{23},\overline{24},\overline{25}\}$. If $m = 5$, it follows immediately that $k_f(\{2\}) = \{1,2\}$.

Else, suppose we would have $m = 6$ and $-1,1 \notin 1-1$ in $F_1 \oplus F_2$. This would imply $f([G_1,G_2 \,|\, G_3,G_4]) \neq 1$ for all $(G_1,G_2,G_3,G_4) \in \mathcal{H}_4^+$ using the definition of $\mathbb{U}^{(0)}(M)$ from Proposition and Definition 3.4. Thus, Lemma 3.3 would yield $f([\overline{21},\overline{23} \,|\, \overline{2i},\overline{26}]) \neq -1,1$ for an $i \in \{4,5\}$ and we would get $k_f(\{2\}) = k_f(\{i'\})$ for a $i' \in \{1,3\}$. Hence, $k_f(\{2\}) = k_f(\{1\})$, a contradiction. $\square$

5.17 Lemma. Let M be a matroid of rank 3 such that $E \nsubseteq H_1 \cup H_2$ for all hyperplanes H_1, H_2 of M, and F_i a partial hyperfield, $i = 1, 2$.

Then M is f-indecomposable for a homomorphism of partial hyperfields $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ if and only if each minor $N \cong U_{3,6}$ is $(f \circ f_N)$-indecomposable, where $f_N\colon \mathbb{U}^{(0)}(N) \to \mathbb{U}^{(0)}(M)$ is the homomorphism of partial hyperfields defined by $f_N(0) \coloneqq 0$, $f_N(-1) \coloneqq -1$, and

$$f_N([H_1, H_2 \,|\, H_3, H_4]) \coloneqq [\overline{H_1}, \overline{H_2} \,|\, \overline{H_3}, \overline{H_4}]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(N)$, where $\overline{H_i} \coloneqq \sigma_M(H_i)$, $i = 1, 2, 3, 4$ (cf. Proposition 3.6).

Proof. Clearly, all minors $N \cong U_{3,6}$ of M are $(f \circ f_N)$-indecomposable if M is f-indecomposable.

Conversely, let F_i be a partial hyperfield, $i = 1, 2$, $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ be a homomorphism of partial hyperfields such that each minor $N \cong U_{3,6}$ of M is $(f \circ f_N)$-indecomposable, and L_1, L_2 be different hyperlines of M such that $k_f(L_i) \neq \{1, 2\}$, $i = 1, 2$. We will show that $k_f(L_1) = k_f(L_2)$.

Let $G_1 \coloneqq H_1 \coloneqq L_1 \vee L_2$. It follows from the proofs of Proposition 3.26 and Theorem 3.28 that there exist pairwise different $H_2, H_3, H_4 \in \mathcal{H}_{L_1} \setminus \{H_1\}$ such that

$$\{f([H_1, H_2 \,|\, H_3, H_4]), f([H_1, H_3 \,|\, H_4, H_2])\} \cap \{-1, 1\} = \emptyset. \tag{5.1}$$

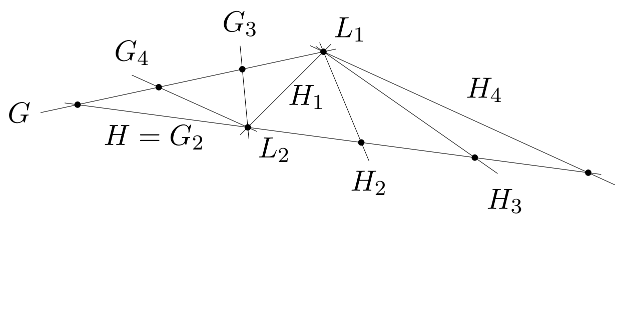
First, we consider the case there exists a hyperplane H such that $H \cap H_i \in \mathcal{L}$ for all $i = 1, 2, 3, 4$. Since Lemma 5.16 yields $k_f(L) = k_f(L_1)$ for all hyperlines $L \nsubseteq H$, let $L_2 \subseteq H \eqqcolon G_2$, as else we have trivially $k_f(L_1) = k_f(L_2)$.

Similar as above there exist different $G_3, G_4 \in \mathcal{H}_{L_2} \setminus \{G_1, G_2\}$ such that

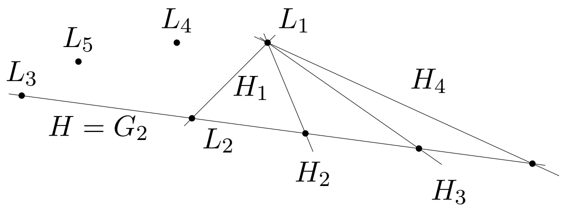
$$\{f([G_1, G_2 \,|\, G_3, G_4]), f([G_1, G_3 \,|\, G_4, G_2])\} \nsubseteq \{-1, 1\}. \tag{5.2}$$

Set $L_3 \coloneqq G_2 \cap H_2$. We will show there exists a hyperline $L \nsubseteq H$ such that $k_f(L) = k_f(L_2)$.

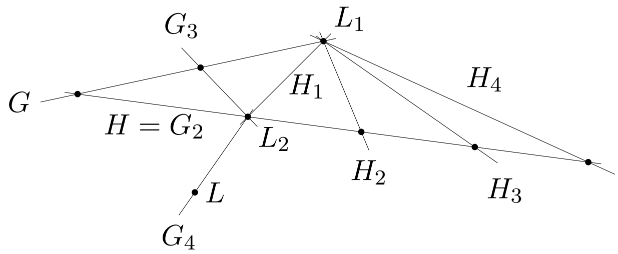
If there exists a hyperplane $G \nsubseteq L_2$ such that $G \cap G_i \in \mathcal{L}$ for all $i = 1, 2, 3, 4$, then there exists an $L \nsubseteq G \cup H$ for which we have $k_f(L) = k_f(L_2)$ using Lemma 5.16.



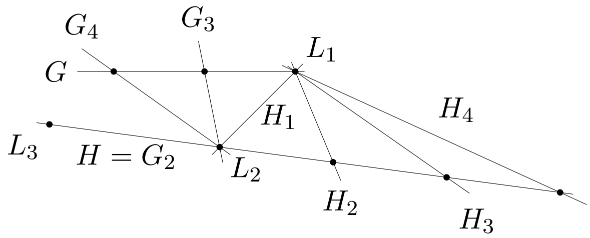
If there exist hyperlines $L_{i+1} \subseteq G_i$, $L_{i+1} \neq L_2$, $i = 3, 4$, such that every hyperline H' contains at most two of the four hyperlines L_1, L_3, L_4, L_5, let E' be a set of five points such that $\varrho(L_i \cap E') = 1$ for all $i = 1, \ldots, 5$. Then $N := M|E' \cong U_{3,5}$ and thus Lemma 5.16 implies that $k_f(L_2) = k_f(L)$, where $L := L_j$ for a suitable $j \in \{4, 5\}$.



Else, there necessarily exist a hyperplane $G \not\supseteq L_2$ and an $s \in \{1, 2, 3, 4\}$ such that $G \cap G_i \in \mathcal{L}$ for all $i = 1, 2, 3, 4$, $i \neq s$, but $G \cap G_s \notin \mathcal{L}$. Then Lemma 5.16 yields that $k_f(L) = k_f(L_2)$ for a hyperline $L \not\subseteq H$ such that $L \not\subseteq G_i$, $i = 1, 2, 3, 4$, $i \neq s$. If additionally $s \neq 2$, we choose a hyperline $L \subseteq G_s$ such that $L \neq L_2$.



If additionally $s = 2$, we choose a hyperline $L \not\subseteq G \cup H$. Since we may assume without loss of generality that $L \not\subseteq G_i$ for $i = 3, 4$ (otherwise we would be in the second subcase), this follows directly if $L \not\subseteq G_1$. Else, we choose any hyperline $L_3 \supseteq P$ such that $L_3 \subseteq H$ and $L_3 \neq L_2$, and get $k_f(L_2) = k_f(L_3) = k_f(L)$.

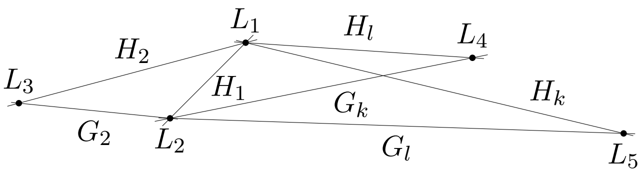


In the remaining case, for all hyperplanes $H \not\supseteq L_1$ there exists an $r \in \{1, 2, 3, 4\}$ such that $H \cap H_r \notin \mathcal{L}$. Choose a hyperline $L_3 \subseteq H_2$, $L_3 \neq L_1$ and set

$G_2 \coloneqq L_2 \vee L_3$. It follows from the proofs of Proposition 3.26 and Theorem 3.28 that there exist different $G_3, G_4 \in \mathcal{H}_{L_2} \setminus \{G_1, G_2\}$ that satisfy (5.2).

Moreover, we may assume without loss of generality that for all hyperplanes $G \not\supseteq L_2$ there exists an $s \in \{1, 2, 3, 4\}$ such that $G \cap G_s \notin \mathcal{L}$ (otherwise exchange the roles of L_1 and L_2 and apply the first case). Thus, using Lemma 3.3, we may further assume that there exist $\{k, l\} = \{3, 4\}$ such that $L_4 \coloneqq G_k \cap H_l \in \mathcal{L}$.

If additionally $L_5 \coloneqq G_l \cap H_k \in \mathcal{L}$, then every hyperplane H' contains at most two of the five hyperplanes $L_1, \ldots, L_5$. Hence, we get $k_f(L_1) = k_f(L_2)$, as in the first case. Thus, for the rest of the proof let $\varrho(G_l \cap H_k) = 0$.

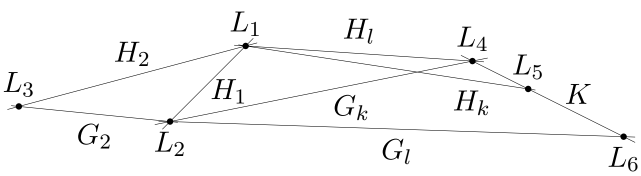


If there exist $L_5 \subseteq H_k$, $L_5 \neq L_1$, $L_6 \subseteq G_l$, $L_6 \neq L_2$, such that each hyperline contains at most two of the five hyperlines $L_1, \ldots, L_4, L_m$, $m \in \{5, 6\}$, then it follows from Lemma 5.16 that for

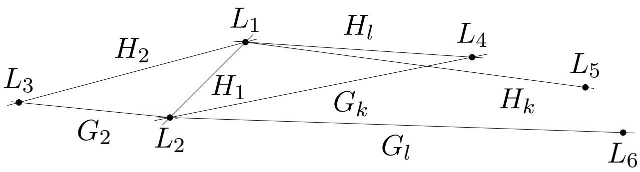
$$S_i \coloneqq \{L \in \{L_1, \ldots, L_4, L_{4+i}\} \mid k_f(L) = k_f(L_i)\}$$

we have $|S_i| \geq 3$, $i = 1, 2$. Clearly, we get $k_f(L_1) = k_f(L_2)$ if $S_1 \cap S_2 \neq \emptyset$. Thus, let $|S_1 \cup S_2| = 6$ and $L_{i+4} \in S_i$, $i = 1, 2$.

If additionally $K \coloneqq L_5 \vee L_6$ contains both L_3 and L_4, then it follows from Lemma 5.16 that $k_f(L_1) = k_f(L_2)$. Similarly, we get $k_f(L_1) = k_f(L_i) = k_f(L_2)$ if K contains L_i but not L_j for $\{i, j\} = \{3, 4\}$.



Else, K contains neither L_3 nor L_4 and since the G_i and the H_i, $i = 1, 2, 3, 4$, are pairwise different all hyperplanes contain at most two of the six hyperplanes $L_1, \ldots, L_6$. Let E' be a set of six points such that $\varrho(L_i \cap E') = 1$ for all $i = 1, \ldots, 6$. Then $N \coloneqq M|E' \cong U_{3,6}$. Since N is $(f \circ f_N)$-indecomposable we obtain $k_f(L_1) = k_f(L_2)$.

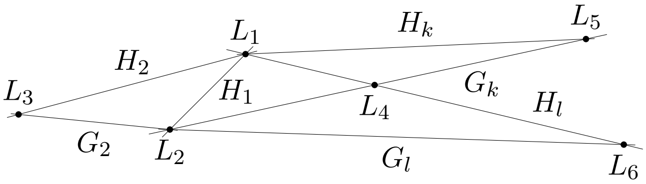


If for every choice of $L_5 \subseteq H_k$, $L_5 \neq L_1$, $L_6 \subseteq G_l$, $L_6 \neq L_2$, there exist an $m \in \{5,6\}$ and a hyperplane of H that contains at least three of the hyperlines $L_1, \dots, L_4, L_m$, we may assume without loss of generality that $m = 5$ and $H \coloneqq L_2 \vee L_5$ contains L_i for $\{i,j\} = \{3,4\}$ (since H_1, H_2, H_3, H_4 are pairwise different). If $G \coloneqq L_1 \vee L_6$ contains also L_i, then Lemma 5.16 yields that $k_f(L_1) = k_f(L_j) = k_f(L_2)$.

Else, if G contains neither L_i nor L_j, we get $k_f(L_1) = k_f(L_m) = k_f(L_2)$ for an $m \in \{j,5\}$.

Finally, let G contain L_j. Then there exists a hyperline $L \nsubseteq G \cup H$. Further, we choose $n \in \{j,5\}$ and $p \in \{i,5\}$ such that $L \nsubseteq L_1 \vee L_p$ and $L_2 \nsubseteq L \vee L_n$. Therefore, using Lemma 5.16, we obtain

$$k_f(L_1) = k_f(L_n) = k_f(L) = k_f(L_p) = k_f(L_2).$$



□

5.18 Proposition. A matroid M such that $E \nsubseteq H_1 \cup H_2$ for all hyperplanes H_1, H_2 with $\varrho(H_1 \cap H_2) \geq \varrho(M) - 3$ is algebraically indecomposable if one of the following conditions is satisfied:

(a) $\varrho(M) = 3$ and $-1, 1 \notin 1 - 1$ in $\mathbb{U}^{(0)}(M)$,

(b) $\varrho(M) = 3$ and M is representable over a field F,

(c) $\varrho(M) \geq 4$, and $-1, 1 \notin 1 - 1$ as well as $-1 \notin 1 + 1$ in $\mathbb{U}^{(0)}(M)$,

(d) $\varrho(M) \geq 4$ and M is representable over a field F of characteristic $\neq 3$.

Proof. Let $n \coloneqq \varrho(M) \geq 3$ and $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ be a homomorphism of partial hyperfields, where F_i is a partial hyperfield, $i = 1, 2$. We will show that if $-1, 1 \notin 1 - 1$ in $F_1 \oplus F_2$, and additionally $-1 \notin 1 + 1$ in $F_1 \oplus F_2$ if $n \geq 4$, then M is f-indecomposable.

This proves our claim, since Lemma 2.5 and Theorem and Definition 3.16 imply that (b) is a special case of (a) and (d) is a special case of (c). If $n = 3$, it follows directly from Lemma 5.16 and Lemma 5.17 that M is f-indecomposable.

Thus, let $n \geq 4$ for the rest of the proof. It follows from the definition of the addition of $\mathbb{U}^{(0)}(M)$ in Proposition and Definition 3.4, the proof of Lemma 3.20, $-1, 1 \notin 1-1$, and $-1 \notin 1+1$ that $f([H_1, H_2 \,|\, H_3, H_4]) \neq -1, 1$ for all hyperlines L and pairwise different $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$. Hence, we obtain from the proof of Lemma 5.15 that $k_f(L) = \{1, 2\}$ if and only if $|\mathcal{H}_L| = 3$ (the case $|\mathcal{H}_L| = 2$ is not possible since otherwise $E = H_1 \cup H_2$ for $\{H_1, H_2\} = \mathcal{H}_L$).

Let L_1, L_2 be distinct hyperlines of M such that $k_f(L_i) \neq \{1, 2\}$, $i = 1, 2$, and $K \coloneqq L_1 \cap L_2$. We will prove by induction on $\varrho(K)$ that $k_f(L_1) = k_f(L_2)$. If $\varrho(K) = n - 3$, this follows from Proposition 3.6 and the proof of (a).

Else, $\varrho(K) < n-3$ and we choose a hyperpoint P of M such that $K \subseteq P \subseteq L_1$. Then $P \vee L_2$ has at least rank $n - 1$ and therefore there exist $a_i \in L_2 \setminus P$, $i = 1, 2$, such that $L_i' \coloneqq P \vee a_i$, $i = 1, 2$ are two distinct hyperlines. Since $L_1 \cap L_i' = P$ and $\varrho(L_2 \cap L_i') \geq \varrho(K) + 1$ by construction of L_i', $i = 1, 2$, we can apply the induction hypothesis if $k_f(L_i') \neq \{1, 2\}$ for an $i \in \{1, 2\}$, and obtain $k_f(L_1) = k_f(L_3) = k_f(L_2)$.

Otherwise, we would have $k_f(L_i') = \{1, 2\}$ and thus $|\mathcal{H}_{L_i'}| = 3$ for all $i = 1, 2$. We will show that this would imply that the simplification of M/P is isomorphic to the Fano matroid, a contradiction to $k_f(L_1) \neq \{1, 2\}$.

Let $G_i, G_{i+2} \supseteq L_i'$ be hyperplanes such that $\mathcal{H}_{L_i'} = \{H, G_i, G_{i+2}\}$, $i = 1, 2$, and $H \coloneqq L_1' \vee L_2'$. For each $i = 1, 2$ every hyperline $L' \supseteq P$ such that $L' \nsubseteq H$ would be contained in G_i or G_{i+2}. Thus, there would exist an $r \in \{4, 5, 6\}$ and pairwise different hyperlines $L_3', \ldots, L_r' \supseteq P$ such that $L_i' \nsubseteq H$, $i = 3, \ldots, r$, and $E = H \cup \bigcup_{i=3}^{r} L_i'$. Further, $r \geq 5$, since $E \nsubseteq H \cup (L_3' \vee L_4')$.

Moreover, every hyperplane $H' \supseteq P$ such that $H' \neq H$ would contain at most three hyperlines that contain P. In particular, the $L_i' \vee L_j'$ for $i, j \in \{3, \ldots, r\}$, $i \neq j$, are s pairwise different hyperplanes, where $s = 3$ if $r = 5$, and $s = 6$ if $r = 6$.

For any hyperline $L' \supseteq P$ with $L' \subseteq H$ there would exist $\{i, j, k\} = \{3, 4, 5\}$ such that $\mathcal{H}_{L'} = \{H, L' \vee L_i, L_j \vee L_k\}$, or $\{i, j, k, l\} = \{3, 4, 5, 6\}$ such that $\mathcal{H}_{L'} = \{H, L_i \vee L_j, L_j \vee L_l\}$. Thus, we obtain that also H would contain at most 3 hyperlines that contain P. Hence, $|\mathcal{H}_{L'}| = 3$ for all hyperlines $L' \supseteq P$, which yields that $\mathrm{s}(M/P)$ is isomorphic to the Fano matroid. $\square$

5.19 Corollary. A matroid M is algebraically indecomposable if one of the following conditions is satisfied:

(a) $\mathrm{s}M$ is uniform; in particular when $\varrho(M) \leq 2$,

(b) $\varrho(M) \geq 3$ and for every hyperpoint P and every hyperplane H of M such that $P \subseteq H$ there exist pairwise different hyperlines L_1, L_2, L_3 such that $P \subseteq L_i \subseteq H$, $i = 1, 2, 3$.[6]

Proof. If M is uniform, it follows from Theorem and Definition 2.20, Proposition 3.26, and Lemma 4.21 that there exist a partial hyperfield F and a homomorphism $f\colon \mathbb{U}^{(0)}(M) \to F$ such that $f([H_1, H_2 \,|\, H_3, H_4]) \neq \pm 1$ for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4^+$. Thus, Proposition 3.11 and the proof of part (c) of Proposition 5.18 imply (a).

In order to prove (b), let $n := \varrho(M) \geq 3$, $f\colon \mathbb{U}^{(0)}(M) \to F_1 \oplus F_2$ be a homomorphism of partial hyperfields, where F_i is a partial hyperfield, $i = 1, 2$, and for every hyperpoint P and every hyperplane H such that $P \subseteq H$ let there exist pairwise different hyperlines L_1, L_2, L_3 such that $P \subseteq L_i \subseteq H$, $i = 1, 2, 3$.

We will show that for every hyperpoint P, every hyperline $L \supseteq P$ and all pairwise different $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ such that $f([H_1, H_2 \,|\, H_3, H_4]) \neq \pm 1$, all hyperlines $L' \supseteq P$ such that $k_f(L') \neq k_f(L)$ are contained in a hyperplane $H \supseteq L$. Then if $L, L' \supseteq P$ are hyperlines such that $k_f(L) \neq \{1, 2\}$ and $k_f(L') \neq k_f(L)$, there exists a hyperline $L'' \supseteq P$ such that $L'' \nsubseteq L \vee L' = H$ and $k_f(L'') = k_f(L)$.

Thus, $k_f(L) = k_f(\tilde{L})$ for all hyperlines $\tilde{L} \supseteq P$ such that $\tilde{L} \nsubseteq H, H' := L \vee L''$, that is $\tilde{L} \neq L'$ and $k_f(L') = \{1, 2\}$. Hence, we get $k_f(L_1) = k_f(L_2)$ by induction on $\varrho(L_1 \cap L_2)$, as in the proof of Proposition 5.18.

Suppose, there existed hyperlines $L_i' \supseteq P$ such that $k_f(L_i') \neq k_f(L)$, $i = 1, 2$, and $L \vee L_1' \neq L \vee L_2'$. Then there would exist hyperlines $L_i \supseteq P$ such that $L_i \subseteq H_i$, $L_i \neq L$, and $L_i \nsubseteq L_1' \vee L_2'$, $i = 1, 2$. Further, choose a hyperline $L_3 \supseteq P$ such that $L_3 \subseteq L_1 \vee L_2 =: G$, $L_3 \neq L_i$, $i = 1, 2$.

Using Lemma 3.3, we can assume without loss of generality that $L_3 \subseteq H_3$. Thus, Lemma 5.16 would imply $G \cap H_4 = P$, $L_1' \subseteq H_{r_0}$, and $L_2 \subseteq H_{s_0}$ for $r_0, s_0 \in \{1, 2, 3\}$ such that $r_0 \neq s_0$. Additionally, it follows from Lemma 5.16 that $k_f(K) = k_f(L)$ for all $K \subseteq H_4$.

Moreover, there would exist hyperlines $K_j \supset P$ with $K_j \subseteq H_4$, $j = 1, 2$, such that K_1, K_2, L are pairwise different. Let $j \in \{1, 2\}$. Then $(G_{1,j}, G_{2,j}, G_{3,j}, G_{4,j})$ and (H_1, H_2, H_3, H_4) are projectively equivalent, where $G_{i,j} := L_i \vee K_j$ for $i = 1, 2, 3$, and $G_{4,j} := H_4$.

[6] In particular, a Sylvester matroid, i. e., a matroid whose lines each contain a circuit, is algebraically indecomposable.

Therefore, applying Lemma 5.16, there exist $r_j, s_j \in \{1,2,3\}$, $r_j \neq s_j$, such that $L_1' \subseteq G_{r_j,j}$ and $L_2' \subseteq G_{s_j,j}$. Since K_1, K_2, L are pairwise different, the same holds for r_0, r_1, r_2 as well as s_0, s_1, s_2. Hence, there exists a $k \in \{1,2\}$ such that $s_k = r_0$.

Finally, since H_1 intersects $G_{r_k,k}$, $G_{s_k,k}$, and $G_{4,k}$ in a hyperline, applying Lemma 5.16 yields $k_f(L_2') = k_f(K_k)$, a contradiction to $k_f(K_k) = k_f(L)$. □

5.2 Decomposition of the canonical real reduced hyperneofield

We will now prove a characterization of the connected components of oriented matroids M where $Q(M)$ is a hyperfield. Additionally, we will apply this as an example for modular combinatorial geometries.

5.20 Lemma. Let F be a partial hyperfield, T a real preordering of F and $x \in F^*$.

(a) $1 + \overline{x}_T = \{\overline{y}_T \in Q(F) \mid \sigma(x) = 1 \Rightarrow \sigma(y) = 1 \; \forall \, \sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})\}$.

(b) If $y, z \in F^*$ such that $\overline{y}_T \in 1 + \overline{x}_T$, we have $\overline{yz}_T \in 1 + \overline{x}_T$ if and only if $\overline{z}_T \in 1 + \overline{x}_T$.

Proof. In order to prove (a), let $y \in F^*$. If $\overline{y}_T \in 1 + \overline{x}_T$ and $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ such that $\sigma(x) = 1$, it follows that $\sigma(y) \in 1 + \sigma(x) = \{1\}$.

Conversely, if $\sigma(y) = 1$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ such that $\sigma(x) = 1$, we have $\sigma(y) = 1 \in 1 + 1 = 1 + \sigma(x)$ if $\sigma(x) = 1$, and $\sigma(y) \in 1 - 1 = 1 + \sigma(x)$ if $\sigma(x) = -1$ for all $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$. Thus, $\overline{y}_T \in 1 + \overline{x}_T$.

Finally, (b) follows directly from (a), since for every $\sigma \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ such that $\sigma(y) = 1$ we have $\sigma(yz) = \sigma(z)$. □

5.21 Proposition and Definition. For any real preordering T of a partial hyperfield F the following statements are equivalent:

(a) the map $\mathrm{Hom}(Q_T(F), \mathbb{S}) \to \mathrm{Hom}(Q_T(F), \mathbb{W})$, $\sigma \mapsto h \circ \sigma$ is surjective, where $h \colon \mathbb{S} \to \mathbb{W}$ is the unique homomorphism (see Lemma 3.20),

(b) $a + b = \{a, b\}$ for all $a, b \in Q_T(F)^*$ such that $a \neq -b$,

(c) $1 + a = \{1, a\}$ for all $a \in Q_T(F) \setminus \{0, -1\}$.

If one (and therefore all) of the statements above is satisfied, we call $Q_T(F)$ a *fan*.[7] Moreover, in this case $Q_T(F)$ is a hyperfield.

[7] This is equivalent to that the corresponding prespaces of orderings is a fan in the sense of Marshall.

Proof. Using Proposition and Definition 2.7, (a) is fulfilled if and only if each group homomorphism $\sigma\colon Q_T(F)^* \to \{-1, 1\}$ such that $\sigma(-1) = -1$ is a homomorphism of partial hyperfields (by setting $\sigma(0) := 0$).

Thus, the equivalence of (a), (b) and (c) is implied by [Mar96, Theorem 3.1.2], Proposition 2.2, and Theorem 4.11.

The last sentence follows from [Mar96, Theorem 3.1.1] and Theorem 4.11. □

5.22 Lemma. Let I be a set, T_i be a real preordering of a partial hyperfield F_i, $i \in I$, $F := \coprod_{i\in I} F_i$, and $\iota_i\colon F \to F_i$ the canonical injection for $i \in I$ as in Corollary 2.12. Then the (multiplicative) submonoid T of F generated by 0 and $\iota_i(T_i)$, $i \in I$, is a real preordering of F. Moreover

(a) the map $\mathrm{Hom}_{T^*}(F, \mathbb{S}) \to \prod_{i\in I} \mathrm{Hom}_{T_i^*}(F_i, \mathbb{S})$, $\sigma \mapsto (\sigma \circ \iota_i)_{i\in I}$ is a bijection,

(b) for any $j \in I$ and $a_j \in F_j$ we have $1 + \overline{\iota_j(a_j)}_T = Q_{T_j,T}(\iota_j)(1 + \overline{a_j}_{T_j})$. Further, $1 + \overline{a}_T = \{1, \overline{a}_T\}$ for any $a \in F$ such that $\overline{a}_T \notin Q_{T_i,T}(\iota_i)$ for all $i \in I$.

(c) $Q_T(F)$ is a hyperfield if and only if there exists a $j \in I$ such that $Q_{T_j}(F_j)$ is a hyperfield and $Q_{T_i}(F_i)$ is a fan for all $i \in I \setminus \{j\}$.

Proof. Applying Corollary 2.12 and Lemma 5.3, we get that T is preordering of F. Further, (a) follows from the universal property of the coproduct of partial hyperfields.

In order to prove (b), let $F'_j := \coprod_{i\neq j} F_i$, $\kappa_i\colon F_i \to F'_j$ the canonical injection $i \in I$, $i \neq j$, and T'_j be the (multiplicative) submonoid of F'_j generated by 0 and $\kappa_i(T_i)$, $i \in I$, $i \neq j$. Using Corollary 2.12, there exists a unique homomorphism $\iota'_j\colon F'_j \to F$ such that $\iota'_j \circ \kappa_i = \iota_i$ for all $i \in I$, $i \neq j$. Further, set

$$H_j(\sigma) := \{\sigma' \in \mathrm{Hom}_{T^*}(F, \mathbb{S}) \mid \sigma \circ \iota_i = \sigma' \circ \iota_i \text{ for all } i \in I,\, i \neq j\}$$

for every $\sigma \in \mathrm{Hom}(F, \mathbb{S})$.

Let $a \in F^*$. Clearly, we always have $1, \overline{a}_T \in 1 + \overline{a}_T$.

Consider the case $a = \iota_j(a_j)$ for an $a_j \in F_j$. Lemma 2.5 implies that $Q_{T_j,T}(\iota_j)(1 + \overline{a_j}_{T_j}) \subseteq 1 + \overline{a}_T$. If there would exists a $b'_j \in F'_j$ such that $\overline{b}_T \in 1 + \overline{a}_T \setminus \{1\}$ for $b = \iota'_j(b'_j)$, there would be a $\sigma' \in \mathrm{Hom}_{T^*}(F, \mathbb{S})$ such that $\sigma'(b) = -1$. Applying (a) would yield the existence of a $\sigma \in H_j(\sigma')$ such that $\sigma(\iota_j(a_j)) = 1$, contradicting $\overline{b}_T \in 1 + \overline{a}_T$. Hence, using Lemma 5.20, we obtain $1 + \overline{a}_T = Q_{T_j,T}(\iota_j)(1 + \overline{a_j}_{T_j})$.

Similarly, we get $1 + \overline{\iota'_j(a'_j)}_T = Q_{T'_j,T}(\iota'_j)(1 + \overline{a'_j}_{T'_j})$. Thus, it remains to show that $1 + \overline{a}_T = \{1, \overline{a}_T\}$ for $a = cc'$, where $c := \iota_j(a_j)$, $c' := \iota'_j(a'_j)$ for $a_j \in F_j$ and $a'_j \in F'_j$, such that $\alpha := \overline{a_j}_{T_j}, \alpha' := \overline{a'_j}_{T'_j} \neq \pm 1$.

Let $b = dd'$ such that $\overline{b}_T \in 1 + \overline{a}_T$, where $d := \iota_j(b_j)$, $d' := \iota'_j(b'_j)$ for $b_j \in F_j$ and $b'_j \in F'_j$, and $\beta := \overline{b_j}_{T_j}, \beta' := \overline{b'_j}_{T'_j} \neq 1$.

First, let $\beta \in 1+\alpha$. If $\alpha = \beta$, then also $\alpha' = \beta'$, as otherwise there would exist a $\sigma' \in \mathrm{Hom}(F, \mathbb{S})$ such that $\sigma'(c') \neq \sigma'(d')$ and (a) would imply there exists a $\sigma \in H_j(\sigma')$ with $\sigma(c) = \sigma'(c')$. Thus, $\sigma(a) = 1$ but $\sigma(b) = -1$, a contradiction to Lemma 5.20.

Otherwise, $\alpha \neq \beta$ and there exists a $\sigma_j \in \mathrm{Hom}(F_j, \mathbb{S})$ such that $\sigma_j(a_j) \neq \sigma_j(b_j)$. Thus, Lemma 5.20 yields $\sigma_j(a_j) = -1$ and $\sigma_j(b_j) = 1$. Let $\sigma' \in \mathrm{Hom}(F, \mathbb{S})$. Then (a) implies the existence of a $\sigma \in H_j(\sigma')$ such that $\sigma(c) = 1$ if $\sigma'(c') = 1$ and $\sigma \circ \iota_j = \sigma_j$ else. Hence, $\sigma(a) = 1$ and therefore $\sigma'(d') = \sigma(b) = 1$, contradicting $\beta' \neq 1$.

Similarly, we get $\overline{a}_T = \overline{b_T}$ if $-\beta \in 1 + \alpha$ or $\pm\beta' \in 1 + \alpha'$.

It remains to consider the case $\beta \notin 1 + \alpha$ and $-\beta' \notin 1 + \alpha'$. Applying Lemma 5.20, we would get a $\sigma' \in \mathrm{Hom}(F, \mathbb{S})$ such that $\sigma'(c') = 1 = \sigma(d')$ and a $\sigma \in H_j(\sigma')$ with $\sigma(c) = 1$ and $\sigma(d) = -1$. Therefore, $\sigma(a) = 1$ but $\sigma(b) = -1$, a contradiction to Lemma 5.20.

Finally, we prove (c). Let $j \in I$ such that $Q_{T_j}(F_j)$ is a hyperfield and $Q_{T_i}(F_i)$ is a fan for all $i \in I$, $i \neq j$, and $a, b, c \in F$. It is sufficient to show $(\overline{a}_T + \overline{b}_T) + \overline{c}_T = \overline{a}_T + (\overline{b}_T + \overline{c}_T)$ for all $a, b, c \in F^*$, since both sides are equal to $\overline{x}_T + \overline{y}_T$ for all $\{x, y, 0\} = \{a, b, c\} \subseteq F$.

Set further $x := ab^{-1}$ and $y := cb^{-1}$. Therefore, we have to show that $(\overline{x}_T + 1) + \overline{y}_T = \overline{x}_T + (1 + \overline{y}_T)$ for all $x, y \in F^*$.

If $\overline{x}_T, \overline{y}_T \in \bar{F}_j$, where $\bar{F}_i := Q_{T_i,T}(\iota_i)(Q_{T_i}(F_i))$ for all $i \in I$, this follows directly from (b).

Otherwise, we can assume without loss of generality that $\overline{x}_T \notin \bar{F}_j$, as addition is commutative. In the case $\overline{y}_T \notin \bar{F}_j$, (b) yields

$$\begin{aligned}(\overline{x}_T + 1) + \overline{y}_T &= \{1, \overline{x}_T\} + \overline{y}_T = (1 + \overline{y}_T) \cup (\overline{x}_T + \overline{y}_T)\\ &= (\overline{x}_T + \overline{y}_T) \cup \{1, \overline{y}_T\} = \overline{x}_T + (1 + \overline{y}_T).\end{aligned}$$

In the remaining case, it follows from (b) that $\overline{x}_T\overline{z}_T \notin \bar{F}_j$ for all $z \in F$ such that $\overline{z}_T \in 1 + \overline{y}_T$, because $\bar{F}_j^*$ is a subgroup of $Q_T(F)^*$. Therefore,

$$\begin{aligned}(\overline{x}_T + 1) + \overline{y}_T &= \{1, \overline{x}_T\} + \overline{y}_T = (1 + \overline{y}_T) \cup \overline{x}_T(1 + \overline{x}_T\overline{y}_T)\\ &= (1 + \overline{y}_T) \cup \{\overline{x}_T\} = \bigcup_{\overline{z}_T \in 1+\overline{y}_T} \overline{x}_T(1 + \overline{x}_T\overline{z}_T)\\ &= \bigcup_{\overline{z}_T \in 1+\overline{y}_T} (\overline{x}_T + \overline{z}_T) = \overline{x}_T + (1 + \overline{y}_T).\end{aligned}$$

Hence, $Q_T(F)$ is hyperfield.

Conversely, if $Q_T(F)$ is a hyperfield, it follows directly from (b) that each $Q_{T_i}(F_i)$ is a hyperfield, $i \in I$. Let $j \in I$ such that $Q_{T_j}(F_j)$ is not a fan and let $k \in I \setminus \{j\}$ such that there exists an $\overline{x}_T \in \bar{F}_k \setminus \bar{F}_j$.

If $\overline{y}_T \in \bar{F}_j$, we obtain as in the last case of the proof of the reverse implication that

$$(\overline{x}_T + 1) + \overline{y}_T = \overline{x}_T + (1 + \overline{y}_T) = \{1, \overline{x}_T\} + \overline{y}_T.$$

Since this set obviously contains each $\overline{z}_T \in 1 + \overline{x}_T$ we have $\overline{z}_T \in 1 + \overline{y}_T$ or $\overline{z}_T \in \overline{x}_T + \overline{y}_T$ for such a $\overline{z}_T$. Since it follows from (a) that $\bar{F}_k \cap \bar{F}_j = \{-1, 0, 1\}$, this implies $\overline{z}_T \in \{1, \overline{x}_T\}$. Hence, (b) yields $Q_{T_k}(F_k)$ is a fan. □

5.23 Remark and Definition. Using the notations of Lemma 5.22, we denote the preordering T of F by $\coprod_{i \in I} T_i$.

Since part (b) Lemma 5.22 yields $Q_{T_j,T}(\iota_j)$ is a strong embedding, for convenience we will identify $Q_{T_j}(F_j)$ with its image under $Q_{T_j,T}(\iota_j)$, $j \in I$.

If $j \in I$ such that each $Q_{T_i}(F_i)$, $i \in I$, $i \neq j$, is a fan and $Q_{T_j}(F_j)$ is a hyperfield, then the corresponding space of orderings via Proposition 4.12 of $Q_T(F)$ is a group extension of the corresponding space of orderings of $Q_{T_j}(F_j)$. Thus, $Q_T(F)$ is a fan if and only if each $Q_{T_i}(F_i)$ is a fan, $i \in I$.

5.24 Theorem. Let M be a matroid and T a real preordering of $\mathbb{U}^{(0)}(M)$. Then we have that $Q_T(M)$ is a hyperfield if and only if $M = S \oplus N$ for matroids S and N such that $Q_{\iota_S^{-1}(T)}(S)$ is a fan and $Q_{\iota_N^{-1}(T)}(N)$ is a hyperfield, where $\iota_X \colon \mathbb{U}^{(0)}(X) \to \mathbb{U}^{(0)}(M)$ is the canonical injection, $X \in \{S, N\}$.[8]

Proof. If $Q_T(M)$ is a fan, then $M \cong S \oplus N$ for $S := N$ and the empty matroid N. Otherwise, let $M_1, \ldots, M_k$ be the connected components of M.

Since Proposition 3.8 implies that $\mathbb{U}^{(0)}(M) \cong \coprod_{i=1}^k \mathbb{U}^{(0)}(M_i)$, we can assume without loss of generality that $Q_{T_1}(M_1)$ is not a fan, using Lemma 5.22, where $T_i := \iota_i^{-1}(T)$ and $\iota_i \colon \mathbb{U}^{(0)}(M_i) \to \mathbb{U}^{(0)}(M)$ is the canonical injection, $i = 1, \ldots, k$.

Moreover, Lemma 5.22 yields $Q_T(M)$ is a hyperfield if and only if $Q_{T_1}(M_1)$ is a hyperfield and $Q_{T_i}(M_i)$ is a fan, $i = 2, \ldots, k$. Our claim thus follows from the fact that $Q_{\iota_S^{-1}(T)}(S)$ is a fan for $S := \bigoplus_{i=2}^k M_i$ if each $Q_{T_i}(M_i)$, $i = 2, \ldots, k$, is a fan (see Remark and Definition 5.23). □

5.25 Corollary. Let M be an orientable modular combinatorial geometry.

(a) $Q(M)$ is a hyperfield which is not a fan if and only if M is the direct sum of an orientable modular combinatorial geometry S such that $Q(S)$ is a fan and a matroid of the following type:

[8]See Corollary 2.12 and Proposition 3.8.

(i) $U_{2,4}$,

(ii) a projective plane Π such that the space of orderings of the planar ternary ring (F,T) coordinatizing it is not a fan,

(iii) a projective geometry $\mathrm{PG}(d,F)$, where F is a skew-field whose space of orderings is not a fan and $d \in \mathbb{N}$, $d \geq 3$.

(b) $Q(M)$ is a fan if and only if it is the direct sum of matroids of the following types:

(i) $U_{0,0}$, $U_{1,1}$, $U_{2,2}$, or $U_{2,3}$,

(ii) a projective plane Π such that the space of orderings of the planar ternary ring (F,T) coordinatizing it is a fan,

(iii) a projective geometry $\mathrm{PG}(d,F)$, where F is a skew-field whose space of orderings is a fan and $d \in \mathbb{N}$, $d \geq 3$.

Proof. As in the proof of Proposition 5.10, using Theorem 5.24, it suffices to examine whether $Q(M)$ is a hyperfield for a projective geometry M and whether it is a fan.

If M has dimension at least 2, this follows from Theorem 3.24, Proposition 3.25 and Proposition 4.10. Otherwise, M has dimension at most 1 and is uniform. Thus, Corollary 4.23 yields our claim. $\square$

5.3 Minors

Definition. We call a set of minors $\mathcal{M}$ of M an *algebraic cover* of M if there exists a map $S\colon \mathcal{M} \multimap E(M)$[9] such that $S(N)$ is an independent set of M, $\varrho_M(S(N)) = \varrho(M) - \varrho(N)$ and $N = (M/S(N))|E(N)$ for each $N \in \mathcal{M}$, and

$$\mathbb{U}^{(0)}(M) = \bigcup_{N\in\mathcal{M}} f_{S(N)}(\mathbb{U}^{(0)}(N)),$$

where $f_{S(N)}\colon \mathbb{U}^{(0)}(N) \to \mathbb{U}^{(0)}(M)$ is the homomorphism of partial hyperfields from Proposition 3.6.

For convenience, we call a minor N an algebraic cover of M if $\{N\}$ is an algebraic cover of M. Furthermore, we say that a minor N *dominates* M if there exists an independent set S of M such that $\varrho_M(S) = \varrho(M) - \varrho(N)$, $N = (M/S)|E(N)$, and f_S is an epimorphism of partial hyperfields.

[9]Where $X \multimap Y$ denotes a map from the set X to the power set of the set Y.

5.26 Remark. Clearly, every dominant minor of a matroid M is an algebraic cover of M, but the converse is false. As a regular matroid, the non-Fano matroid is an algebraic cover of the Fano matroid which does not dominate the Fano matroid.

5.27 Lemma. Let M be a finite matroid.

(a) A set $\mathcal{M}$ of minors of M is an algebraic cover of M if and only if the set $\mathcal{M}^* := \{N^* \mid N \in \mathcal{M}\}$ of minors of M^* is an algebraic cover of M^*.

(b) A minor N of M dominates M if and only if N^* dominates M^*.

Proof. To prove (a), let $E := E(M) = E(M^*)$ and $\mathcal{M}$ be an algebraic cover of M. We define a map $S^*\colon \mathcal{M}^* \multimap E$ by $S^*(N^*) := E \setminus (E(N) \cup S(N))$ for all $N \in \mathcal{M}$. Thus, $E = E(N) \mathbin{\dot\cup} S(N) \mathbin{\dot\cup} S^*(N^*)$ and Proposition 3.11 implies that $S^*(N^*)$ is an independent set of M^* for all $N \in \mathcal{M}$.

Further, let $\varphi_M\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(M^*)$ and $\varphi_N\colon \mathbb{U}^{(0)}(N) \to \mathbb{U}^{(0)}(N^*)$ for each $N \in \mathcal{M}$ be the isomorphisms of Proposition 3.11, which also yields

$$\begin{aligned}\mathbb{U}^{(0)}(M^*) = \varphi_M(\mathbb{U}^{(0)}(M)) &= \bigcup_{N\in\mathcal{M}} \varphi_M(f_{S(N)}(\mathbb{U}^{(0)}(N)))\\ &= \bigcup_{N\in\mathcal{M}} f_{S^*(N^*)}(\varphi_N(\mathbb{U}^{(0)}(N))) = \bigcup_{N^*\in\mathcal{M}^*} f_{S^*(N^*)}(\mathbb{U}^{(0)}(N^*)).\end{aligned}$$

Therefore, $\mathcal{M}^*$ is an algebraic cover of M^*.

In order to prove (b), let N be a minor of M that dominates M. As this implies $\{N\}$ is an algebraic cover of M, we will reuse the definitions and notations from the proof of part (a). Set $S := S(N)$ and $S^* := S^*(N^*)$. It follows from Proposition 3.11 that $f_{S^*} = \varphi_M \circ f_S \circ \varphi_N^{-1}$.

Since f_S is an epimorphism of partial hyperfields and φ_M as well as φ_N are isomorphisms, it follows that f_{S^*} is also an epimorphism of partial hyperfields. Hence, N^* dominates M^*. □

5.28 Proposition. Let M be a matroid and $\mathcal{M}$ be an algebraic cover of M. Then M is semiartinian resp. almost artinian resp. artinian if each $N \in \mathcal{M}$ has this property.

Proof. Let $S\colon \mathcal{M} \multimap E(M)$ be a map such that $S(N)$ is an independent set of M, $\varrho_M(S(N)) = \varrho(M) - \varrho(N)$ and $N = (M/S(N))|E(N)$ for each $N \in \mathcal{M}$, as well as $\mathbb{U}^{(0)}(M) = \bigcup_{N\in\mathcal{M}} f_{S(N)}(\mathbb{U}^{(0)}(N))$.

Since $f_{S(N)}(\mathcal{F}(N)) \subseteq \mathcal{F}(M)$ by Lemma 2.5 for each $N \in \mathcal{M}$, we have $\bigcup_{N\in\mathcal{M}} f_{S(N)}(\alpha\mathcal{F}(N)) \subseteq \alpha\mathcal{F}(M)$ for $\alpha \in \{-1, 1\}$. Thus, our claim follows from the definitions of semiartinian, almost artinian, and artinian matroids. □

5.29 Proposition. Let M be a matroid which has a modular hyperplane U. Then $M|U$ is an algebraic cover of M if and only if either $\varrho(M\backslash U) \neq 2$ or $\mathrm{s}(M\backslash U) \cong U_{2,2}$.

Proof. If $\varrho(M\backslash U) \leq 2$, then our claim follows from the proof of Proposition 5.13. Let $\varrho(M\backslash U) \geq 3$ and $E \coloneqq E(M)$. Then Proposition 3.6 implies that the map $f\colon \mathbb{U}^{(0)}(M|U) \to \mathbb{U}^{(0)}(M)$ defined by $f(0) \coloneqq 0$, $f(-1) \coloneqq -1$, and

$$f([H_1, H_2 \,|\, H_3, H_4]) \coloneqq [H_1 \vee p, H_2 \vee p \,|\, H_3 \vee p, H_4 \vee p]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M|U)$, where $p \subseteq E \setminus U$ is a flat of M of rank 1, is a homomorphism of partial hyperfields, since $f = f_{\{e\}}$ for any maximal independent set $\{e\} \subseteq p$.

Set $\Sigma \coloneqq \sigma_M(E \setminus U) \cap U$, let L be a hyperline of M and $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ be pairwise different. To show $[H_1, H_2 \,|\, H_3, H_4] \in f(\mathbb{T}^{(0)}(M))$ and thus f is surjective, we consider three cases:

First, let $L \nsubseteq U$. Then $L' \coloneqq L \cap U$ is a hyperline of $M|U$ and each $H_i' \coloneqq H_i \cap U$ is a hyperplane of $M|U$ containing L', $i = 1, 2, 3, 4$. As U is modular, (H_1, H_2, H_3, H_4) and $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4)$ are projectively equivalent, where $\tilde{H}_i \coloneqq H_i' \vee p$, $i = 1, 2, 3, 4$. Therefore, Proposition and Definition 3.29 yields

$$[H_1, H_2 \,|\, H_3, H_4] = [\tilde{H}_1, \tilde{H}_2 \,|\, \tilde{H}_3, \tilde{H}_4] = f([H_1', H_2' \,|\, H_3', H_4']).$$

Second, let $L \subseteq U$ and $\varrho_M(L \cap \Sigma) > 0$. Using Lemma 3.3, we can assume without loss of generality that $H_2 = U$. Let $\{s\}$ be an independent set of $L \cap \Sigma$ and choose a flat P such that $\sigma_M(P \cup \{s\}) = L \eqqcolon L_1'$. Then P is a hyperpoint. As $H_3, H_4 \neq U$ there exist hyperlines L_3, L_4 of M such that $P \subseteq L_i \subseteq H_i$ and $H_i = L \vee L_i$, $i = 3, 4$. Further, $H \coloneqq L_3 \vee L_4$ intersects U in a hyperline $L_2' \supseteq P$.

If there exists a hyperline $L_3' \neq L$ of M such that $P \subseteq L_3' \subseteq H_1$ and $L_3' \nsubseteq H$, then $H_i' \coloneqq L_j' \vee L_k'$ for all $\{i, j, k\} = \{1, 2, 3\}$, $H_{i+3}' \coloneqq L_i' \vee L_3$, and $H_{i+6}' \coloneqq L_i' \vee L_4$ for all $i = 1, 2, 3$ satisfy the conditions (i) – (v) from (CR6). Since $H_1 = L \vee L_3' = H_2'$, $H_2 = U = L \vee L_2' = H_3'$, $H_3 = L \vee L_3 = H_4'$, $H_4 = L \vee L_4 = H_7'$, and $H_5' = H_8' = H$, it follows from the first case

$$[H_1, H_2 \,|\, H_3, H_4] = [H_2', H_3' \,|\, H_4', H_7'] = [H_2', H_1' \,|\, H_6', H_9'] \in f(\mathbb{T}^{(0)}(M|U)).$$

Otherwise, $H \cap H_1$ is a hyperline of M and $H_1 = L \cup (H_1 \cap H)$. Therefore, there exists a hyperline L' of M such that $P \subseteq L'$ and neither $L' \subseteq U$ nor $L' \subseteq H$, because $E \setminus U \nsubseteq H$ (otherwise we would have $\Sigma \subseteq P$). Set $H' \coloneqq L_1 \vee L'$. Using the previous subcase, we can conclude

$$[H_1, H_2 \,|\, H_3, H_4] = [H_1, H' \,|\, H_3, H_4] \cdot [H', H_2 \,|\, H_3, H_4] \in f(\mathbb{T}^{(0)}(M|U)).$$

Finally, let $L \subseteq U$ and $\varrho(L \cap \Sigma) = 0$. We will first show by induction that for every $k \in \{0, \ldots, \varrho(M) - 3\}$ there exist flats $K_1 \subseteq L$ of rank k and $K_2 \subseteq U$ of rank $k+1$ with $K_1 \subseteq K_2$, $\Sigma \nsubseteq K_2$, and $\varrho(K_2 \cap \Sigma) > 0$. If $k = 0$, this follows from the fact that $\varrho(M \backslash U) \geq 3$ implies $\varrho(\Sigma) \geq 2$.

Let $k > 0$. By induction hypothesis there exist flats $K_1 \subseteq L$ of rank $k-1$ and $K_2 \subseteq U$ of rank k such that $K_1 \subseteq K_2$, $\Sigma \nsubseteq K_2$, and $\varrho(K_2 \cap \Sigma) > 0$. Since $k < \varrho(L)$ there exist two different flats $K_{1,i}$ of rank k such that $K_1 \subseteq K_{1,i} \subseteq L$, $i = 1, 2$. As $\varrho(K_2 \cap \Sigma) > 0$ and therefore $K_2 \nsubseteq L$ we get $K_{1,i} \cap K_2 = K_1$, $i = 1, 2$. It follows $K_{2,i} := K_{1,i} \vee K_2$, $i = 1, 2$, are two different flats of rank $k+1$ such that $K_{2,i} \subseteq U$ and $K_{1,i} \subseteq K_{2,i}$, $i = 1, 2$.

Suppose $\Sigma \subseteq K_{2,1} \cap K_{2,2}$. Then $K_1 \vee \Sigma \subseteq K_{2,1} \cap K_{2,2} = K_2$, yielding the contradiction $\Sigma \subseteq K_2$. Thus, for a suitable $j \in \{1, 2\}$, $K_1' := K_{1,j} \subseteq L$ is a flat of rank k and $K_2' := K_{2,j} \subseteq U$ a flat of rank $k+1$ with $K_1' \subseteq K_2'$, $\Sigma \nsubseteq K_2'$ and $\varrho(K_2' \cap \Sigma) \geq \varrho(K_2 \cap \Sigma) > 0$.

Applying this in the case $k = \varrho(M) - 3$, there exists a hyperpoint P of M such that $P \subseteq L$ and a hyperline L_2' of M such that $P \subseteq L_2' \subseteq U$, $\Sigma \nsubseteq L_2'$ and $\varrho(L_2' \cap \Sigma) > 0$. Further, set $L_1' := L$ and choose hyperlines $L_3', L_i \supseteq P$, $i = 3, 4$, with $H = L_1' \vee L_3'$ and $H_i = L \vee L_i$, $i = 3, 4$.

Then the hyperplanes $H_i' := L_j' \vee L_k'$ for all $\{i, j, k\} = \{1, 2, 3\}$, $H_{i+3}' := L_i' \vee L_3$, and $H_{i+6}' := L_i' \vee L_4$ for all $i = 1, 2, 3$ satisfy the conditions (i) – (v) from (CR6) and – since $H_1 = L \vee L_3' = H_2'$, $H_2 = U = L \vee L_2' = H_3'$, $H_3 = L \vee L_3 = H_4'$, $H_4 = L \vee L_4 = H_7'$ — we get using the previous cases

$$[H_1, H_2 \mid H_3, H_4] = [H_1', H_3' \mid H_5', H_8'] \cdot [H_2', H_1' \mid H_6', H_9'] \in f(\mathbb{T}^{(0)}(M|U)). \qquad \square$$

5.30 Remark. Let $M_i = (E_i, \mathcal{H}_i)$, $i = 1, 2$, be matroids such that $U := E_1 \cap E_2$ is a modular hyperplane of M_1 and $M_1|U = M_2|U$. Then E_2 is a modular hyperplane of the generalized parallel connection $P_U(M_1, M_2)$ of M_1 and M_2.

Moverover, if M is a matroid which has a modular hyperplane U, and additionally $\varrho(M \backslash U) = 1$ or $\mathrm{s}(M \backslash U) \cong U_{2,2}$, then $M|U$ dominates M (we even have $\mathbb{U}^{(0)}(M) \cong \mathbb{U}^{(0)}(M|U)$). However, this is not necessarily the case if $\varrho(M \backslash U) = 3$.

Proof. First, let $M := P_U(M_1, M_2)$, $n := \varrho(M_2)$ and $k := \varrho(U)$. Then we have $\varrho(M_1) = k + 1$ and $\varrho(M) = k + 1 + n - k = n + 1$. Further,

$$\varrho_M(E_2) = \varrho_{M_1}(E_2 \cap E_1) + \varrho_{M_2}(E_2) - \varrho(U) = k + n - k = n.$$

Therefore, E_2 is a hyperplane of M. If E_2 was not modular, there would exist a line $\ell \subseteq E_1 \subseteq E$ of M such that $\varrho_M(\ell \cap E_2) = 0$. Thus, $\varrho_M(\ell \cap U) = 0$ and U would not be a modular hyperplane of M_1, a contradiction.

Furthermore, that $M|U$ dominates M (as well as $\mathbb{U}^{(0)}(M) \cong \mathbb{U}^{(0)}(M|U)$) if U is a modular hyperplane of M, and $\varrho(M\backslash U) = 1$ or $\mathrm{s}(M\backslash U) \cong U_{2,2}$, follows directly from Proposition 5.13.

Finally, if M_1 is the Fano matroid and M_2 is the non-Fano matroid, then $M \coloneqq P_U(M_1, M_2)$, where U is a common 3-point line of M_1 and M_2, is a binary matroid which is not regular, but $M_2 = M|E_2$ is regular. Thus, Corollary 3.21 yields $\mathbb{U}^{(0)}(M) \cong \mathbb{F}_2$ and $\mathbb{U}^{(0)}(M_2) \cong \mathbb{U}_0$. Hence, M_2 does not dominate M, completing our proof. □

5.31 Example. Choose a line ℓ of $\mathrm{PG}(2,2)$ and identify its points with the points of a line of $\mathrm{AG}(2,3)$. Then Proposition 5.29 and Remark 5.30 imply that $\mathrm{AG}(2,3)$ is an algebraic cover of $M \coloneqq P_\ell(\mathrm{PG}(2,2), \mathrm{AG}(2,3))$. Since $\mathrm{AG}(2,3)$ is semiartinian and M contains the Fano matroid $\mathrm{PG}(2,2)$ as a minor Proposition 5.28 yields that M is artinian.

Further, $\mathrm{AG}(2,3)$ is representable over $\mathbb{F}_4$ and thus a restriction of $\mathrm{PG}(3,4)$. If we choose any hyperplane of $\mathrm{PG}(2,4)$ which intersects this restriction in three collinear points, two of these points can be extended to a quadrangle such that the remaining point is one of its diagonal points. As the restriction of the chosen hyperplane to the quadrangle and the diagonal points is isomorphic to $\mathrm{PG}(2,2)$ it follows that M is representable over $\mathbb{F}_4$. Hence, $\mathbb{U}^{(0)}(M) \cong \mathbb{F}_4$ by Proposition 5.2.

5.32 Lemma (cf. [Wen89, Proposition 2.9]). Let M be a matroid.

(a) If $\varrho(M) \neq 2$, then $\{M/e \mid e \in E\}$ is an algebraic cover of M.

(b) If $|E| \neq \varrho(M) + 2$, then $\{M\backslash e \mid e \in E\}$ is an algebraic cover of M.

Proof. Let $E \coloneqq E(M)$. First, we prove (a). If $\varrho(M) \in \{0, 1\}$, then M and all of its minors are regular. Thus, (a) follows from Lemma 3.20 in this case.

Otherwise, let $\varrho(M) \geq 3$. Then every hyperline L has rank at least 1. If $H_1, H_2, H_3, H_4 \supseteq L$ are pairwise different hyperplanes and $\{e\} \subseteq L$ is independent, we have

$$f_{\{e\}}([H_1 \setminus \{e\}, H_2 \setminus \{e\} \mid H_3 \setminus \{e\}, H_4 \setminus \{e\}]) = [H_1, H_2 \mid H_3, H_4]$$

for the homomorphism of partial hyperfields $f_{\{e\}} \colon \mathbb{U}^{(0)}(M/e) \to \mathbb{U}^{(0)}(M)$ from Proposition 3.6. Thus, $\{M/e \mid e \in E\}$ is an algebraic cover of M.

Using Lemma 5.27, it remains to prove (b) in the case that M is infinite, since otherwise we have $|E| \neq \varrho(M) + 2$ if and only if $\varrho(M^*) = |E| - \varrho(M) \neq 2$ and $(M/e)^* = M^*\backslash e$ for all $e \in E$.

Let $L \in \mathcal{L}$ and $H_1, H_2, H_3, H_4 \supseteq L$ pairwise different hyperplanes. Choose a maximal independent set $I \subseteq L$ and $a_i \in H_i \setminus L$, $i = 1, 2, 3, 4$. As M is infinite there exists an $e \in E \setminus (I \cup \{a_1, a_2, a_3, a_4\})$. Therefore, $\sigma_M(L \setminus \{e\}) = L$ and $\sigma_M(H_i \setminus \{e\}) = H_i$ for all $i = 1, 2, 3, 4$. Thus, using the homomorphism of partial hyperfields $f_\emptyset \colon \mathbb{U}^{(0)}(M\backslash e) \to \mathbb{U}^{(0)}(M)$ from Proposition 3.11, we obtain

$$f_\emptyset([H_1 \setminus \{e\}, H_2 \setminus \{e\} \,|\, H_3 \setminus \{e\}, H_4 \setminus \{e\}]) = [H_1, H_2 \,|\, H_3, H_4],$$

proving our claim. □

5.33 Theorem. *A matroid M is artinian if M/e is artinian for all $e \in E$, or if $M\backslash e$ is artinian for all $e \in E$.*

Proof. First, note if $\varrho(M) = 2$ resp. $|E| = \varrho(M) + 2$, we have $\varrho(M/e) = 1$ resp. $|E \setminus \{e\}| = \varrho(M\backslash e) + 1$ for all $e \in E$. Such matroids are regular and therefore not artinian.

Thus, applying Lemma 5.32, if M/e is artinian for all $e \in E$ resp. $M\backslash e$ is artinian for all $e \in E$, we get that $\{M/e \mid e \in E\}$ resp. $\{M\backslash e \mid e \in E\}$ is an algebraic cover of M where each minor is artinian. Hence, Proposition 5.28 implies that M is artinian. □

5.34 Remark. The statement above is no longer true if we replace artinian by almost artinian or semiartinian, e. g. $U_{2,4}$ is not semiartinian despite $U_{2,4}/e$ and $U_{2,4}\backslash e$ are regular and thus almost artinian for all $e \in \{1, 2, 3, 4\}$.

5.35 Theorem. *Let N be a minor of the matroid M which dominates M.*

(a) *$\mathbb{U}^{(0)}(M)$ is a hyperfield if $\mathbb{U}^{(0)}(N)$ is a hyperfield.*

(b) *$Q(M)$ is a hyperfield if $Q(N)$ is a hyperfield.*

Proof. Follows from Remark and Definition 2.10, Lemma 4.9, and Proposition 4.10. □

Definition. We call an artinian matroid *extremally artinian*, if for every $e \in E$ neither M/e nor $M\backslash e$ is artinian.

5.36 Proposition. *A finite matroid is extremally artinian if and only if its dual is extremally artinian.*

Proof. Follows from Proposition 3.11 and the fact that $(M/e)^* = M^*\backslash e$ and $(M\backslash e)^* = M^*/e$. □

5.37 Theorem. The Fano matroid $\mathrm{PG}(2,2)$, the ternary Reid geometry R_9 and their duals are extremally artinian.

Proof. Using Proposition 5.36, it suffices to show that $\mathrm{PG}(2,2)$ and R_9 are extremally artinian. Since $\mathrm{PG}(2,2)/e$ and R_9/e have rank 2 for every point e, Proposition 3.10 and Proposition 3.26 imply they are not artinian.

Furthermore, for every point e of $\mathrm{PG}(2,2)$ the matroid $\mathrm{PG}(2,2)\backslash e$ is a binary matroid which does not have $\mathrm{PG}(2,2)$ as a minor and therefore is regular. Thus, $\mathrm{PG}(2,2)$ is extremally artinian.

To prove that R_9 is extremally artinian we use the result from [Kun90] that there are three isomorphy types of ternary combinatorial geometries on 9 points, which are obtained by removing 4 points from $\mathrm{PG}(2,3)$:

We get the ternary affine plane $\mathrm{AG}(2,3)$ by removing a line, the ternary Reid geometry R_9 by removing 3 points on a common line ℓ and a point $\omega \notin \ell$, and the ternary Dowling geometry over the cyclic group C_2 of two elements by removing a circuit.

Let e be a point of R_9. We will show that $R_9\backslash e$ is representable over $\mathbb{C}$. This implies $R_9\backslash e$ is not artinian, as if $R_9\backslash e$ were artinian, then Proposition 3.6 would imply the existence of a homomorphism $f\colon \mathbb{F}_3 \to \mathbb{U}^{(0)}(M)$ of partial hyperfields. Hence, Theorem and Definition 3.16 would yield that there would exist a homomorphism $\mathbb{F}_3 \to \mathbb{C}$ (which would be a classical field homomorphism), a contradiction.

Let ℓ be a line and $\omega \notin \ell$ a point of $\mathrm{PG}(2,3)$ such that $R_9 = \mathrm{PG}(2,3)\backslash(\ell \cup \{\omega\})$. If $e \in \ell$, then $R_9\backslash e$ is also a minor of $\mathrm{AG}(2,3)$ and thus Corollary 3.35 yields it is representable over $\mathbb{C}$.

Otherwise, if $e \notin \ell$, there exits a circuit C containing e, ω and two points from ℓ. Therefore, $R_9\backslash e$ is a minor of the ternary Dowling geometry. It follows from [Dow73, Theorem 11] that $R_9\backslash e$ is representable over $\mathbb{C}$. $\square$

6 Affine and projective like matroids

In this chapter we will present additional classes of examples for artinian matroids whose universal partial hyperfield is a hyperfield.

First, we will introduce a generalization of vector matroids over skew fields (and thus also a generalization of projective geometries) and of affine geometries. In particular, we will show that each affine geometry of dimension at least 3 has this property.

Second, we will prove that the universal partial hyperfield of an affine translation plane, whose kernel contains at least 4 elements, is isomorphic to that of its projective closure.

6.1 Vectorlike matroids

By generalizing a construction from Kalhoff in [Kal96], we will construct a series of artinian matroids over quasifields Q of rank greater or equal to three, whose universal partial hyperfield is the isomorphic to the one of the projective plane over Q. Although no longer modular in general, these matroids are supersolvable, i. e., they contain a maximal chain of flats, where each of them is modular. Their simplifications are thus a generalization of projective geometries of dimension at least 3 to the quasifields case.

Moreover, we will construct generalizations of affine geometries of dimension at least 3, whose universal partial field is isomorphic to the one of the projective plane over Q. In particular, the universal partial hyperfield of an affine geometry of dimension greater or equal to 3 is isomorphic to the one of their projective closure.

Definition. A *quasifield*[1] is a set Q with two binary operations $+\colon Q\times Q \to Q$, $\cdot\colon Q \times Q \to Q$ that satisfy the following axioms:

(Q1) $(Q, +)$ is a group with neutral element $0 \in Q$,[2]

(Q2) $(Q \setminus \{0\}, \cdot)$ is a loop[3] with neutral element $1 \in Q$,

[1] We consider only left quasifields here. Every quasifield is a planar ternary ring via $T(m, x, c) := mx + c$ for all $m, x, c \in Q$.

[2] This group is always abelian, cf. [Pic, Satz 31, p. 91].

[3] As usual, we denote by a/b and $a\backslash b$ the unique elements such that $(a/b)b = a$ and $a(a\backslash b) = b$.

(Q3) $a(b+c) = ab + ac$ for all $a, b, c \in Q$,

(Q4) for all $a, b, c \in Q$ such that $a \neq b$ the equation $ax - bx = c$ has exactly one solution $x \in Q$.

For each quasifield Q we define the *left nucleus*, the *right nucleus* and the *kernel* of Q by

$$\begin{aligned}
\mathcal{N}_l &:= \mathcal{N}_l(Q) := \{q \in Q \mid q(ab) = (qa)b \text{ for all } a, b \in Q\},\\
\mathcal{N}_r &:= \mathcal{N}_r(Q) := \{q \in Q \mid (ab)q = a(bq) \text{ for all } a, b \in Q\},\\
K &:= \operatorname{Ker} Q := \{q \in \mathcal{N}_r \mid (a+b)q = aq + bq \text{ for all } a, b \in Q\}.
\end{aligned}$$

In particular, Q is a right-vector space over K.

For any quasifield Q and $n \in \mathbb{N}$, $n \geq 2$, we call a tuple $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ a *system of coordinates for Q and n* if $\Lambda_1 = \Lambda_2 = Q$ and $\Lambda_n \subseteq \dots \subseteq \Lambda_3 \subseteq K$ is a chain of skew fields. For any system of coordinates Λ for Q and n, we set $E_{n,\Lambda} := \{(x_1, \dots, x_n) \in Q^n \mid x_i \in \Lambda_i\}$ and for all $a_1, \dots, a_n \in Q$ let

$$[a_1, \dots, a_n]_\Lambda := \left\{ x \in E_{n,\Lambda} \;\middle|\; \sum_{i=1}^{n} a_i x_i = 0 \right\}.$$

Further, we set $\mu(0) := 0$ and

$$\mu(x) := \max_{\preceq}\{i \in \{1, \dots, n\} \mid x_i \neq 0\}$$

for any $x \in E_{n,\Lambda} \setminus \{0\}$, where $\preceq$ is the total order on the set $\{1, \dots, n\}$ defined by

$$2 \prec 1 \prec 3 \prec 4 \prec \dots \prec n.$$

Moreover, if $\mu(x) \neq 0$, then $\lambda := x_{\mu(x)} \neq 0$ and $\bar{x} := (x_1/\lambda, \dots, x_n/\lambda) \in E_{n,\Lambda}$ is an element such that $\mu(\bar{x}) = \mu(x)$ and $\bar{x}_{\mu(\bar{x})} = \lambda/\lambda = 1$. We call the elements $x \in E_{n,\Lambda} \setminus \{0\}$ such that $x_{\mu(x)} = 1$ the *canonical elements* of $E_{n,\Lambda}$.

Furthermore, we call a subgroup V of $(E_{n,\Lambda}, +)$ a Λ-*subspace* of $E_{n,\Lambda}$ if $\bar{x}\Lambda_{\mu(x)} = \{\bar{x}\lambda | \lambda \in \Lambda_{\mu(x)}\} \subseteq V$ for all $x \in V \setminus \{0\}$. The *dimension* of a Λ-subspace V of $E_{n,\Lambda}$ is defined as supremum of the length k of chains $V_0 \subsetneq \dots \subsetneq V_k \subseteq V$ of Λ-subspaces V_i of $E_{n,\Lambda}$, $i = 1, \dots, k$, and is denoted by $\dim V$.

For the rest of this section let Q be quasifield.

6.1 Lemma. Let Λ be a system of coordinates for Q and $n \in \mathbb{N}$, $n \geq 2$. For any Λ-subspace V of $E_{n,\Lambda}$ we have $\dim V \leq n$.

Furthermore, the Λ-subspaces of $E_{n,\Lambda}$ of dimension $k \in \{0, \dots, n\}$ are exactly the sets of the form $\sum_{i=1}^{k} x_i \Lambda_{\mu(x_i)}$ for canonical elements $x_1, \dots, x_k \in E_{n,\Lambda}$ such that $\mu(x_1) \prec \dots \prec \mu(x_k)$. In particular, $\dim V = n$ if and only if $V = E_{n,\Lambda}$.

Proof. We will first show that for canonical elements $x_1, \ldots, x_k \in E_{n,\Lambda}$ with $\mu(x_1) \prec \cdots \prec \mu(x_k)$ the set $V := \sum_{i=1}^k x_i \Lambda_{\mu(x_i)}$ is a Λ-subspace of $E_{n,\Lambda}$.

To prove that V is a subgroup of $(E_{n,\Lambda}, +)$, let $x, y \in V$. Then there exist $\lambda_i, \mu_i \in \Lambda_{\mu(x_i)}$, $i = 1, \ldots, k$, with $x = \sum_{i=1}^k x_i \lambda_i$ and $y = \sum_{i=1}^k x_i \mu_i$. Using (Q3), we obtain $x_i \lambda_i + x_i \mu_i = x_i(\lambda_i + \mu_i)$ for all $i = 1, \ldots, k$. Thus, $x + y \in V$.

Moreover, let $x \in V \setminus \{0\}$ and $\lambda \in \Lambda_m$, where $m := \mu(x)$. Then there exist $\lambda_i \in \Lambda_{\mu(x_i)}$, $i = 1, \ldots, k$, such that $x = \sum_{i=1}^k x_i \lambda_i$. We show $\bar{x}\lambda \in V$.

For $k = 1$ we have $\bar{x} = x_1$, so there is nothing to prove. If $m \geq 3$, then Λ_m is a skew field contained in K and therefore

$$\bar{x}\lambda = \left(\sum_{i=1}^k x_i \lambda_i \lambda_m^{-1} \right) \lambda = \sum_{i=1}^k x_i (\lambda_i \lambda_m^{-1} \lambda) \in V.$$

The only remaining case is $k = 2$ and $m = 1$. Since $\lambda_i = 0$ for all $i = 3, \ldots, n$ we can assume without loss of generality that $n = 2$. Then there exists a $u \in Q$ such that $x_1 = (0, 1)$ and $x_2 = (1, u)$. It follows that $\bar{x} = (1, v)$, where $v := (\lambda_1 + u\lambda_2)/\lambda_2$. Setting $w := v\lambda - u\lambda$ we obtain

$$\bar{x}\lambda = (0, w) + (\lambda, u\lambda) = x_1 w + x_2 \lambda \in V.$$

Conversely, let V be a Λ-subspace of $E_{n,\Lambda}$. We set

$$k(V) := |\{\mu(x) \mid x \in V \setminus \{0\}\}| \in \{0, \ldots, n\}.$$

We will prove by induction on $k(V)$ that $\dim V = k(V)$ and there exist canonical elements $x_1, \ldots, x_{k(V)} \in V$ with $V = \sum_{i=1}^{k(V)} x_i \Lambda_{\mu(x_i)}$.

Clearly, $k(V) = 0$ implies $V = \{0\}$. If $k := k(V) > 0$, we choose a canonical element $x_k \in V$ such that $\mu(x_k)$ is maximal among all $\mu(x)$ with respect to $\preceq$, $x \in V$, and set $\tilde{V} := \{x \in V \mid \mu(x) \prec \mu(x_k)\}$.

As $\mu(x + y) \preceq \max\{\mu(x), \mu(y)\}$ and $\mu(\bar{x}\lambda) \preceq \mu(x)$ for all $x, y \in \tilde{V}$ and $\lambda \in \Lambda_{\mu(x)}$, $\tilde{V}$ is a Λ-subspace of $E_{n,\Lambda}$. By construction, we have $k(\tilde{V}) = k(V) - 1$. Thus, the induction hypothesis yields $\dim(\tilde{V}) = k - 1$ and there exist canonical elements $x_1, \ldots, x_{k-1} \in \tilde{V}$ such that $\tilde{V} = \sum_{i=1}^{k-1} x_i \Lambda_{\mu(x_i)}$.

It follows that $\sum_{i=1}^k x_i \Lambda_{\mu(x_i)} \subseteq V$, because V is a Λ-subspace of $E_{n,\Lambda}$. For all $x \in V \setminus \tilde{V}$, we have $\mu(x) = \mu(x_k)$ and therefore $\bar{x} - x_k \in \tilde{V}$. Hence,

$$\bar{x} \in \tilde{V} + x_k \Lambda_{\mu(x_k)} = \sum_{i=1}^k x_i \Lambda_{\mu(x_i)}.$$

Since $\sum_{i=1}^k x_i \Lambda_{\mu(x_i)}$ is a Λ-subspace of $E_{n,\Lambda}$ it also contains x. Therefore, $V = \sum_{i=1}^k x_i \Lambda_{\mu(x_i)}$.

Further, $\dim(V) \geq \dim(\tilde{V}) + 1 = k$. If there would exist a Λ-subspace $V' \subsetneq V$ of $E_{n,\Lambda}$ such that $k(V') = k$, then for $\tilde{V}' := \{x \in V' \mid \mu(x) \prec \mu(x_k)\}$ we would have $\{\mu(x) \mid x \in \tilde{V}'\} = \{\mu(x) \mid x \in \tilde{V}\}$. Using the same argument as above, we would get $\tilde{V}' = \sum_{i=1}^{k-1} x_i \Lambda_{\mu(x_i)} = \tilde{V}$ and thus $V' = V$, a contradiction. Hence, using the induction hypothesis, we get $\dim(V) = k$, as desired. □

Definition. Let $n \in \mathbb{N}$, $n \geq 2$. We call $a_1, \ldots, a_n \in Q$ *admissible* for a system of coordinates Λ for Q and n, if $a_2 \in \mathcal{N}_l$, and for all $j = 1, \ldots, n$ we have

$$a_j \in \sum_{i \prec j} a_i \Lambda_i, \qquad \text{or} \qquad a_j \neq 0 \text{ and } a_i = 0 \text{ for all } i \prec j. \tag{6.1}$$

Further, let $\mathcal{H}_{n,\Lambda}$ be the set of all $[a_1, \ldots, a_n]_\Lambda$, where $a_1, \ldots, a_n \in Q$ are admissible for Λ and $a_i \neq 0$ for at least one $i \in \{1, \ldots, n\}$.

6.2 Lemma. Let $n \in \mathbb{N}$, $n \geq 3$, $a_1, \ldots, a_n \in Q$ be admissible for a system of coordinates Λ for Q and n, and $k \in \{3, \ldots, n\}$. Then $a_1, \ldots, \widehat{a_k}, \ldots, a_n$ are admissible for $\Lambda^{(k)} := (\Lambda_1, \ldots, \widehat{\Lambda_k}, \ldots, \Lambda_n)$.

Proof. Clearly, (6.1) is satisfied if $j = 1, \ldots, k-1$, or $a_k = 0$, or $a_i = 0$ for all $i \prec j$. Let $j \in \{k+1, \ldots, n\}$, $a_k \neq 0$, and $a_l \neq 0$ for an $l \in \{1, \ldots, j-1\}$. Since $a_1, \ldots, a_n$ are admissible for Λ there exist $\lambda_i \in \Lambda_i$, $i = 1, \ldots, j-1$ and $\mu_i \in \Lambda_i$, $i = 1, \ldots, k-1$ such that $a_j \in \sum_{i=1}^{j-1} a_i \lambda_i$ and $a_k \in \sum_{i=1}^{k-1} a_i \mu_i$. Thus,

$$a_j \in \sum_{i=1}^{k-1} a_i(\lambda_i + \mu_i \lambda_k) + \sum_{i=k+1}^{j-1} a_i \lambda_i.$$

Hence, $a_1, \ldots, \widehat{a_k}, \ldots, a_n$ are admissible for $\Lambda^{(k)}$. □

6.3 Lemma and Definition. Let Λ be a system of coordinates for Q and $n \in \mathbb{N}$, $n \geq 2$. For each $H := [a_1, \ldots, a_n]_\Lambda \in \mathcal{H}_{n,\Lambda}$ we set

$$\mu(H) := \min_{\preceq}\{k \in \{1, \ldots, n\} \mid a_k \neq 0\}.$$

Then we have $H \subseteq H' := [a_1', \ldots, a_n']_\Lambda \in \mathcal{H}_{n,\Lambda}$ if and only if $m := \mu(H) = \mu(H')$ and $a_m \backslash a_i = a_m' \backslash a_i'$ for all $i = 1, \ldots, n$ (and thus $H = H'$).

In particular, the a_i are uniquely determined by H if $a_m = 1$, and we call these coefficients the *canonical representation of H*.

Proof. We will first show that we can assume without loss of generality that $a_m = 1 = a'_{m'}$, where $m' := \mu(H')$. If $m = 2$, since $a_2 \in \mathcal{N}_l$, it follows for all $x \in E_{n,\Lambda}$

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} (a_2 \cdot a_2 \backslash a_i) x_i = \sum_{i=1}^{n} a_2 \cdot (a_2 \backslash a_i \cdot x_i) = a_2 \left(\sum_{i=1}^{n} a_2 \backslash a_i \cdot x_i \right).$$

If $m \succeq 1$, set $j := \max_{\preceq}(3, m+1)$. For all $x \in E_{n,\Lambda}$ we obtain, as $x_i \in K$ for all $i \succeq m'$,

$$\begin{aligned} a_m x_m + \sum_{i=j}^{n} a_i x_i &= a_m x_m + \sum_{i=j}^{n} (a_m \cdot a_m \backslash a_i) x_i \\ &= a_m x_m + \sum_{i=j}^{n} a_m ((a_m \backslash a_i) x_i) \\ &= a_m \cdot \left(\sum_{i=1}^{n} (a_m \backslash a_i) \cdot x_i \right). \end{aligned}$$

Further, $a_m \backslash a_2 = 0 = a_2 \in \mathcal{N}_l$.

Hence, in both cases, $a_m \backslash a_1, \ldots, a_m \backslash a_n$ are admissible for Λ by definition, and $x \in [a_1, \ldots, a_n]_\Lambda$ if and only if $x \in [a_m \backslash a_1, \ldots, a_m \backslash a_n]_\Lambda$. Thus, we can assume without loss of generality that $a_m = 1 = a'_{m'}$.

It remains to show $H \subseteq H'$ implies $a_i = a'_i$ for all $i \in \{1, \ldots, n\}$. If $m = n$, we obtain $a_1 = \ldots = a_{n-1} = 0$ and $a_n = 1$. Since for every $k \in \{1, \ldots, n-1\}$ the element $x \in E_{n,\Lambda}$ such that $x_i = \delta_{ik}$ for all $i = 1, \ldots, n$ is contained in $H' \supseteq H$ it follows that $a'_i = 0$ for all $i = 1, \ldots, n-1$.

Otherwise, let $m < n$. We will prove first there exists an $x \in H$ such that $\mu(x) = n$ and thus $m' < n$. If $a_n = 0$, set $x_i = \delta_{in}$ for all $i = 1, \ldots, n$.

If $a_n \neq 0$, there exist $\lambda_i \in \Lambda_i$, $i = 1, \ldots, n-1$, such that $a_n = \sum_{i=1}^{n-1} a_i \lambda_i$, as $a_1, \ldots, a_n$ are admissible for Λ, so we set $x_i := -\lambda_i$ and $x_n := 1$.

Now, we will proceed by induction on n. If $n = 2$, it follows that $a_2 = 1 = a'_2$ and $a_1 x_1 + x_2 = 0 = a'_1 x_1 + x_2$. Thus, $a_1 = -x_2/x_1 = a'_1$.

Else, we have $n \geq 3$ and $a_1, \ldots, a_{n-1}$ as well as $a'_1, \ldots, a'_{n-1}$ are admissible for $\Lambda^{(n)}$. Additionally, $H^{(n)} := [a_1, \ldots, a_{n-1}]_{\Lambda^{(n)}}$ and $H'^{(n)} := [a'_1, \ldots, a'_{n-1}]_{\Lambda^{(n)}}$ are the canonical representations of two hyperplanes such that $H^{(n)} \subseteq H'^{(n)}$. Therefore, the induction hypothesis implies $a_i = a'_i$ for $i = 1, \ldots, n-1$.

Moreover, our construction of x yields $a_n = 0$ if and only if $a'_n = 0$ and $a_n = \sum_{i=1}^{n-1} a_i \lambda_i = \sum_{i=1}^{n-1} a'_i \lambda_i = a'_n$ otherwise. □

6.4 Proposition. For any system of coordinates Λ for Q and $n \in \mathbb{N}$, $n \geq 2$, the sets of $\mathcal{H}_{n,\Lambda}$ are the hyperplanes of a matroid $M(n, \Lambda)$ of rank n on the set $E_{n,\Lambda}$.

Proof. First, (H2) follows directly from Lemma and Definition 6.3. To complete the proof we will show by induction on n that the set of Λ-subspaces of $E_{n,\Lambda}$ of dimension $n-1$ is $\mathcal{H}_{n,\Lambda}$, which directly implies (H1).

To prove that it yields also (H3), let $H_1, H_2 \in \mathcal{H}_{n,\Lambda}$ with $H_1 \neq H_2$ and $x \in E_{n,\Lambda} \setminus (H_1 \cup H_2)$. Then $H_1 \cap H_2$ is subspace of dimension $k \in \mathbb{N}_0$, $k \leq n-2$. Clearly, the Λ-subspace V generated by $H_1 \cap H_2$ and x has dimension $k+1$. Therefore, there exists a Λ-subspace H_3 of rank $n-1$ for which we have $(H_1 \cap H_2) \cup \{x\} \subseteq H_3$.

Finally, for any subset $X \subseteq E_{n,\Lambda}$ we have $X \nsubseteq H$ for all $H \in \mathcal{H}_{n,\Lambda}$ if and only if for each $k = 1, \dots, n$ the set X contains an $x \in E_{n,\Lambda}$ such that $\mu(x) = k$. Thus, $\mathcal{H}_{n,\Lambda}$ satisfies (H0) and (H1), and the resulting matroid has rank n.

If $n = 2$, then $H \in \mathcal{H}_{n,\Lambda}$ if and only if either $H = [1, 0]_\Lambda = \{(0, y) \mid y \in Q\}$ or $H = [a, 1]_\Lambda = \{(x, -ax) \mid x \in Q\}$ for an $a \in Q$. In this case our claim follows directly from Lemma 6.1.

Else, let $n \geq 3$ and $H = [a_1, \dots, a_n]_\Lambda \in \mathcal{H}_{n,\Lambda}$ be in canonical representation. If $a_1 = \cdots = a_{n-1} = 0$ and $a_n = 1$, then $H = \sum_{i=1}^{n-1} x_i \Lambda_i$ for any canonical elements $x_i \in E_{n,\Lambda}$ with $\mu(x_i) = i$, $i = 1, \dots, n-1$. Hence, H is a Λ-subspace of $E_{n,\Lambda}$ of dimension $n-1$ using Lemma 6.1.

Otherwise, there exist $\lambda_i \in \Lambda_i$, $i = 1, \dots, n-1$, such that $a_n = \sum_{i=1}^{n-1} a_i \lambda_i$, since $a_1, \dots, a_{n-1}$ are admissible for Λ. Lemma 6.2 yields that $a_1, \dots, a_{n-1}$ are admissible for $\Lambda^{(n)}$, where $\Lambda^{(n)} = (\Lambda_1, \dots, \Lambda_{n-1})$, and therefore by induction hypothesis there exist canonical elements $y_1, \dots, y_{n-2} \in E_{n-1,\Lambda^{(n)}}$ such that $\mu(y_1) \prec \cdots \prec \mu(y_{n-2}) \leq n-1$ and $[a_1, \dots, a_{n-1}]_{\Lambda^{(n-1)}} = \sum_{i=1}^{n-2} y_i \Lambda_{\mu(y_i)}$.

Thus, $H = \sum_{i=1}^{n-1} x_i \Lambda_{\mu(x_i)}$ and $\mu(x_1) \prec \cdots \prec \mu(x_n)$ for $x_i := (y_i, 0) \in E_{n,\Lambda}$, $i = 1, \dots, n-2$ and $x_{n-1} := (\lambda_1, \dots, \lambda_n, -1)$. Hence, Lemma 6.1 implies H is a Λ-subspace of $E_{n,\Lambda}$ of dimension $n-1$.

Conversely, let $V = \sum_{i=1}^{n-1} x_i \Lambda_{\mu(x_i)}$ be a Λ-subspace of $E_{n,\Lambda}$ of dimension $n-1$, where $x_1, \dots, x_{n-1}$ are canonical elements of $E_{n,\Lambda}$ with $\mu(x_1) \prec \cdots \prec \mu(x_{n-1})$.

If $\mu(x_{n-1}) < n-1$, then $V = [0, \dots, 0, 1]_\Lambda$. Else, set $y_i = (x_{i,1}, \dots, x_{i,n-1})$, where $x_i = (x_{i,1}, \dots, x_{i,n})$, $i = 1, \dots, n-2$. Then $\mu(y_i) = \mu(x_i)$, $i = 1, \dots, n-2$, and $\tilde{V} := \sum_{i=1}^{n-2} y_i \Lambda_{\mu(y_i)}$ is a $\Lambda^{(n)}$-subspace of $E_{n-1,\Lambda^{(n)}}$.

By induction hypothesis, there exist $a_1, \dots, a_{n-1} \in Q$ which are admissible for $\Lambda^{(n)}$ such that $\tilde{V} = [a_1, \dots, a_{n-1}]_{\Lambda^{(n)}}$. Since $\mu(x_n) = n$ and $-1 \in K$ we have $a_n = \sum_{i=1}^{n-1} a_i(-x_i)$. Thus, $V \subseteq [a_1, \dots, a_n]_\Lambda =: H$. As both have dimension $n-1$ we get $V = H$. $\square$

6.5 Corollary. For any system of coordinates Λ for Q and $n \in \mathbb{N}$, $n \geq 2$, the flats of rank k of $M(n, \Lambda)$ are the Λ-subspaces of $E_{n,\Lambda}$ of dimension k.

Proof. Clearly, every intersection of Λ-subspaces of $E_{n,\Lambda}$ is also an Λ-subspace of $E_{n,\Lambda}$. Thus, the proof of Proposition 6.4 implies that every flat of $M(n, \Lambda)$ is a Λ-subspace of $E_{n,\Lambda}$.

Conversely, let V be a Λ-subspace of dimension $k \in \{0, \dots, n\}$. We will prove by induction on k that V is a flat of rank k of $M(n, \Lambda)$. If $k = n$, then $V = E$ is a flat of $M(n, \Lambda)$ of rank n.

Otherwise, $k < n$ and it follows from the proofs of Lemma 6.1 and Proposition 6.4 there exist $H \in \mathcal{H}_{n,\Lambda}$ such that $V \subseteq H$ and a canonical element $y \in E_{n,\Lambda}$ with $y \notin H$. Thus, $V' := y\Lambda_{\mu(y)} + V$ is a Λ-subspace of $E_{n,\Lambda}$ of dimension $k+1$. Therefore, the induction hypothesis yields that V' is a flat of rank $k+1$. Hence, $V = V' \cap H$ is a flat of rank k. $\square$

6.6 Lemma. Let Λ be a system of coordinates for Q and $n \in \mathbb{N}$, $n \geq 4$. For each $k \in \{1, \dots, n\}$ the hyperplane $U^k_{n,\Lambda} := [\delta_{1k}, \dots, \delta_{nk}]_\Lambda$ of $M(n, \Lambda)$ is modular if and only if $k \in \{n-1, n\}$ or $\Lambda_k = \ldots = \Lambda_{n-1}$.

Proof. Throughout this proof, we will use the fact that a hyperplane of a matroid M is modular if and only if has a non-trivial intersection with each line of M (see [Bry75, Corollary 3.4]).

Let $\ell = x\Lambda_{\mu(x)} + y\Lambda_{\mu(y)}$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in E_{n,\Lambda}$ are canonical elements such that $\mu(x) \prec \mu(y)$. Since $\mu(x) < n$ it follows that $x \in U^n_{n,\Lambda}$. Furthermore, either $\mu(x) < n-1$ and $x \in U^{n-1}_{n,\Lambda}$, or $xy_{n-1} - y$ is a non-zero element which is contained in both ℓ and $U^{n-1}_{n,\Lambda}$. Therefore, $U^k_{n,\Lambda}$ is a modular hyperplane for $k \in \{n-1, n\}$.

If $\Lambda_k = \ldots = \Lambda_{n-1}$, then $x_k = 0$ and $x \in U^k_{n,\Lambda}$, or $y_k = 0$ and $y \in U^k_{n,\Lambda}$, or $x_k, y_k \in \Lambda^*_k$. In the last case we obtain $x\lambda - y \in \ell \cap U^k_{n,\Lambda}$, where $\lambda := x_k \backslash y_k$.

Conversely, if $U^k_{n,\Lambda}$ is modular and $k \leq n-2$, we will show that the dimension of Λ_k as a right Λ_{n+1}-vector space is 1, which implies $\Lambda_k = \ldots = \Lambda_{n-1}$.

Let $a, b \in \Lambda_k$ and set $x_{n-1} := 1 =: y_n$, $x_k := a$, $y_k := b$ and $x_i := 0 =: y_j$ for all $i, j = 1, \dots, n$ such that $i \neq k, n-1$, $j \neq k, n$. Then $\ell_{a,b} := x\Lambda_{n-1} + y\Lambda_n$ is a line of $M(n, \Lambda)$ with $\varrho(\ell_{a,b} \cap U^k_{n,\Lambda}) = 1$.

Thus, there exist $\lambda_i \in \Lambda_i$, $i = n-1, n$, such that $z = x\lambda_{n-1} + y\lambda_n \in U^k_{n,\Lambda}$. In particular, we have $a\lambda_{n-1} + b\lambda_n = 0$. $\square$

6.7 Proposition. Let Λ be a system of coordinates for Q and $n \in \mathbb{N}$, $n \geq 3$. For every $H \in \mathcal{H}_{n,\Lambda}$ we set

$$f_H \colon E_{n,\Lambda} \to Q/_{*}R_a, \quad x \mapsto R_a\left(\sum_{i=1}^{n} a_i x_i\right),$$

where $H = [a_1, \dots, a_n]_\Lambda$ is the canonical representation of H. Then $(f_H)_{H\in\mathcal{H}_{n,\Lambda}}$ is a system of hyperplane functions for $M(n,\Lambda)$ and $Q/_{*}R_a$.

Proof. By construction, we have $f_H(e) = 0$ if and only if $e \in H$ for all $H \in \mathcal{H}_{n,\Lambda}$ and $e \in E_{n,\Lambda}$. We will prove our claim by induction on n.

First, let $n = 3$ and $\Pi = (E, \mathcal{H})$ be the projective plane over Q. The map $\varphi\colon \mathrm{s}E_{3,\Lambda} \to E$ defined by

$$\varphi(\mathrm{s}(x, y, 1)) := (x, y), \quad \varphi(\mathrm{s}(1, m, 0)) := (m), \quad \varphi(\mathrm{s}(0, 1, 0)) := (\infty)$$

for all $m, x, y \in Q$ is an isomorphism, since we have

$$\varphi([-m, 1, -c]_\Lambda) = [m, c], \quad \varphi([1, 0, -d]_\Lambda) = [d], \quad \varphi([0, 0, 1]_\Lambda) = [\infty]$$

for all $c, d, m \in Q$.

Let $(g_H)_{H\in\mathcal{H}}$ be the system of hyperplane functions for Π and $Q/_{*}R_a$ from Theorem 3.24. For all $c, d, m, n, x, y \in Q$ we obtain

$$\begin{aligned}
g_{[m,c]}((x, y)) &= R_a(mx + c - y) = -f_{[-m,1,-c]_\Lambda}((x, y, 1)),\\
g_{[m,c]}((n)) &= R_a(m - n) = -f_{[-m,1,-c]_\Lambda}((1, n, 0)),\\
g_{[m,c]}((\infty)) &= R_a = f_{[-m,1,-c]_\Lambda}((0, 1, 0)),\\
g_{[d]}((x, y)) &= R_a(x - d) = f_{[1,0,-d]_\Lambda}((x, y, 1)),\\
g_{[d]}((n)) &= R_a = f_{[1,0,-d]_\Lambda}((1, n, 0)),\\
g_{[\infty]}((x, y)) &= R_a = f_{[0,0,1]_\Lambda}((x, y, 1)).
\end{aligned}$$

Therefore, $f_H(e) = \eta(H)\eta(e)g_{\varphi(H)}(\varphi(e))$, where

$$\begin{aligned}
\eta(H) &:= -1 \quad \Leftrightarrow \quad H = [-m, 1, -c] \text{ for } m, c \in Q,\\
\eta(e) &:= -1 \quad \Leftrightarrow \quad e = (0, 1, 0)
\end{aligned}$$

for all $H \in \mathcal{H}_{n,\Lambda}$ and $e \in E_{n,\Lambda}$. Thus, Proposition and Definition 3.14 and Theorem and Definition 3.16 yield that $(f_H)_{\mathcal{H}_{n,\Lambda}}$ is a system of hyperplane functions for $M(n,\Lambda)$ and $Q/_{*}R_a$ in this case.

Now, let $n > 3$ and H_1, H_2, H_3 be pairwise different hyperplanes of $M(n,\Lambda)$ which contain a common hyperline L and let $H_i = [a_{i,1}, \dots, a_{i,n}]_\Lambda$ be their canonical representation, $i = 1, 2, 3$. Applying Proposition and Definition 3.14, we have to show there exist $\alpha_2, \alpha_3 \in (Q/_{*}R_a)^*$ such that $0 \in f_{H_1} + \alpha_2 f_{H_2} + \alpha_3 f_{H_3}$.

If $L = U^{n-1}_{n,\Lambda} \cap U^{n}_{n,\Lambda}$, we choose canonical elements $x_i \in H_i \setminus L$, $i = 2, 3$. We can assume without loss of generality that $\mu(x_2) = \mu(x_3) = n$. Further, set $\alpha_2 := -f_{H_1}(x_3)f_{H_2}(x_3)^{-1}$ and $\alpha_3 := -f_{H_1}(x_2)f_{H_3}(x_2)^{-1}$.

As $\mu(x_2 - x_3) = n - 1$ (otherwise $x_2 - x_3 \in L$ and thus $H_2 = H_3$), it follows from Lemma 6.1 that for every $x \in E_{n,\Lambda}$ there exist unique $y \in L$ and $\lambda_i \in \Lambda_i$, $i = n-1, n$ with $x = y + (x_2 - x_3)\lambda_{n-1} + x_3\lambda_n$. This implies

$$\begin{aligned} 0 \in f_{H_1}(x) - f_{H_1}(x) &\subseteq f_{H_1}(x) - f_{H_1}(x_2)\lambda_{n-1} - f_{H_1}(x_3)(\lambda_n - \lambda_{n-1}) \\ &\subseteq f_{H_1}(x) + \alpha_2 f_{H_2}(x_3)(\lambda_n - \lambda_{n-1}) + \alpha_3 f_{H_3}(x_2)\lambda_{n-1} \\ &\subseteq f_{H_1}(x) + \alpha_2 f_{H_2}(x) + \alpha_3 f_{H_3}(x). \end{aligned}$$

Otherwise, there exists a $j \in \{n-1, n\}$ such that $L \not\subseteq U^j_{n,\Lambda}$. Moreover, we set $G_i \coloneqq [a_{i,1}, \ldots, \widehat{a_{i,j}}, \ldots, a_{i,n}]_\Lambda$, $i = 1, 2, 3$. Since $U^j_{n,\Lambda}$ is modular by Lemma 6.6 and $M(n-1, \Lambda^{(j)})$ is isomorphic to $M(n, \Lambda)|U^j_{n,\Lambda}$, G_1, G_2, G_3 are pairwise different hyperplanes of $M(n-1, \Lambda^{(j)})$ which intersect in the hyperline $K \coloneqq \{(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \mid x \in L\}$.

Applying the induction hypothesis, we obtain $\alpha_1, \alpha_2, \alpha_3 \in Q/_*R_a$ such that $0 \in \alpha_1 f_{G_1} + \alpha_2 f_{G_2} + \alpha_3 f_{G_3}$. Let $x \in E_{n,\Lambda}$. Using Lemma 6.1, there exist $y \in U^j_{n,\Lambda}$ and $z \in L \setminus U^j_{n,\Lambda}$ with $x = y + z$. Setting $w \coloneqq (y_1, \ldots, \widehat{y_j}, \ldots, y_n)$ we get

$$\begin{aligned} 0 \in \alpha_1 f_{G_1}(w) + \alpha_2 f_{G_2}(w) + \alpha_3 f_{G_3}(w) \\ \subseteq \alpha_1 f_{H_1}(y) + \alpha_2 f_{H_1}(y) + \alpha_3 f_{H_3}(y) \\ \subseteq \alpha_1 f_{H_1}(x) + \alpha_2 f_{H_2}(x) + \alpha_3 f_{H_3}(x), \end{aligned}$$

completing our proof. □

6.8 Theorem. For any system of coordinates Λ for Q and $n \in \mathbb{N}$, $n \geq 3$, the universal partial hyperfield of $M(n, \Lambda)$ is isomorphic to $Q/_*R_a$.

Proof. Let $M \coloneqq M(n, \Lambda)$ and $(f_H)_{H \in \mathcal{H}_{n,\Lambda}}$ be the system of hyperplane functions for M and $Q/_*R_a$ from Proposition 6.7. Applying Theorem and Definition 3.16, it induces a homomorphism $f\colon \mathbb{U}^{(0)}(M) \to Q/_*R_a$. We will show by induction on n that f is an isomorphism. For $n = 3$ we obtain this directly from Theorem 3.24 and the proof of Proposition 6.7.

Let $n \geq 4$, $\Lambda' \coloneqq \Lambda^{(n)}$, $M' \coloneqq M(n-1, \Lambda')$, $(f'_H)_{H \in \mathcal{H}_{n-1,\Lambda'}}$ be the system of hyperplane functions for M' and $Q/_*R_a$, and $f'\colon \mathbb{U}^{(0)}(M') \to Q/_*R_a$ be the induced homomorphism.

Since it follows from Lemma 6.6 that $U^n_{n,\Lambda}$ is a modular hyperplane and M' is isomorphic to $M|U^n_{n,\Lambda}$, the proof of Proposition 5.29 yields that the map $g\colon \mathbb{U}^{(0)}(M') \to \mathbb{U}^{(0)}(M)$ defined by $g(0) \coloneqq 0$, $g(-1) \coloneqq -1$, and

$$g([H_1, H_2 \mid H_3, H_4]) \coloneqq [H_1 \vee e, H_2 \vee e \mid H_3 \vee e, H_4 \vee e]$$

for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M')$, where $e \coloneqq (\delta_{1n}, \dots, \delta_{nn}) \in E_{n,\Lambda}$, is a surjective homomorphism of partial hyperfields.

As $[a_1, \dots, a_{n-1}]_{\Lambda'} \vee e = [a_1, \dots, a_{n-1}, 0]_\Lambda$ for all $a_1, \dots, a_{n-1} \in Q$ which are admissible for Λ' and $n-1$, Proposition and Definition 3.14 and Proposition 6.7 imply $f \circ g = f'$. Using the induction hypothesis, f' is an isomorphism, and therefore g is bijective. Hence, also f is bijective.

For all $a_1', a_2' \in Q/_{*}R_a$ and $a_3' \in a_1' + a_2'$ set $a_i \coloneqq g(f'^{-1}(a_i')) \in \mathbb{U}^{(0)}(M)$, $i = 1, 2, 3$. Then $a_i \in f^{-1}(a_i')$ for all $i = 1, 2, 3$ and $a_3 \in a_1 + a_2$, as f' is an isomorphism and g a homomorphism of partial hyperfields. Thus, f is an epimorphism and it follows from Proposition 2.14 that f is an isomorphism. $\square$

6.9 Remark. If F is a skew field, $n \geq 3$ and $\Lambda = (\Lambda_1, \dots, \Lambda_n)$, where $\Lambda_i = F$, $i = 1, \dots, n$, then $M(n, \Lambda)$ is the usual vector space matroid of rank n on F^n. Thus, $\mathrm{s}(M(n, \Lambda))$ is isomorphic to the projective geometry $\mathrm{PG}(n-1, F)$ and we obtain another proof of the result of Proposition 3.25.

Further, in contrast to the case of projective geometries, $\mathrm{s}(M(n, \Lambda))$ can contain two lines ℓ_1 and ℓ_2 with $|\ell_1| \neq |\ell_2|$.

In particular, for all natural numbers $k, m, r_1, \dots, r_k$ such that there exist a prime number p and natural numbers $s_1, \dots, s_k$ with $s_{i-1}|s_i$, $i = 2, \dots, k$, and $r_i = p^{s_i}$, $i = 1, \dots, k$, and infinite cardinal numbers $\kappa_1 < \dots < \kappa_m$, the matroid $\mathrm{s}M(k+m, \Lambda)$ for $\Lambda = (\Lambda_1, \dots, \Lambda_{m+k})$, where $\Lambda_i = \mathbb{F}_{r_i}$ for $i = 1, \dots, k$ and Λ_{k+i} is a field of characteristic p of cardinality κ_i, $i = 1, \dots, m$, such that $\Lambda_{k+i-1} \supseteq \Lambda_{k+i}$, $i = 1, \dots, m$, is a combinatorial geometry which has lines of cardinality $r_i + 1$, $i = 1, \dots, k$, and lines of cardinality κ_i, $i = 1, \dots, m$.

6.10 Proposition. Let Λ be a system of coordinates for Q and $n \in \mathbb{N}$, $n \geq 3$. The empty set and the sets $x + V$, where $x \in E_{n,\Lambda}$ and V is a Λ-subspace of $E_{n,\Lambda}$, are the flats of a matroid $N(n, \Lambda)$ of rank $n+1$ on the set $E_{n,\Lambda}$.

Proof. We will show that the bijection

$$\iota \colon E_{n,\Lambda} \to E', \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1),$$

where $E' \coloneqq \{z \in E_{n+1,\hat{\Lambda}} \mid z_{n+1} = 1\}$ and $\hat{\Lambda} = (\Lambda_1, \dots, \Lambda_n, \Lambda_{n+1})$ for a fixed skew field $\Lambda_{n+1} \subseteq \Lambda_n$, induces a bijection between the sets $x + V$, $x \in E_{n,\Lambda}$ and V a Λ-subspace of $E_{n,\Lambda}$, and the flats of $M \coloneqq M(n+1, \hat{\Lambda})|E'$. Then our claim follows from Proposition 6.4.

Let $x \in E_{n,\Lambda}$ and V be an Λ-subspace of $E_{n,\Lambda}$. Using Lemma 6.1, there exist $k \in \mathbb{N}_0$ and canonical elements $y_1, \dots, y_k$ of $E_{n,\Lambda}$ such that $V = \sum_{i=1}^{k} y_i \Lambda_{\mu(y_i)}$

and $\mu(y_1) \prec \cdots \prec \mu(y_k)$. Thus, for $\hat{M} := M(n+1, \hat{\Lambda})$ we obtain

$$\sigma_{\hat{M}}(\iota(x+V)) = \sum_{i=1}^{k} z_i \Lambda_{\mu(z_i)} + y\Lambda_{n+1} =: W,$$

where $z_i = (y_{i,1}, \ldots, y_{i,n}, 0)$, $y = (x_1, \ldots, x_n, 1) \in E_{n+1,\hat{\Lambda}}$, $i = 1, \ldots, k$. It follows from Corollary 6.5 that W is a flat of $\hat{M}$. Since $W \cap E' \neq \emptyset$ we obtain $\iota(x+V) = W \cap E'$ is a flat of M.

Conversely, let $W \cap E'$ be a flat of M, where $W \neq \{0\}$ is a flat of $\hat{M}$. Then there exist $k \in \mathbb{N}$ and canonical elements $z_1, \ldots, z_k$ such that $W = \sum_{i=1}^{k} z_i \Lambda_{\mu(z_i)}$ and $\mu(z_1) \prec \cdots \prec \mu(z_k) = n+1$.

Hence, $W \cap E' = \iota(y_k + V)$, where $y_i = (z_{i,1}, \ldots, z_{i,n}) \in E_{n,\Lambda}$ for $i = 1, \ldots, k$, and $V := \sum_{i=1}^{k-1} y_i \Lambda_{\mu(y_i)}$. This proves $N(n, \Lambda)$ is a matroid isomorphic to M.

Moreover, $\varrho(N(n,\Lambda)) = \varrho(M) = n+1$, as a hyperplane $H := [a_1, \ldots, a_{n+1}]_{\hat{\Lambda}}$ of $\hat{M}$ that contained all $b_k := (\delta_{1k}, \ldots, \delta_{nk}, 1) \in E'$, $k = 1, \ldots, n+1$, would also contain $b_k - b_{n+1} = (\delta_{1k}, \ldots, \delta_{nk}, 0)$, $k = 1, \ldots, n$. This would imply $a_k = 0$ for all $k = 1, \ldots, n+1$, a contradiction. □

Definition. We call a subset $A \subseteq Q$ a *subring* of a quasifield Q if $(A, +)$ is a subgroup of $(Q, +)$, $1 \in A$ and $ab \in A$ for all $a, b \in A$.

Moreover, for any subring A of a quasifield Q, $n \in \mathbb{N}$, $n \geq 3$, and a system of coordinates Λ for Q and n we call a tuple $\Xi = (\Xi_1, \ldots, \Xi_n)$ a *subsystem* of Λ for A and n if $\Xi_1 = \Xi_2 = A$, each Ξ_i is a subring of Λ_i, $i = 1, \ldots, n$, $\Xi_{i-1} \supseteq \Xi_i$ for $i = 2, \ldots, n$ and for each $\lambda_i \in \Lambda_i$ there exists a $\xi \in \Xi_n$ such that $\lambda_i \xi \in \Xi_i$, $i = 1, \ldots, n$.

For any subring A of Q and any subsystem Ξ for A and n of a system of coordinates Λ for Q and n we define $E_{n,\Lambda} := \{x \in E_{n,\Lambda} \mid x_i \in \Xi_i,\ i = 1, \ldots, n\}$ and let $N(n, \Xi) := N(n, \Lambda)|E_{n,\Xi}$ be the restriction of $N(n, \Lambda)$ to $E_{n,\Xi}$.

6.11 Lemma. Let Λ be system of coordinates for Q and $n \in \mathbb{N}$, $n \geq 3$, and Ξ be a subsystem of Λ for A and n, where A is a subring of Q, and $p, q \in E_{n,\Lambda}$. Further, let $L \ni p$ be a hyperline of $N(n, \Xi)$ and $H_1, H_2, H_3, H_4 \in \mathcal{H}_L$ be pairwise different such that $q \in H_1$.

Then there exist a hyperline $L' \ni q'$ with $L' \subseteq H_1$ and pairwise different $H_2', H_3', H_4' \in \mathcal{H}_{L'} \setminus \{L_1\}$ such that (H_1, H_2, H_3, H_4) and (H_1, H_2', H_3', H_4') are projectively equivalent.

Proof. Since obviously for every $v \in E_{n,\Xi}$ the map $\tau_v \colon E_{n,\Xi} \to E_{n,\Xi}$, $x \mapsto x + v$ is a matroid isomorphism of $N(n, \Lambda)$, we may assume without loss of generality that $o \in L$, where $o \in E_{n,\Xi}$ with $o_i = 0$, $i = 1, \ldots, n$, and $\mu(x) \preceq \mu(p)$ for all $x \in L$. Let $W^k := U^k_{n,\Lambda}$ for $k = 1, \ldots, n$. We consider two cases:

First, let there exist a $k \in \{n-1, n\}$ such that $L \not\subseteq W^k =: Z$. Then $H_i \neq Z$ for all $i = 1, 2, 3, 4$, and it follows from Lemma 6.6 that $P := L \cap Z$ is a hyperpoint, and $L_i := H_i \cap Z$ is a hyperline for all $i = 1, 2, 3, 4$. Thus, in this case our claim follows for $H'_i := L_i \vee q$, $i = 2, 3, 4$.

Second, let $L = W^{n-1} \cap W^n$. Applying a suitable τ_v, $v \in E_{n,\Xi}$, we may further assume without loss of generality that $q \in W^n$ and $H_1 = W^n$. Set $Z := [a_1, \dots, a_n]_\Lambda$, where $a_i = 0$ for all $i = 1, \dots, n-3$ and $a_i = 1$ for $i = n-2, n-1, n$.

Let $i \in \{2, 3, 4\}$. If $H_i = [a_{i,1}, \dots, a_{i,n}]_\Lambda$ is the canonical representation, then $a_{i,j} = 0$ for all $j = 1, \dots, n-2$ and $a_{i,n-1} = 1$. As $a_{i,1}, \dots, a_{i,n}$ are admissible for Λ there exists a $x_{n-1} \in \Lambda_{n-1}$ with $a_{i,n} = a_{i,n-1}x_{n-1}$.

Let $x_n \in \Xi_n$ such that $x_{n-1}x_n =: y_{n-1} \in \Xi_n$. Set $y_n := -x_n$, $y_{n-2} := x_n - y_{n-1}$ and $y_j := 0$ for all $j = 0, \dots, n-3$. For $P := L \cap W^{n-2}$ thus the hyperline $P \vee y$ is the intersection of H_i and Z.

Hence, for $H'_i := (H_i \cap Z) \vee q$, $i = 2, 3, 4$, we obtain our claim. □

6.12 Theorem. For any system of coordinates Λ for Q and $n \in \mathbb{N}$, $n \geq 3$, the universal partial hyperfield of $N(n, \Lambda)$ is isomorphic to $Q/_*R_a$.

Moreover, if A is a subring of Q and Ξ is a subsystem of Λ for A and n, the universal partial hyperfield of the restriction $N(n, \Xi)$ of $N(n, \Lambda)$ to the points $x \in E_{n,\Lambda}$ such that $x_i \in \Xi_i$, $i = 1, \dots, n$, is also isomorphic to $Q/_*R_a$.

Proof. It suffices to prove that the universal partial hyperfield of $N := N(n, \Xi)$ is isomorphic to $Q/_*R_a$, since Λ is itself a subsystem of Λ for Q and n.

Let $o \in E_{n,\Xi}$ be defined by $o_i := 0$ for all $i = 1, \dots, n$. We will show first that the map

$$\varphi\colon \mathrm{s}(E_{n,\Xi} \setminus \{o\}) \to E, \quad \mathrm{s}y \mapsto \bar{y},$$

where E is the set of canonical elements of $E_{n,\Lambda}$, is an isomorphism from N/o to $M := M(n, \Lambda)|E$.

Clearly, it follows from the definition of N, Lemma 6.1, Corollary 6.5, and Proposition 6.10 that for all $x, y \in E_{n,\Xi} \setminus \{o\}$ we have $\mathrm{s}x = \mathrm{s}y$ if and only if $\bar{x} = \bar{y}$. Thus, φ is well-defined and injective.

To prove that φ is surjective, let $z_0 \in E_{n,\Lambda}$ be a canonical element. As Ξ is a subsystem of Λ for A and n there exist $\xi_1, \dots, \xi_n \in \Xi_n$ such that $z_i := z_{i-1}\xi_i$ satisfies $z_{i,i} \in \Xi_i$, $i = 1, \dots, n$. As each Ξ_i is a subring of Λ_i, $i = 1, \dots, n$, we get $z_n \in \Xi_i$, $i = 1, \dots, n$. Thus, $z_o = \varphi(\mathrm{s}z_n)$.

Applying Proposition 3.6 and Proposition 3.11, we obtain that the map $\beta\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(N)$ defined by $\beta(0) := 0$, $\beta(-1) := -1$, and

$$\beta([H_1, H_2 \,|\, H_3, H_4]) := [H'_1, H'_2 \,|\, H'_3, H'_4],$$

where $H_i' := H_i \cap E_{n,\Xi}$, $i = 1,2,3,4$, for all $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$, is an homomorphism of partial hyperfields. It follows from Lemma 3.3 and Lemma 6.11 that β is surjective.

Moreover, using the proof of Proposition 6.10, we can view N as restriction of the matroid $\hat{M} := M(n+1, \hat{\Lambda})$, where $\hat{\Lambda} = (\Lambda_1, \ldots, \Lambda_{n+1})$ and $\Lambda_{n+1} \subseteq \Lambda_n$ is a skew field. Thus, Proposition 3.6 yields that the map $\alpha\colon \mathbb{U}^{(0)}(N) \to \mathbb{U}^{(0)}(\hat{M})$ defined by $\alpha(0) := 0$, $\alpha(-1) := -1$, and

$$\alpha([x + H_1', x + H_2' \,|\, x + H_3', x + H_4']) := [\hat{H}_1, \hat{H}_2 \,|\, \hat{H}_3, \hat{H}_4],$$

where $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$ and $x \in E_{n,\Xi}$, as well as $\hat{H}_i := H_i + \hat{x}\Lambda_{n+1}$ and $\hat{x} = (x_1, \ldots, x_n, 1)$ for all $i = 1,2,3,4$, is a well-defined homomorphism of partial hyperfields.

Using the isomorphism $g\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(\hat{M})$ from the proof of Theorem 6.8, we obtain $\alpha \circ \beta = g$ by straightforward computation. Therefore, β is bijective and thus also α. As in the proof of Theorem 6.8 we get that α is an epimorphism. Hence, α and also β are isomorphism, which yields our claim. □

Definition. A (not necessarily commutative) integral domain R is said to be a *right Ore domain* if $aR \cap bR \neq \emptyset$ for all $a, b \in R^*$, where R^* is the set of units of R.

For any right Ore domain R there exist (unique up to isomorphism) a skew field F, called *field of right fractions*, and an embedding $\iota\colon R \to F$ such that for each $f \in F$ there exist $r, s \in R$, $s \neq 0$, satisfying $f = \iota(r)\iota(s)^{-1}$ (see [Coh08, Proposition 1.3.4]).

6.13 Corollary. Let $d \in \mathbb{N}$ be at least 3.

(a) For any skew field F the universal partial hyperfield of the affine geometry $\mathrm{AG}(d, F)$ of dimension d over F is isomorphic to $F/_{*}F^{*\prime}$, where $F^{*\prime}$ is the commutator subgroup of F^*.

(b) For any right Ore domain R and its field of right fractions F the universal partial field of the restriction of $\mathrm{AG}(d, F)$ to the elements R^d is isomorphic to $F/_{*}F^{*\prime}$.

Proof. Let R be a right Ore domain and F be its field of right fractions. Set $\Xi = (\Xi_1, \ldots, \Xi_d)$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_d)$, where $\Xi_i := R$ and $\Lambda_i := F$ for all $i = 1, \ldots, d$. Since $\mathrm{AG}(d, F) = N(d, \Lambda)$ and $\mathrm{AG}(d, F)|R^d = N(d, \Xi)$, our claim follows from Theorem 6.8 and Theorem 6.12 (see Remark 6.9). □

6.2 Affine translation planes

The systems of hyperplanes functions for an affine plane M and $\mathbb{S}$ which extend to their projective closure were characterized by Karzel ([Kar63]) to satisfy two conditions. Later, Kroll ([Kro86]) proved that these two conditions are true in case of affine translation planes, whose kernel contains at least four elements.

Based on the methods Kroll used in his proof we will show that the homomorphism of partial hyperfields from the universal partial hyperfield $\mathbb{U}^{(0)}(M)$ of an affine translation plane to that of its projective closure, which we obtain from Proposition 3.6, is an isomorphism.

For the theory of affine planes and especially translation planes we refer the reader to [Pic].

Definition. An *affine plane* is a combinatorial geometry $M = (E, \mathcal{H})$ of rank 3 such that for each $H \in \mathcal{H}$ and $a \in E \setminus H$ there exists a unique $H' \in \mathcal{H}$ with $a \in H'$ and $H \cap H' = \emptyset$.

Two hyperplanes $H, H' \in \mathcal{H}$ are called *parallel* (denoted by $H \parallel H'$) if $H = H'$ or $H \cap H' = \emptyset$. For any $a \in E$ and any $H \in \mathcal{H}$ we denote the unique $H' \in \mathcal{H}$ such that $a \in H'$ and $H \parallel H'$ by $\{a \parallel H\}$.

6.14 Lemma. Let $M = (E, \mathcal{H})$ be a matroid. For $H \in \mathcal{H}$ and $a_1, a_2 \in E \setminus H$ we set $[H \,|\, a_1, a_2] := H(a_1) \cdot H(a_2)^{-1} \in \mathbb{T}^{\mathcal{H}}(M)$. Then we have

(a) $[H \,|\, a_1, a_2] \cdot [H \,|\, a_2, a_3] = [H \,|\, a_1, a_3]$ for all $H \in \mathcal{H}$ and $a_1, a_2, a_3 \in E \setminus H$,

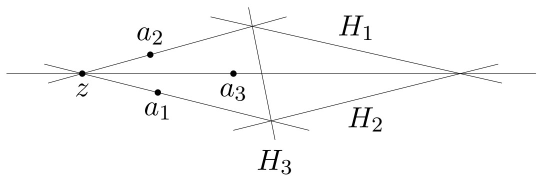
(b) $[H_1 \,|\, a_1, a_2] = [H_2 \,|\, a_1, a_2]$ for all $H_1, H_2 \in \mathcal{H}$ and $a_1, a_2 \in E \setminus (H_1 \cup H_2)$ such that $L := H_1 \cap H_2$ is a hyperline of M and $L \vee a_1 = L \vee a_2$,

(c) $[H_1 \,|\, a_2, a_3] \cdot [H_2 \,|\, a_3, a_1] \cdot [H_3 \,|\, a_1, a_2] = \varepsilon$ for $H_1, H_2, H_3 \in \mathcal{H}$ which are pairwise different such that $L := H_1 \cap H_2 \cap H_3$ is a hyperline of M and $a_i \in H_i \setminus L$, $i = 1, 2, 3$.

Proof. Follows from the definition and Lemma 3.1. □

6.15 Lemma (cf. [Kro86, (1)]). Let $C = \{z, a_1, a_2, a_3\}$ be a circuit of an affine plane $M = (E, \mathcal{H})$ and $H_1, H_2, H_3 \in \mathcal{H}$ three hyperplanes such that $a_1 \notin H_2$, $H_i \parallel a_j \vee a_k$ and $H_i \cap H_j \in z \vee a_k$ for all $\{i, j, k\} = \{1, 2, 3\}$. Then

$$[H_1 \,|\, a_2, a_3] \cdot [H_2 \,|\, a_3, a_1] \cdot [H_3 \,|\, a_1, a_2] = 1.$$



Proof. Clearly, $a_i, a_j \notin H_k$ for all $\{i,j,k\} = \{1,2,3\}$, as otherwise we would have $H_k = a_i \vee a_j$ for all $\{i,j,k\} = \{1,2,3\}$, a contradiction to $a_1 \notin H_2$.

Further, we have either $z \in H_i$ for all $i = 1,2,3$, or $z \notin H_i$ for all $i = 1,2,3$. In the former case $H_i = z \vee a_i$, $i = 1,2,3$, and our claim follows from Lemma 3.1.

In ther latter case, since $H_i \cap H_j, a_k, z$ are collinear it follows from Lemma 6.14 that $[H_i \mid a_k, z] = [H_j \mid a_k, z]$ for all $\{i,j,k\} = \{1,2,3\}$. Thus,

$$\begin{aligned} &[H_1 \mid a_2, a_3] \cdot [H_2 \mid a_3, a_1] \cdot [H_3 \mid a_1, a_2] \\ &= [H_1 \mid a_2, z] \cdot [H_1 \mid z, a_3] \cdot [H_2 \mid a_3, z] \\ &\quad \cdot [H_2 \mid z, a_1] \cdot [H_3 \mid a_1, z] \cdot [H_3 \mid z, a_2] = 1. \end{aligned}$$ □

Definition. An automorphism σ of an affine plane $M = (E, \mathcal{H})$ is called *dilation* if $\sigma(H) \parallel H$ for all $H \in \mathcal{H}$. Any dilation σ which is not the identity map has at most one fixpoint, thus σ is called a *translation* if it is the identity map or has no fixpoint, and *dilation with center z* if it has $z \in E$ as fixpoint.

An affine plane $M = (E, \mathcal{H})$ is said to be an *affine translation plane* if for all $x, y \in E$ there exists a translation τ such that $\tau(x) = y$. The group of translations of an affine translation plane is abelian. Its endomorphism ring, called the *kernel* of M, is a skew field.

Moreover, for every $z \in E$ there exists a bijection between the dilations whose center is z and the elements of the kernel of M.

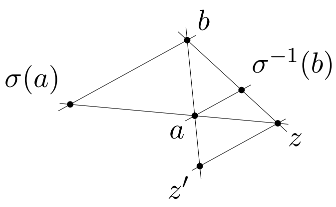
6.16 Lemma (cf. [Kro86, (3)]). Let $\{a, b, z\}$ be a basis of an affine translation plane $M = (E, \mathcal{H})$. For every non-trivial dilation σ with center z let

$$z' := (a \vee b) \cap \{z \parallel a \vee \sigma^{-1}(b)\},$$

and τ the unique translation such that $\tau(z) = z'$. Then $\sigma' := \tau\sigma\tau^{-1}$ is a dilation with center z' and $\sigma'(a) = b$.

Furthermore, the mapping $\sigma \mapsto \sigma'$ is a bijection between the non-trivial dilations with center z and the dilations with a center mapping a to b.

In particular, there are at least two dilations mapping a to b with different centers if the kernel of M contains at least four elements.



Proof. Clearly $\sigma'(z') = z'$. Let $c := \sigma^{-1}(b)$. Since $\sigma'(z' \vee a) || z' \vee a$ we have $\sigma'(a) \in \sigma'(z' \vee a) = z' \vee a$. As $a \vee c \parallel z \vee z'$, it follows $\tau^{-1}(a) \in a \vee c$. Therefore, $(\sigma\tau^{-1})(a) \in \sigma(a) \vee b \parallel a \vee c$. Thus, $\sigma'(a) \in \sigma(a) \vee b$. Because $b \in z' \vee a$, we obtain $\sigma'(a) = b$.

In order to prove that this an injective mapping, let σ_1, σ_2 be non-trivial dilations with center z with $\sigma_1' = \sigma_2'$. By construction, σ_i' has center

$$z_i' := (a \vee b) \cap \{z \parallel a \vee \sigma_i^{-1}(b)\},$$

$i = 1, 2$. As $z_1' = z_2'$ we obtain $a \vee \sigma_1^{-1}(b) = a \vee \sigma_2^{-1}(b)$. Thus, $\sigma_i^{-1}(b) \in z \vee b$ yields $\sigma_1^{-1}(b) = \sigma_2^{-1}(b)$. Hence, $\sigma_1 = \sigma_2$.

Finally, if $\tilde{\sigma}$ is a dilation with center $\tilde{z}$ such that $\tilde{\sigma}(a) = b$, let τ be a translation with $\tau(\tilde{z}) = z$ and $\sigma := \tau\tilde{\sigma}\tau^{-1}$. Then $\sigma(z) = z$ and therefore σ' is a dilation with center z such that $\sigma' = \tilde{\sigma}$, which proves that the mapping is bijective.

The last sentence follows from [Pic, 4. on p. 203]. □

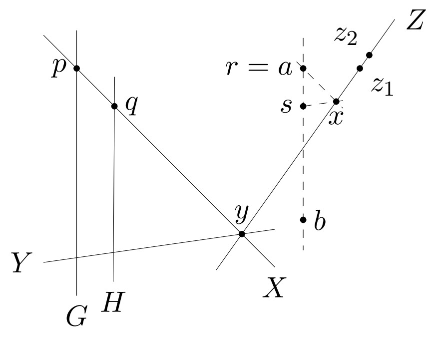
6.17 Lemma (cf. [Kro86, Satz (4)]). Let $M = (E, \mathcal{H})$ be an affine translation plane whose kernel contains at least four elements, $G, H \in \mathcal{H}$, and $a, b \in E \setminus (G \cup H)$ such that $G \parallel H \parallel a \vee b$. Then

$$[G \,|\, a, b] = [H \,|\, a, b].$$

Proof. Let $r \in \{a, b\}$ and $p \in G$. Using Lemma 6.16 there exists a dilation σ_1 with center z_1 such that $\sigma_1(r) = p$.

Further, we choose an $X \in \mathcal{H}_p \setminus \{G, r \vee p\}$ and set $q := X \cap H$. Again applying Lemma 6.16, there exist two dilations σ with $\sigma(r) = q$. Let σ_2 be the one whose center $z_2 \notin \{z_1 \parallel X\}$.

Since p, q, r are not collinear by choice of X, $z_1 \neq z_2$. Set $Z := z_1 \vee z_2$ and let $x := Z \cap \{r \parallel X\}$ ($X \parallel Z$ would imply $z_2 \in Z = \{z_1 \parallel X\}$). Moreover, we choose an $s \in (a \vee b) \setminus (Z \cup \{r\})$, set $y := X \cap Z$ and $Y := \{y \parallel x \vee s\}$.



By construction, $\{z_i, r, x, s\}$, $i = 1, 2$, are circuits and $r \notin G \cup H$. Furthermore, $Y \parallel x \vee s$, $G \parallel H \parallel r \vee s$, $X \parallel r \vee x$ and $X \cap Y = y \in z_i \vee x$, $i = 1, 2$. As $\sigma_i(r) \in X$, it follows

$$\sigma_i(x) = \sigma_i(Z \cap (x \vee r)) = Z \cap X = y$$

for $i = 1, 2$. Therefore, we have

$$\begin{aligned} Y \cap G &= \sigma_1((x \vee s) \cap (a \vee b)) = \sigma_1(s) \in z_1 \vee s, \\ X \cap G &= \sigma_1((x \vee r) \cap (a \vee b)) = \sigma_1(r) \in z_1 \vee r, \\ Y \cap H &= \sigma_2((x \vee s) \cap (a \vee b)) = \sigma_2(s) \in z_2 \vee s, \\ X \cap H &= \sigma_2((x \vee r) \cap (a \vee b)) = \sigma_2(r) \in z_2 \vee r. \end{aligned}$$

Thus, applying Lemma 6.15 twice, we get

$$[Y \,|\, s, x] \cdot [X \,|\, x, r] \cdot [G \,|\, r, s] = 1 = [Y \,|\, s, x] \cdot [X \,|\, x, r] \cdot [H \,|\, r, s].$$

We conclude $[G \,|\, r, s] = [H \,|\, r, s]$.

If in one of the cases $r = a$ or $r = b$, the constructed line Z does not contain any point of $\{a, b\}$ we can choose the point s above such that $\{r, s\} = \{a, b\}$ and immediately get $[G \,|\, a, b] = [H \,|\, a, b]$.

Otherwise, in both cases $Z \cap \{a, b\} \neq \emptyset$ and we can choose $s \in (a \vee b) \setminus \{a, b\}$ independently of $r \in \{a, b\}$. Hence,

$$[G \,|\, a, b] = [G \,|\, a, s] \cdot [G \,|\, s, b] = [H \,|\, a, s] \cdot [H \,|\, s, b] = [H \,|\, a, b]. \qquad \square$$

6.18 Lemma. Let $M = (E, \mathcal{H})$ be an affine translation plane whose kernel contains at least four elements. For (H_1, H_2, H_3, H_4), $(H'_1, H'_2, H'_3, H'_4) \in \mathcal{H}_4$ we have

$$[H_1, H_2 \,|\, H_3, H_4] = [H'_1, H'_2 \,|\, H'_3, H'_4]$$

if there is an $H \in \mathcal{H}$ such that $p, p' \notin H$, where $p := \bigcap_{i=1}^4 H_i$ and $p' := \bigcap_{i=1}^4 H'_i$, and for all $i = 1, 2, 3, 4$ either $H_i \cap H = H \cap H'_i$ or $H_i \parallel H \parallel H'_i$.

Proof. If $H_i \cap H = H'_i \cap H$ for all $i = 1, 2, 3, 4$, our claim follows directly from Proposition and Definition 3.29.

Otherwise, using Lemma 3.3 we can assume without loss of generality that $H_1 \parallel H \parallel H'_1$. Obviously, we have $H_1 = H_2$ if and only if $H'_1 = H'_2$ and $H_3 = H_4$ if and only if $H'_3 = H'_4$. In both cases

$$[H_1, H_2 \,|\, H_3, H_4] = 1 = [H'_1, H'_2 \,|\, H'_3, H'_4].$$

Finally, let H_1, H_2, H_3, H_4 resp. H'_1, H'_2, H'_3, H'_4 be pairwise different and set $a_i := H_i \cap H = H'_i \cap H$, $i = 3, 4$. Since $a_3, a_4, H_2 \cap H'_2$ are collinear and H_1, $H'_1, a_3 \vee a_4$ are parallel it follows from Lemma 3.1, Lemma 6.17 and the identification of $\mathbb{T}^{(0)}(M)$ as a subgroup of $\mathbb{T}^{\mathcal{H}}(M)$ from Proposition 3.2 that

$$\begin{aligned}[][H_1, H_2 \,|\, H_3, H_4] &= [H_1 \,|\, a_3, a_4] \cdot [H_2 \,|\, a_4, a_3] \\ &= [H'_1 \,|\, a_3, a_4] \cdot [H'_2 \,|\, a_4, a_3] = [H'_1, H'_2 \,|\, H'_3, H'_4]. \qquad \square\end{aligned}$$

6.19 Proposition. Let $M = (E, \mathcal{H})$ be an affine translation plane whose kernel contains at least four elements. For (H_1, H_2, H_3, H_4), $(H'_1, H'_2, H'_3, H'_4) \in \mathcal{H}_4$ we have

$$[H_1, H_2 \,|\, H_3, H_4] = [H'_1, H'_2 \,|\, H'_3, H'_4]$$

if $H_i \parallel H'_i$ for all $i = 1, 2, 3, 4$.

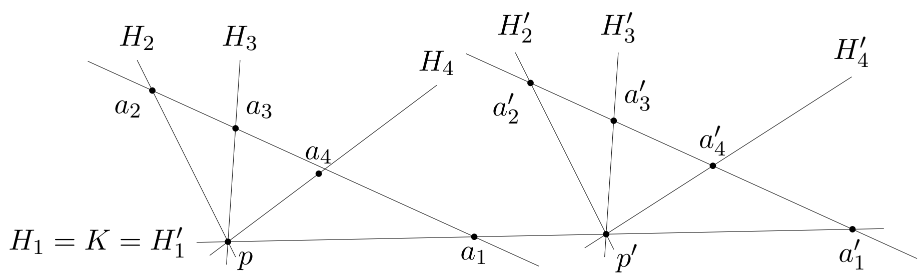
Proof. If $p = p'$, where $p := \bigcap_{i=1}^4 H_i$ and $p' := \bigcap_{i=1}^4 H'_i$, we have $H_i = H'_i$ for all $i = 1, 2, 3, 4$ and thus our claim follows trivially. Therefore, let $p \neq p'$.

It suffices to show our claim in the case $H_i = H'_i$ for a suitable $i \in \{1, 2, 3, 4\}$, because otherwise we obtain using Lemma 3.3

$$[H_1, H_2 \,|\, H_3, H_4] = [H_1, K \,|\, H_3, H_4] \cdot [K, H_2 \,|\, H_3, H_4],$$

where $K := p \vee p'$. We can further assume without loss of generality that $H_1 = H'_1$, and thus $H_1 = H'_1 = K$ and $H_i \neq H'_i$ for all $i = 2, 3, 4$.

Since M is of order at least 4 there exists a hyperplane Z of M such that $p \notin Z$ and $Z \nparallel H_i$ for all $i = 1, 2, 3, 4$. Set $a_i := Z \cap H_i$, $i = 1, 2, 3, 4$, and let τ be the unique translation of M such that $\tau(p) = p'$. Then for $Z' := \tau(Z)$ and $a'_i := \tau(a_i)$, $i = 1, 2, 3, 4$, we have $p' \notin Z'$, $Z' \nparallel H'_i$ and $a'_i = Z' \cap H'_i$ for all $i = 1, 2, 3, 4$.



Moreover, as τ is a non-trivial translation, $Z \parallel Z'$ and $Z \neq Z'$. Thus, a_i, a'_j, $i, j = 1, 2, 3, 4$, are eight pairwise different points and $G_i := a_i \vee a'_i \parallel p \vee p'$ for all $i = 1,2,3,4$. Hence, using Lemma 6.17, we get $[G_1 \,|\, a_i, a_i'] = [G_2 \,|\, a_i, a_i']$ for $i = 3,4$, and $H_1 = G_1 = H_1'$. Therefore, Lemma 3.1 and Lemma 6.14 yield

$$\begin{aligned}
[H_1 \,|\, a_3, a_4] \cdot [H_2 \,|\, a_4, a_3] &= [G_1 \,|\, a_3, a_4] \cdot [G_2 \,|\, a_4, a_3] \\
&= [G_1 \,|\, a_3, a_3'] \cdot [G_1 \,|\, a_3', a_4'] \cdot [G_1 \,|\, a_4', a_4] \\
&\quad \cdot [G_2 \,|\, a_4, a_4'] \cdot [G_2 \,|\, a_4', a_3'] \cdot [G_2 \,|\, a_3', a_3] \\
&= [G_1 \,|\, a_3', a_4'] \cdot [G_2 \,|\, a_4', a_3'] \\
&= [H_1' \,|\, a_3', a_4'] \cdot [H_2' \,|\, a_4', a_3'].
\end{aligned}$$

Thus, we get $[H_1, H_2 \,|\, H_3, H_4] = [H_1', H_2' \,|\, H_3', H_4']$ similarly as in the proof of Lemma 6.18. □

6.20 Lemma. Let $M = (E, \mathcal{H})$ be an affine translation plane whose kernel has at least four elements. Then for any (H_1, H_2, H_3, H_4), $(H_1', H_2', H_3', H_4') \in \mathcal{H}_4$ we have

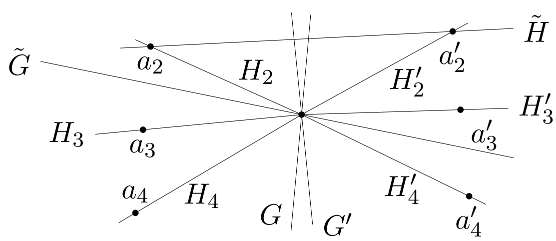
$$[H_1, H_2 \,|\, H_3, H_4] = [H_1', H_2' \,|\, H_3', H_4']$$

if $\bigcap_{i=1}^4 H_i = \bigcap_{i=1}^4 H_i' =: p$ and there exist $K \in \mathcal{H}_p$ and $H, H' \in \mathcal{H} \setminus \mathcal{H}_p$ such that for each $i = 1,2,3,4$ either $H_i \cap H$ and $H_i' \cap H'$ are both points which lie on a common line parallel to K, or $H \parallel H_i$ and $H' \parallel H_i'$.

Proof. If $H = H'$, then $H_i = H_i'$ for all $i = 1,2,3,4$ and thus our claim follows trivially. Therefore, let $H \neq H'$.

Furthermore, as $H_i = H_{i+3}$ if and only if $H_i' = H_{i+3}'$ for all $i = 1,3$, and Lemma 3.3 implies that both cross-ratios are equal to 1 in this case, we can assume that H_1, H_2, H_3, H_4 as well as H_1', H_2', H_3', H_4' are pairwise different.

Let $G := \{p \parallel H\}$ and $G' := \{p \parallel H'\}$. As $G \in \{H_1, H_2, H_3, H_4\}$ if and only if $G' \in \{H_1', H_2', H_3', H_4'\}$ using Lemma 3.3 we can further assume without loss of generality that $H_1 = G$ and $H_1' = G'$.



Let $a_i := H_i \cap H$ and $a_i' := H_i' \cap H'$, $i = 2,3,4$, as well as $\tilde{H} := a_2 \vee a_2'$ and $\tilde{G} := \{p \parallel a_3 \vee a_4'\}$. Since $G \parallel a_3 \vee a_4$, $G' \parallel a_3' \vee a_4'$, $\tilde{H} \parallel a_3 \vee a_3' \parallel a_4 \vee a_4'$, and

$\tilde{G} \parallel a_3 \vee a'_4$, it follows from Lemma 6.14 and Lemma 6.17

$$[G \mid a_3, a_4] \cdot [\tilde{H} \mid a_4, a'_4] \cdot [\tilde{G} \mid a'_4, a_3] = 1,$$
$$[G' \mid a'_3, a'_4] \cdot [\tilde{G} \mid a'_4, a_3] \cdot [\tilde{H} \mid a_3, a'_3] = 1.$$

Using Lemma 6.14 and Lemma 6.17, we therefore obtain

$$\begin{aligned}
[H_1 \mid a_3, a_4] \cdot [H_2 \mid a_4, a_3] &= [G \mid a_3, a_4] \cdot [\tilde{H} \mid a_4, a_3] \\
&= [\tilde{G} \mid a_3, a'_4] \cdot [\tilde{H} \mid a'_4, a_4] \cdot [\tilde{H} \mid a_4, a_3] \\
&= [\tilde{G} \mid a_3, a'_4] \cdot [\tilde{H} \mid a'_3, a_3] \cdot [\tilde{H} \mid a'_4, a'_3] \\
&= [G' \mid a'_3, a'_4] \cdot [\tilde{H} \mid a'_4, a'_3] \\
&= [H_1 \mid a'_3, a'_4] \cdot [H_2 \mid a'_4, a'_3].
\end{aligned}$$

Hence, our claim follows as in the proof of Lemma 6.18. □

For convenience of the reader we repeat the definition of the projective closure of an affine plane (cf. [Pic, Satz 7, p. 11]).

Definition. Let $M = (E, \mathcal{H})$ be an affine plane. For a hyperplane $H \in \mathcal{H}$ we set $[H] := \{H' \in \mathcal{H} \mid H' \parallel H\}$ and $\overline{H} := H \cup \{[H]\}$.

Further, let $U := \{[H] \mid H \in \mathcal{H}\}$, $\mathcal{P} := E \cup U$, and $\overline{\mathcal{H}} := \{\overline{H} \mid H \in \mathcal{H}\} \cup \{U\}$. Then $\Pi = (\mathcal{P}, \overline{\mathcal{H}})$ is a projective plane called the *projective closure* of M.

6.21 Lemma ([Kal92b, p. 6]). For any projective plane $\Pi = (E, \mathcal{H})$ of order at least 3 its inner Tutte group is isomorphic to $\mathbb{F}^{(0)}(M)/U$, where U is the subgroup of $\mathbb{F}^{(0)}(M)$ generated by the elements of the form (CR2) and the elements of the form (CR6').

6.22 Theorem. For any affine translation plane $M = (E, \mathcal{H})$ whose kernel contains at least four elements and its projective closure $\Pi = (\mathcal{P}, \overline{\mathcal{H}})$, the map $\alpha\colon \mathbb{T}^{(0)}(M) \to \mathbb{T}^{(0)}(\Pi)$ defined by $\alpha(0) := 0$, $\alpha(-1) := -1$ and

$$\alpha([H_1, H_2 \mid H_3, H_4]) := [\bar{H}_1, \bar{H}_2 \mid \bar{H}_3, \bar{H}_4],$$

where $\bar{H}_i = \sigma_\Pi(H_i)$, $i = 1, 2, 3, 4$, and $(H_1, H_2, H_3, H_4) \in \mathcal{H}_4(M)$, is an isomorphism of partial hyperfields.

Proof. As $M = \Pi|E$, Proposition 3.6 implies that $\alpha = f_\emptyset$ is a homomorphism of partial hyperfields. In order to show that α is an isomorphism, we construct the inverse homomorphism α^{-1}.

First, we define a group homomorphism $\beta\colon \mathbb{F}^{(0)}(\Pi) \to \mathbb{T}^{(0)}(M)$, whose kernel contains the subgroup $U \subseteq \mathbb{F}^{(0)}(\Pi)$ from Lemma 6.21. We set $\beta(\varepsilon) := \varepsilon$. Further, let $(G_1, G_2, G_3, G_4) \in \mathcal{H}_4(\Pi)$ and $p := \bigcap_{i=1}^4 G_i$.

If $p \in E$, we set

$$\beta((G_1, G_2 \,|\, G_3, G_4)) := [G_1 \setminus U, G_2 \setminus U \,|\, G_3 \setminus U, G_4 \setminus U].$$

Otherwise, $p \notin E$. We choose an $e \in G_k \setminus U$, where $k \in \{1, 2, 3, 4\}$ is minimal such that $G_k \neq U$, and a $Z \in \mathcal{H}$ with $e \notin Z$ and $Z \nparallel G_k \setminus U$, and define

$$\beta((G_1, G_2 \,|\, G_3, G_4)) := [G_1(e, Z), G_2(e, Z) \,|\, G_3(e, Z), G_4(e, Z)] =: v(e, Z),$$

where $G_i(e, Z) := \sigma_M((Z \cap G_i) \cup \{e\})$ if $G_k \cap Z \in E$, and $G_i(e, Z) := \{e \parallel Z\}$ else, $i = 1, 2, 3, 4$.

This definition is independent of the choice of e and Z, since for $e_1, e_2 \in G_k \setminus U$ and $Z_1, Z_2 \in \mathcal{H}$ such that $e_i \notin Z_i$ and $Z_i \nparallel G_k \setminus U$, $i = 1, 2$, there exists an $e \in G_k \setminus (U \cup Z_1 \cup Z_2)$ (M has order at least 4).

As the $G_i(e_j, Z_j)$ and the $G_i(e, Z_j)$, $i = 1, 2, 3, 4$ and $j = 1, 2$ satisfy the precondition of Lemma 6.18 by construction, we have $v(e_j, Z_j) = v(e, Z_j)$ for $j = 1, 2$. Similarly, the $G_i(e, Z_1)$ and the $G_i(e, Z_2)$, $i = 1, 2, 3, 4$, satisfy the precondition of Lemma 6.20. Thus, we get $v(e, Z_1) = v(e, Z_2)$, which yields the desired result $v(e_1, Z_1) = v(e_2, Z_2)$.

Clearly, for all $p \in \mathcal{P}$ and $G_1, G_2, G_3, G_4, G_5 \in \mathcal{H}_p$ such that $G_i \neq G_j$, $i = 1, 2$, $j = 3, 4, 5$ we have

$$(G_1, G_2 \,|\, G_3, G_4) \cdot (G_1, G_2 \,|\, G_4, G_5) \cdot (G_1, G_2 \,|\, G_5, G_3) \in \ker \beta.$$

Therefore, $\ker \beta$ contains all elements of $\mathbb{F}^{(0)}(M)$ of the form (CR2).

To show that $\ker \beta$ also contains all elements of $\mathbb{F}^{(0)}(M)$ of the form (CR6'), let (G_1, G_2, G_3, G_4), $(G_1', G_2', G_3', G_4') \in \mathcal{H}_4(\Pi)$ satisfy (i) and (ii) of (CR6') and set $p = \bigcap_{i=1}^4 G_i$, $p' = \bigcap_{i=1}^4 G_i'$. Thus, there exists a $G \in \overline{\mathcal{H}}$ such that $p, p' \notin G$ and $G \cap G_i = G \cap G_i'$ is a point of Π.

If $p, p' \in E$, let $H_i := G_i \setminus U$ and $H_i' := G_i' \setminus U$, $i = 1, 2, 3, 4$.

In the subcase $G = U$ we have $H_i \parallel H_i'$ for all $i = 1, 2, 3, 4$. Therefore, Proposition 6.19 implies

$$[H_1, H_2 \,|\, H_3, H_4] = [H_1', H_2' \,|\, H_3', H_4'].$$

This is also true for the subcase $G \neq U$. Since $H := G \setminus U$ is a line such that for all $i = 1, 2, 3, 4$, we have either that $H \cap H_i = H \cap H_i'$ is a point or $H_i \parallel H_i'$, and it follows from Lemma 6.18.

If either $p \in U$ or $p' \in U$, say $p' \in U$, we set $H_i \coloneqq G_i \setminus U$ and $H_i' \coloneqq G_i'(e, Z)$ for all $i = 1, 2, 3, 4$, where $e \coloneqq p$ and $Z \coloneqq G \setminus U$. By choice of e and Z, we get $H_i = H_i'$ for all $i = 1, 2, 3, 4$.

Otherwise, $p, p' \in U$. We choose an $e \in E \setminus G$ and set $Z \coloneqq G \setminus U$. Moreover, let $H_i \coloneqq G_i(e, Z)$ and $H_i' \coloneqq G_i'(e, Z)$ for all $i = 1, 2, 3, 4$. As in the previous case we get $H_i = H_i'$ for all $i = 1, 2, 3, 4$.

Hence, in all cases we get

$$(G_1, G_2 \,|\, G_3, G_4) \cdot (G_1', G_2' \,|\, G_4', G_3') \in \ker \beta.$$

Thus, by the homorphism theorem for groups and Lemma 6.21, there exists a multiplicative homomorphism $\hat{\beta}\colon \mathbb{T}^{(0)}(\Pi) \to \mathbb{T}^{(0)}(M)$ such that $\hat{\beta}(\varepsilon) \coloneqq \varepsilon$ and

$$\hat{\beta}([G_1, G_2 \,|\, G_3, G_4]) = \beta((G_1, G_2 \,|\, G_3, G_4))$$

for all $(G_1, G_2, G_3, G_4) \in \mathcal{H}_4(\Pi)$.

The definition of β, Lemma 2.5, and Proposition and Definition 3.4 imply $\hat{\beta}\colon \mathbb{U}^{(0)}(\Pi) \to \mathbb{U}^{(0)}(M)$ is a homomorphism of partial hyperfields such that $\hat{\beta} \circ \alpha = \mathrm{id}_{\mathbb{U}^{(0)}(M)}$ if we extend it by $\hat{\beta}(0) \coloneqq 0$.

In order to prove that $\alpha \circ \hat{\beta} = \mathrm{id}_{\mathbb{U}^{(0)}(\Pi)}$, let $(G_1, G_2, G_3, G_4) \in \mathcal{H}_4(\Pi)$ and $p \coloneqq \bigcap_{i=1}^4 G_i$. If $p \in E$, we have $\overline{H_i} = G_i$, where $H_i \coloneqq G_i \setminus U$, for all $i = 1, 2, 3, 4$.

Otherwise, let $e \in G_k \setminus U$, where $k \in \{1, 2, 3, 4\}$ is minimal with $G_k \neq U$ and $Z \in \mathcal{H}$ such that $e \notin Z$ and $Z \nparallel G_k \setminus U$. Then (G_1, G_2, G_3, G_4) and $(\overline{H_1}, \overline{H_2}, \overline{H_3}, \overline{H_4})$ satisfy (i) and (ii) of (CR6'), where $H_i \coloneqq G_i(e, Z)$, $i = 1, 2, 3, 4$.

Thus, in both cases we have

$$\alpha(\hat{\beta}([G_1, G_2 \,|\, G_3, G_4]) = [\overline{H_1}, \overline{H_2} \,|\, \overline{H_3}, \overline{H_4}] = [G_1, G_2 \,|\, G_3, G_4],$$

which proves our claim. □

6.23 Corollary. The universal partial hyperfield of an affine translation plane $M = (E, \mathcal{H})$ whose kernel contains at least four elements is isomorphic to $Q/_{*}R_a$, where Q is the quasifield coordinatizing its projective closure $\Pi = (\mathcal{P}, \overline{\mathcal{L}})$ with respect to a quadrangle (o, u, v, e), where $o, e \in E$ and $u, v \in U$.

Proof. Our choice of the points o, u, e, v implies that the planar ternary ring coordinatizing Π is indeed a quasifield[4], see [Pic, §8].

Thus, our claim follows from Theorem 3.24 and Theorem 6.22. □

[4]The kernel of this quasifield is isomorphic to the kernel of M.

6.24 Remark. For affine planes M of order at least 3 and their projective closure Π the map $\alpha\colon \mathbb{U}^{(0)}(M) \to \mathbb{U}^{(0)}(\Pi)$ defined in Theorem 6.22 is an epimorphism.

However, in general it is not necessarily injective. For example, it follows from Corollary 3.35 that α is not injective if M is the ternary affine plane.

This is even the case for orientable affine planes M. Joussen ([Jou63] and [Jou66]) proved that if Π is a free projective plane, then M and Π are orientable, but there exists a system of hyperplane functions for M and $\mathbb{S}$ which does not extend to Π, and proves α is not injective in this case.

Bibliography

[ABR96] C. Andradas, L. Bröcker, and J. M. Ruiz. *Constructible sets in real geometry.* English. Vol. 33. Ergeb. Math. Grenzgeb., 3. Folge. Berlin: Springer, 1996.

[AHS06] J. Adámek, H. Herrlich, and G. E. Strecker. "Abstract and Concrete Categories: The Joy of Cats". English. In: *Repr. Theory Appl. Categ.* 2006.17 (2006).

[BB19] M. Baker and N. Bowler. "Matroids over partial hyperstructures". English. In: *Adv. Math.* 343 (2019), pp. 821–863.

[Bjö+99] A. Björner et al. *Oriented matroids.* English. 2nd ed. Vol. 46. Cambridge: Cambridge University Press, 1999.

[Bry75] T. Brylawski. "Modular constructions for combinatorial geometries". English. In: *Trans. Am. Math. Soc.* 203 (1975), pp. 1–44.

[CC11] A. Connes and C. Consani. "The hyperring of adèle classes." English. In: *J. Number Theory* 131.2 (2011), pp. 159–194.

[Coh08] P. M. Cohn. *Skew fields. Theory of general division rings.* English. Vol. 57. Cambridge: Cambridge University Press, 2008.

[Dow73] T. A. Dowling. "A class of geometric lattices based on finite groups". English. In: *J. Comb. Theory, Ser. B* 14 (1973), pp. 61–86.

[DS06] B. Davvaz and A. Salasi. "A realization of hyperrings." English. In: *Commun. Algebra* 34.12 (2006), pp. 4389–4400.

[DW89] A. W. Dress and W. Wenzel. "Geometric algebra for combinatorial geometries." English. In: *Adv. Math.* 77.1 (1989), pp. 1–36.

[DW90] A. W. Dress and W. Wenzel. "On combinatorial and projective geometry." English. In: *Geom. Dedicata* 34.2 (1990), pp. 161–197.

[GRS95] I. M. Gelfand, G. L. Rybnikov, and D. A. Stone. "Projective orientations of matroids." English. In: *Adv. Math.* 113.1 (1995), pp. 118–150.

[JK90] W. Junkers and F. Kalhoff. "Zur algebraischen Kennzeichnung mehrwertiger Halbordnungen. (An algebraic characterization of multivalued halforderings)". German. In: *J. Geom.* 37.1-2 (1990), pp. 95–104.

[Jou63] J. Joussen. "Über die projektive Erweiterungsfähigkeit affiner Ordnungsfunktionen". German. In: *Abh. Math. Semin. Univ. Hamb.* 26 (1963), pp. 23–38.

[Jou66] J. Joussen. "Die Anordnungsfähigkeit der freien Ebenen." German. In: *Abh. Math. Semin. Univ. Hamb.* 29 (1966), pp. 137–184.

[Kal88] F. Kalhoff. "Eine Kennzeichnung anordnungsfähiger Ternärkörper. (A characterization of orderable ternary fields)". German. In: *J. Geom.* 31.1-2 (1988), pp. 100–113.

[Kal89] F. Kalhoff. *Anordnungsräume und Wittringe projektiver Ebenen.* German. Dortmund: Univ. Dortmund, FB Math., 1989.

[Kal92a] F. Kalhoff. "On multiple-valued ordered projective planes". English. In: *J. Geom.* 44.1-2 (1992), pp. 87–101.

[Kal92b] F. Kalhoff. "The Tutte group of projective planes." English. In: *Geom. Dedicata* 43.2 (1992), pp. 225–231.

[Kal96] F. Kalhoff. "Realization of rank 3 matroids." English. In: *Appl. Math. Lett.* 9.1 (1996), pp. 55–58.

[Kar63] H. Karzel. "Zur Fortsetzung affiner Ordnungsfunktionen". German. In: *Abh. Math. Semin. Univ. Hamb.* 26 (1963), pp. 17–22.

[KK88] H. Karzel and H.-J. Kroll. *Geschichte der Geometrie seit Hilbert. (The history of geometry since Hilbert).* German. Darmstadt: Wissenschaftliche Buchgesellschaft, 1988.

[Kro86] H.-J. Kroll. "Zur Fortsetzbarkeit von Ordnungsfunktionen einer affinen Ebene. (Extension of order functions of affine planes)". German. In: *J. Geom.* 26 (1986), pp. 182–185.

[Kun90] J. P. S. Kung. "Combinatorial geometries representable over GF(3) and GF(q). I: The number of points". English. In: *Discrete Comput. Geom.* 5.1 (1990), pp. 83–95.

[Mar06] M. A. Marshall. "Real reduced multirings and multifields." English. In: *J. Pure Appl. Algebra* 205.2 (2006), pp. 452–468.

[Mar96] M. A. Marshall. *Spaces of orderings and abstract real spectra.* English. Vol. 1636. Berlin: Springer, 1996.

[Mas85a] C. G. Massouros. “Methods of constructing hyperfields”. English. In: *Int. J. Math. Math. Sci.* 8 (1985), pp. 725–728.

[Mas85b] C. G. Massouros. “Theory of hyperrings and hyperfields”. English. In: *Algebra Logic* 24 (1985), pp. 477–485.

[Oxl11] J. G. Oxley. *Matroid theory.* English. 2nd ed. Vol. 21. Oxf. Grad. Texts Math. Oxford: Oxford University Press, 2011.

[Pic] G. Pickert. *Projektive Ebenen.* 2nd ed. Berlin: Springer.

[PV10] R. Pendavingh and S. Van Zwam. “Lifts of matroid representations over partial fields.” English. In: *J. Comb. Theory, Ser. B* 100.1 (2010), pp. 36–67.

[Sem98] C. A. Semple. “k-Regular Matroids”. PhD thesis. Victoria University of Wellington, 1998.

[SW96] C. Semple and G. Whittle. “Partial fields and matroid representation.” English. In: *Adv. Appl. Math.* 17.2 (1996), pp. 184–208.

[Szp30] E. Szpilrajn. “Sur l’extension de l’ordre partiel.” French. In: *Fundam. Math.* 16 (1930), pp. 386–389.

[Tut65] W. T. Tutte. “Lectures on matroids”. English. In: *J. Res. Natl. Bur. Stand., Sect. B* 69 (1965), pp. 1–47.

[Vir10] O. Viro. *Hyperfields for Tropical Geometry I. Hyperfields and dequantization.* 2010. arXiv: `1006.3034 [math.AG]`.

[Wen89] W. Wenzel. “A group-theoretic interpretation of Tutte’s homotopy theory.” English. In: *Adv. Math.* 77.1 (1989), pp. 37–75.

[Whi86] N. White, ed. *Theory of Matroids.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1986.

Notation

$\mathrm{AG}(d, F)$	Affine geometry of dimension d over F
$\mathcal{B}(M)$, $\mathcal{B}$	Set of bases of matroid M
$F_1 \oplus F_2$	Coproduct of partial hyperfields F_1, F_2
$F/_{\!*}U$	Quotient of partial hyperfield F by subgroup U
$\mathbb{F}_q$	Finite field of order q
$F/_{\!*}R_a$	Hyperfield associated to planar ternary ring (F, T)
$F_{\|U}$	Restriction of partial hyperfield F to subgroup U
$K \vee a$	Join of the flat K and the flat generated by a
$\mathcal{H}(M)$, $\mathcal{H}$	Set of hyperplanes of matroid M
$[H_1, H_2 \,\|\, H_3, H_4]$	Cross-ratio of H_1, H_2, H_3, H_4
$\mathbb{K}$	Krasner hyperfield
$K_1 \vee K_2$	Join of the flats K_1 and K_2
$\mathcal{L}(M)$, $\mathcal{L}$	Set of hyperlines of the matroid M
$M \backslash F$	Restriction of M onto $E \setminus F$
$M \| F$	Restriction of M onto F
M/F	Contraction of M onto $E \setminus F$
$\mathrm{PG}(d, F)$	Projective geometry of dimension d over F
$\bigoplus_{i=1}^{k} M_i$	Direct sum the matroids M_i, $i = 1, \ldots, k$
$\coprod_{i \in I} F_i$	Coproduct of partial hyperfields F_i, $i \in I$
$Q(F)$	Canonical real reduced hyperneofield of F w.r.t. $\sum F^2$

$Q_T(F)$	Canonical real reduced hyperneofield of F w.r.t. T
R_9	Ternary Reid geometry
R_a	Extended radical of planar ternary ring (F,T)
$\varrho_M(A),\ \varrho(A)$	rank of the set A as subset of the matroid M
$\varrho(M)$	Rank of the matroid M
$\mathbb{S}$	Hyperfield of signs
$\sigma_M(A),\ \sigma(A)$	Closure of the set A as subset of the matroid M
$\mathrm{s}M$	Simplification of matroid M
$\mathbb{T}^{(0)}(M)$	Inner tutte group of matroid M
$\mathbb{U}_0$	Regular partial field
$\mathbb{U}_{\mathcal{A}}$	$\mathcal{A}$-regular partial field
$\mathbb{U}_\kappa$	κ-regular partial field
$\mathbb{U}^{(0)}(M)$	The universal partial hyperfield of a matroid M
$\mathbb{U}^{\mathcal{B}}(M)$	Extension of $\mathbb{U}^{(0)}(M)$ to the bases Tutte group
$\mathbb{U}^{\mathcal{H}}(M)$	Extension of $\mathbb{U}^{(0)}(M)$ to the extended Tutte group
$U_{n,E}$	Uniform matroid of rank n on the set E
$U_{n,k}$	Uniform matroid of rank k on the set $\{1,\ldots,k\}$

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